Asymptotic Proximity Between Equations with Mean Curvature Operator and Linear Equation

Zuzana Došlá

Department of Mathematics and Statistics, Masaryk University, Brno, Czech Republic
E-mail: dosla@math.muni.cz

Mauro Marini, Serena Matucci

Department of Mathematics and Informatics "U. Dini", University of Florence, Italy
E-mails: mauro.marini@unifi.it; serena.matucci@unifi.it

1 Introduction

Consider the nonlinear equations

$$\big(a(t)\Phi_E(x')\big)' + b(t)F(x) = 0, \ \ t \in I = [1,\infty), \tag{1.1}$$

and

$$\big(a(t)\Phi_R(x')\big)' + b(t)F(x) = 0, \ \ t \in I = [1,\infty), \tag{1.2}$$

where the functions a and b are continuous and positive on $[1,\infty)$, the function F is continuous on $\mathbb{R}$ with $F(u)u > 0$ for $u \neq 0$, and the functions $\Phi_E : \mathbb{R} \to (-1,1)$ and $\Phi_R : (-1,1) \to \mathbb{R}$ are defined as

$$\Phi_E(u) = \frac{u}{\sqrt{1+u^2}}\,, \quad \Phi_R(u) = \frac{u}{\sqrt{1-u^2}}\,.$$

The operator Φ_E is called the *Euclidean mean curvature operator*. It arises in the search for radial solutions to partial differential equations which model fluid mechanics problems, in particular capillarity-type phenomena for compressible and incompressible fluids. The operator Φ_R is called the *Minkowski mean curvature operator* or, sometimes, the relativity operator. It originates from studying certain extrinsic properties of the mean curvature of hypersurfaces in the relativity theory, see e.g., [1,2] and the references therein.

The operators Φ_E and Φ_R are strictly related: the inverse of Φ_E is Φ_R and vice-versa. This fact plays an important role in the study of equations (1.1), (1.2), as we show below.

Here we consider the problem associated with (1.1) and (1.2) to find necessary and sufficient conditions for the existence of solutions such that

$$\lim_{t\to\infty} x(t) = \infty, \quad \lim_{t\to\infty} a(t)x'(t) = 0. \tag{1.3}$$

Observe that sometimes such solutions are called *intermediate solutions*, see, e.g., [3]. Other boundary value problems concerning Kneser-type boundary value problems for (1.1), or (1.2), are in [7]. More details on Kneser boundary value problems can be found in [11, Sections 13.1, 13.2 and 16.1].

Denote by J_a, J_b, J_1 the following integrals

$$J_a = \int\limits_1^\infty \frac{1}{a(t)}\,dt, \quad J_b = \int\limits_1^\infty b(t)\,dt, \quad J_{ab} = \int\limits_1^\infty b(t)\bigg(\int\limits_1^t \frac{1}{a(s)}\,ds\bigg)\,dt.$$

If the nonlinearity F is odd and satisfies the conditions

$$\liminf_{u\to\infty} \frac{F(u)}{u} > 0, \quad \limsup_{u\to\infty} \frac{F(u)}{u} < \infty, \tag{1.4}$$

that, is, roughly speaking, F has a linear growth near infinity, we show that equations (1.1) and (1.2) are closely related with the linear equation

$$\big(a(t)x'\big)' + b(t)x = 0. \tag{1.5}$$

Indeed, the well-known Leighton criterion states that (1.5) is oscillatory if $J_a = J_b = \infty$, see, e.g., [6] or [12, Theorem 2.24]. This oscillation result is valid also for equations with the curvature operator, see below. Further, the qualitative similarity between equations with the curvature operator and the linear case continues to hold also when (1.5) is nonoscillatory. More precisely, concerning the intermediate solutions in the linear case, the following holds, see, e.g., [5, Theorems 1 and 2].

Theorem 1.1. *Assume that $J_a = \infty, J_b < \infty$. If the linear equation* (1.5) *is nonoscillatory, then* (1.5) *has eventually positive solutions x satisfying* (1.3) *if and only if $J_{ab} = \infty$.*

In the following we illustrate how Theorem 1.1 continues to hold for equations (1.1) and (1.2).

2 Main results

We start by considering equation (1.1). The following oscillation result can be viewed as an extension of the quoted Leighton criterion.

Theorem 2.1. *Let $J_a = \infty$, $J_b = \infty$ and $\liminf_{u\to\infty} F(u) > 0$. Then any continuable solution at infinity of equation* (1.1) *is oscillatory.*

Theorem 2.1 is proved in [3, Theorem 2.1 (ii)], see also [8, Theorem 4.1], by using a different argument to the one in [6] or [12, Theorem 2.24] for linear equation.

The next result concerns the asymptotic proximity between the intermediate solutions to equations (1.1) and (1.5). The following holds.

