



# THE HADWIGER THEOREM ON CONVEX FUNCTIONS, I

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**Abstract.** A complete classification of all continuous, epi-translation and rotation invariant valuations on the space of super-coercive convex functions on  $\mathbb{R}^n$  is established. The valuations obtained are functional versions of the classical intrinsic volumes. For their definition, singular Hessian valuations are introduced.

## 1 Introduction and statement of results

Intrinsic volumes play a fundamental role in Euclidean geometry. For a convex body  $K$  (that is, a non-empty, compact convex set) in  $n$ -dimensional Euclidean space,  $\mathbb{R}^n$ , and  $j \in \{0, \dots, n\}$ , the  $j$ th intrinsic volume,  $V_j(K)$ , measures, in a translation and rotation invariant way, the  $j$ -dimensional “size” of  $K$ . In particular,  $V_n(K)$  is the  $n$ -dimensional volume of  $K$  and  $V_0(K) = 1$  is the Euler characteristic. For  $1 \leq j \leq n - 1$ , the  $j$ th intrinsic volume can be defined in various ways, for example, via the Steiner formula, via integral geometric formulas, or, for a convex body  $K$  with smooth boundary, as

$$V_j(K) = \frac{1}{\omega_{n-j}} \int_{\mathbb{S}^{n-1}} [D^2 h_K(y)]_j d\mathcal{H}^{n-1}(y),$$

where  $h_K$  is the support function of  $K$  (see Sect. 2.1 for the definition) and  $D^2 h_K$  the Hessian matrix of  $h_K$ . Here, we write  $[A]_j$  for the  $j$ th elementary symmetric function of the eigenvalues of a symmetric matrix  $A$  and use the convention that  $[A]_0 = 1$ . We write  $\mathcal{H}^{n-1}$  for  $(n - 1)$ -dimensional Hausdorff measure and  $\omega_k$  for the  $(k - 1)$ -dimensional Hausdorff measure of the unit sphere,  $\mathbb{S}^{k-1}$ , in  $\mathbb{R}^k$ . The  $j$ th intrinsic volume is homogeneous of degree  $j$ , that is,  $V_j(\lambda K) = \lambda^j V_j(K)$  for all convex bodies  $K$  in  $\mathbb{R}^n$  and all  $\lambda \geq 0$ . The intrinsic volumes are continuous functionals on the space of convex bodies,  $\mathcal{K}^n$ , in  $\mathbb{R}^n$  equipped with the standard topology induced by the Hausdorff metric (see [Sch14] for information regarding intrinsic volumes).

One of the most important results in geometry is Hadwiger’s characterization theorem [Had57] for intrinsic volumes. Hadwiger’s theorem classifies all continuous, translation and rotation invariant valuations on convex bodies. Here a functional  $Z: \mathcal{K}^n \rightarrow \mathbb{R}$  is called a valuation if

$$Z(K) + Z(L) = Z(K \cup L) + Z(K \cap L) \tag{1.1}$$

for every  $K, L \in \mathcal{K}^n$  such that  $K \cup L \in \mathcal{K}^n$ . Recall that  $Z$  is translation invariant if  $Z(\tau K) = Z(K)$  for all  $K \in \mathcal{K}^n$  and translations  $\tau$  on  $\mathbb{R}^n$ , and it is rotation invariant if  $Z(\vartheta K) = Z(K)$  for all  $K \in \mathcal{K}^n$  and  $\vartheta \in \text{SO}(n)$ .

**Theorem 1.1** (Hadwiger [Had57]). *A functional  $Z: \mathcal{K}^n \rightarrow \mathbb{R}$  is a continuous, translation and rotation invariant valuation if and only if there are constants  $\zeta_0, \dots, \zeta_n \in \mathbb{R}$  such that*

$$Z(K) = \zeta_0 V_0(K) + \dots + \zeta_n V_n(K)$$

for every  $K \in \mathcal{K}^n$ .

Hadwiger's theorem leads to effortless proofs of numerous results in integral geometry and geometric probability (see [Had57] and [KR97]). It is the first culmination of a program initiated by Blaschke and the starting point of the geometric theory of valuations, where classification theorems for valuations invariant under various groups are fundamental (see [Sch14, Chap. 6]). Interesting new valuations keep arising (see, for example, [B+20, H+16, Lud03]) and see [Ale99, Ale01, AF14, BF17, BF11, Hab12, HP14, LM17, Lud10, LR99, LR10, PW13] for some recent classification results.

Currently, a geometric theory of valuations on function spaces is developed. A functional  $Z$  defined on a space of (extended) real-valued functions,  $X$ , is called a *valuation* if

$$Z(u) + Z(v) = Z(u \vee v) + Z(u \wedge v)$$

for all  $u, v \in X$  such that the pointwise maximum  $u \vee v$  and the pointwise minimum  $u \wedge v$  belong to  $X$ . When  $X$  is the space of indicator functions of convex bodies in  $\mathbb{R}^n$ , we recover the classical notion of valuation on convex bodies. The first classification results were obtained for  $L_p$  and Sobolev spaces, for Lipschitz functions and for definable functions (see, for example, [BGW13, C+20, LM17, Lud11, Lud12, Tsa10]).

Spaces of convex functions play a special role because of their close connection to convex bodies. Here classification results were obtained for  $\text{SL}(n)$  invariant valuations in [CLM17, CLM19, Mus19, Mus21], the connection to valuations on convex bodies was explored in [Ale19] and first structural results were established in [CLM201, Kno21, Kno24]. The general space of extended real-valued convex functions on  $\mathbb{R}^n$  is defined as

$$\text{Conv}(\mathbb{R}^n)$$

$$:= \{u: \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}: u \text{ is convex and lower semicontinuous, } u \not\equiv +\infty\}.$$

It is equipped with the topology induced by epi-convergence (see Sect. 2.2), and the continuity of valuations defined on  $\text{Conv}(\mathbb{R}^n)$  (and its subsets) will always be with respect to this topology.

The aim of this paper is to establish the Hadwiger theorem on convex functions. A valuation  $Z$  defined on (a subset of) the space  $\text{Conv}(\mathbb{R}^n)$  is said to be *epi-translation*

*invariant* if  $Z(u \circ \tau^{-1} + \alpha) = Z(u)$  for every translation  $\tau$  on  $\mathbb{R}^n$  and  $\alpha \in \mathbb{R}$ . It is said to be *rotation invariant* if  $Z(u \circ \vartheta^{-1}) = Z(u)$  for every  $\vartheta \in \text{SO}(n)$ . We will identify a convex function with its epi-graph,

$$\text{epi}(u) := \{(x, t) \in \mathbb{R}^{n+1} : t \geq u(x)\}.$$

Note that an epi-translation invariant valuation on  $\text{Conv}(\mathbb{R}^n)$  corresponds to a translation invariant valuation on the set of epi-graphs in the sense of (1.1). We say that  $Z$  is *epi-homogeneous* of degree  $j$ , if the associated valuation on epi-graphs is homogeneous of degree  $j$ . It was shown in [CLM201] that a continuous and epi-translation invariant valuation on  $\text{Conv}(\mathbb{R}^n)$  is always constant, and the corresponding statement was proved on the space of coercive convex functions. Thus, we look at valuations on the smaller space of *super-coercive convex functions*,

$$\text{Conv}_{\text{sc}}(\mathbb{R}^n) := \left\{ u \in \text{Conv}(\mathbb{R}^n) : \lim_{|x| \rightarrow +\infty} \frac{u(x)}{|x|} = +\infty \right\},$$

where  $|\cdot|$  is the Euclidean norm. For  $n = 1$ , a complete classification of continuous and epi-translation invariant valuations was obtained in [CLM201]. So, let  $n \geq 2$ . For  $\zeta \in C_c((0, \infty))$  and  $j \in \{0, \dots, n\}$ , it follows from results in [CLM202] that a valuation, defined for  $u \in \text{Conv}_{\text{sc}}(\mathbb{R}^n) \cap C_+^2(\mathbb{R}^n)$  (that is, with a positive definite Hessian matrix  $D^2u$ ) by

$$V_{j,\zeta}(u) := \int_{\mathbb{R}^n} \zeta(|\nabla u(x)|) [D^2u(x)]_{n-j} dx, \quad (1.2)$$

can be extended to a continuous, epi-translation and rotation invariant valuation that is epi-homogeneous of degree  $j$  on  $\text{Conv}_{\text{sc}}(\mathbb{R}^n)$ . The valuation defined in (1.2) is a *Hessian valuation*, and the extension of the integral (1.2) uses the notion of Hessian measures. These measures play an important role in the study of a class of fully non-linear elliptic PDEs, the so-called Hessian equations, and in this context were introduced by Trudinger and Wang [TW97, TW99]. In the case of convex functions, these measures have been studied, for example, in [CH00] (see Sect. 2.3).

As we will show, the functionals  $V_{j,\zeta}$  also extend to continuous valuations on  $\text{Conv}_{\text{sc}}(\mathbb{R}^n)$  for functions  $\zeta \in C((0, \infty))$  with a certain type of singularity at the origin. We thus obtain *singular Hessian valuations*. Let  $C_b((0, \infty))$  be the set of continuous functions on  $(0, \infty)$  with bounded support. For  $0 \leq j \leq n - 1$ , let

$$D_j^n := \left\{ \zeta \in C_b((0, \infty)) : \lim_{s \rightarrow 0^+} s^{n-j} \zeta(s) = 0, \lim_{s \rightarrow 0^+} \int_s^\infty t^{n-j-1} \zeta(t) dt \text{ exists and is finite} \right\}.$$

In addition, let  $D_n^n$  be the set of functions  $\zeta \in C_b((0, \infty))$  where  $\lim_{s \rightarrow 0^+} \zeta(s)$  exists and is finite and set  $\zeta(0) := \lim_{s \rightarrow 0^+} \zeta(s)$ . Note that  $V_{0,\zeta}$  is constant on  $\text{Conv}_{\text{sc}}(\mathbb{R}^n)$  for  $\zeta \in D_0^n$  (see Sect. 2.4).

Our first main result establishes the existence of the extension of singular Hessian valuations for functions  $\zeta \in D_j^n$ .

**Theorem 1.2.** *For  $j \in \{0, \dots, n\}$  and  $\zeta \in D_j^n$ , there exists a unique, continuous, epi-translation and rotation invariant valuation  $V_{j,\zeta}: \text{Conv}_{\text{sc}}(\mathbb{R}^n) \rightarrow \mathbb{R}$  such that*

$$V_{j,\zeta}(u) = \int_{\mathbb{R}^n} \zeta(|\nabla u(x)|) [D^2 u(x)]_{n-j} dx$$

for every  $u \in \text{Conv}_{\text{sc}}(\mathbb{R}^n) \cap C_+^2(\mathbb{R}^n)$ .

The proof of this result will use analytic tools developed within the theory of Hessian equations (see Sect. 3.2 for details). Given a singular function  $\zeta$ , the integral (1.2) converges absolutely for sufficiently regular functions. For the general extension argument, Hessian valuations, which were introduced in [CLM202], will be used, but singular functions are, in general, not integrable with respect to Hessian measures. For the extension, we will use the Moreau–Yosida approximation and the existence of a homogeneous decomposition for epi-translation invariant valuations that was established in [CLM201].

Our second main result shows that the singular Hessian valuations that were introduced above span the space of continuous, epi-translation and rotation invariant valuations on  $\text{Conv}_{\text{sc}}(\mathbb{R}^n)$ . Let  $n \geq 2$ .

**Theorem 1.3.** *A functional  $Z: \text{Conv}_{\text{sc}}(\mathbb{R}^n) \rightarrow \mathbb{R}$  is a continuous, epi-translation and rotation invariant valuation if and only if there exist functions  $\zeta_0 \in D_0^n, \dots, \zeta_n \in D_n^n$  such that*

$$Z(u) = V_{0,\zeta_0}(u) + \dots + V_{n,\zeta_n}(u)$$

for every  $u \in \text{Conv}_{\text{sc}}(\mathbb{R}^n)$ .

The functions  $\zeta_j$  are uniquely determined by  $Z$  for  $j \in \{1, \dots, n\}$  (see Lemma 3.23 and Lemma 4.9) whereas  $V_{0,\zeta_0}$  is constant on  $\text{Conv}_{\text{sc}}(\mathbb{R}^n)$  and does not uniquely determine  $\zeta_0$ . Theorem 1.2 and Theorem 1.3 show that the singular Hessian valuations  $V_{j,\zeta}$  with  $\zeta \in D_j^n$  are functional versions of the intrinsic volumes  $V_j$  and that, from the point of view of geometric valuation theory, they are the canonical functional versions of intrinsic volumes.

The space of super-coercive convex functions is related to another subspace of  $\text{Conv}(\mathbb{R}^n)$ , formed by convex functions with finite values,

$$\text{Conv}(\mathbb{R}^n; \mathbb{R}) := \left\{ v \in \text{Conv}(\mathbb{R}^n) : v(x) < +\infty \text{ for all } x \in \mathbb{R}^n \right\}.$$

A function  $v$  belongs to  $\text{Conv}(\mathbb{R}^n; \mathbb{R})$  if and only if its standard conjugate or Legendre transform  $v^*$  belongs to  $\text{Conv}_{\text{sc}}(\mathbb{R}^n)$  (see Sect. 2.2). It was proved in [CLM202] that  $Z$  is a continuous valuation on  $\text{Conv}(\mathbb{R}^n; \mathbb{R})$  if and only if  $Z^*: \text{Conv}_{\text{sc}}(\mathbb{R}^n) \rightarrow \mathbb{R}$ , defined by

$$Z^*(u) := Z(u^*),$$

is a continuous valuation on  $\text{Conv}_{\text{sc}}(\mathbb{R}^n)$ . This fact permits us to transfer results valid for valuations on  $\text{Conv}(\mathbb{R}^n; \mathbb{R})$  to results for valuations on  $\text{Conv}_{\text{sc}}(\mathbb{R}^n)$  and vice versa. A valuation  $Z$  on  $\text{Conv}(\mathbb{R}^n; \mathbb{R})$  is called *homogeneous* of degree  $j \in \mathbb{R}$  if  $Z(\lambda v) = \lambda^j Z(v)$  for all  $v \in \text{Conv}(\mathbb{R}^n; \mathbb{R})$  and  $\lambda \geq 0$ . It is *dually epi-translation invariant* if

$$Z(v + \ell + \alpha) = Z(v)$$

for every  $v \in \text{Conv}(\mathbb{R}^n; \mathbb{R})$  and every linear function  $\ell: \mathbb{R}^n \rightarrow \mathbb{R}$  and  $\alpha \in \mathbb{R}$ . The valuation  $Z$  is dually epi-translation invariant and homogeneous of degree  $j$  on  $\text{Conv}(\mathbb{R}^n; \mathbb{R})$  if and only if  $Z^*$  is epi-translation invariant and epi-homogeneous of degree  $j$  on  $\text{Conv}_{\text{sc}}(\mathbb{R}^n)$ . Using results on Hessian valuations for conjugate functions, we will obtain the following extension theorem from Theorem 1.2.

**Theorem 1.4.** *For  $j \in \{0, \dots, n\}$  and  $\zeta \in D_j^n$ , the functional  $V_{j, \zeta}^*: \text{Conv}(\mathbb{R}^n; \mathbb{R}) \rightarrow \mathbb{R}$  is a continuous, dually epi-translation and rotation invariant valuation such that*

$$V_{j, \zeta}^*(v) = \int_{\mathbb{R}^n} \zeta(|x|) [D^2 v(x)]_j dx \quad (1.3)$$

for every  $v \in \text{Conv}(\mathbb{R}^n; \mathbb{R}) \cap C_+^2(\mathbb{R}^n)$ .

Theorem 1.3 has the following dual version. Let  $n \geq 2$ .

**Theorem 1.5.** *A functional  $Z: \text{Conv}(\mathbb{R}^n; \mathbb{R}) \rightarrow \mathbb{R}$  is a continuous, dually epi-translation and rotation invariant valuation if and only if there exist functions  $\zeta_0 \in D_0^n, \dots, \zeta_n \in D_n^n$  such that*

$$Z(v) = V_{0, \zeta_0}^*(v) + \dots + V_{n, \zeta_n}^*(v)$$

for every  $v \in \text{Conv}(\mathbb{R}^n; \mathbb{R})$ .

In the proof of Theorem 1.3 and Theorem 1.5, we will use both the primal setting of epi-translation invariant valuations on  $\text{Conv}_{\text{sc}}(\mathbb{R}^n)$  and the dual setting of dually epi-translation invariant valuations on  $\text{Conv}(\mathbb{R}^n; \mathbb{R})$ . We follow the original approach of Hadwiger [Had57]. We introduce rotational epi-symmetrization to establish a classification of valuations that are epi-homogeneous of degree 1 on  $\text{Conv}_{\text{sc}}(\mathbb{R}^n)$  and induction on the dimension for the general case. The homogeneous decomposition of epi-translation invariant valuations established in [CLM201] will be an important tool in the proof.

An alternate proof of Hadwiger's theorem is due to Klain [Kla95]. We will discuss elements of Klain's proof and how to apply Theorem 1.3 and Theorem 1.5 in integral geometry in subsequent papers (see [CLM22, CLM23, CLM]).

## 2 Preliminaries

We work in  $n$ -dimensional Euclidean space  $\mathbb{R}^n$ , with  $n \geq 1$ , endowed with the Euclidean norm  $|\cdot|$  and the usual scalar product  $\langle \cdot, \cdot \rangle$ . We write  $e_1, \dots, e_n$  for the

canonical basis vectors of  $\mathbb{R}^n$  and use coordinates,  $x = (x_1, \dots, x_n)$ , for  $x \in \mathbb{R}^n$  with respect to this basis. We write  $\text{span}$  for linear span. Let  $B^n := \{x \in \mathbb{R}^n : |x| \leq 1\}$  be the Euclidean unit ball and  $\mathbb{S}^{n-1}$  the unit sphere in  $\mathbb{R}^n$ . We denote by  $\kappa_n$  the  $n$ -dimensional volume of  $B^n$  and by  $\omega_n$  the  $(n-1)$ -dimensional Hausdorff measure of  $\mathbb{S}^{n-1}$ .

**2.1 Convex sets.** A basic reference on convex sets is the book by Schneider [Sch14]. Let  $K \subset \mathbb{R}^n$  be a closed convex set that is non-empty and such that  $K \neq \mathbb{R}^n$ . Its *support function*  $h_K : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  is defined as

$$h_K(y) := \sup_{x \in K} \langle x, y \rangle.$$

It is a one-homogeneous and convex function that determines  $K$ . If  $K \subset \mathbb{R}^n$  is a convex body with boundary of class  $C_+^2$  and  $y \in \mathbb{S}^{n-1}$ , then the eigenvalues of its Hessian matrix,  $D^2 h_K(y)$ , are the principal radii of curvature of  $K$  at  $y$ . Hence  $[D^2 h_K(y)]_i$  is the  $i$ th elementary symmetric function of the principal radii of curvature of  $K$  at  $y$  (see [Sch14, Sect. 2.5]).

**2.2 Convex functions.** We collect some properties of convex functions. Basic references are the books by Rockafellar [Roc97] and Rockafellar & Wets [RW98].

Let  $u \in \text{Conv}(\mathbb{R}^n)$ . For  $t \in (-\infty, +\infty]$ , its *sublevel sets*

$$\{u < t\} := \{x \in \mathbb{R}^n : u(x) < t\}, \quad \{u \leq t\} := \{x \in \mathbb{R}^n : u(x) \leq t\}$$

are convex, and the latter is closed by the lower semicontinuity of  $u$ . In addition, also its *domain*,

$$\text{dom}(u) := \{x \in \mathbb{R}^n : u(x) < +\infty\}$$

is a convex set. We equip  $\text{Conv}(\mathbb{R}^n)$  and its subspaces with the topology associated to epi-convergence. Here a sequence  $u_k$  of functions from  $\text{Conv}(\mathbb{R}^n)$  is *epi-convergent* to  $u \in \text{Conv}(\mathbb{R}^n)$  if the following conditions hold for all  $x \in \mathbb{R}^n$ :

- (i) For every sequence  $x_k$  that converges to  $x$ , we have  $u(x) \leq \liminf_{k \rightarrow \infty} u_k(x_k)$ .
- (ii) There exists a sequence  $x_k$  that converges to  $x$  such that  $u(x) = \lim_{k \rightarrow \infty} u_k(x_k)$ .

Note that a sequence  $v_k$  of functions from  $\text{Conv}(\mathbb{R}^n; \mathbb{R})$  is epi-convergent to a function  $v \in \text{Conv}(\mathbb{R}^n; \mathbb{R})$  if and only if  $v_k$  converges pointwise to  $v$  and uniformly on compact sets. We will just say that  $v_k$  converges to  $v$  in this case. We also require the following fact. If  $u_k$  is a sequence of functions from  $\text{Conv}(\mathbb{R}^n) \cap C^1(\mathbb{R}^n)$  that epi-converges to  $u \in \text{Conv}(\mathbb{R}^n) \cap C^1(\mathbb{R}^n)$ , then  $\nabla u_k$  converges uniformly to  $\nabla u$  on compact sets (see [Roc97, Theorem 25.7]).

Given  $u \in \text{Conv}(\mathbb{R}^n)$ , the *convex conjugate*  $u^* : \mathbb{R}^n \rightarrow (-\infty, \infty]$  is defined by

$$u^*(y) := \sup_{x \in \mathbb{R}^n} (\langle y, x \rangle - u(x)).$$

Since  $u \in \text{Conv}(\mathbb{R}^n)$ , we also have  $u^* \in \text{Conv}(\mathbb{R}^n)$  and  $u^{**} = u$ . Moreover, we have  $u \in \text{Conv}_{\text{sc}}(\mathbb{R}^n) \cap C_+^2(\mathbb{R}^n)$  if and only if  $u^* \in \text{Conv}_{\text{sc}}(\mathbb{R}^n) \cap C_+^2(\mathbb{R}^n)$ . Furthermore,  $\text{Conv}_{\text{sc}}(\mathbb{R}^n) \cap C_+^2(\mathbb{R}^n)$  is a dense subset of  $\text{Conv}(\mathbb{R}^n; \mathbb{R}) \cap C_+^2(\mathbb{R}^n)$ . Given a subset  $A \subset \mathbb{R}^n$ , let  $\mathbf{I}_A: \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  denote the (convex) indicatrix function of  $A$ ,

$$\mathbf{I}_A(x) := \begin{cases} 0 & \text{if } x \in A, \\ +\infty & \text{if } x \notin A. \end{cases}$$

Note that for a convex body  $K \subset \mathbb{R}^n$ , we have  $\mathbf{I}_K \in \text{Conv}_{\text{sc}}(\mathbb{R}^n)$ . Let  $t \geq 0$ . We will often use the following pair of dual functions,

$$u_t(x) := t|x| + \mathbf{I}_{B^n}(x) \tag{2.1}$$

and

$$v_t(x) := \begin{cases} 0 & \text{if } |x| \leq t, \\ |x| - t & \text{if } |x| > t. \end{cases} \tag{2.2}$$

Note that  $u_t \in \text{Conv}_{\text{sc}}(\mathbb{R}^n)$  and  $v_t \in \text{Conv}(\mathbb{R}^n; \mathbb{R})$  while  $u_t^* = v_t$ .

For  $u, v \in \text{Conv}(\mathbb{R}^n)$ , let

$$(u \square v)(x) := \inf_{x=y+z} (u(y) + v(z))$$

denote the *infimal convolution* of  $u$  and  $v$  at  $x \in \mathbb{R}^n$ . Note, that

$$\text{epi}(u \square v) = \text{epi } u + \text{epi } v,$$

where on the right, we have the Minkowski sum of the sets  $\text{epi } u$  and  $\text{epi } v$ , and if  $u \square v > -\infty$  pointwise, then

$$(u \square v)^* = u^* + v^*. \tag{2.3}$$

We define the *epi-multiplication* of  $u \in \text{Conv}(\mathbb{R}^n)$  and  $\lambda > 0$  by setting

$$\lambda \cdot u(x) := \lambda u\left(\frac{x}{\lambda}\right)$$

for  $x \in \mathbb{R}^n$ . Note that the epi-graph of  $\lambda \cdot u$  is obtained by rescaling the epi-graph of  $u$  by the factor  $\lambda$ . It is easy to see that  $u \in \text{Conv}_{\text{sc}}(\mathbb{R}^n)$  implies  $\lambda \cdot u \in \text{Conv}_{\text{sc}}(\mathbb{R}^n)$  for  $\lambda \geq 0$ . A functional  $Z: \text{Conv}_{\text{sc}}(\mathbb{R}^n) \rightarrow \mathbb{R}$  is called *epi-homogeneous* of degree  $j \in \mathbb{R}$  if

$$Z(\lambda \cdot u) = \lambda^j Z(u)$$

for all  $u \in \text{Conv}_{\text{sc}}(\mathbb{R}^n)$  and  $\lambda > 0$ . In the following, corresponding definitions will be used for  $\text{Conv}(\mathbb{R}^n)$  and its subspaces.

Let  $E$  and  $F$  be orthogonal and complementary subspaces of  $\mathbb{R}^n$  with  $\dim E, \dim F \geq 1$ . We can identify  $E$  with  $\mathbb{R}^k$  when  $k = \dim E$  and write  $\text{Conv}(E)$  for

$\text{Conv}(\mathbb{R}^k)$  in this case. We use corresponding definitions for subspaces of  $\text{Conv}(E)$ . For  $x \in \mathbb{R}^n$ , we write  $x = (x_E, x_F)$  with  $x_E \in E$  and  $x_F \in F$ . If  $u \in \text{Conv}_{\text{sc}}(\mathbb{R}^n)$  is such that  $\text{dom}(u) \subset E$ , then  $v = u^*$  does not depend on  $x_F$ . The restriction of  $v$  to  $E$ , which we denote by  $v_E \in \text{Conv}(E; \mathbb{R})$ , is equal to the Legendre transform (with respect to the ambient space  $E$ ) of the restriction of  $u$  to  $E$ , which we denote by  $u_E \in \text{Conv}_{\text{sc}}(E)$ . In this case, we will identify  $u$  with  $u_E$  and  $v$  with  $v_E$ . Hence, for  $u_E \in \text{Conv}_{\text{sc}}(E)$  and  $u_F \in \text{Conv}_{\text{sc}}(F)$ , we can use this identification and define  $u = u_E \square u_F \in \text{Conv}_{\text{sc}}(\mathbb{R}^n)$  where we use the usual infimal convolution on  $\mathbb{R}^n$ . Setting  $v_E = u_E^*$  and  $v_F = u_F^*$  and using (2.3), we obtain that

$$(u_E \square u_F)^* = v_E^* + v_F^*. \quad (2.4)$$

For  $u \in \text{Conv}(\mathbb{R}^n)$  and  $\lambda > 0$ , we define the *Moreau–Yosida envelope*,  $\text{env}_\lambda u$ , of  $u$  as

$$\text{env}_\lambda u := u \square \frac{|\cdot|^2}{2\lambda}.$$

If  $u \in \text{Conv}_{\text{sc}}(\mathbb{R}^n)$ , then  $\text{env}_\lambda u \in \text{Conv}_{\text{sc}}(\mathbb{R}^n) \cap C^1(\mathbb{R}^n)$ , and in particular,  $\text{dom}(\text{env}_\lambda u) = \mathbb{R}^n$  (see [RW98, Theorem 2.26]). If additionally,  $u \in C^2(\mathbb{R}^n)$ , then so is  $\text{env}_\lambda u$  (see [RW98, Theorem 7.37]). If  $u_k \in \text{Conv}_{\text{sc}}(\mathbb{R}^n)$  and  $u_k$  epi-converges to  $u$ , then  $\text{env}_\lambda u_k$  epi-converges to  $\text{env}_\lambda u$ .

**2.3 Hessian measures.** In this part, we briefly recall the notion of Hessian measures of convex functions. For a more detailed presentation, the reader is referred to [CH00, CLM202].

We begin with recalling the definition of the subdifferential of  $u \in \text{Conv}(\mathbb{R}^n)$ . For  $x \in \mathbb{R}^n$ , the subdifferential of  $u$  at  $x$  is defined by

$$\partial u(x) := \{y \in \mathbb{R}^n : u(z) \geq u(x) + \langle y, z - x \rangle \text{ for } z \in \mathbb{R}^n\}.$$

We set

$$\Gamma_u := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : y \in \partial u(x)\},$$

that is,  $\Gamma_u$  is the graph of the subdifferential map.

The Hessian measures of  $u$ , that we denote by  $\Theta_j^n(u, \cdot)$  for  $j = 0, \dots, n$ , are non-negative measures defined on the Borel subsets of  $\mathbb{R}^n \times \mathbb{R}^n$  that can be introduced in the following way. Given a Borel subset  $A$  of  $\mathbb{R}^n \times \mathbb{R}^n$  and  $s \geq 0$ , we consider the set,

$$P_s(u, A) := \{x + sy : (x, y) \in \Gamma_u \cap A\}.$$

Its  $n$ -dimensional Hausdorff measure is a polynomial in  $s$ ; in other words, there exist  $(n + 1)$  non-negative coefficients  $\Theta_0^n(u, A), \dots, \Theta_n^n(u, A)$ , such that

$$\mathcal{H}^n(P_s(u, A)) = \sum_{j=0}^n s^j \Theta_{n-j}^n(u, A)$$



for every Borel set  $A$  in  $\mathbb{R}^n \times \mathbb{R}^n$  and  $s \geq 0$ . The previous relation defines the Hessian measures of  $u$ . They are locally finite Borel measures on  $\mathbb{R}^n \times \mathbb{R}^n$ .

We will frequently use marginals of Hessian measures. Given  $j \in \{0, \dots, n\}$  and  $u \in \text{Conv}(\mathbb{R}^n)$ , we set

$$\Phi_j^n(u, B) := \Theta_{n-j}^n(u, B \times \mathbb{R}^n) \quad \text{for every Borel subset } B \subseteq \mathbb{R}^n,$$

and

$$\Psi_j^n(u, B) := \Theta_j^n(u, \mathbb{R}^n \times B) \quad \text{for every Borel subset } B \subseteq \mathbb{R}^n.$$

Note that

$$d\Phi_j^n(u, x) = [D^2u(x)]_j dx \quad (2.5)$$

for  $u \in \text{Conv}(\mathbb{R}^n) \cap C^2(\mathbb{R}^n)$ . If  $B$  is a Borel set in  $\mathbb{R}^n$  and  $u \in \text{Conv}(\mathbb{R}^n)$ , then

$$\Phi_0^n(u, B) = \mathcal{H}^n(B) \quad (2.6)$$

and

$$\Phi_j^n(\lambda u, B) = \lambda^j \Phi_j^n(u, B)$$

for  $j \in \{0, \dots, n\}$  and  $\lambda > 0$ .

Let  $u_k$  be a sequence in  $\text{Conv}(\mathbb{R}^n)$ . If  $u_k$  epi-converges to  $u \in \text{Conv}(\mathbb{R}^n)$ , then the sequence of measures  $\Theta_j^n(u_k, \cdot)$  converges weakly to the measure  $\Theta_j^n(u, \cdot)$  (see [CLM202, Theorem 7.3]). In particular, we obtain the following result for a function  $\zeta \in C_c(\mathbb{R}^n \times \mathbb{R}^n)$  with compact support in the second variable. If  $B$  is a bounded Borel set in  $\mathbb{R}^n$  and  $\Theta_j^n(u, \partial B) = 0$ , then

$$\lim_{k \rightarrow \infty} \int_{B \times \mathbb{R}^n} \zeta(x, y) d\Theta_j^n(u_k, (x, y)) = \int_{B \times \mathbb{R}^n} \zeta(x, y) d\Theta_j^n(u, (x, y)). \quad (2.7)$$

The interplay of Hessian measures and convex conjugation is well understood. Let  $u \in \text{Conv}(\mathbb{R}^n)$  and  $j \in \{0, \dots, n\}$ . It was shown in [CLM202, Theorem 8.2] that

$$\Theta_j^n(u, A) = \Theta_{n-j}^n(u^*, \hat{A})$$

for every Borel subset  $A$  of  $\mathbb{R}^n \times \mathbb{R}^n$  where  $\hat{A} = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : (y, x) \in A\}$ . As an immediate consequence, we obtain

$$\int_B \zeta(y) d\Psi_j^n(u, y) = \int_B \zeta(x) d\Phi_j^n(u^*, x) \quad (2.8)$$

for every  $u \in \text{Conv}_{\text{sc}}(\mathbb{R}^n)$  and Borel subset  $B \subseteq \mathbb{R}^n$ , when  $\zeta : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}$  is such that one of the two integrals above and therefore both exist. In particular,

$$\int_{\mathbb{R}^n} \zeta(\nabla u(x)) [D^2u(x)]_{n-j} dx = \int_{\mathbb{R}^n} \zeta(x) [D^2u^*(x)]_j dx \quad (2.9)$$

for every  $u \in \text{Conv}_{\text{sc}}(\mathbb{R}^n) \cap C_+^2(\mathbb{R}^n)$  and such  $\zeta$ .

**2.4 Valuations on convex functions.** The following homogeneous decomposition result was established in [CLM201, Theorem 1].

**Theorem 2.1.** *If  $Z : \text{Conv}_{\text{sc}}(\mathbb{R}^n) \rightarrow \mathbb{R}$  is a continuous and epi-translation invariant valuation, then there are continuous and epi-translation invariant valuations  $Z_i : \text{Conv}_{\text{sc}}(\mathbb{R}^n) \rightarrow \mathbb{R}$  that are epi-homogeneous of degree  $i$  such that  $Z = Z_0 + \dots + Z_n$ .*

We say that a valuation  $Z : \text{Conv}_{\text{sc}}(\mathbb{R}^n) \rightarrow \mathbb{R}$  is epi-additive if

$$Z(\alpha \cdot u \square \beta \cdot v) = \alpha Z(u) + \beta Z(v)$$

for every  $\alpha, \beta > 0$  and  $u, v \in \text{Conv}_{\text{sc}}(\mathbb{R}^n)$ . The following result is a consequence of Theorem 2.1.

**LEMMA 2.2** ([CLM201], Corollary 22). *If  $Z : \text{Conv}_{\text{sc}}(\mathbb{R}^n) \rightarrow \mathbb{R}$  is a continuous and epi-translation invariant valuation that is epi-homogeneous of degree 1, then  $Z$  is epi-additive.*

For  $\zeta \in D_0^n$  and  $u \in \text{Conv}_{\text{sc}}(\mathbb{R}^n) \cap C_+^2(\mathbb{R}^n)$ , the substitution  $y = \nabla u(x)$  shows that

$$V_{0,\zeta}(u) = \int_{\mathbb{R}^n} \zeta(|\nabla u(x)|) [D^2 u(x)]_n \, dx = \int_{\mathbb{R}^n} \zeta(|y|) \, dy$$

does not depend on  $u$ . Hence its extension to  $\text{Conv}_{\text{sc}}(\mathbb{R}^n)$  is constant. We have the following more general result.

**Theorem 2.3** ([CLM201], Theorem 25). *A functional  $Z : \text{Conv}_{\text{sc}}(\mathbb{R}^n) \rightarrow \mathbb{R}$  is a continuous and epi-translation invariant valuation that is epi-homogeneous of degree 0, if and only if  $Z$  is constant.*

For the other extremal degree, we have the following classification result.

**Theorem 2.4** ([CLM201], Theorem 2). *A functional  $Z : \text{Conv}_{\text{sc}}(\mathbb{R}^n) \rightarrow \mathbb{R}$  is a continuous and epi-translation invariant valuation that is epi-homogeneous of degree  $n$ , if and only if there exists  $\zeta \in C_c(\mathbb{R}^n)$  such that*

$$Z(u) = \int_{\text{dom}(u)} \zeta(\nabla u(x)) \, dx$$

for every  $u \in \text{Conv}_{\text{sc}}(\mathbb{R}^n)$ .

As a simple consequence, we obtain the following result. Here, we say that  $Z$  is reflection invariant if  $Z(u) = Z(u^-)$ , where  $u^-(x) := u(-x)$  for every  $x \in \mathbb{R}^n$ .

**COROLLARY 2.5.** *For  $n \geq 2$ , a functional  $Z : \text{Conv}_{\text{sc}}(\mathbb{R}^n) \rightarrow \mathbb{R}$  is a continuous, epi-translation and rotation invariant valuation that is epi-homogeneous of degree  $n$ , if and only if there exists  $\zeta \in C_c([0, \infty))$  such that*

$$Z(u) = \int_{\text{dom}(u)} \zeta(|\nabla u(x)|) \, dx$$

for every  $u \in \text{Conv}_{\text{sc}}(\mathbb{R}^n)$ . For  $n = 1$ , the same representation holds if we replace rotation invariance with reflection invariance.

*Proof.* Let  $Z$  be given. By Theorem 2.4 there exists  $\xi \in C_c(\mathbb{R}^n)$  such that

$$Z(u) = \int_{\text{dom}(u)} \xi(\nabla u(x)) \, dx$$

for every  $u \in \text{Conv}_{\text{sc}}(\mathbb{R}^n)$ . Fix  $y \in \mathbb{R}^n$  and let  $u(x) = \mathbf{I}_{B^n}(x) + \langle y, x \rangle$  for  $x \in \mathbb{R}^n$ . Note, that for  $\vartheta \in \text{SO}(n)$

$$u \circ \vartheta^{-1}(x) = \mathbf{I}_{B^n}(x) + \langle \vartheta^{-t}y, x \rangle.$$

Hence, using the  $\text{SO}(n)$  invariance of  $Z$ , we obtain

$$\kappa_n \xi(y) = Z(u) = Z(u \circ \vartheta^{-1}) = \kappa_n \xi(\vartheta^{-t}y)$$

for every  $\vartheta \in \text{SO}(n)$ . Since  $y \in \mathbb{R}^n$  was arbitrary, it follows that there exists a function  $\zeta \in C_c([0, \infty))$  such that  $\xi(y) = \zeta(|y|)$  for every  $y \in \mathbb{R}^n$ . In case  $n = 1$ , we simply need to choose  $\vartheta(x) = -x$  in the last argument.  $\square$

Combining this corollary with Theorem 2.3 gives the following result.

**COROLLARY 2.6.** *A functional  $Z : \text{Conv}_{\text{sc}}(\mathbb{R}) \rightarrow \mathbb{R}$  is a continuous, epi-translation and reflection invariant valuation, if and only if there exist  $\zeta_0 \in D_0^1$  and  $\zeta_1 \in D_1^1$  such that*

$$Z(u) = V_{0, \zeta_0}(u) + V_{1, \zeta_1}(u)$$

for every  $u \in \text{Conv}_{\text{sc}}(\mathbb{R}^n)$ .

We remark that this implies that Theorem 1.5 is also true in the one-dimensional case if we use the additional assumption of reflection invariance.

All of the previous results have dual versions. The following result is Theorem 4 from [CLM201].

**Theorem 2.7.** *If  $Z : \text{Conv}(\mathbb{R}^n; \mathbb{R}) \rightarrow \mathbb{R}$  is a continuous and dually epi-translation invariant valuation, then there are continuous and dually epi-translation invariant valuations  $Z_i : \text{Conv}(\mathbb{R}^n; \mathbb{R}) \rightarrow \mathbb{R}$  that are homogeneous of degree  $i$  such that  $Z = Z_0 + \dots + Z_n$ .*

**Theorem 2.8** ([CLM201], Theorem 25). *A functional  $Z : \text{Conv}(\mathbb{R}^n; \mathbb{R}) \rightarrow \mathbb{R}$  is a continuous and dually epi-translation invariant valuation that is homogeneous of degree 0, if and only if  $Z$  is constant.*

**Theorem 2.9** ([CLM201], Theorem 5). *A functional  $Z : \text{Conv}(\mathbb{R}^n; \mathbb{R}) \rightarrow \mathbb{R}$  is a continuous and dually epi-translation invariant valuation that is homogeneous of degree  $n$ , if and only if there exists  $\zeta \in C_c(\mathbb{R}^n)$  such that*

$$Z(v) = \int_{\mathbb{R}^n} \zeta(x) \, d\Phi_n^n(v, x)$$

for every  $v \in \text{Conv}(\mathbb{R}^n; \mathbb{R})$ .

**COROLLARY 2.10.** *For  $n \geq 2$ , a functional  $Z : \text{Conv}(\mathbb{R}^n; \mathbb{R}) \rightarrow \mathbb{R}$  is a continuous, dually epi-translation and rotation invariant valuation that is homogeneous of degree  $n$ , if and only if there is  $\zeta \in C_c([0, \infty))$  such that*

$$Z(v) = \int_{\mathbb{R}^n} \zeta(|x|) d\Phi_n^n(v, x)$$

for every  $v \in \text{Conv}_{\text{sc}}(\mathbb{R}^n)$ . For  $n = 1$ , the same representation holds if we replace rotation invariance with reflection invariance.

The dual to Corollary 2.6 can be written in the following way. The integrals are well-defined by the definition of  $D_0^1$  together with (2.6), as well as, Theorem 2.9.

**COROLLARY 2.11.** *If  $Z : \text{Conv}(\mathbb{R}; \mathbb{R}) \rightarrow \mathbb{R}$  is a continuous, dually epi-translation and reflection invariant valuation, then there exist  $\zeta_0 \in D_0^1$  and  $\zeta_1 \in D_1^1$  such that*

$$Z(v) = \int_{\mathbb{R}} \zeta_0(|x|) d\Phi_0^1(v, x) + \int_{\mathbb{R}} \zeta_1(|x|) d\Phi_1^1(v, x)$$

for every  $v \in \text{Conv}(\mathbb{R}; \mathbb{R})$ .

The following was shown in [CLM201, Theorem 17].

**LEMMA 2.12.** *Let  $\zeta \in C(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n)$  have compact support with respect to the second variable. For  $j \in \{0, 1, \dots, n\}$ ,*

$$Z(v) = \int_{\mathbb{R}^n \times \mathbb{R}^n} \zeta(v(x), x, y) d\Theta_j^n(v, (x, y))$$

is well-defined for every  $v \in \text{Conv}(\mathbb{R}^n; \mathbb{R})$  and defines a continuous valuation on  $\text{Conv}(\mathbb{R}^n; \mathbb{R})$ .

We conclude this part with a result that will be repeatedly used throughout this paper. It is a direct consequence of (2.8) and Lemma 2.12.

**LEMMA 2.13.** *If  $\zeta \in C_c([0, \infty))$  and  $j \in \{0, \dots, n\}$ , then  $Z : \text{Conv}_{\text{sc}}(\mathbb{R}^n) \rightarrow \mathbb{R}$ , defined by*

$$Z(u) = \int_{\mathbb{R}^n} \zeta(|y|) d\Psi_j^n(u, y),$$

is a continuous, epi-translation and rotation invariant valuation that is epi-homogeneous of degree  $j$ .