Theorem 2.2. *Let $J_a = \infty$, $\liminf_{t\to\infty} a(t) > 0$, conditions* (1.4) *hold and $F_M = \sup_{u\geq 1} F(u)/u$.*

If the linear equation

$$\Big(\frac{\sqrt{3}}{2}\, a(t)w'\Big)' + F_M\, b(t)w = 0 \tag{2.1}$$

is nonoscillatory, then equation (1.1) *has infinitely many solutions x satisfying* (1.3) *if and only if*

$$J_b < \infty, \ \ J_{ab} = \infty. \tag{2.2}$$

Theorem 2.2 follows from [8, Theorem 3.1, Theorem 4.2]. Observe that Theorem 2.2 requires the existence of a suitable nonoscillatory linear equation (2.1) which, roughly speaking, can be viewed with respect to (1.1), as a dominant equation. This assumption can be verified by comparing (2.1) with known linear auxiliary equations such as, for instance, the Euler equation or the Riemann–Weber equation. More precisely, consider the Euler equation $w'' + 4^{-1}t^{-2}w = 0$. Using the substitution $z(t) = t^{-\lambda}w$ we get that the linear equation

$$(c(t)z')' + d(t)z = 0, \tag{2.3}$$

where $c(t) = t^{2\lambda}$, $d(t) = (\lambda - 2^{-1})^2 t^{2(\lambda-1)}$, is nonoscillatory. If $\lambda < 2^{-1}$, then $J_c = \infty$, $J_d < \infty$ and $J_{cd} = \infty$. Hence, from Theorem 2.2 we have the following, see [8, Corollary 5.1].

Corollary 2.1. *Let* (1.4) *be verified. Assume that there exists* $\lambda \in (0, 2^{-1})$ *such that for large* t

$$a(t) \geq \frac{2}{\sqrt{3}}\, t^{2\lambda}, \quad b(t) \leq \frac{(\lambda - 2^{-1})^2}{F_M}\, t^{2(\lambda-1)},$$

where F_M *is given in Theorem 2.2. Then equation* (1.1) *has a solution* x *satisfying* (1.3).

Clearly, any other nonoscillatory linear equation of type (2.3) satisfying $J_c = \infty$, $J_d < \infty$ and $J_{cd} = \infty$ can be used as majorant equation.

Now, we study the qualitative similarity between (1.2) and (1.5). The oscillation for (1.2) is a more subtle problem, see, e.g., [3]. The following holds.

Theorem 2.3. *Let* $J_b = \infty$, $\liminf_{u\to\infty} F(u) > 0$ *and for any* $\lambda > 0$

$$\int_1^\infty \Phi_E\Big(\frac{\lambda}{a(t)}\Big)\, dt = \infty.$$

Then any continuable solution at infinity of equation (1.2) *is oscillatory.*

Theorem 2.3 follows, with minor changes, from a more general result stated in [3, Theorem 2.1]. Concerning the existence of intermediate solutions to (1.2), the following holds.

Theorem 2.4. *Let* $J_a = \infty$, $J_b < \infty$, $J_{ab} = \infty$, $\liminf\limits_{t\to\infty} a(t) > 0$, *conditions* (1.4) *hold and* $F_M = \sup\limits_{u\geq 1} F(u)/u$. *If* (2.2) *holds and the linear equation*

$$\big(a(t)w'\big)' + F_M\, b(t)w = 0 \tag{2.4}$$

is nonoscillatory, then equation (1.2) *has infinitely many solutions* x *satisfying* (1.3).

Theorem 2.4 is proved in [8, Theorem 5.1]. Moreover, in [8, Section 5] some necessary conditions for existence of intermediate solutions to (1.2) are given too.

3 Concluding remarks

We start by presenting the idea of the proof of Theorem 2.2. It is based on an important feature on the operator Φ_E and its inverse Φ_R. Setting

$$w = x, \quad z = a(t)\Phi_E(x'),$$

an easy calculation shows that the problem (1.1), (1.3) is equivalent to the problem

$$\begin{cases} w' = \Phi_R\Big(\dfrac{z}{a(t)}\Big) = \dfrac{z}{\sqrt{a^2(t) - z^2}}\,, \quad z' = -b(t)F(w), \quad t \in I, \\[2ex] \lim\limits_{t\to\infty} w(t) = \infty, \quad \lim\limits_{t\to\infty} \Phi_R\Big(\dfrac{z(t)}{a(t)}\Big) = 0. \end{cases} \tag{3.1}$$

For solving (3.1), we use a fixed point result, which originates from [4, Theorem 1.4], jointly with some asymptotic properties of the principal solution of a linear equation, see, e.g., [10, Chapter 11, Section 6]. We briefly describe our approach.