**2.5 The Abel transform.** For  $\zeta \in C_b((0, \infty))$ , define its *Abel transform* for  $t > 0$  as

$$\mathcal{A}\zeta(t) := 2 \int_t^\infty \frac{s \zeta(s) ds}{\sqrt{s^2 - t^2}} = \int_{-\infty}^\infty \zeta(\sqrt{s^2 + t^2}) ds.$$

If  $\xi \in C_b^1((0, \infty))$ , then the inverse transform is given by

$$\mathcal{A}^{-1}\xi(s) = -\frac{1}{\pi} \int_s^\infty \frac{\xi'(t) dt}{\sqrt{t^2 - s^2}}.$$

In particular,  $\mathcal{A}\zeta \equiv 0$  implies that  $\zeta \equiv 0$ . More generally, we have the following fact.

LEMMA 2.14. *Let  $\zeta \in C_b((0, \infty))$  and  $k \in \mathbb{N} \cup \{0\}$ . If*

$$\int_0^\infty \zeta(\sqrt{r^2 + t^2}) r^k dr = 0 \quad (2.10)$$

for every  $t > 0$ , then  $\zeta \equiv 0$ .

*Proof.* By the definition of the Abel transform, we have

$$\mathcal{A}^{k+1}\zeta(t) = \int_{\mathbb{R}^{k+1}} \zeta(\sqrt{|x|^2 + t^2}) dx = (k+1)\kappa_{k+1} \int_0^{+\infty} \zeta(\sqrt{r^2 + t^2}) r^k dr$$

for every  $t > 0$ . Thus, it follows from (2.10) that  $\mathcal{A}^{k+1}\zeta \equiv 0$ , which implies that  $\zeta \equiv 0$ .  $\square$

**2.6 The classes  $D_j^n$ .** Let  $j \in \{1, \dots, n-1\}$  and let  $\zeta \in D_j^n$ . We will associate with  $\zeta$  two functions  $\eta, \rho \in C_c([0, \infty))$  where

$$\eta(t) = \int_t^\infty s^{n-j-1} \zeta(s) ds \quad (2.11)$$

and

$$\rho(t) = t^{n-j} \zeta(t) + (n-j)\eta(t) \quad (2.12)$$

for every  $t > 0$ . Observe that it follows from the definition of  $D_j^n$  that  $\eta(0) = \lim_{t \rightarrow 0^+} \eta(t)$  exists and is finite and that  $\rho(0) = \lim_{t \rightarrow 0^+} \rho(t) = (n-j)\eta(0)$ .

The following result gives a geometric interpretation of the function  $\rho$  using the function  $u_t$  and  $v_t$  defined in (2.1) and (2.2).

LEMMA 2.15. *Let  $j \in \{1, \dots, n-1\}$  and  $\zeta \in D_j^n$ . For every  $t \geq 0$ ,*

$$\int_{\mathbb{R}^n} \zeta(|y|) d\Psi_j^n(u_t, y) = \int_{\mathbb{R}^n} \zeta(|x|) d\Phi_j^n(v_t, x) = \kappa_n \binom{n}{j} \rho(t).$$

*Proof.* Recall that  $v_t = (u_t)^*$ . Hence, the first equality follows from (2.8). For  $t > 0$ , all Hessian measures of  $v_t$ , with the exception of the measure  $\Theta_n^n(v_t, \cdot)$ , vanish in the set  $\{x \in \mathbb{R}^n : |x| < t\} \times \mathbb{R}^n$ . In particular, we have  $\Theta_j^n(v_t, B) = 0$  for every Borel subset  $B$  of  $\{x \in \mathbb{R}^n : |x| < t\} \times \mathbb{R}^n$ . Moreover, the function  $v_t$  is of class  $C^2$  in the set  $\{x \in \mathbb{R}^n : |x| > t\}$  and the Hessian matrix of  $v_t$  at a point  $x$  in this set has  $(n-1)$  eigenvalues equal to  $1/|x|$  and the last eigenvalue equal to zero. Hence

$$[\mathbf{D}^2 v_t(x)]_j = \binom{n-1}{j} \frac{1}{|x|^j}$$

for  $|x| > t$ . The set  $t\mathbb{S}^{n-1}$  is the set of singular points for  $v_t$ . For  $x \in t\mathbb{S}^{n-1}$ , we have

$$\partial v_t(x) = \left\{ r \frac{x}{|x|} : r \in [0, 1] \right\}.$$

Consequently,  $P_s(v_t, t\mathbb{S}^{n-1} \times \mathbb{R}^n) = \{(t+r)x : x \in \mathbb{S}^{n-1}, r \in [0, s]\}$  for  $s \geq 0$ . Thus,

$$\mathcal{H}^n(P_s(v_t, t\mathbb{S}^{n-1} \times \mathbb{R}^n)) = \kappa_n((t+s)^n - t^n)$$

and therefore

$$\Phi_j^n(v_t, t\mathbb{S}^{n-1}) = \Theta_{n-j}^n(v_t, t\mathbb{S}^{n-1} \times \mathbb{R}^n) = \kappa_n \binom{n}{j} t^{n-j}.$$

Summing up, we obtain

$$\begin{aligned} \int_{\mathbb{R}^n} \zeta(|x|) d\Phi_j^n(v_t, x) &= \kappa_n \binom{n}{j} t^{n-j} \zeta(t) + \binom{n-1}{j} \int_{\{x \in \mathbb{R}^n : |x| > t\}} \zeta(|x|) \frac{1}{|x|^j} dx \\ &= \kappa_n \binom{n}{j} t^{n-j} \zeta(t) + n\kappa_n \binom{n-1}{j} \int_t^{+\infty} s^{n-j-1} \zeta(s) ds, \end{aligned}$$

which implies the result for  $t > 0$ . Note that the set of singular points of  $v_0$  is  $\{0\}$  and that  $\partial v_0(0) = B^n$ . Hence  $\Theta_{n-j}^n(v_0, \{0\} \times B^n) = 0$  which implies that the result holds for  $t = 0$ .  $\square$

### 3 Singular Hessian valuations

This section is devoted to the proof of Theorem 1.2 and Theorem 1.4. We start with functions in  $\text{Conv}(\mathbb{R}^n; \mathbb{R}) \cap C_+^2(\mathbb{R}^n)$  and show that on this class of functions, the valuations from (1.3) are well-defined and finite for  $\zeta \in D_j^n$  with  $j \in \{0, \dots, n\}$ . The dual result implies that (1.2) is well-defined and finite for such  $\zeta$  on  $\text{Conv}_{\text{sc}}(\mathbb{R}^n) \cap C_+^2(\mathbb{R}^n)$ . The next step is an extension to functions in  $\text{Conv}_{\text{sc}}(\mathbb{R}^n) \cap C^1(\mathbb{R}^n)$ . For this, we collect results related to the theory of Hessian equations and establish uniform estimates for regular functions. In the last step of the proof, we use Moreau–Yosida envelopes and the polynomiality of epi-translation invariant valuations, which was proved in [CLM201], to extend the valuations to general functions in  $\text{Conv}_{\text{sc}}(\mathbb{R}^n)$ . In the last part, we apply Theorem 1.2 to establish a representation formula for  $V_{j,\zeta}$  on a certain class of functions.

**3.1 The smooth case.** Let  $\text{Conv}_{(0)}(\mathbb{R}^n; \mathbb{R})$  denote the set of  $v \in \text{Conv}(\mathbb{R}^n; \mathbb{R})$  that are of class  $C^2$  in a neighborhood of the origin.

LEMMA 3.1. *Let  $j \in \{1, \dots, n\}$  and  $\zeta \in D_j^n$ . If  $v \in \text{Conv}_{(0)}(\mathbb{R}^n; \mathbb{R})$ , then*

$$\int_{\mathbb{R}^n} |\zeta(|x|)| d\Phi_j^n(v, x) \tag{3.1}$$

*is well-defined and finite.*

*Proof.* Let  $\delta > 0$  be such that  $v$  is of class  $C^2$  on  $\{x \in \mathbb{R}^n : |x| < \delta\}$ . We rewrite (3.1) as

$$\int_{\{x: |x| < \delta/2\}} |\zeta(|x|)| \, d\Phi_j^n(v, x) + \int_{\{x: |x| \geq \delta/2\}} |\zeta(|x|)| \, d\Phi_j^n(v, x).$$

The second term is bounded, as  $\zeta \in C_b((0, \infty))$  and  $\Phi_j^n(v, \cdot)$  is locally finite. Concerning the first term, let  $\gamma > 0$  be such that  $[D^2v(x)]_j \leq \gamma$  for every  $x$  such that  $|x| \leq \delta/2$ . Using (2.5), we get

$$\begin{aligned} \int_{\{x: |x| < \delta/2\}} |\zeta(|x|)| \, d\Phi_j^n(v, x) &= \int_{\{x: |x| < \delta/2\}} |\zeta(|x|)| [D^2v(x)]_j \, dx \\ &\leq \gamma \int_{\{x: |x| < \delta/2\}} |\zeta(|x|)| \, dx \\ &= \omega_n \gamma \int_0^{\delta/2} r^{n-1} |\zeta(r)| \, dr. \end{aligned}$$

As  $\zeta \in D_j^n$ , the function  $r^{n-1} |\zeta(r)|$  can be extended to  $r = 0$  as a continuous function. Hence the last integral is finite.  $\square$

We remark that as a consequence of this lemma, we obtain that (1.3) in Theorem 1.4 holds for every  $v \in \text{Conv}(\mathbb{R}^n; \mathbb{R}) \cap C^2(\mathbb{R}^n)$ . A special case of the dual result of Lemma 3.1 is the content of the following lemma.

**LEMMA 3.2.** *Let  $j \in \{1, \dots, n\}$  and  $\zeta \in D_j^n$ . If  $u \in \text{Conv}_{\text{sc}}(\mathbb{R}^n) \cap C_+^2(\mathbb{R}^n)$ , then*

$$\int_{\mathbb{R}^n} \zeta(|\nabla u(x)|) [D^2u(x)]_{n-j} \, dx$$

*is well-defined and finite.*

*Proof.* Note that  $u^* \in \text{Conv}_{(0)}(\mathbb{R}^n; \mathbb{R})$  for  $u \in \text{Conv}_{\text{sc}}(\mathbb{R}^n) \cap C_+^2(\mathbb{R}^n)$ . Hence, the statement follows from Lemma 3.1 combined with (2.9).  $\square$

**3.2 Preparatory results for Theorem 1.2.** We introduce the following class of functions,

$$\text{Conv}_{\text{rg}}(\mathbb{R}^n) := \{u \in \text{Conv}_{\text{sc}}(\mathbb{R}^n) : u \text{ is regular, } 0 = u(0) < u(x) \text{ for every } x \neq 0\},$$

where  $u$  is *regular* if  $u \in C^2(\mathbb{R}^n)$  and the boundary of its sublevel sets  $\{u \leq t\}$  is of class  $C^2$  with positive Gaussian curvature for every  $t > 0$ .

Let  $u \in \text{Conv}_{\text{rg}}(\mathbb{R}^n)$ . Note that  $\nabla u(x) \neq 0$  for every  $x \neq 0$  and that, for  $t > 0$ , the sublevel set  $\{u \leq t\}$  is a convex body with non-empty interior and

$$\partial\{u \leq t\} = \{u = t\}.$$

Given  $x \neq 0$  and  $j \in \{0, \dots, n-1\}$ , we denote by  $\tau_j(u, x)$  the  $j$ th elementary symmetric function of the principal curvatures at  $x$  of  $\{u \leq t\}$  for  $t = u(x)$ . For every  $t \geq 0$ , we denote by  $h(u, \cdot, t) : \mathbb{S}^{n-1} \rightarrow \mathbb{R}$ , the support function of  $\{u \leq t\}$ . If  $t > 0$ , the

regularity properties of the sublevel set  $\{u \leq t\}$  imply that  $h(u, \cdot, t) \in C^2(\mathbb{S}^{n-1})$ . For  $y \in \mathbb{S}^{n-1}$ , let  $\sigma_j(u, y, t)$  denote the  $j$ th elementary symmetric function of the principal radii of curvature of  $\{u = t\}$  at  $\nu_t^{-1}(y)$ , where  $\nu_t: \{u = t\} \rightarrow \mathbb{S}^{n-1}$  denotes the Gauss map of  $\{u \leq t\}$  and  $\nu_t^{-1}: \mathbb{S}^{n-1} \rightarrow \{u = t\}$  its inverse.

**3.2.1 Hessian operators and Reilly-type lemmas.** Let  $A = (a_{ij})$  be a symmetric  $n \times n$  matrix and  $k \in \{0, \dots, n\}$ . We denote by  $[A]_k$  the  $k$ th elementary symmetric function of the eigenvalues of  $A$ , that is,

$$[A]_k := \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k}$$

(with the usual convention  $[A]_0 \equiv 1$ ), where  $\lambda_1, \dots, \lambda_n$  are the eigenvalues of  $A$ . For  $l, m \in \{1, \dots, n\}$ , we set

$$[A]_k^{lm} := \frac{\partial [A]_k}{\partial a_{lm}}. \quad (3.2)$$

The  $n \times n$  matrix formed by the entries  $[A]_k^{lm}$  is called the *cofactor matrix* of order  $k$  of the matrix  $A$ . When  $k = n$ , this is the usual cofactor matrix. From the homogeneity of the map  $A \mapsto [A]_k$ , we deduce

$$[A]_k^{lm} a_{lm} = k [A]_k \quad (3.3)$$

(here and throughout this section, we adopt the convention of summation over repeated indices). For a  $n \times j$  matrix  $(b^{lm})$  with elements of class  $C^1$ , we define

$$\operatorname{div}_m((b^{lm}))$$

as the divergence of the vector field whose components are the elements of the  $l$ th column of the matrix  $(b^{lm})$ .

Let  $u \in C^2(\mathbb{R}^n)$ . For  $k \in \{0, \dots, n\}$ , the  $k$ th *Hessian operator* applied to  $u$  and evaluated at a point  $x \in \mathbb{R}^n$  is given by

$$[D^2u(x)]_k.$$

The following result is due to Reilly [Rei73, Proposition 2.1].

**LEMMA 3.3.** *Let  $k \in \{0, 1, \dots, n\}$  and  $l \in \{1, \dots, n\}$ . Then*

$$\operatorname{div}_m([D^2u(x)]_k^{lm}) = 0$$

for every  $u \in C^3(\mathbb{R}^n)$  and  $x \in \mathbb{R}^n$ .

We will also need a corresponding result for functions defined on the unit sphere. Let  $h \in C^2(\mathbb{S}^{n-1})$ . Given  $x \in \mathbb{S}^{n-1}$ , for  $i, j \in \{1, \dots, n-1\}$  we denote by  $h_i$  and  $h_{ij}$  the first and second covariant derivatives of  $h$  with respect to a local orthonormal frame defined in a neighborhood of  $x$ . Let  $\delta_{ij}$  be the usual Kronecker symbols. The following result was proved in [CY76] in the case  $m = n - 1$  and then extended to the general case in [CS10, Lemma 3.1].



LEMMA 3.4. *Let  $k \in \{0, \dots, n-1\}$  and  $l \in \{1, \dots, n-1\}$ . Then*

$$\operatorname{div}_m \left( [h(x)_{lm} + h(x)\delta_{lm}]_k^{lm} \right) = \sum_{m=1}^{n-1} \left( [h(x)_{lm} + h(x)\delta_{lm}]_k^{lm} \right)_m = 0$$

for every  $h \in C^3(\mathbb{S}^{n-1})$  and  $x \in \mathbb{S}^{n-1}$ , where  $([h(x)_{lm} + h(x)\delta_{lm}]_k^{lm})_m$  denotes the derivative of  $[h(x)_{lm} + h(x)\delta_{lm}]_k^{lm}$  with respect to the  $m$ th variable of a local orthonormal frame defined in a neighborhood of  $x$ .

As a consequence, we obtain the following facts.

COROLLARY 3.5. *Let  $k \in \{0, \dots, n-1\}$ . If  $f, g, h \in C^2(\mathbb{S}^{n-1})$ , then*

$$\int_{\mathbb{S}^{n-1}} f [h_{lm} + h\delta_{lm}]_k^{lm} g_{lm} d\mathcal{H}^{n-1} = - \int_{\mathbb{S}^{n-1}} [h_{lm} + h\delta_{lm}]_k^{lm} f_l g_m d\mathcal{H}^{n-1}.$$

*Proof.* When  $h \in C^3(\mathbb{S}^{n-1})$ , the statement follows from the divergence theorem on the sphere and Lemma 3.4:

$$\begin{aligned} 0 &= \int_{\mathbb{S}^{n-1}} \sum_{m=1}^{n-1} \left( f [h_{lm} + h\delta_{lm}]_k^{lm} g_l \right)_m d\mathcal{H}^{n-1} \\ &= \int_{\mathbb{S}^{n-1}} f [h_{lm} + h\delta_{lm}]_k^{lm} g_{lm} d\mathcal{H}^{n-1} + \int_{\mathbb{S}^{n-1}} [h_{lm} + h\delta_{lm}]_k^{lm} f_l g_m d\mathcal{H}^{n-1}. \end{aligned}$$

The general case is obtained by approximation. □

COROLLARY 3.6. *Let  $k \in \{0, \dots, n-1\}$ . If  $f, g, h \in C^2(\mathbb{S}^{n-1})$ , then*

$$\int_{\mathbb{S}^{n-1}} f [h_{lm} + h\delta_{lm}]_k^{lm} (g_{lm} + g\delta_{lm}) d\mathcal{H}^{n-1} = \int_{\mathbb{S}^{n-1}} g [h_{lm} + h\delta_{lm}]_k^{lm} (f_{lm} + f\delta_{lm}) d\mathcal{H}^{n-1}.$$

3.2.2 *Geometric statements.* The following result is Proposition 2.4 from [LS07].

LEMMA 3.7. *For  $u \in \operatorname{Conv}_{\operatorname{rg}}(\mathbb{R}^n)$  the function  $h(u, \cdot, \cdot)$  is of class  $C^2(\mathbb{S}^{n-1} \times (0, \infty))$  and, for every  $(y, t) \in \mathbb{S}^{n-1} \times (0, \infty)$ ,*

$$\frac{1}{|\nabla u(\nu_t^{-1}(y))|} = \frac{\partial}{\partial t} h(u, y, t).$$

The next result follows from formulas (4.9), (5.55), and (5.56) in [Sch14].

LEMMA 3.8. *Let  $1 \leq i \leq n-1$  and  $t > 0$ . For  $u \in \operatorname{Conv}_{\operatorname{rg}}(\mathbb{R}^n)$ ,*

$$\int_{\{u=t\}} \tau_{n-i-1}(u, x) d\mathcal{H}^{n-1}(x) = \alpha V_i(\{u \leq t\}),$$

and

$$\int_{\mathbb{S}^{n-1}} h(u, y, t) \sigma_{i-1}(u, y, t) d\mathcal{H}^{n-1}(y) = \alpha V_i(\{u \leq t\}),$$

where  $\alpha$  is a positive constant depending only on  $n$  and  $i$ .

LEMMA 3.9. *Let  $1 \leq i \leq n-1$  and  $0 < t_1 < t_2$ . For  $u \in \text{Conv}_{\text{rg}}(\mathbb{R}^n)$ ,*

$$\int_{\{t_1 < u \leq t_2\}} \tau_{n-i}(u, x) dx = \alpha \left( V_i(\{u \leq t_2\}) - V_i(\{u \leq t_1\}) \right),$$

where  $\alpha$  is a positive constant depending only on  $n$  and  $i$ .

*Proof.* Note that  $\nabla u(x) \neq 0$  for every  $x \in \{t_1 < u \leq t_2\}$ . By the coarea formula, we have

$$\int_{\{t_1 < u \leq t_2\}} \tau_{n-i}(u, x) dx = \int_{t_1}^{t_2} \int_{\{u=t\}} \frac{1}{|\nabla u(x)|} \tau_{n-i}(u, x) d\mathcal{H}^{n-1}(x) dt.$$

Using the change of variable  $y = \nu_t(x)$  in the inner integral and Lemma 3.7, we get

$$\begin{aligned} & \int_{\{u=t\}} \frac{1}{|\nabla u(x)|} \tau_{n-i}(u, x) d\mathcal{H}^{n-1}(x) \\ &= \int_{\mathbb{S}^{n-1}} \frac{1}{|\nabla u(\nu_t^{-1}(y))|} \sigma_{i-1}(u, y, t) d\mathcal{H}^{n-1}(y) \\ &= \int_{\mathbb{S}^{n-1}} \frac{\partial}{\partial t} h(u, y, t) \sigma_{i-1}(u, y, t) d\mathcal{H}^{n-1}(y). \end{aligned}$$

Next, we prove that

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{S}^{n-1}} h(u, y, t) \sigma_{i-1}(u, y, t) d\mathcal{H}^{n-1}(y) \\ &= i \int_{\mathbb{S}^{n-1}} \frac{\partial}{\partial t} h(u, y, t) \sigma_{i-1}(u, y, t) d\mathcal{H}^{n-1}(y). \end{aligned} \tag{3.4}$$

We have

$$\sigma_{i-1}(u, y, t) = [h(u, y, t)_{lm} + h(u, y, t) \delta_{lm}]_{i-1} \tag{3.5}$$

(see, for example, [Sch14, Sect. 2.5]). We start with the formula,

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{S}^{n-1}} h(u, y, t) \sigma_{i-1}(u, y, t) d\mathcal{H}^{n-1}(y) \\ &= \int_{\mathbb{S}^{n-1}} \frac{\partial}{\partial t} h(u, y, t) \sigma_{i-1}(u, y, t) d\mathcal{H}^{n-1}(y) \\ & \quad + \int_{\mathbb{S}^{n-1}} h(u, y, t) \frac{\partial \sigma_{i-1}}{\partial t}(u, y, t) d\mathcal{H}^{n-1}(y). \end{aligned} \tag{3.6}$$

By (3.5), (3.2), Corollary 3.6, and (3.5),

$$\begin{aligned} & \int_{\mathbb{S}^{n-1}} h(u, y, t) \frac{\partial}{\partial t} \sigma_{i-1}(u, y, t) d\mathcal{H}^{n-1}(y) \\ &= \int_{\mathbb{S}^{n-1}} h(u, y, t) [h(u, y, t)_{lm} + h(u, y, t) \delta_{lm}]_{i-1}^{lm} \end{aligned}$$

$$\begin{aligned}
& \left( \left( \frac{\partial}{\partial t} h(u, y, t) \right)_{lm} + \frac{\partial}{\partial t} h(u, y, t) \delta_{lm} \right) d\mathcal{H}^{n-1}(y) \\
&= \int_{\mathbb{S}^{n-1}} \frac{\partial}{\partial t} h(u, y, t) [h(u, y, t)_{lm} + h(u, y, t) \delta_{lm}]_{i-1}^{lm} \\
&\quad \left( h(u, y, t)_{lm} + h(u, y, t) \delta_{lm} \right) d\mathcal{H}^{n-1}(y) \\
&= (i-1) \int_{\mathbb{S}^{n-1}} \frac{\partial}{\partial t} h(u, y, t) [h(u, y, t)_{lm} + h(u, y, t) \delta_{lm}]_{i-1} d\mathcal{H}^{n-1}(y) \\
&= (i-1) \int_{\mathbb{S}^{n-1}} \frac{\partial}{\partial t} h(u, y, t) \sigma_{i-1}(u, y, t) d\mathcal{H}^{n-1}(y).
\end{aligned}$$

The last chain of equations and (3.6) imply (3.4). Hence, by the latter relation,

$$\begin{aligned}
& \int_{\{t_1 < u \leq t_2\}} \tau_{n-i}(u, x) dx \\
&= \int_{t_1}^{t_2} \int_{\mathbb{S}^{n-1}} \frac{\partial}{\partial t} h(u, y, t) \sigma_{i-1}(u, y, t) d\mathcal{H}^{n-1}(y) dt \\
&= \frac{1}{i} \int_{t_1}^{t_2} \frac{d}{dt} \int_{\mathbb{S}^{n-1}} h(u, t, y) \sigma_{i-1}(u, y, t) d\mathcal{H}^{n-1}(y) dt \\
&= \frac{1}{i} \left( \int_{\mathbb{S}^{n-1}} h(u, y, t_2) \sigma_{i-1}(u, y, t_2) d\mathcal{H}^{n-1}(y) \right. \\
&\quad \left. - \int_{\mathbb{S}^{n-1}} h(u, y, t_1) \sigma_{i-1}(u, y, t_1) d\mathcal{H}^{n-1}(y) \right) \\
&= \alpha \left( V_i(\{u \leq t_2\}) - V_i(\{u \leq t_1\}) \right),
\end{aligned}$$

where we have used Lemma 3.8 in the last equation.  $\square$

**3.2.3 Integration by parts.** We start with a lemma which can be found in [B+08] (see formula (11)). We write  $u_k$  for the partial derivative of  $u$  with respect to  $x_k$ .

**LEMMA 3.10** ([B+08]). *Let  $i \in \{1, \dots, n-1\}$  and  $u \in \text{Conv}_{\text{rg}}(\mathbb{R}^n)$ . Then*

$$[D^2 u(x)]_{n-i} = |\nabla u(x)|^{n-i} \tau_{n-i}(u, x) + \frac{[D^2 u(x)]_{n-i}^{lm} u_k(x) u_l(x) u_{km}(x)}{|\nabla u(x)|^2}$$

for every  $x \neq 0$ .

We will also need the following lemma for which we refer, for instance, to [Rei73].

**LEMMA 3.11.** *Let  $i \in \{0, \dots, n-1\}$  and  $u \in \text{Conv}_{\text{rg}}(\mathbb{R}^n)$ . Then*

$$[D^2 u(x)]_{n-i}^{lm} u_l(x) u_m(x) = |\nabla u(x)|^{n-i+1} \tau_{n-i-1}(u, x)$$

for every  $x \neq 0$ .

PROPOSITION 3.12. *Let  $i \in \{1, \dots, n-1\}$  and  $u \in \text{Conv}_{\text{rg}}(\mathbb{R}^n)$ . If  $\gamma : (0, \infty) \rightarrow \mathbb{R}$  is differentiable, then*

$$\begin{aligned} & \text{div}_m \left( \gamma(|\nabla u(x)|) [D^2 u(x)]_{n-i}^{lm} u_l(x) \right) \\ &= [D^2 u(x)]_{n-i} \left( (n-i) \gamma(|\nabla u(x)|) + \gamma'(|\nabla u(x)|) |\nabla u(x)| \right) \\ & \quad - \gamma'(|\nabla u(x)|) |\nabla u(x)|^{n-i+1} \tau_{n-i}(u, x) \end{aligned}$$

for every  $x \neq 0$ .

*Proof.* We have

$$\text{div}_m \left( \gamma(|\nabla u|) [D^2 u]_{n-i}^{lm} u_l \right) = \gamma(|\nabla u|) \text{div}_m \left( [D^2 u]_{n-i}^{lm} u_l \right) + \frac{\gamma'(|\nabla u|)}{|\nabla u|} [D^2 u]_{n-i}^{lm} u_k u_l u_{km}.$$

Note that, by Lemma 3.3 and (3.3),

$$\text{div}_m \left( [D^2 u]_{n-i}^{lm} u_l \right) = [D^2 u]_{n-i}^{lm} u_{lm} = (n-i) [D^2 u]_{n-i}.$$

The conclusion follows immediately from Lemma 3.10.  $\square$

Recall that to a given function  $\zeta \in D_i^n$  for  $1 \leq i \leq n-1$ , we associate two functions  $\eta, \rho \in C_c([0, \infty))$ , defined by

$$\eta(t) = \int_t^\infty s^{n-i-1} \zeta(s) \, ds, \quad \rho(t) = t^{n-i} \zeta(t) + (n-i) \eta(t)$$

for  $t \in [0, \infty)$ .

The main result of this part is the following proposition.

PROPOSITION 3.13. *Let  $1 \leq i \leq n-1$  and  $\zeta \in D_i^n$ . For every  $u \in \text{Conv}_{\text{rg}}(\mathbb{R}^n)$  and  $t_1, t_2$  with  $0 < t_1 < t_2$ ,*

$$\begin{aligned} \int_{\{t_1 < u \leq t_2\}} \zeta(|\nabla u(x)|) [D^2 u(x)]_{n-i} \, dx &= \int_{\{t_1 < u \leq t_2\}} \rho(|\nabla u(x)|) \tau_{n-i}(u, x) \, dx \\ & \quad - \int_{\{u=t_2\}} \eta(|\nabla u(x)|) \tau_{n-i-1}(u, x) \, d\mathcal{H}^{n-1}(x) \\ & \quad + \int_{\{u=t_1\}} \eta(|\nabla u(x)|) \tau_{n-i-1}(u, x) \, d\mathcal{H}^{n-1}(x). \end{aligned}$$

*Proof.* For  $r > 0$ , we set

$$\gamma(r) := -\frac{1}{r^{n-i}} \int_r^\infty s^{n-i-1} \zeta(s) \, ds.$$

Then

$$\gamma'(r) = \frac{n-i}{r^{n-i+1}} \int_r^\infty s^{n-i-1} \zeta(s) \, ds + \frac{\zeta(r)}{r}$$

and

$$(n-i)\gamma(r) + r\gamma'(r) = \zeta(r)$$

for every  $r > 0$ . Moreover,

$$r^{n-i+1}\gamma'(r) = (n-i)\eta(r) + r^{n-i}\zeta(r) = \rho(r)$$

for every  $r > 0$ . By the previous relations, Proposition 3.12 and the divergence theorem, we have

$$\begin{aligned} & \int_{\{t_1 < u \leq t_2\}} \zeta(|\nabla u|) [D^2 u]_{n-i} \, dx \\ &= \int_{\{t_1 < u \leq t_2\}} \left( (n-i)\gamma(|\nabla u|) + |\nabla u| \gamma'(|\nabla u|) \right) [D^2 u]_{n-i} \, dx \\ &= \int_{\{t_1 < u \leq t_2\}} |\nabla u|^{n-i+1} \gamma'(|\nabla u|) \tau_{n-i} \, dx \\ &\quad + \int_{\{t_1 < u \leq t_2\}} \operatorname{div}_m \left( \gamma(|\nabla u|) [D^2 u]_{n-i}^{lm} u_l \right) \, dx \\ &= \int_{\{t_1 < u \leq t_2\}} \rho(|\nabla u|) \tau_{n-i} \, dx + \int_{\{u=t_2\}} \frac{\gamma(|\nabla u|)}{|\nabla u|} [D^2 u]_{n-i}^{lm} u_l u_m \, d\mathcal{H}^{n-1} \\ &\quad - \int_{\{u=t_1\}} \frac{\gamma(|\nabla u|)}{|\nabla u|} [D^2 u]_{n-i}^{lm} u_l u_m \, d\mathcal{H}^{n-1}. \end{aligned}$$

In the last equation, we have used the fact that the outer unit normal to the set  $\{t_1 < u \leq t_2\}$  at a point  $x \in \{u = t_2\}$  is given by

$$\frac{\nabla u(x)}{|\nabla u(x)|},$$

while at a point  $x \in \{u = t_1\}$  it is given by

$$-\frac{\nabla u(x)}{|\nabla u(x)|}.$$

The conclusion follows from Lemma 3.11.  $\square$

**3.2.4 An estimate.** Given a real-valued function  $u$  defined in a subset  $V$  of  $\mathbb{R}^n$ , the Lipschitz constant of  $u$  in  $U \subset V$  is defined as

$$\operatorname{Lip}_U(u) := \sup \left\{ \frac{|u(x) - u(y)|}{|x - y|} : x, y \in U, x \neq y \right\}.$$

For  $u \in \operatorname{Conv}_{\text{sc}}(\mathbb{R}^n)$  and  $t > 0$ , we set

$$\operatorname{Lip}(u, t) := \operatorname{Lip}_{\{u \leq t\}}(u)$$

and note that for  $u \in \text{Conv}_{\text{sc}}(\mathbb{R}^n) \cap C^1(\mathbb{R}^n)$ ,

$$\lim_{t \rightarrow 0^+} \text{Lip}(u, t) = 0 \quad (3.7)$$

if  $0 = u(0) \leq u(x)$  for  $x \in \mathbb{R}^n$ .

LEMMA 3.14. *Let  $1 \leq i \leq n - 1$  and  $\zeta \in D_i^n$ . If  $u \in \text{Conv}_{\text{rg}}(\mathbb{R}^n)$ , then*

$$\left| \int_{\{t_1 < u \leq t_2\}} \zeta(|\nabla u(x)|) [D^2 u(x)]_{n-i} dx \right| \leq \alpha V_i(\{u \leq t_2\}) \left( \max_{[0, \text{Lip}(u, t_2)]} |\rho| + \max_{[0, \text{Lip}(u, t_2)]} |\eta| \right)$$

for every  $0 < t_1 < t_2$  where  $\alpha$  is a positive constant depending only on  $n$  and  $i$ .

*Proof.* By Lemma 3.9 (and the monotonicity of intrinsic volumes)

$$\begin{aligned} \left| \int_{\{t_1 < u \leq t_2\}} \rho(|\nabla u(x)|) \tau_{n-i}(u, x) dx \right| &\leq \alpha (V_i(\{u \leq t_2\}) - V_i(\{u \leq t_1\})) \max_{[0, \text{Lip}(u, t_2)]} |\rho| \\ &\leq \alpha V_i(\{u \leq t_2\}) \max_{[0, \text{Lip}(u, t_2)]} |\rho|, \end{aligned}$$

where  $\alpha > 0$  only depends on  $n$  and  $i$ . Similarly, by Lemma 3.8

$$\left| \int_{\{u=t_j\}} \eta(|\nabla u(x)|) \tau_{n-i-1}(u, x) d\mathcal{H}^{n-1}(x) \right| \leq \alpha V_i(\{u \leq t_2\}) \max_{[0, \text{Lip}(u, t_2)]} |\eta|$$

for  $j \in \{1, 2\}$ . The conclusion now follows from Proposition 3.13.  $\square$

**3.3 Proof of Theorem 1.2.** Throughout this section, fix  $j \in \{1, \dots, n - 1\}$  and  $\zeta \in D_j^n$ . We will associate the two functions  $\rho, \eta \in C_c([0, \infty))$  defined in (2.11) and (2.12) with  $\zeta$ .

First, we prove that we may reduce to the case

$$\eta(0) = \rho(0) = 0.$$

Indeed, let

$$\eta_0 := \lim_{r \rightarrow 0^+} \eta(r),$$

and choose  $\zeta_0 \in C_c([0, \infty))$  such that

$$\int_0^\infty s^{n-j-1} \zeta_0(s) ds = \eta_0.$$

Then  $\bar{\zeta}: (0, \infty) \rightarrow \mathbb{R}$ , defined by

$$\bar{\zeta}(r) := \zeta(r) - \zeta_0(r),$$

belongs to  $D_j^n$ . The function  $\bar{\eta}$ , defined by

$$\bar{\eta}(r) := \int_r^\infty s^{n-j-1} \bar{\zeta}(s) ds,$$

verifies  $\bar{\eta}(0) = 0$ . Moreover, as  $\zeta_0 \in C_c([0, \infty))$ , the functional  $Z_{\zeta_0} : \text{Conv}_{\text{sc}}(\mathbb{R}^n) \rightarrow \mathbb{R}$ , given by

$$Z_{\zeta_0}(u) := \int_{\mathbb{R}^n \times \mathbb{R}^n} \zeta_0(|y|) d\Theta_j^n(u, y),$$

defines, by Lemma 2.13, a continuous, epi-translation and rotation invariant valuation on  $\text{Conv}_{\text{sc}}(\mathbb{R}^n)$ . We have then proved that  $\zeta$  can be written as

$$\zeta = \bar{\zeta} + \zeta_0,$$

where  $\zeta_0$  gives rise to a continuous valuation on  $\text{Conv}_{\text{sc}}(\mathbb{R}^n)$ , while  $\bar{\zeta} \in D_j^n$ , and the function  $\bar{\eta}$  associated to  $\bar{\zeta}$  verifies  $\bar{\eta}(0) = 0$ . Finally, note that it follows from the relation

$$\bar{\rho}(r) = (n - j)\bar{\eta}(r) + r^{n-j}\bar{\zeta}(r),$$

that  $\bar{\rho}(0) = 0$ .

In the following, let

$$\text{Conv}_{\text{sc},0}(\mathbb{R}^n) := \{u \in \text{Conv}_{\text{sc}}(\mathbb{R}^n) : u(0) = 0 \leq u(x) \text{ for } x \in \mathbb{R}^n\}$$

and

$$T_j(u) := \{t > 0 : \Theta_j^n(u, \{u = t\} \times \mathbb{R}^n) = 0\}.$$

Since  $\Theta_j^n(u, \cdot)$  is a locally finite measure, the complement of  $T_j(u)$  in  $(0, \infty)$  has at most countably many points.

LEMMA 3.15. *Let  $u \in \text{Conv}_{\text{sc},0}(\mathbb{R}^n) \cap C^1(\mathbb{R}^n)$ . Then there exists a sequence  $u_k$  of functions from  $\text{Conv}_{\text{rg}}(\mathbb{R}^n)$  such that  $u_k$  epi-converges to  $u$  and*

$$\lim_{k \rightarrow \infty} \int_{\{t_1 < u_k \leq t_2\}} \zeta(|\nabla u_k(x)|) [D^2 u_k(x)]_{n-j} dx = \int_{\{(x,y) : t_1 < u(x) \leq t_2\}} \zeta(|y|) d\Theta_j^n(u, (x, y))$$

for all  $0 < t_1 < t_2$  with  $t_1, t_2 \in T_j(u)$ .

*Proof.* The proof is divided into several steps.

(i) There exists a sequence  $u_k$  of functions from  $\text{Conv}_{\text{rg}}(\mathbb{R}^n)$  that epi-converges to  $u$ . Indeed, by a standard mollification procedure, we first find a sequence of convex functions  $\bar{u}_k$  of class  $C^2(\mathbb{R}^n)$  such that  $0 = \bar{u}_k(0) \leq \bar{u}_k(x)$  for every  $x \in \mathbb{R}^n$  and  $k \in \mathbb{N}$ , and  $\bar{u}_k$  converges to  $u$  on compact sets (in particular,  $\bar{u}_k$  epi-converges to  $u$ ). Then we define, for every  $x \in \mathbb{R}^n$  and  $k \in \mathbb{N}$ ,

$$u_k(x) := \bar{u}_k(x) + \frac{1}{k}|x|^2.$$

Then clearly  $u_k \in \text{Conv}_{\text{sc},0}(\mathbb{R}^n) \cap C^2(\mathbb{R}^n)$ , and  $u_k$  epi-converges to  $u$ . Moreover,  $D^2 u_k(x) > 0$  for every  $x \neq 0$  and  $k \in \mathbb{N}$ , and this implies that the boundary of  $\{u_k \leq t\}$

is of class  $C^2$  with positive curvature for every  $t > 0$ . Moreover, as  $u \in C^1(\mathbb{R}^n)$ , the sequence  $\nabla u_k$  converges to  $\nabla u$  uniformly on compact sets.

(ii) For  $t_1, t_2 \in T_j(u)$  such that  $0 < t_1 < t_2$ ,

$$\begin{aligned} & \lim_{k \rightarrow \infty} \int_{\{(x,y): t_1 < u(x) \leq t_2\}} \zeta(|y|) d\Theta_j^n(u_k, (x, y)) \\ &= \int_{\{(x,y): t_1 < u(x) \leq t_2\}} \zeta(|y|) d\Theta_j^n(u, (x, y)). \end{aligned} \quad (3.8)$$

For  $\bar{t} > 0$  fixed, there exists  $\alpha > 0$  such that

$$|\nabla u(x)| \geq 2\alpha \quad \text{for every } x \text{ such that } u(x) > \bar{t}.$$

Since the sequence  $u_k$  converges to  $u$  in the  $C^1$ -norm on compact sets, we may assume that

$$|\nabla u_k(x)| \geq \alpha \quad \text{for every } x \text{ such that } u_k(x) > \bar{t}$$

for  $k$  sufficiently large (independent of  $x$ ). Let  $\bar{\zeta}$  be a continuous function in  $[0, \infty)$ , which coincides with  $\zeta$  in  $[\alpha, \infty)$  and hence has compact support. We have

$$\int_{\{(x,y): t_1 < u(x) \leq t_2\}} \zeta(|y|) d\Theta_j^n(u_k, (x, y)) = \int_{\{(x,y): t_1 < u(x) \leq t_2\}} \bar{\zeta}(|y|) d\Theta_j^n(u_k, (x, y)),$$

for every  $\bar{t} < t_1 < t_2$  and for every  $k$  sufficiently large, and the corresponding statement holds for  $u_k$  replaced by  $u$ . As  $u_k$  epi-converges to  $u$  and as  $t_1, t_2 \in T_j(u)$ , it follows from (2.7) that

$$\lim_{k \rightarrow \infty} \int_{\{t_1 < u \leq t_2\} \times \mathbb{R}^n} \zeta(|y|) d\Theta_j^n(u_k, (x, y)) = \int_{\{t_1 < u \leq t_2\} \times \mathbb{R}^n} \zeta(|y|) d\Theta_j^n(u, (x, y)).$$

This proves that (3.8) holds for all  $t_1, t_2 \in T_j(u)$  such that  $t_2 > t_1$  and  $t_1, t_2 > \bar{t}$ . As  $\bar{t} > 0$  was arbitrary, the claim is proved.

(iii) For every  $t > 0$ ,

$$\begin{aligned} & \lim_{k \rightarrow \infty} \left| \int_{\{u \geq t\}} \zeta(|\nabla u_k(x)|) [D^2 u_k(x)]_{n-j} dx \right. \\ & \quad \left. - \int_{\{u_k \geq t\}} \zeta(|\nabla u_k(x)|) [D^2 u_k(x)]_{n-j} dx \right| = 0. \end{aligned} \quad (3.9)$$

We first note that in the previous relation, we may replace  $\zeta$  by  $\bar{\zeta}$  (where  $\bar{\zeta}$  is chosen as in the previous step). Let  $\bar{\zeta}_+$  and  $\bar{\zeta}_-$  denote the positive and negative parts of  $\bar{\zeta}$ ; these are continuous functions in  $[0, \infty)$ , with compact support. As  $\bar{\zeta} = \bar{\zeta}_+ - \bar{\zeta}_-$ , it is enough to prove (3.9) for  $\bar{\zeta}_+$  and  $\bar{\zeta}_-$  separately. In other words, we may reduce to prove (3.9) under the assumption that  $\zeta$  is non-negative and belongs to  $C_c([0, \infty))$ .