Let Ω be a nonempty, closed, convex and bounded subset of $C([1,\infty),\mathbb{R}^2)$ and for any $(u,v)\in\Omega$ consider the linear boundary value problem

$$\begin{cases} \xi' = \dfrac{\eta}{\sqrt{a^2(t)-v^2(t)}}\,, \quad \eta' = -b(t)\,\dfrac{F(u)}{u(t)}\,\xi, \quad t\in I, \\ \lim\limits_{t\to\infty}\xi(t)=\infty, \quad \lim\limits_{t\to\infty}\eta(t)=0. \end{cases} \tag{3.2}$$

For any $(u,v)\in\Omega$ denote by (ξ_{uv},η_{uv}) the principal solution of the linear system in (3.2) such that $\eta_{uv}(1)=k_a$, where k_a is a suitable positive fixed constant. Let T be the operator which maps (u,v) into (ξ_{uv},η_{uv}). Defining in an appropriate way the set Ω and using some comparison results on the behavior of the principal solution, it is easy to show that T has a fixed point $(\widehat{\xi},\widehat{\eta})$, which clearly is a solution of (3.1).

Observe that the linear system in (3.2) is equivalent to the second order linear equation

$$\Big(\sqrt{a^2(t)-v^2(t)}\,y'\Big)' + b(t)\,\frac{F(u)}{u(t)}y = 0 \tag{3.3}$$

and so the principal solution of the linear system in (3.2) coincides with the principal solution y_0 of (3.3). Thus, roughly speaking, this approach reduces the solvability of (3.1) to the solvability of a boundary value problem for a suitable associated second order linear equation. Clearly, a similar approach, with minor changes, is valid for proving the existence of intermediate solutions to (1.2).

Using the disconjugacy theory and some comparison results for principal solutions of linear equations, we can extend Theorems 2.2 and 2.4 by obtaining the so-called global positiveness of intermediate solutions, that is their positiveness on the whole interval I. Observe that, in general, this fact does not occur, because nonoscillatory solutions can have an arbitrary finite number of zeros, also in the linear case. This result is a consequence of a more general criterion in the forthcoming paper [9] and reads as follows.

Theorem 3.1. *Let the assumptions of Theorem 2.2 [Theorem 2.4] be valid. In addition, if the linear equation* (2.1) [(2.4)] *has the principal solution which is positive on I, then* (1.1) [(1.2)] *has infinitely many solutions x which are positive nondecreasing on I and satisfy* (1.3).

References

[1] C. Bereanu, P. Jebelean and J. Mawhin, Radial solutions for some nonlinear problems involving mean curvature operators in Euclidean and Minkowski spaces. *Proc. Amer. Math. Soc.* **137** (2009), no. 1, 161–169.

[2] A. Boscaggin and G. Feltrin, Positive periodic solutions to an indefinite Minkowski-curvature equation. *J. Differential Equations* **269** (2020), no. 7, 5595–5645.

[3] M. Cecchi, Z. Došlá and M. Marini, Oscillation of a class of differential equations with generalized phi-Laplacian. *Proc. Roy. Soc. Edinburgh Sect. A* **143** (2013), no. 3, 493–506.

[4] M. Cecchi, M. Furi and M. Marini, On continuity and compactness of some nonlinear operators associated with differential equations in noncompact intervals. *Nonlinear Anal.* **9** (1985), no. 2, 171–180.

[5] M. Cecchi, M. Marini and G. Villari, Integral criteria for a classification of solutions of linear differential equations. *J. Differential Equations* **99** (1992), no. 2, 381–397.

[6] W. J. Coles, A simple proof of a well-known oscillation theorem. *Proc. Amer. Math. Soc.* **19** (1968), 507.

[7] Z. Došlá, M. Marini and S. Matucci, Positive decaying solutions to BVPs with mean curvature operator. *Rend. Istit. Mat. Univ. Trieste* **49** (2017), 147–164.

[8] Z. Došlá, M. Marini and S. Matucci, On unbounded solutions for differential equations with mean curvature operator. *Czech. Math. J.*, 2023, 20 pp.; DOI: 10.21136/CMJ.2023.0111-23.

[9] Z. Došlá, M. Marini and S. Matucci, On some properties of the mean curvature equations. (*in preparation*).

[10] Ph. Hartman, *Ordinary Differential Equations*. Reprint of the second edition. Birkhäuser, Boston, MA, 1982.

[11] I. T. Kiguradze and T. A. Chanturia, *Asymptotic Properties of Solutions of Nonautonomous Ordinary Differential Equations*. Mathematics and its Applications (Soviet Series), 89. Kluwer Academic Publishers Group, Dordrecht, 1993.

[12] C. A. Swanson, *Comparison and Oscillation Theory of Linear Differential Equations*. Mathematics in Science and Engineering, Vol. 48. Academic Press, New York–London, 1968.