For every  $k \in \mathbb{N}$ , let

$$t_{k,1} = \max\{r > 0: \{u_k \leq r\} \subset \{u \leq t\}\},$$



and

$$t_{k,2} = \min\{r > 0: \{u_k \leq r\} \supset \{u \leq t\}\}.$$

Equivalently,  $\{u_k \leq t_{k,1}\}$  is the largest sublevel set of  $u_k$  contained in  $\{u \leq t\}$ , while  $\{u_k \leq t_{k,2}\}$  is the smallest sublevel set of  $u_k$  containing  $\{u \leq t\}$ . Clearly

$$\{u_k \leq t_{k,1}\} \subset \{u \leq t\} \subset \{u_k \leq t_{k,2}\}$$

for every  $k \in \mathbb{N}$ . Moreover, by uniform convergence, we have

$$\lim_{k \rightarrow \infty} \{u_k \leq t_{k,1}\} = \lim_{k \rightarrow \infty} \{u_k \leq t_{k,2}\} = \{u \leq t\}$$

(here, convergence is in the Hausdorff metric). As  $\zeta$  is non-negative,

$$\begin{aligned} & \int_{\mathbb{R}^n \setminus \{u_k \leq t_{k,2}\}} \zeta(|\nabla u_k(x)|) [D^2 u_k(x)]_{n-j} dx \\ & \leq \int_{\mathbb{R}^n \setminus \{u \leq t\}} \zeta(|\nabla u_k(x)|) [D^2 u_k(x)]_{n-j} dx \\ & \leq \int_{\mathbb{R}^n \setminus \{u_k \leq t_{k,1}\}} \zeta(|\nabla u_k(x)|) [D^2 u_k(x)]_{n-j} dx. \end{aligned}$$

Therefore, the difference

$$\alpha_k := \int_{\{u \geq t\}} \zeta(|\nabla u_k(x)|) [D^2 u_k(x)]_{n-j} dx - \int_{\{u_k \geq t\}} \zeta(|\nabla u_k(x)|) [D^2 u_k(x)]_{n-j} dx$$

is bounded from below by

$$\int_{\{a_k \leq u_k \leq b_k\}} \zeta(|\nabla u_k(x)|) [D^2 u_k(x)]_{n-j} dx, \quad (3.10)$$

where  $a_k = \min\{t, t_{k,2}\}$  and  $b_k = \max\{t, t_{k,2}\}$ . Similarly,  $\alpha_k$  is bounded from above by

$$\int_{\{c_k \leq u_k \leq d_k\}} \zeta(|\nabla u_k(x)|) [D^2 u_k(x)]_{n-j} dx, \quad (3.11)$$

where  $c_k = \min\{t, t_{k,1}\}$  and  $d_k = \max\{t, t_{k,1}\}$ . Note that the sequences  $a_k$ ,  $b_k$ ,  $c_k$  and  $d_k$  converge to  $t$  as  $k \rightarrow \infty$ . We prove that the integral in (3.10) converges to zero as  $k \rightarrow \infty$ . The same, with a similar proof, holds for (3.11). This will complete the proof of this step.

By Proposition 3.13, we have

$$\begin{aligned} & \int_{\{a_k \leq u_k \leq b_k\}} \zeta(|\nabla u_k(x)|) [D^2 u_k(x)]_{n-j} dx \\ & = \int_{\{a_k \leq u_k \leq b_k\}} \rho(|\nabla u_k(x)|) \tau_{n-j}(u_k, x) dx \end{aligned}$$

$$\begin{aligned}
& - \int_{\{u_k=b_k\}} \eta(|\nabla u_k(x)|) \tau_{n-j-1}(u_k, x) \, d\mathcal{H}^{n-1}(x) \\
& + \int_{\{u_k=a_k\}} \eta(|\nabla u_k(x)|) \tau_{n-j-1}(u_k, x) \, d\mathcal{H}^{n-1}(x),
\end{aligned}$$

where the functions  $\rho$  and  $\eta$  are associated to  $\zeta$  as in Proposition 3.13. For the first term on the right side, we may write, using the continuity of  $\rho$  and Lemma 3.9,

$$\left| \int_{\{a_k \leq u_k \leq b_k\}} \rho(|\nabla u_k(x)|) \tau_{n-j}(u, x) \, dx \right| \leq \alpha (V_j(\{u_k \leq b_k\}) - V_j(\{u_k \leq a_k\})),$$

where  $\alpha > 0$  depends on  $n$ ,  $j$  and  $\rho$ . By the uniform convergence and the continuity of intrinsic volumes, the right side of the previous inequality tends to 0 as  $k \rightarrow \infty$ .

Next, we consider

$$\int_{\{u_k=b_k\}} \eta(|\nabla u_k(x)|) \tau_{n-j-1}(u_k, x) \, d\mathcal{H}^{n-1}(x).$$

This integral can be written as

$$\binom{n-1}{j} \int_{\mathbb{R}^n} \eta(|\nabla u_k(x)|) \, dC_j(\{u_k \leq b_k\}, x),$$

where  $C_j(\{u_k \leq b_k\}, \cdot)$  is the  $j$ th curvature measure of  $\{u_k \leq b_k\}$  (see [Sch14, Chap. 4] for the definition of curvature measures). Note that by the convergence of  $\{u_k \leq b_k\}$  to  $\{u \leq t\}$  and [Sch14, Theorem 4.2.1], the curvature measures  $C_j(\{u_k \leq b_k\}, \cdot)$  converge weakly to  $C_j(\{u \leq t\}, \cdot)$ . Moreover, the support of the curvature measures of a convex body is contained in the boundary of the convex body. Hence there exists a compact set  $C \subset \mathbb{R}^n$ , such that the supports of  $C_j(\{u_k \leq b_k\}, \cdot)$  and of  $C_j(\{u \leq t\}, \cdot)$  are contained in  $C$  for every  $k$ . Taking the continuity of  $\eta$  and the uniform convergence of  $\nabla u_k$  to  $\nabla u$  on compact sets into account, we deduce that

$$\begin{aligned}
& \lim_{k \rightarrow \infty} \int_{\{u_k=b_k\}} \eta(|\nabla u_k(x)|) \tau_{n-j-1}(u_k, x) \, d\mathcal{H}^{n-1}(x) \\
& = \binom{n-1}{j} \int_{\mathbb{R}^n} \eta(|\nabla u(x)|) \, dC_j(\{u \leq t\}, x).
\end{aligned}$$

By a similar argument, we may prove that

$$\begin{aligned}
& \lim_{k \rightarrow \infty} \int_{\{u_k=a_k\}} \eta(|\nabla u_k(x)|) \tau_{n-j-1}(u_k, x) \, d\mathcal{H}^{n-1}(x) \\
& = \binom{n-1}{j} \int_{\mathbb{R}^n} \eta(|\nabla u(x)|) \, dC_j(\{u \leq t\}, x).
\end{aligned}$$

Thus, the integral in (3.10) converges to zero as  $k \rightarrow \infty$ , which concludes the proof of this step.

(iv) By the previous step we have for every  $t_1, t_2 \in T_j(u)$  such that  $0 < t_1 < t_2$ ,

$$\lim_{k \rightarrow \infty} \left| \int_{\{t_1 < u \leq t_2\}} \zeta(|\nabla u_k(x)|) [D^2 u_k(x)]_{n-j} dx - \int_{\{t_1 < u_k \leq t_2\}} \zeta(|\nabla u_k(x)|) [D^2 u_k(x)]_{n-j} dx \right| = 0.$$

This, together with (3.8), concludes the proof of this lemma.  $\square$

LEMMA 3.16. *Let  $u \in \text{Conv}_{\text{sc},0}(\mathbb{R}^n) \cap C^1(\mathbb{R}^n)$ . For  $t_1, t_2 \in T_j(u)$  such that  $0 < t_1 < t_2$ , we have*

$$\begin{aligned} & \left| \int_{\{(x,y): t_1 < u(x) \leq t_2\}} \zeta(|y|) d\Theta_j^n(u, (x, y)) \right| \\ & \leq \alpha V_j(\{u \leq t_2\}) \left( \max_{[0, \text{Lip}(u, t_2)]} |\rho| + \max_{[0, \text{Lip}(u, t_2)]} |\eta| \right), \end{aligned}$$

where  $\alpha$  is a positive constant only depending on  $n$  and  $j$ .

*Proof.* The validity of the claimed inequality follows from Lemma 3.14, Lemma 3.15, and the convergence of Lipschitz constants and of sublevel sets under uniform convergence.  $\square$

LEMMA 3.17. *For every  $u \in \text{Conv}_{\text{sc},0}(\mathbb{R}^n) \cap C^1(\mathbb{R}^n)$ , the limit*

$$\lim_{\substack{t \rightarrow 0 \\ t \in T_j(u)}} \int_{\{(x,y): u(x) > t\}} \zeta(|y|) d\Theta_j^n(u, (x, y))$$

*exists and is finite.*

*Proof.* From Lemma 3.16, we obtain that for every  $t_1, t_2 \in T_j(u)$  such that  $0 < t_1 < t_2$ ,

$$\begin{aligned} & \left| \int_{\{(x,y): t_1 < u(x) \leq t_2\}} \zeta(|y|) d\Theta_j^n(u, (x, y)) \right| \\ & \leq \alpha V_j(\{u \leq t_2\}) \left( \max_{[0, \text{Lip}(u, t_2)]} |\rho| + \max_{[0, \text{Lip}(u, t_2)]} |\eta| \right). \end{aligned}$$

The conclusion now follows from (3.7) and the fact that both  $\eta$  and  $\rho$  are continuous on  $[0, \infty)$  and vanish at  $t = 0$ .  $\square$

Lemma 3.17 and the fact that  $\text{env}_\lambda u \in \text{Conv}_{\text{sc}}(\mathbb{R}^n) \cap C^1(\mathbb{R}^n)$  allow us to make the following definition. For  $u \in \text{Conv}_{\text{sc},0}(\mathbb{R}^n)$  and  $\lambda > 0$ , we set

$$V_{j,\zeta}^{(\lambda)}(u) := \lim_{t \rightarrow 0} \int_{\{(x,y): \text{env}_\lambda u(x) > t\}} \zeta(|y|) d\Theta_j^n(\text{env}_\lambda u, (x, y)). \quad (3.12)$$

An immediate consequence of this definition and Lemma 3.16 is the following result.

LEMMA 3.18. *Let  $u \in \text{Conv}_{\text{sc},0}(\mathbb{R}^n)$  and  $\lambda > 0$ . If  $t \in T_j(\text{env}_\lambda u)$ , then*

$$\left| V_{j,\zeta}^{(\lambda)}(u) - \int_{\{(x,y): \text{env}_\lambda u(x) > t\}} \zeta(|y|) d\Theta_j^n(\text{env}_\lambda u, (x,y)) \right| \leq \alpha V_j(\{\text{env}_\lambda u \leq t\}) \left( \max_{[0, \text{Lip}(\text{env}_\lambda u, t)]} |\rho| + \max_{[0, \text{Lip}(\text{env}_\lambda u, t)]} |\eta| \right),$$

where  $\alpha$  is a positive constant only depending on  $n$  and  $j$ .

We introduce the following auxiliary functions. For  $r > 0$ , we define  $\zeta_r : [0, \infty) \rightarrow \mathbb{R}$  as

$$\zeta_r(t) := \begin{cases} \zeta(t) & \text{for } t \geq r, \\ \zeta(r) & \text{for } 0 \leq t < r. \end{cases}$$

Note that we have  $\zeta_r \in C_c([0, \infty))$ . We also introduce the corresponding functions  $\eta_r, \rho_r : [0, \infty) \rightarrow \mathbb{R}$ , defined as

$$\eta_r(t) := \int_t^\infty s^{n-j+1} \zeta_r(s) ds, \quad \rho_r(t) := t^{n-1} \zeta_r(t) + (n-j)\eta_r(t).$$

Clearly,  $\eta_r, \rho_r \in C_c([0, \infty))$ . For  $t > r$ , we have  $\eta_r(t) = \eta(t)$ , and, for  $0 \leq t < r$ ,

$$\eta_r(t) = \int_t^r s^{n-j-1} \zeta(r) ds + \int_r^\infty s^{n-j-1} \zeta(s) ds = \zeta(r) \frac{r^{n-j} - t^{n-j}}{n-j} + \eta(r).$$

Hence

$$\eta_r(t) = \begin{cases} \eta(t) & \text{for } t \geq r, \\ \zeta(r) \frac{r^{n-j} - t^{n-j}}{n-j} + \eta(r) & \text{for } 0 \leq t < r. \end{cases}$$

Concerning the function  $\rho_r$ , we have  $\rho_r(t) = \rho(t)$  for  $t \geq r$  and

$$\begin{aligned} \rho_r(t) &= t^{n-j} \zeta(r) + (n-j) \zeta(r) \frac{r^{n-j} - t^{n-j}}{n-j} + (n-j)\eta(r) \\ &= r^{n-j} \zeta(r) + (n-j)\eta(r) = \rho(r) \end{aligned}$$

for  $0 \leq t < r$ . Hence

$$\rho_r(t) = \begin{cases} \rho(t) & \text{for } t \geq r, \\ \rho(r) & \text{for } 0 \leq t < r. \end{cases}$$

For every  $\delta > 0$ , we obtain

$$\begin{aligned} \max_{[0, \delta]} |\eta_r| &\leq |\eta(r)| + \frac{2}{n-j} |r^{n-j} \zeta(r)| + \max_{[0, \delta]} |\eta|, \\ \max_{[0, \delta]} |\rho_r| &\leq |\rho(r)| + \max_{[0, \delta]} |\rho|. \end{aligned} \tag{3.13}$$

These auxiliary functions are used in the following two lemmas.

LEMMA 3.19. *If  $u_k \in \text{Conv}_{\text{sc},0}(\mathbb{R}^n)$  epi-converges to  $u \in \text{Conv}_{\text{sc},0}(\mathbb{R}^n)$ , then*

$$\lim_{k \rightarrow \infty} V_{j,\zeta}^{(\lambda)}(u_k) = V_{j,\zeta}^{(\lambda)}(u)$$

for every  $\lambda > 0$ .

*Proof.* Fix  $\lambda > 0$  and set  $\bar{u} = \text{env}_\lambda u$  and  $\bar{u}_k = \text{env}_\lambda u_k$ . For any fixed  $\bar{t} > 0$ , there exists a constant  $\alpha > 0$  (depending on  $u$  and  $\bar{t}$ ) such that

$$|\nabla \bar{u}(x)| > 2\alpha \quad \text{for every } x \text{ such that } \bar{u}(x) \geq \bar{t}.$$

As  $u_k$  epi-converges to  $u$ , the sequence  $\bar{u}_k$  converges to  $\bar{u}$  and the sequence  $\nabla \bar{u}_k$  to  $\nabla \bar{u}$ . In both cases, the convergence is uniform on compact sets. Therefore, we have

$$|\nabla \bar{u}_k(x)| > \alpha \quad \text{for every } x \text{ such that } \bar{u}_k(x) \geq \bar{t}$$

for  $k$  sufficiently large.

Set  $r = \alpha$ . We have

$$\left| V_{j,\zeta}^{(\lambda)}(u) - V_{j,\zeta}^{(\lambda)}(u_k) \right| \leq \beta_0 + \dots + \beta_4,$$

where

$$\begin{aligned} \beta_0 &= \left| V_{j,\zeta_r}^{(\lambda)}(u) - V_{j,\zeta_r}^{(\lambda)}(u_k) \right|, \\ \beta_1 &= \left| V_{j,\zeta}^{(\lambda)}(u) - \int_{\{(x,y): \bar{u}(x) > \bar{t}\}} \zeta(|y|) d\Theta_j^n(\bar{u}, (x, y)) \right|, \\ \beta_2 &= \left| V_{j,\zeta_r}^{(\lambda)}(u) - \int_{\{(x,y): \bar{u}(x) > \bar{t}\}} \zeta_r(|y|) d\Theta_j^n(\bar{u}, (x, y)) \right|, \\ \beta_3 &= \left| V_{j,\zeta}^{(\lambda)}(u_k) - \int_{\{(x,y): \bar{u}_k(x) > \bar{t}\}} \zeta(|y|) d\Theta_j^n(\bar{u}_k, (x, y)) \right|, \\ \beta_4 &= \left| V_{j,\zeta_r}^{(\lambda)}(u_k) - \int_{\{(x,y): \bar{u}_k(x) > \bar{t}\}} \zeta_r(|y|) d\Theta_j^n(\bar{u}_k, (x, y)) \right|. \end{aligned}$$

These quantities depend, in general, on  $\bar{t}$  and  $k$  (note that  $r$  depends on  $\bar{t}$ ). We fix  $\varepsilon > 0$  and first consider the terms  $\beta_1$  and  $\beta_3$ . By Lemma 3.18, the continuity of  $\eta$  and  $\rho$  at  $t = 0$ , the relations  $\eta(0) = \rho(0) = 0$ , and the uniform convergence of  $\bar{u}_k$  to  $\bar{u}$  on compact sets, we may choose  $\bar{t}_1 > 0$  so that  $\beta_1, \beta_3 \leq \varepsilon$  for every

$$\bar{t} \in (0, \bar{t}_1) \cap T_j(\bar{u}) \cap \bigcap_{k \geq k_1} T_j(\bar{u}_k)$$

with  $k_1$  sufficiently large. Note that this set has full measure in  $(0, \bar{t}_1)$ .

Next, we deal with the term  $\beta_2$ . For  $t > 0$ , set

$$r(t) := \inf \left\{ |\nabla \bar{u}(x)| : x \text{ such that } \bar{u}(x) > t \right\}.$$

Note that  $r(t)$  tends to zero as  $t \rightarrow 0$ . By Lemma 3.18, we have for  $\bar{t} \in T_j(\bar{u})$ ,

$$\beta_2 \leq \alpha V_j(\{\bar{u} \leq \bar{t}\}) \left( \max_{[0, \text{Lip}(\bar{u}, \bar{t})]} |\rho_{r(\bar{t})}| + \max_{[0, \text{Lip}(\bar{u}, \bar{t})]} |\eta_{r(\bar{t})}| \right).$$

By (3.7) and (3.13) combined with the conditions on  $\rho$  and  $\eta$ , we deduce that there exists  $\bar{t}_2 > 0$  so that  $\beta_2 \leq \varepsilon$  for every  $\bar{t} \in (0, \bar{t}_2) \cap T_j(\bar{u})$ .

We proceed in a similar way for  $\beta_4$ . For  $k \in \mathbb{N}$  and  $t > 0$ , let

$$r_k(t) := \inf \left\{ |\nabla \bar{u}_k(x)| : x \text{ such that } \bar{u}_k(x) > t \right\}.$$

As before,  $r_k(t)$  tends to zero as  $t \rightarrow 0$ . Moreover, for  $\bar{t} \in T_j(\bar{u}_k)$ ,

$$\beta_4 \leq \alpha V_j(\{\bar{u}_k \leq \bar{t}\}) \left( \max_{[0, \text{Lip}(\bar{u}_k, \bar{t})]} |\rho_{r_k(\bar{t})}| + \max_{[0, \text{Lip}(\bar{u}_k, \bar{t})]} |\eta_{r_k(\bar{t})}| \right).$$

By (3.7) and (3.13) combined with the uniform convergence of  $\bar{u}_k$  to  $\bar{u}$  on compact sets, there exists  $\bar{t}_3 > 0$  such that  $\beta_4 \leq \varepsilon$  for every  $\bar{t} \in (0, \bar{t}_3) \cap \bigcap_{k \geq k_3} T_j(\bar{u}_k)$  with  $k_3$  sufficiently large.

Finally, for every fixed  $r > 0$ , and hence for every fixed  $\bar{t} > 0$ , the term  $\beta_0 \rightarrow 0$  as  $k \rightarrow \infty$ , since  $\zeta_r \in C_c([0, \infty))$  and by Lemma 2.12. This concludes the proof.  $\square$

LEMMA 3.20. *For every  $u \in \text{Conv}_{\text{sc},0}(\mathbb{R}^n)$  and  $\lambda > 0$ ,*

$$\lim_{r \rightarrow 0^+} V_{j, \zeta_r}^{(\lambda)}(u) = V_{j, \zeta}^{(\lambda)}(u).$$

*Proof.* For  $r > 0$ , let

$$t(r) := \inf \left\{ t > 0 : |\nabla \text{env}_\lambda u(x)| \geq r \text{ for all } x \text{ such that } \text{env}_\lambda u(x) > t \right\}.$$

This defines a monotone function of  $r$ , and

$$\lim_{r \rightarrow 0} t(r) = 0$$

since  $\text{env}_\lambda u$  is of class  $C^1$ .

For every  $r > 0$ , let  $\bar{t}(r) \in T_j(\text{env}_\lambda u)$  be such that  $t(r) \leq \bar{t}(r) \leq t(r) + r$ . We have

$$\begin{aligned} \left| V_{j, \zeta}^{(\lambda)}(u) - V_{j, \zeta_r}^{(\lambda)}(u) \right| &\leq \left| V_{j, \zeta}^{(\lambda)}(u) - \int_{\{(x,y) : \text{env}_\lambda u(x) > \bar{t}(r)\}} \zeta(|y|) d\Theta_j^n(\text{env}_\lambda u, (x, y)) \right| \\ &\quad + \left| V_{j, \zeta_r}^{(\lambda)}(u) - \int_{\{(x,y) : \text{env}_\lambda u(x) > \bar{t}(r)\}} \zeta_r(|y|) d\Theta_j^n(\text{env}_\lambda u, (x, y)) \right|. \end{aligned}$$

Let  $r \rightarrow 0^+$ . The conclusion follows from Lemma 3.18, (3.7), and (3.13).  $\square$

We extend  $V_{j,\zeta}^{(\lambda)}$  from a functional defined on  $\text{Conv}_{\text{sc},0}(\mathbb{R}^n)$  to a functional defined on  $\text{Conv}_{\text{sc}}(\mathbb{R}^n)$  in the following way. For every  $u \in \text{Conv}_{\text{sc}}(\mathbb{R}^n)$ , there exists a function  $u_0 \in \text{Conv}_{\text{sc},0}(\mathbb{R}^n)$  such that  $\text{epi}(u_0)$  is a translate of  $\text{epi}(u)$ . Indeed, if  $x_0 \in \mathbb{R}^n$  is such that  $u$  attains its absolute minimum at  $x_0$ , it is sufficient to define  $u_0$  for  $x \in \mathbb{R}^n$  as

$$u_0(x) := u(x + x_0) - \min_{\mathbb{R}^n} u.$$

We note that  $u_0$  is not uniquely determined (as  $x_0$  is not), but any two functions of this form can be obtained from each other by a translation of the epi-graph. Therefore,

$$V_{j,\zeta}^{(\lambda)}(u) := V_{j,\zeta}^{(\lambda)}(u_0)$$

is independent of the choice of the particular  $u_0$ . Clearly,  $V_{j,\zeta}^{(\lambda)}$  extended in this way is epi-translation invariant and, as  $V_{j,\zeta}^{(\lambda)}$  is rotation invariant on  $\text{Conv}_{\text{sc},0}(\mathbb{R}^n)$ , the same holds for its extension to  $\text{Conv}_{\text{sc}}(\mathbb{R}^n)$ . By Lemma 3.19, this extension is also continuous.

For every  $r > 0$ , we have  $\zeta_r \in C_c([0, \infty))$ . Hence, it follows from Lemma 3.20, that the functional  $V_{j,\zeta_r}^{(\lambda)}$ , defined in (3.12), is

$$V_{j,\zeta_r}^{(\lambda)}(u) = \int_{\mathbb{R}^n \times \mathbb{R}^n} \zeta_r(|y|) d\Theta_j^n(\text{env}_\lambda u, (x, y))$$

for  $u \in \text{Conv}_{\text{sc},0}(\mathbb{R}^n)$ , extends by this representation to  $\text{Conv}_{\text{sc}}(\mathbb{R}^n)$  and defines there an epi-translation invariant valuation. Hence, using the epi-translation invariance of both  $V_{j,\zeta}^{(\lambda)}$  and  $V_{j,\zeta_r}^{(\lambda)}$  on  $\text{Conv}_{\text{sc}}(\mathbb{R}^n)$ , we deduce from Lemma 3.20 that also  $V_{j,\zeta}^{(\lambda)}$  is a valuation on  $\text{Conv}_{\text{sc}}(\mathbb{R}^n)$ . We conclude that  $V_{j,\zeta}^{(\lambda)}$  is a continuous, epi-translation invariant and rotation invariant valuation on  $\text{Conv}_{\text{sc}}(\mathbb{R}^n)$ .

The inclusion  $D_j^n \subset D_i^n$  for  $i \in \{1, \dots, j\}$  ensures that we may repeat this construction replacing  $j$  by any  $i \in \{1, \dots, j\}$ .

**LEMMA 3.21.** *For every  $i \in \{1, \dots, j\}$  and  $\lambda > 0$ , the function  $V_{i,\zeta}^{(\lambda)}$  is a continuous, epi-translation and rotation invariant valuation on  $\text{Conv}_{\text{sc}}(\mathbb{R}^n)$ .*

For  $u \in \text{Conv}_{\text{sc}}(\mathbb{R}^n) \cap C_+^2(\mathbb{R}^n)$ , Proposition 3.2 shows that

$$V_{j,\zeta}(u) := \int_{\mathbb{R}^n} \zeta(|\nabla u(x)|) [D^2 u(x)]_{n-j} dx$$

is well-defined. Moreover, as  $D_j^n \subset D_i^n$  for  $i \in \{1, \dots, j\}$ , we may also set

$$V_{i,\zeta}(u) := \int_{\mathbb{R}^n} \zeta(|\nabla u(x)|) [D^2 u(x)]_{n-i} dx$$

for  $i = 1, \dots, j$  and  $u \in \text{Conv}_{\text{sc}}(\mathbb{R}^n) \cap C_+^2(\mathbb{R}^n)$ . Since  $\text{env}_\lambda u \in \text{Conv}_{\text{sc}}(\mathbb{R}^n) \cap C_+^2(\mathbb{R}^n)$  for such  $u$  and  $\lambda > 0$ , we have

$$V_{i,\zeta}(\text{env}_\lambda u) = V_{i,\zeta}^{(\lambda)}(u) \tag{3.14}$$

by Proposition 3.2.

For the final step of the proof of Theorem 1.2, we require the following statement.

LEMMA 3.22. *There exist  $\alpha_1, \dots, \alpha_{j+1} \in \mathbb{R}$  such that*

$$V_{j,\zeta}(u) = \sum_{i=1}^{j+1} \alpha_i V_{j,\zeta}^{(i)}(u) \quad (3.15)$$

for every  $u \in \text{Conv}_{\text{sc}}(\mathbb{R}^n) \cap C_+^2(\mathbb{R}^n)$ .

*Proof.* Let  $u \in \text{Conv}_{\text{sc}}(\mathbb{R}^n) \cap C_+^2(\mathbb{R}^n)$  and let  $\lambda > 0$ . As usual, we denote by  $u^*$  the conjugate of  $u$ . Note that

$$(\text{env}_\lambda u)^*(y) = u^*(y) + \lambda \frac{|y|^2}{2}$$

and that

$$[\mathbb{D}^2(\text{env}_\lambda u)^*(y)]_j = [\mathbb{D}^2(u^*(y) + \lambda \frac{|y|^2}{2})]_j = \sum_{i=0}^j \binom{n-i}{j-i} \lambda^{j-i} [\mathbb{D}^2 u^*(y)]_i$$

for every  $y \in \mathbb{R}^n$ . Hence, it follows from (3.14) and (2.9) that

$$\begin{aligned} V_{j,\zeta}^{(\lambda)}(u) &= \int_{\mathbb{R}^n} \zeta(|\nabla(\text{env}_\lambda u)(x)|) [\mathbb{D}^2(\text{env}_\lambda u)(x)]_{n-j} dx \\ &= \int_{\mathbb{R}^n} \zeta(|y|) [\mathbb{D}^2(\text{env}_\lambda u)^*(y)]_j dy \\ &= \sum_{i=0}^j \binom{n-i}{j-i} \lambda^{j-i} \int_{\mathbb{R}^n} \zeta(|y|) [\mathbb{D}^2 u^*(y)]_i dx \\ &= \sum_{i=0}^j \binom{n-i}{j-i} \lambda^{j-i} \int_{\mathbb{R}^n} \zeta(|\nabla u(x)|) [\mathbb{D}^2 u(x)]_{n-i} dx \\ &= \sum_{i=0}^j \binom{n-i}{j-i} \lambda^{j-i} V_{i,\zeta}(u). \end{aligned}$$

Inverting the system that we obtain by writing the previous equation for  $\lambda = 1, \dots, n+1$ , we deduce (3.15). We use that the matrix is a Vandermonde matrix and, therefore, invertible.  $\square$

Combined with Lemma 3.21, Lemma 3.22 implies that  $V_{j,\zeta}$  continuously extends to  $\text{Conv}_{\text{sc}}(\mathbb{R}^n)$ . This completes the proof of Theorem 1.2.

**3.4 Representation formulas.** As an application of Theorem 1.4, we derive representation formulas for  $V_{j,\zeta}$  and  $V_{j,\zeta}^*$  on convex functions with certain regularity properties.



LEMMA 3.23. *Let  $j \in \{0, \dots, n\}$  and  $\zeta \in D_j^n$ . We have*

$$V_{j,\zeta}^*(v) = \int_{\mathbb{R}^n} \zeta(|x|) d\Phi_j^n(v, x)$$

for  $v \in \text{Conv}_{(0)}(\mathbb{R}^n; \mathbb{R})$ .

*Proof.* By Theorem 1.4 and (2.5), we know that for every  $v \in \text{Conv}(\mathbb{R}^n; \mathbb{R}) \cap C_+^2(\mathbb{R}^n)$ ,

$$V_{j,\zeta}^*(v) = \int_{\mathbb{R}^n} \zeta(|x|) d\Phi_j^n(v, x) = \int_{\mathbb{R}^n} \zeta(|x|) [D^2v(x)]_j dx.$$

Now, let  $v \in \text{Conv}_{(0)}(\mathbb{R}^n; \mathbb{R})$ . There exists a sequence  $v_k$  of functions from  $\text{Conv}(\mathbb{R}^n; \mathbb{R}) \cap C_+^2(\mathbb{R}^n)$  that converges to  $v$ . Indeed, by a standard mollification procedure, we first find a sequence of convex functions  $\bar{v}_k$  of class  $C^2(\mathbb{R}^n)$  that converges to  $v$  on compact sets. Then we define, for every  $x \in \mathbb{R}^n$  and  $k \in \mathbb{N}$ ,

$$v_k(x) := \bar{v}_k(x) + \frac{1}{k}|x|^2.$$

Then clearly  $v_k \in \text{Conv}(\mathbb{R}^n; \mathbb{R}) \cap C_+^2(\mathbb{R}^n)$ , and  $v_k$  converges to  $v$ . Moreover, if  $\bar{r} > 0$  is such that  $v$  is of class  $C^2$  in a neighborhood of  $\bar{r}B^n$ , then the convergence is in  $C^2$ -norm in this set.

Since  $V_{j,\zeta}^*$  is continuous, it is now sufficient to prove that

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}^n} \zeta(|x|) d\Phi_j^n(v_k, x) = \int_{\mathbb{R}^n} \zeta(|x|) d\Phi_j^n(v, x).$$

By (2.7) and since  $\zeta$  is continuous on  $(0, \infty)$ , we have

$$\lim_{k \rightarrow \infty} \int_{\{x: |x| \geq r\}} \zeta(|x|) d\Phi_j^n(v_k, x) = \int_{\{x: |x| \geq r\}} \zeta(|x|) d\Phi_j^n(v, x) \quad (3.16)$$

for  $r > 0$  with  $\Phi_j^n(v, \{v = r\}) = 0$ . Since  $\Phi_j^n(v, \cdot)$  is locally finite, (3.16) holds for a.e.  $r > 0$ . On the other hand, for every  $r < \bar{r}$  and for every  $k$  we have

$$\int_{\{x: |x| < r\}} \zeta(|x|) d\Phi_j^n(v_k, x) = \int_{\{x: |x| < r\}} \zeta(|x|) [D^2v_k(x)]_j dx$$

and a corresponding relation for  $v$ . Using the convergence of  $v_k$  to  $v$  in  $\bar{r}B^n$  in  $C^2$ -norm, polar coordinates and the fact that for  $\delta > 0$ ,

$$\int_0^\delta r^{n-1} \zeta(r) dr$$

is finite, we obtain that for every  $\varepsilon > 0$  there exists  $0 < \delta < \bar{r}$  such that

$$\left| \int_{\{x: |x| < \delta\}} \zeta(|x|) [D^2v(x)]_j dx \right|, \left| \int_{\{x: |x| < \delta\}} \zeta(|x|) [D^2v_m(x)]_j dx \right| \leq \varepsilon \quad (3.17)$$

for every  $k \in \mathbb{N}$ . The conclusion follows from (3.16) and (3.17).  $\square$

An immediate consequence is the following result.

LEMMA 3.24. *Let  $j \in \{0, \dots, n\}$  and  $\zeta \in D_j^n$ . We have*

$$V_{j,\zeta}(u) = \int_{\mathbb{R}^n} \zeta(|y|) d\Psi_j^n(u, y)$$

for  $u^* \in \text{Conv}_{(0)}(\mathbb{R}^n; \mathbb{R})$ .

## 4 The classification result

In the proof of Theorem 1.3 and Theorem 1.5, we follow the original approach by Hadwiger [Had57]. First, we establish a classification of valuations on  $\text{Conv}_{\text{sc}}(\mathbb{R}^n)$  that are epi-homogeneous of degree 1. For this, we introduce rotational epi-symmetrization and reduce the classification problem to the solution of an integral equation. We immediately obtain, by duality, a classification of valuations on  $\text{Conv}(\mathbb{R}^n; \mathbb{R})$  that are homogeneous of degree 1. Next, we introduce orthogonal cylinder functions on  $\text{Conv}_{\text{sc}}(\mathbb{R}^n)$  and show that if a valuation vanishes on this class of functions, then it has to be epi-homogeneous of degree 1. By duality, we also obtain that if a valuation vanishes on dual orthogonal cylinder functions on  $\text{Conv}(\mathbb{R}^n; \mathbb{R})$ , then it has to be homogeneous of degree 1. In the final step, we work in the dual setting and use induction on the dimension to establish a representation formula of valuations on dual orthogonal cylinder functions. Combined with the results established in the first part and the classification of one-homogeneous valuations, it completes the proof in the dual setting, that is, the proof of Theorem 1.5. The result in the primal setting, that is, Theorem 1.3, follows by duality.

**4.1 The case of epi-homogeneity of degree 1.** Let  $n \geq 2$ . A classical result by Hadwiger [Had57, §4.5.3] (or see, [Sch14, Theorem 3.3.5]) states that for every convex body  $K \subset \mathbb{R}^n$ , that is at least one-dimensional, there exists a sequence of rotation means of  $K$  that converges to a ball. Here a *rotation mean* of  $K$  is given, for  $k \geq 1$  and rotations  $\vartheta_1, \dots, \vartheta_k \in \text{SO}(n)$ , as

$$\frac{1}{k} (\vartheta_1 K + \dots + \vartheta_k K).$$

For  $u \in \text{Conv}_{\text{sc}}(\mathbb{R}^n)$  and rotations  $\vartheta_1, \dots, \vartheta_k \in \text{SO}(n)$ , we call

$$\frac{1}{k} \cdot (u \circ \vartheta_1^{-1} \square \dots \square u \circ \vartheta_k^{-1})$$

a *rotation epi-mean* of  $u$ . A function  $u \in \text{Conv}_{\text{sc}}(\mathbb{R}^n)$  is called *radial* if  $u = u \circ \vartheta^{-1}$  for every  $\vartheta \in \text{SO}(n)$ .

For  $u \in \text{Conv}_{\text{sc}}(\mathbb{R}^n)$ , define its *rotational epi-symmetrization*  $u^\circ$  via the support function of its epi-graph for  $y \in \mathbb{R}^n$  and  $t \in \mathbb{R}$  by

$$h_{\text{epi } u^\circ}(y, t) := \int_{\text{SO}(n)} h_{\text{epi } u}(\vartheta^{-1}y, t) d\vartheta,$$

where we integrate with respect to the Haar probability measure on  $\mathrm{SO}(n)$ . The following result is obtained by suitably approximating the above integral.

LEMMA 4.1. *For every  $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$  with at least one-dimensional domain, there is a sequence of rotation epi-means of  $u$  epi-converging to its rotation epi-mean  $u^\circ \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ .*

Recall that for given  $t \geq 0$ , we have  $u_t(x) := t|x| + \mathbf{I}_{B^n}(x)$  for  $x \in \mathbb{R}^n$ .

LEMMA 4.2. *If  $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$  is radial, non-negative, and such that  $u(0) = 0$ , then there exists a sequence  $u_k$  which epi-converges to  $u$  and such that each  $u_k$  is the finite epi-sum of functions of the form  $r \cdot u_t$  with  $r > 0$  and  $t \geq 0$ .*

*Proof.* Since the function  $u$  is radial, this is a one-dimensional problem. Any piecewise affine function  $\bar{u}: [0, \infty) \rightarrow [0, \infty)$  with positive derivative at 0 can be written as an epi-sum of finitely many functions of the form  $s \mapsto t s + \mathbf{I}_{[0,r]}(s)$  with suitable  $r > 0$  and  $t \geq 0$ . Hence the statement follows from the fact that such piecewise affine functions are dense.  $\square$

The main result of this section is the following result.

PROPOSITION 4.3. *Let  $n \geq 2$ . If  $Z: \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \rightarrow \mathbb{R}$  is a continuous, epi-translation and rotation invariant valuation that is epi-homogeneous of degree 1, then there exists  $\zeta \in D_1^n$  such that*

$$Z(u) = V_{1,\zeta}(u)$$

for every  $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ .

*Proof.* Since  $Z$  is a continuous valuation that is epi-homogeneous of degree 1, Lemma 2.2 implies that it is epi-additive. Let  $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$  and let  $u_k$  be any rotation epi-mean of  $u$ , that is, there are rotations  $\vartheta_1, \dots, \vartheta_{m_k}$  such that

$$u_k = \frac{1}{m_k} \cdot \left( u \circ \vartheta_1^{-1} \square \dots \square u \circ \vartheta_{m_k}^{-1} \right).$$

Since  $Z$  is epi-additive, rotation invariant, and epi-homogeneous of degree 1, we see that

$$Z(u) = Z(u_k).$$

By Lemma 4.1, there is a sequence of rotation epi-means of  $u$  that converges to the rotation epi-symmetral  $u^\circ$ . Since  $Z$  is continuous, it follows that

$$Z(u) = Z(u^\circ).$$

Hence we obtain that  $Z$  is determined by its values on radial functions.

Since  $Z$  is epi-translation invariant,  $Z$  is even determined by radial functions  $u$  that are non-negative with  $u(0) = 0$ . Combined with the epi-additivity and the epi-homogeneity of degree 1 of  $Z$ , Lemma 4.2 implies that  $Z$  is already determined by

its values on the one-parameter family  $u_t$  for  $t \geq 0$ . Thus, the function  $t \mapsto Z(u_t)$  on  $[0, \infty)$  uniquely determines  $Z$  and it suffices to show that there is  $\zeta \in D_1^n$  such that

$$Z(u_t) = V_{1,\zeta}(u_t) \quad (4.1)$$

for  $t \geq 0$ .

Note that the continuity of  $Z$  implies that  $t \mapsto Z(u_t)$  is continuous in  $[0, \infty)$ . Let us prove that it has compact support. Assume that this is not true. Hence, there exists a sequence  $t_k$  such that  $t_k \rightarrow \infty$  as  $k \rightarrow \infty$ , and

$$\alpha_k := Z(u_{t_k}) > 0$$

for all  $k \in \mathbb{N}$ . Define  $\bar{u}_k \in \text{Conv}_{\text{sc}}(\mathbb{R}^n)$  by

$$\bar{u}_k(x) := \frac{1}{\alpha_k} \cdot u_{t_k}(x) = t_k |x| + \mathbf{I}_{\frac{1}{\alpha_k} B^n}(x).$$

By the epi-homogeneity of  $Z$ , we have  $Z(\bar{u}_k) = 1$  for every  $k \in \mathbb{N}$ . On the other hand, the sequence  $\bar{u}_k$  epi-converges to  $\mathbf{I}_{\{0\}}$ , and it is easy to prove, using once more the fact that  $Z$  is epi-homogeneous of degree 1, that  $Z(\mathbf{I}_{\{0\}}) = 0$ . Hence we have a contradiction to the continuity of  $Z$ .

To prove (4.1), we solve an integral equation. Let  $u_t^*$  be the dual function of  $u_t$  and note that we have  $u_t^* \in \text{Conv}_{(0)}(\mathbb{R}^n; \mathbb{R})$ . By the definition of  $V_{1,\bar{\zeta}}$  combined with Lemma 3.24, we can calculate  $V_{1,\bar{\zeta}}(u_t)$  for  $\bar{\zeta} \in D_1^n$  by Lemma 2.15 and obtain

$$V_{1,\bar{\zeta}}(u_t) = \omega_n \left( \bar{\zeta}(t) t^{n-1} + (n-1) \int_t^\infty r^{n-2} \bar{\zeta}(r) dr \right).$$

Hence, we need to solve the integral equation,

$$\zeta(t) + \frac{n-1}{t^{n-1}} \int_t^\infty r^{n-2} \zeta(r) dr = \frac{Z(u_t)}{\omega_n t^{n-1}},$$

to determine  $\zeta$  when  $Z(u_t)$  is given. We claim that

$$\zeta(t) := \frac{Z(u_t)}{\omega_n t^{n-1}} - \frac{(n-1)}{\omega_n} \int_t^\infty \frac{Z(u_r)}{r^n} dr \quad (4.2)$$

is a solution. Indeed, using integration by parts and that  $t \mapsto Z(u_t)$  has compact support, we obtain that

$$(n-1) \int_t^\infty r^{n-2} \int_r^\infty \frac{Z(u_s)}{s^n} ds dr = -t^{n-1} \int_t^\infty \frac{Z(u_r)}{r^n} dr + \int_t^\infty \frac{Z(u_r)}{r} dr, \quad (4.3)$$

and the result follows. Since  $t \mapsto Z(u_t)$  is continuous with compact support, also  $\zeta$  is continuous on  $(0, \infty)$  and it has bounded support.

Finally, we show that the solution  $\zeta$  defined in (4.2) is in  $D_1^n$ . First, we show that

$$\lim_{t \rightarrow 0^+} \int_t^\infty r^{n-2} \zeta(r) dr \quad \text{exists and is finite.} \quad (4.4)$$

Note, that for  $t > 0$ ,

$$\omega_n \int_t^\infty r^{n-2} \zeta(r) \, dr = \int_t^\infty \left( \frac{Z(u_r)}{r} - (n-1)r^{n-2} \int_r^\infty \frac{Z(u_s)}{s^n} \, ds \right) \, dr$$

and therefore, by (4.3),

$$\begin{aligned} \omega_n \int_t^\infty r^{n-2} \zeta(r) \, dr &= \int_t^\infty \frac{Z(u_r)}{r} \, dr + t^{n-1} \int_t^\infty \frac{Z(u_r)}{r^n} \, dr - \int_t^\infty \frac{Z(u_r)}{r} \, dr \\ &= t^{n-1} \int_t^\infty \frac{Z(u_r)}{r^n} \, dr. \end{aligned}$$

Since  $r \mapsto Z(u_r)$  is in  $C_c([0, \infty))$  it follows from L'Hospital's rule that

$$\lim_{t \rightarrow 0^+} t^{n-1} \int_t^\infty \frac{Z(u_r)}{r^n} \, dr = \frac{Z(u_0)}{n-1} \quad (4.5)$$

and we have shown (4.4).

Second, since

$$\begin{aligned} \omega_n t^{n-1} \zeta(t) &= \omega_n t^{n-1} \left( \frac{Z(u_t)}{\omega_n t^{n-1}} - \frac{n-1}{\omega_n} \int_t^\infty \frac{Z(u_r)}{r^n} \, dr \right) \\ &= Z(u_t) - (n-1)t^{n-1} \int_t^\infty \frac{Z(u_r)}{r^n} \, dr, \end{aligned}$$

it follows from (4.5) that

$$\lim_{t \rightarrow 0^+} t^{n-1} \zeta(t) = 0.$$

Thus,  $\zeta \in D_1^n$  and we have proved (4.1).  $\square$

We immediately obtain the following dual result.

**PROPOSITION 4.4.** *Let  $n \geq 2$ . If  $Z: \text{Conv}(\mathbb{R}^n; \mathbb{R}) \rightarrow \mathbb{R}$  is a continuous, dually epi-translation and rotation invariant valuation that is homogeneous of degree 1, then there exists  $\zeta \in D_1^n$  such that*

$$Z(v) = V_{1, \zeta}^*(v)$$

for every  $v \in \text{Conv}(\mathbb{R}^n; \mathbb{R})$ .

**4.2 Orthogonal cylinder functions.** Let  $P \subset \mathbb{R}^n$  be an  $n$ -dimensional convex polytope. We call the  $n$ -dimensional convex polytopes  $P_1, \dots, P_m \subset \mathbb{R}^n$  a *dissection* of  $P$ , if

$$P = \bigcup_{i=1}^m P_i$$

and the interiors of the polytopes  $P_i$  and  $P_j$  are disjoint for  $i \neq j$ . In this case, we write

$$P = \bigsqcup_{i=1}^m P_i.$$

Two  $n$ -dimensional convex polytopes  $P$  and  $Q$  are *translatively equi-dissectable*, written  $P \sim Q$ , if there are dissections  $P = \bigsqcup_{i=1}^m P_i$  and  $Q = \bigsqcup_{i=1}^m Q_i$  such that  $P_i$  is a translate of  $Q_i$  for  $i = 1, \dots, m$ .

A valuation  $Z : \mathcal{P}^n \rightarrow \mathbb{R}$  is *simple* if it vanishes on lower dimensional polytopes. For a simple valuation  $Z : \mathcal{P}^n \rightarrow \mathbb{R}$ , we have

$$Z\left(\bigsqcup_{i=1}^m P_i\right) = \sum_{i=1}^m Z(P_i) \quad (4.6)$$

(see, for example, [Sch14, Sect. 6.2]).

An  $n$ -dimensional simplex  $S$  is the convex hull of  $(n+1)$  affinely independent points  $p_0, \dots, p_n$ . Set  $x_i = p_i - p_{i-1}$ . We say that  $S$  is orthogonal if the vectors  $x_1, \dots, x_n$  are pairwise orthogonal. We set  $x_0 = p_0$  and write  $S = \langle x_0; x_1, \dots, x_n \rangle$ . For  $0 < t < 1$ , the *canonical simplex dissection* is

$$S = \bigsqcup_{k=0}^n ((1-t)\underline{S}_k + t\overline{S}_{n-k}), \quad (4.7)$$

where

$$\underline{S}_k := \langle x_0; x_1, \dots, x_k \rangle, \quad \overline{S}_{n-k} := \langle x_0 + \sum_{i=1}^k x_i; x_{k+1}, \dots, x_n \rangle$$

while  $+$  in (4.7) denotes Minkowski addition (see Hadwiger [Had57]).

Let  $\mathcal{P}^n$  denote the set of convex polytopes in  $\mathbb{R}^n$ . We say that  $P \in \mathcal{P}^n$  is a *proper orthogonal cylinder* if there are orthogonal and complementary subspaces  $E$  and  $F$  with  $\dim E, \dim F \geq 1$  and polytopes  $P_E \subset E$  and  $P_F \subset F$  such that  $P = P_E + P_F$ . If  $S$  is an orthogonal simplex, then the terms in (4.7) for  $1 < k < n$  are proper orthogonal cylinders.

The following result is due to Hadwiger [Had57, Sect. 1.3.4, Theorem VII].

LEMMA 4.5. *For an  $n$ -dimensional polytope  $P \in \mathcal{P}^n$ , there are orthogonal simplices  $S_1, \dots, S_m, S'_1, \dots, S'_{m'}$  such that*

$$P \sqcup \bigsqcup_{i=1}^m S_i \sim \bigsqcup_{j=1}^{m'} S'_j.$$

Let  $Z : \text{Conv}_{\text{sc}}(\mathbb{R}^n) \rightarrow \mathbb{R}$  be a valuation. We say that it is *simple* if  $Z$  vanishes on functions with lower dimensional domain. The following analog of (4.6) is a consequence of the inclusion-exclusion principle established in [CLM201]. Let

$u_1, \dots, u_m \in \text{Conv}_{\text{sc}}(\mathbb{R}^n)$  be such that the domains of  $u_i \vee u_j$  are lower dimensional for  $i \neq j$ . If  $Z : \text{Conv}_{\text{sc}}(\mathbb{R}^n) \rightarrow \mathbb{R}$  is a simple valuation, then

$$Z\left(\bigwedge_{i=1}^m u_i\right) = \sum_{i=1}^m Z(u_i). \quad (4.8)$$

A function  $u \in \text{Conv}_{\text{sc}}(\mathbb{R}^n)$  is called *piecewise affine* if its domain is a convex polytope and  $u$  is the pointwise maximum of finitely many affine functions on its domain. Hence there is a dissection of  $\text{dom}(u)$  into polytopes  $P_1, \dots, P_m$  such that

$$u = \bigwedge_{i=1}^m (\ell_i + \alpha_i + \mathbf{I}_{P_i})$$

with linear functions  $\ell_i : \mathbb{R}^n \rightarrow \mathbb{R}$  and  $\alpha_i \in \mathbb{R}$  for  $i = 1, \dots, m$ . If  $Z : \text{Conv}_{\text{sc}}(\mathbb{R}^n) \rightarrow \mathbb{R}$  is an epi-translation invariant and simple valuation, we obtain by (4.8) that

$$Z(u) = \sum_{i=1}^m Z(\ell_i + \mathbf{I}_{P_i}). \quad (4.9)$$

Note that piecewise affine functions are dense in  $\text{Conv}_{\text{sc}}(\mathbb{R}^n)$ .

We say that the function  $u \in \text{Conv}_{\text{sc}}(\mathbb{R}^n)$  is an *orthogonal cylinder function* if there are orthogonal and complementary subspaces  $E$  and  $F$  with  $\dim E, \dim F \geq 1$  and  $u_E \in \text{Conv}_{\text{sc}}(\mathbb{R}^n)$  with  $\text{dom}(u_E) \subset E$  and  $u_F \in \text{Conv}_{\text{sc}}(\mathbb{R}^n)$  with  $\text{dom}(u_F) \subset F$  such that  $u = u_E \square u_F$ .

**PROPOSITION 4.6.** *If  $Z : \text{Conv}_{\text{sc}}(\mathbb{R}^n) \rightarrow \mathbb{R}$  is a continuous and epi-translation invariant valuation that vanishes on orthogonal cylinder functions, then it is epi-homogeneous of degree 1.*

*Proof.* For  $y \in \mathbb{R}^n$ , set  $\ell_y(x) = \langle y, x \rangle$ . Define  $\tilde{Z} : \mathcal{P}^n \rightarrow \mathbb{R}$  by

$$\tilde{Z}(P) := Z(\ell_y + \mathbf{I}_P).$$

Since  $Z$  is continuous, so is  $\tilde{Z}$ .

Let  $S$  be an  $n$ -dimensional orthogonal simplex in  $\mathbb{R}^n$  and  $0 < t < 1$ . In the canonical simplex dissection

$$S = \bigsqcup_{k=0}^n ((1-t)\underline{S}_k + t\overline{S}_{n-k}),$$

the simplices  $\underline{S}_k$  and  $\overline{S}_{n-k}$  are orthogonal and lie in orthogonal subspaces. Since  $Z$  vanishes on orthogonal cylinder functions, it is simple. Consequently,  $\tilde{Z}$  vanishes on orthogonal cylinders and is simple. By (4.6), this implies

$$\tilde{Z}(S) = \tilde{Z}((1-t)S) + \tilde{Z}(tS).$$

Let  $r, s > 0$ . Setting  $\alpha(r) = \tilde{Z}(rS)$  and  $t = r/(r+s)$ , we obtain

$$\alpha(r+s) = \alpha(r) + \alpha(s)$$

for  $r, s > 0$ . Since  $\tilde{Z}$  is continuous, so is  $\alpha : (0, \infty) \rightarrow \mathbb{R}$ . Hence,  $\alpha$  is a continuous solution of Cauchy's functional equation, and we obtain that

$$\tilde{Z}(tS) = t\tilde{Z}(S)$$

for any  $t > 0$  and any orthogonal simplex  $S$ .

For  $P \in \mathcal{P}^n$ , by Lemma 4.5 there are orthogonal simplices such that

$$P \sqcup \bigsqcup_{i=1}^m S_i \sim \bigsqcup_{j=1}^{m'} S'_j.$$

Therefore, for every  $t > 0$ ,

$$\tilde{Z}(tP) = \tilde{Z}\left(\bigsqcup_{j=1}^{m'} tS'_j\right) - \tilde{Z}_y\left(\bigsqcup_{i=1}^m tS_i\right) = t\tilde{Z}\left(\bigsqcup_{j=1}^{m'} S'_j\right) - t\tilde{Z}_y\left(\bigsqcup_{i=1}^m S_i\right) = t\tilde{Z}(P).$$

Hence,  $\tilde{Z}$  is homogeneous of degree 1 on polytopes. Consequently,  $Z$  is epi-homogeneous of degree 1 on functions of the form

$$u = \ell_y + \mathbf{I}_P$$

for every  $y \in \mathbb{R}^n$  and  $P \in \mathcal{P}^n$ . Since  $Z$  is epi-translation invariant and simple, (4.9) now implies that  $Z$  is epi-homogeneous of degree 1 on piecewise affine functions. Since piecewise affine functions are dense in  $\text{Conv}_{\text{sc}}(\mathbb{R}^n)$ , the continuity of  $Z$  now implies that  $Z$  is epi-homogeneous of degree 1 on  $\text{Conv}_{\text{sc}}(\mathbb{R}^n)$ .  $\square$

We say that  $v \in \text{Conv}(\mathbb{R}^n; \mathbb{R})$  is a *dual orthogonal cylinder function* if there are orthogonal and complementary subspaces  $E$  and  $F$  with  $\dim E, \dim F \geq 1$  as well as functions  $v_E \in \text{Conv}(E; \mathbb{R})$  and  $v_F \in \text{Conv}(F; \mathbb{R})$  such that  $v = v_E + v_F$ . We use (2.4) and obtain the following result as an immediate consequence of Proposition 4.6.

**PROPOSITION 4.7.** *If  $Z: \text{Conv}(\mathbb{R}^n; \mathbb{R}) \rightarrow \mathbb{R}$  is a continuous and dually epi-translation invariant valuation that vanishes on dual orthogonal cylinder functions, then it is homogeneous of degree 1.*

**4.3 Additional results on Hessian measures and valuations.** On a special class of functions, we write the measure  $\Phi_l^n$  as a sum of product measures.

**LEMMA 4.8.** *Let  $E$  and  $F$  be orthogonal and complementary subspaces of  $\mathbb{R}^n$  such that  $1 \leq k < n$  with  $k = \dim E$ . If  $v_E \in \text{Conv}(E; \mathbb{R})$  and  $v_F \in \text{Conv}(F; \mathbb{R})$ , then*

$$\Phi_l^n(v_E + v_F, \cdot) = \sum_{i=0 \vee (l+k-n)}^{k \wedge l} \Phi_i^k(v_E, \cdot) \otimes \Phi_{l-i}^{n-k}(v_F, \cdot)$$

for every  $0 \leq l \leq n$ .



*Proof.* Let  $v = v_E + v_F$ . Observe that  $y \in \partial v(x)$  if and only if  $y_E \in \partial v_E(x_E)$  and  $y_F \in \partial v_F(x_F)$ . Therefore, for  $s \geq 0$  and Borel subsets  $B_E \subset E$  and  $B_F \subset F$ ,

$$\begin{aligned} P_s(v, B_E \times B_F \times \mathbb{R}^n) &= \{x + sy : x \in B_E \times B_F, y \in \partial v(x)\} \\ &= \{(x_E, x_F) + s(y_E, y_F) : x_E \in B_E, x_F \in B_F, y_E \in \partial v_E(x_E), y_F \in \partial v_F(x_F)\} \\ &= P_s(v_E, B_E \times E) \times P_s(v_F, B_F \times F). \end{aligned}$$

Thus,

$$\begin{aligned} \sum_{l=0}^n s^l \Phi_l^n(v, B_E \times B_F) &= \mathcal{H}^n(P_s(v, B_E \times B_F \times \mathbb{R}^n)) \\ &= \mathcal{H}^k(P_s(v_E, B_E \times E)) \mathcal{H}^{n-k}(P_s(v_F, B_F \times F)) \\ &= \sum_{i=0}^k s^i \Phi_i^k(v_E, B_E) \sum_{j=0}^{n-k} s^j \Phi_j^{n-k}(v_F, B_F). \end{aligned}$$

Consequently, by comparing coefficients, we obtain

$$\Phi_l^n(v, \cdot) = \sum_{i=0 \vee (l-(n-k))}^{k \wedge l} \Phi_i^k(v_E, \cdot) \otimes \Phi_{l-i}^{n-k}(v_F, \cdot)$$

on product sets. This completes the proof since the statement only needs to be checked on such sets.  $\square$

We repeatedly evaluate Hessian measures at piecewise affine functions. Fix a point  $\bar{x} = (\bar{x}_1, \dots, \bar{x}_n) \in \mathbb{R}^n$  and define  $\bar{v} \in \text{Conv}(\mathbb{R}^n; \mathbb{R})$  as

$$\bar{v}(x_1, \dots, x_n) := \frac{1}{2} \sum_{i=1}^n |x_i - \bar{x}_i|. \quad (4.10)$$

Note, that if  $\bar{x}_1, \dots, \bar{x}_n \neq 0$ , then  $\bar{v} \in \text{Conv}_{(0)}(\mathbb{R}^n; \mathbb{R})$ . If  $\bar{u} = \bar{v}^* \in \text{Conv}_{\text{sc}}(\mathbb{R}^n)$ , then

$$\bar{u}(x) = \mathbf{I}_{[-\frac{1}{2}, \frac{1}{2}]^n}(x) + \langle \bar{x}, x \rangle$$

for  $x \in \mathbb{R}^n$ . It is now easy to see by (2.8) that

$$\int_{\mathbb{R}^n} \zeta(x) d\Phi_n^n(\bar{v}, x) = \int_{\mathbb{R}^n} \zeta(y) d\Psi_n^n(\bar{u}, y) = \int_{[-\frac{1}{2}, \frac{1}{2}]^n} \zeta(\bar{x}) dx = \zeta(\bar{x})$$

for every continuous function  $\zeta : \mathbb{R}^n \rightarrow \mathbb{R}$  with compact support (for the second equality, compare [CLM202, Sect. 10.4]). Hence, we conclude that

$$\Phi_n^n(\bar{v}, \cdot) = \delta_{\bar{x}}, \quad (4.11)$$

where  $\delta_{\bar{x}}$  denotes the Dirac point measure concentrated at  $\bar{x}$ .

Next, for  $1 \leq j < n$ , let

$$v(x_1, \dots, x_n) := \frac{1}{2} \sum_{i=1}^j |x_i - \bar{x}_i|$$

with  $\bar{x}_1, \dots, \bar{x}_j \in \mathbb{R}$ . Observe that  $v$  is of the form

$$v(x_1, \dots, x_n) = v_1(x_1, \dots, x_j) + v_2(x_{j+1}, \dots, x_n)$$

where  $v_1$  is of the same form as (4.10) and  $v_2 \equiv 0$ . In particular,  $\Phi_i^{n-j}(v_2, A) = 0$  for every  $1 \leq i \leq n - j$  and Borel subset  $A \subseteq \mathbb{R}^{n-j}$ . Hence, it follows from Lemma 4.8, (2.6) and (4.11) that

$$\begin{aligned} d\Phi_j^n(v, (x_1, \dots, x_n)) &= d\Phi_j^j(v_1, (x_1, \dots, x_j)) d\Phi_0^{n-j}(v_2, (x_{j+1}, \dots, x_n)) \\ &= d\delta_{(\bar{x}_1, \dots, \bar{x}_j)}((x_1, \dots, x_j)) dx_{j+1} \cdots dx_n \end{aligned} \quad (4.12)$$

and

$$\Phi_i^n(v, B) = 0 \quad (4.13)$$

for every  $j < i \leq n$  and Borel subset  $B \subseteq \mathbb{R}^n$ .

We require the following result on vanishing integrals. We remark that the condition (4.14) implies, in particular, that  $x \mapsto \zeta(|x|)$  is in  $L^1(\mathbb{R}^n, \Phi_j^n(v, \cdot))$  for  $v \in \text{Conv}_{(0)}(\mathbb{R}^n; \mathbb{R})$ .

LEMMA 4.9. *Let  $\zeta \in C_b((0, \infty))$  and  $j \in \{1, \dots, n\}$ . If*

$$\int_{\mathbb{R}^n} \zeta(|x|) d\Phi_j^n(v, x) = 0 \quad (4.14)$$

for every  $v \in \text{Conv}_{(0)}(\mathbb{R}^n; \mathbb{R})$ , then  $\zeta \equiv 0$ .

*Proof.* Fix  $\bar{x}_1, \dots, \bar{x}_j \in \mathbb{R} \setminus \{0\}$ , set  $t = \sqrt{\bar{x}_1^2 + \dots + \bar{x}_j^2} \neq 0$  and  $v(x_1, \dots, x_n) = \frac{1}{2} \sum_{i=1}^j |x_i - \bar{x}_i|$ . For  $j = n$ , the measure  $\Phi_n^n(v, \cdot)$  is the Dirac point mass concentrated at  $(\bar{x}_1, \dots, \bar{x}_n)$  by (4.11). Hence

$$0 = \int_{\mathbb{R}^n} \zeta(|x|) d\Phi_j^n(v, x) = \zeta(t),$$

and the result follows. For  $1 \leq j < n$ , we use (4.12) and obtain

$$\begin{aligned} 0 &= \int_{\mathbb{R}^n} \zeta(|x|) d\Phi_j^n(v, x) \\ &= \int_{\mathbb{R}^{n-j}} \zeta(|(\bar{x}_1, \dots, \bar{x}_j, x_{j+1}, \dots, x_n)|) dx_{j+1} \cdots dx_n \\ &= \omega_{n-j} \int_0^\infty \zeta(\sqrt{r^2 + t^2}) r^{n-j-1} dr. \end{aligned}$$

Since  $\bar{x}_1, \dots, \bar{x}_j \in \mathbb{R} \setminus \{0\}$  and hence  $t > 0$  are arbitrary, the result now follows from Lemma 2.14.  $\square$

LEMMA 4.10. *Let  $j \in \{1, \dots, n\}$  and  $\zeta_m, \zeta \in D_j^n$ . If*

$$\lim_{m \rightarrow \infty} \int_{\mathbb{R}^n} \zeta_m(|x|) d\Phi_j^n(v_m, x) = \int_{\mathbb{R}^n} \zeta(|x|) d\Phi_j^n(v, x) \quad (4.15)$$

for every  $v_m, v \in \text{Conv}_{(0)}(\mathbb{R}^n; \mathbb{R})$  with  $v_m$  epi-convergent to  $v$ , then  $\lim_{m \rightarrow \infty} \zeta_m(t) = \zeta(t)$  for every  $t > 0$ .

*Proof.* In case  $j = n$ , we may simply choose  $v_m(x) = \bar{v}(x) = \frac{1}{2} \sum_{j=1}^n |x_j - \bar{x}_j|$  with  $\bar{x}_j \in \mathbb{R} \setminus \{0\}$ . It now follows from (4.11) and (4.15) that  $\lim_{m \rightarrow \infty} \zeta_m(t) = \zeta(t)$  for  $t = \sqrt{\bar{x}_1^2 + \dots + \bar{x}_n^2} > 0$ .

So let  $1 \leq j < n$ . For  $t > 0$ , we will consider the functions  $v_t$  defined in (2.2). Note, that  $v_t \equiv 0$  on  $tB^n$  and hence  $v_t \in \text{Conv}_{(0)}(\mathbb{R}^n; \mathbb{R})$ . It follows from Lemma 2.15 that

$$\int_{\mathbb{R}^n} \zeta_m(|x|) d\Phi_j^n(v_t, x) = \kappa_n \binom{n}{j} \rho_m(t),$$

where  $\rho_m \in C_c([0, \infty))$  is defined by

$$\rho_m(t) := (n - j)\eta_m(t) - t\eta'_m(t) \quad (4.16)$$

and  $\eta_m \in C_c([0, \infty))$  is in turn defined by

$$\eta_m(t) := \int_t^\infty r^{n-j-1} \zeta_m(r) dr.$$

We will denote by  $\rho$  and  $\eta$  the corresponding functions associated to  $\zeta$  according to the same relations. For every  $t > 0$  and for every sequence  $t_m$ , with  $t_m > 0$ , converging to  $t$ , we have  $v_{t_m} \rightarrow v_t$  and hence, by the assumptions of the present lemma,

$$\begin{aligned} \lim_{m \rightarrow \infty} \kappa_n \binom{n}{j} \rho_m(t_m) &= \lim_{m \rightarrow \infty} \int_{\mathbb{R}^n} \zeta_m(|x|) d\Phi_j^n(v_{t_m}, x) = \int_{\mathbb{R}^n} \zeta(|x|) d\Phi_j^n(v_t, x) \\ &= \kappa_n \binom{n}{j} \rho(t). \end{aligned}$$

In other words, the sequence of functions  $\rho_m$  converges uniformly to  $\rho$  on compact subsets of  $(0, \infty)$ .

Next, we will show that there exists  $\bar{t} > 0$  such that

$$\rho_m(t) = 0 \quad \text{for every } t > \bar{t} \text{ and } m \in \mathbb{N}, \quad (4.17)$$

which means that the supports of the functions  $\rho_m$  are uniformly bounded. If this is not the case, then we can, without loss of generality, assume that there exists a sequence  $t_m \rightarrow \infty$  such that

$$\rho_m(t_m) > 0 \quad \text{for every } m \in \mathbb{N}.$$

For  $m \in \mathbb{N}$ , set

$$r_m := \rho_m(t_m)^{-\frac{1}{j}}$$

and

$$v_m := r_m v_{t_m}.$$

Since  $v_{t_m}(x) = 0$  whenever  $|x| \leq t_m$ , it follows that  $v_m$  converges to  $v \equiv 0$ , which is in  $\text{Conv}_{(0)}(\mathbb{R}^n; \mathbb{R})$ , as  $m \rightarrow \infty$ . On the other hand, by homogeneity,

$$\int_{\mathbb{R}^n} \zeta_m(|x|) d\Phi_j^n(v_m, x) = \kappa_n \binom{n}{j} r_m^j \rho_m(t_m) = \kappa_n \binom{n}{j} \quad \text{for } m \in \mathbb{N},$$

which contradicts the convergence to  $v \equiv 0$ . Thus (4.17) holds.

Let  $\bar{t} > 0$  be such that  $\rho_m(t) = 0$  for every  $t \geq \bar{t}$  and  $m \in \mathbb{N}$ . Since  $\eta_m$  has compact support, there exists  $t_m > \bar{t}$  such that  $\eta_m(t_m) = 0$ . Moreover,  $\eta_m$  solves the Cauchy problem,

$$\begin{aligned} (n-j)\eta_m(t) - t\eta'_m(t) &= 0 & \text{in } (\bar{t}, \infty), \\ \eta_m(t_m) &= 0. \end{aligned}$$

This implies that  $\eta_m \equiv 0$  in  $(\bar{t}, \infty)$ . As  $\bar{t}$  is independent of  $m$ , we deduce that the supports of the functions  $\eta_m$  are uniformly bounded as well.

Equation (4.16) implies

$$-\frac{d}{dt} \left( \frac{\eta_m(t)}{t^{n-j}} \right) = \frac{\rho_m(t)}{t^{n-j+1}} \quad \text{for } m \in \mathbb{N} \text{ and } t > 0.$$

Therefore,

$$-\frac{d}{dt} \left( \frac{\eta_m(t)}{t^{n-j}} \right)$$

converges uniformly to

$$\frac{\rho(t)}{t^{n-j+1}}$$

on compact subsets of  $(0, \infty)$ . Set

$$\bar{\eta}_m(t) := \frac{\eta_m(t)}{t^{n-j}}$$

for  $t > 0$  and  $m \in \mathbb{N}$ . As the supports of the functions  $\eta_m$  are uniformly bounded, there exists  $t_1 > 0$  such that  $\bar{\eta}_m(t_1) = 0$  and  $\rho_m(t_1) = 0$  for every  $m \in \mathbb{N}$ . We deduce that  $\bar{\eta}_m$  converges to a primitive of

$$-\frac{\rho(t)}{t^{n-j+1}}$$

which vanishes at  $t_1$ . But since

$$\frac{\eta(t)}{t^{n-j}}$$

has the same properties, it follows that  $\eta_m$  converges to  $\eta$  uniformly on compact subsets of  $(0, \infty)$ . We also have, from the definition of  $\eta_m$  and  $\rho_m$ ,

$$t^{n-j}\zeta_m(t) = \rho_m(t) - (n-j)\eta_m(t) \quad (4.18)$$

for every  $t > 0$ . Analogously,

$$t^{n-j}\zeta(t) = \rho(t) - (n-j)\eta(t)$$

for every  $t > 0$ . Hence, passing to the limit as  $m \rightarrow \infty$  in (4.18), we obtain the statement of the lemma.  $\square$

**4.4 Proof of Theorem 1.5.** For  $\zeta_0 \in D_0^n, \dots, \zeta_n \in D_n^n$ , we obtain from Theorem 1.4 that

$$v \mapsto V_{0,\zeta_0}^*(v) + \dots + V_{n,\zeta_n}^*(v)$$

defines a continuous, dually epi-translation and rotation invariant valuation on  $\text{Conv}(\mathbb{R}^n; \mathbb{R})$ .

Conversely, let  $Z : \text{Conv}(\mathbb{R}^n; \mathbb{R}) \rightarrow \mathbb{R}$  be a continuous, dually epi-translation and rotation invariant valuation. We want to show that there are  $\zeta_0 \in D_0^n, \dots, \zeta_n \in D_n^n$  such that

$$Z(v) = V_{0,\zeta_0}^*(v) + \dots + V_{n,\zeta_n}^*(v)$$

for every  $v \in \text{Conv}(\mathbb{R}^n; \mathbb{R})$ . By Theorem 2.7, there is a homogeneous decomposition of  $Z$ . Therefore, we may assume that  $Z$  is homogeneous of degree  $l$  with  $l \in \{0, \dots, n\}$ . Since the cases  $l = 0, 1, n$  are settled in Theorem 2.8, Proposition 4.4 and Theorem 2.9, respectively, we may assume that  $2 \leq l \leq n-1$ . In particular, we already have a complete classification for the case  $n = 2$ . Therefore, we assume that  $n \geq 3$ .

Recall that  $v \in \text{Conv}_{(0)}(\mathbb{R}^n; \mathbb{R})$  if  $v \in \text{Conv}(\mathbb{R}^n; \mathbb{R})$  and  $v$  is of class  $C^2$  in a neighborhood of the origin. This space is dense in  $\text{Conv}(\mathbb{R}^n; \mathbb{R})$  and by Lemma 3.23, we have for  $\zeta \in D_l^n$ ,

$$V_{l,\zeta}^*(v) = \int_{\mathbb{R}^n} \zeta(|x|) d\Phi_l^n(v, x) \quad (4.19)$$

for every  $v \in \text{Conv}_{(0)}(\mathbb{R}^n; \mathbb{R})$ .

We proceed by induction on the dimension  $n$ . By (4.19), we may use the induction hypothesis in the following form. Recall that the statement is true in the two-dimensional case.

**INDUCTION HYPOTHESIS** *Let  $2 \leq k \leq n - 1$ . If  $Z : \text{Conv}(\mathbb{R}^k; \mathbb{R}) \rightarrow \mathbb{R}$  is a continuous, dually epi-translation and rotation invariant valuation, then there exist  $\zeta_i \in D_i^k$ , for  $0 \leq i \leq k$ , such that*

$$Z(v) = \sum_{i=0}^k \int_{\mathbb{R}^n} \zeta_i(|x|) d\Phi_i^k(v, x)$$

for every  $v \in \text{Conv}_{(0)}(\mathbb{R}^k; \mathbb{R})$ .

The main ingredient of our proof is the following result (which we prove using the induction hypothesis).

**PROPOSITION 4.11.** *If  $Z : \text{Conv}(\mathbb{R}^n; \mathbb{R}) \rightarrow \mathbb{R}$  is a continuous, dually epi-translation and rotation invariant valuation that is homogeneous of degree  $l$  with  $2 \leq l \leq n - 1$ , then there exists  $\zeta \in D_l^n$  such that*

$$Z(v_E + v_F) = \int_{\mathbb{R}^n} \zeta(|x|) d\Phi_l^n(v_E + v_F, x)$$

for every pair of orthogonal and complementary subspaces  $E, F$  with  $1 \leq \dim E < n$  and every pair of functions  $v_E \in \text{Conv}_{(0)}(E; \mathbb{R})$  and  $v_F \in \text{Conv}_{(0)}(F; \mathbb{R})$ .

Before proving this result, we complete the proof of Theorem 1.5. Define

$$\tilde{Z}(v) := Z(v) - V_{l, \zeta}^*(v)$$

for  $v \in \text{Conv}(\mathbb{R}^n; \mathbb{R})$ . Note that  $\tilde{Z}$  is a continuous, dually epi-translation and rotation invariant valuation. From Proposition 4.11 and the continuity of  $\tilde{Z}$ , we obtain that

$$\tilde{Z}(v) = 0$$

for all dual orthogonal cylinder functions  $v \in \text{Conv}(\mathbb{R}^n; \mathbb{R})$ . Thus, it follows from Proposition 4.7 that  $\tilde{Z}$  is homogeneous of degree 1. Since by construction  $\tilde{Z}$  is also homogeneous of degree  $l > 1$ , this implies that  $\tilde{Z} \equiv 0$  and thus,

$$Z(v) = V_{l, \zeta}^*(v)$$

for every  $v \in \text{Conv}(\mathbb{R}^n; \mathbb{R})$ . This completes the proof of Theorem 1.5.

**4.5 Proof of Proposition 4.11.** Since  $Z$  is rotation invariant and the roles of  $E$  and  $F$  can be interchanged, we may assume that  $E = \text{span}\{e_1, \dots, e_k\}$ ,  $F = \text{span}\{e_{k+1}, \dots, e_n\}$ , and  $\lceil \frac{n}{2} \rceil \leq k \leq n - 1$ . Since  $n \geq 3$ , this also implies that  $k \geq 2$ .

**LEMMA 4.12.** *Let  $v_F \in \text{Conv}(F; \mathbb{R})$ . There exist  $\zeta_{i, v_F} \in D_i^k$  for  $0 \leq i \leq k$  such that*

$$Z(v_E + v_F) = \sum_{i=0}^k \int_E \zeta_{i, v_F}(|x_E|) d\Phi_i^k(v_E, x_E) \quad (4.20)$$

for every  $v_E \in \text{Conv}_{(0)}(E; \mathbb{R})$ . Moreover, the map

$$v_F \mapsto \int_E \zeta_{0, v_F}(|x_E|) d\Phi_0^k(v_E, x_E)$$

and the maps  $v_F \mapsto \zeta_{i, v_F}(s)$  for  $1 \leq i \leq k$  and  $s > 0$  are continuous, dually epi-translation and  $O(n - k)$  invariant valuations for every  $v_E \in \text{Conv}_{(0)}(E; \mathbb{R})$ .

*Proof.* For  $v_F \in \text{Conv}(F; \mathbb{R})$ , the map

$$v_E \mapsto Z(v_E + v_F)$$

is a continuous, dually epi-translation and  $SO(k)$  invariant valuation on  $\text{Conv}(E; \mathbb{R})$ . Since  $k \geq 2$ , it follows from the induction hypothesis that there exist  $\zeta_{i, v_F} \in D_i^k$  for  $0 \leq i \leq k$ , depending on  $v_F \in \text{Conv}(F; \mathbb{R})$ , such that (4.20) holds for every function  $v_E \in \text{Conv}_{(0)}(E; \mathbb{R})$ .

Note, that the map  $v_E \mapsto \int_E \zeta_{i, v_F}(|x_E|) d\Phi_i^k(v_E, x_E)$  is homogeneous of degree  $i$ . Since the map  $v_F \mapsto Z(v_E + v_F)$  is a continuous, dually epi-translation and  $SO(n - k)$  invariant valuation on  $\text{Conv}(F; \mathbb{R})$  for every  $v_E \in \text{Conv}_{(0)}(E; \mathbb{R})$ , it follows from (4.20) and homogeneity that also

$$v_F \mapsto \int_E \zeta_{i, v_F}(|x_E|) d\Phi_i^k(v_E, x_E)$$

is a valuation with the same properties on  $\text{Conv}(F; \mathbb{R})$  for every  $v_E \in \text{Conv}_{(0)}(E; \mathbb{R})$  and  $0 \leq i \leq k$ . This map is actually  $O(n - k)$  invariant, since  $Z$  is rotation invariant and since (4.20) shows that the map  $v_E \mapsto Z(v_E + v_F)$  is  $O(k)$  invariant.

Fix  $i$  with  $1 \leq i \leq k$  and let  $v_{F,1}, v_{F,2} \in \text{Conv}(F; \mathbb{R})$  be such that we also have  $v_{F,1} \wedge v_{F,2} \in \text{Conv}(F; \mathbb{R})$ . By the valuation property, we have

$$\int_E (\zeta_{i, v_{F,1} \vee v_{F,2}}(|x_E|) + \zeta_{i, v_{F,1} \wedge v_{F,2}}(|x_E|) - \zeta_{i, v_{F,1}}(|x_E|) - \zeta_{i, v_{F,2}}(|x_E|)) d\Phi_i^k(v_E, x_E) = 0$$

for every  $v_E \in \text{Conv}_{(0)}(E; \mathbb{R})$ . Hence, it follows from Lemma 4.9 that  $v_F \mapsto \zeta_{i, v_F}(s)$  defines a valuation on  $\text{Conv}(F; \mathbb{R})$  for every  $s > 0$ . Similarly, it can be seen that this valuation is dually epi-translation and  $O(n - k)$  invariant. To prove that it is also continuous, note that the map

$$(v_E, v_F) \mapsto \int_E \zeta_{i, v_F}(|x_E|) d\Phi_i^k(v_E, x_E)$$

is jointly continuous in the two variables  $v_E \in \text{Conv}_{(0)}(E; \mathbb{R})$  and  $v_F \in \text{Conv}(F; \mathbb{R})$ . Hence continuity follows from Lemma 4.10.  $\square$

In order to avoid an unnecessary distinction of cases, we set  $\Phi_j^m \equiv 0$  if  $j > m$  or  $j < 0$ .

LEMMA 4.13. For  $1 \leq i \leq k$  with  $l + k - n \leq i \leq l$ , there exist  $\zeta_{i,l-i} : (0, \infty)^2 \rightarrow \mathbb{R}$  and  $\tilde{\zeta}_{0,l} \in D_l^{n-k}$  such that

$$\begin{aligned} Z(v_E + v_F) &= \int_F \tilde{\zeta}_{0,l}(|x_F|) d\Phi_l^{n-k}(v_F, x_F) \\ &\quad + \sum_{i=1 \vee (l+k-n)}^{k \wedge l} \int_E \int_F \zeta_{i,l-i}(|x_E|, |x_F|) d\Phi_{l-i}^{n-k}(v_F, x_F) d\Phi_i^k(v_E, x_E) \end{aligned}$$

for every  $v_E \in \text{Conv}_{(0)}(E; \mathbb{R})$  and  $v_F \in \text{Conv}_{(0)}(F; \mathbb{R})$ . Moreover,  $\zeta_{i,l-i}(s, \cdot) \in D_{l-i}^{n-k}$  for  $s > 0$  and

$$\int_F \zeta_{i,l-i}(\cdot, |x_F|) d\Phi_{l-i}^{n-k}(v_F, x_F) \in D_i^k$$

for every  $v_F \in \text{Conv}_{(0)}(F; \mathbb{R})$ . For  $i = l + k - n$  with  $i \geq 1$ , the statements even hold for every  $v_F \in \text{Conv}(F; \mathbb{R})$ .

*Proof.* By Lemma 4.12 combined with the induction hypothesis and Corollary 2.11, there exist functions  $\tilde{\zeta}_{i,j,s} \in D_j^{n-k}$  for every  $1 \leq i \leq k$  and  $0 \leq j \leq n - k$ , depending on  $s > 0$ , such that

$$\zeta_{i,v_F}(s) = \sum_{j=0}^{n-k} \int_F \tilde{\zeta}_{i,j,s}(|x_F|) d\Phi_j^{n-k}(v_F, x_F)$$

for every  $v_F \in \text{Conv}_{(0)}(F; \mathbb{R})$ . Since the function  $s \mapsto \zeta_{i,v_F}(s)$  belongs to  $D_i^k$  and each of the summands on the right side is homogeneous of a different degree in  $v_F$ , it follows that also the functions

$$s \mapsto \int_F \tilde{\zeta}_{i,j,s}(|x_F|) d\Phi_j^{n-k}(v_F, x_F)$$

belong to  $D_i^k$  for every  $v_F \in \text{Conv}_{(0)}(F; \mathbb{R})$  and  $0 \leq j \leq n - k$ . Note, that since both  $\zeta_{i,v_F}$  and the last integral for the case  $j = n - k$  are well-defined for every  $v_F \in \text{Conv}(F; \mathbb{R})$ , the last statement also holds for every  $v_F \in \text{Conv}(\mathbb{R}^n; \mathbb{R})$  in this case. We remark that this also follows from Theorem 2.7 and Corollary 2.10. Hence, there exist functions  $\zeta_{i,j} : (0, \infty)^2 \rightarrow \mathbb{R}$  with  $\zeta_{i,j}(s, \cdot) \in D_j^{n-k}$  for every  $s > 0$  and  $\int_F \zeta_{i,j}(\cdot, |x_F|) d\Phi_j^{n-k}(v_F, x_F) \in D_i^k$  for every  $v \in \text{Conv}_{(0)}(F; \mathbb{R})$  such that

$$\tilde{\zeta}_{i,j,s}(t) = \zeta_{i,j}(s, t)$$

for every  $s, t > 0$ . Moreover, we have  $\int_F \zeta_{i,n-k}(\cdot, |x_F|) d\Phi_{n-k}^{n-k}(v_F, x_F) \in D_i^k$  for every  $v \in \text{Conv}(F; \mathbb{R})$ .

For the case  $i = 0$ , note that

$$\int_E \zeta_{0,v_F}(|x_E|) d\Phi_0^k(v_E, x_E) = \int_E \zeta_{0,v_F}(|x_E|) dx_E = \tilde{\zeta}_{0,v_F}$$



is a constant that is independent of  $v_E$ . By Lemma 4.12 the map  $v_F \mapsto \tilde{\zeta}_{0,v_F}$  is a continuous, dually epi-translation and  $O(n-k)$  invariant valuation on  $\text{Conv}(F; \mathbb{R})$ . Therefore, by the induction hypothesis, there exist functions  $\tilde{\zeta}_{0,j} \in D_j^{n-k}$  for every  $0 \leq j \leq n-k$  such that

$$\tilde{\zeta}_{0,v_F} = \sum_{j=0}^{n-k} \int_F \tilde{\zeta}_{0,j}(|x_F|) d\Phi_j^{n-k}(v_F, x_F)$$

for every  $v_F \in \text{Conv}_{(0)}(F; \mathbb{R})$ .

Combining the cases  $i > 0$  and  $i = 0$ , we obtain that

$$\begin{aligned} Z(v_E + v_F) &= \sum_{j=0}^{n-k} \int_F \tilde{\zeta}_{0,j}(|x_F|) d\Phi_j^{n-k}(v_F, x_F) \\ &\quad + \sum_{i=1}^k \sum_{j=0}^{n-k} \int_E \int_F \zeta_{i,j}(|x_E|, |x_F|) d\Phi_j^{n-k}(v_F, x_F) d\Phi_i^k(v_E, x_E) \end{aligned}$$

for every  $v_E \in \text{Conv}_{(0)}(E; \mathbb{R})$  and  $v_F \in \text{Conv}_{(0)}(F; \mathbb{R})$ . As  $Z$  is homogeneous of degree  $l$ , and  $\Phi_i^k(v_E, \cdot)$  and  $\Phi_j^{n-k}(v_F, \cdot)$  in the sums on the right side of the previous equation are homogeneous of degree  $i$  and  $j$ , respectively, only those terms for which  $i + j = l$  may appear. Moreover, we must have

$$i \leq l, \quad i \leq k, \quad l - i \leq n - k$$

which gives the desired representation of  $Z$ .  $\square$

Recall that  $\lceil \frac{n}{2} \rceil \leq k \leq n-1$  while  $E = \text{span}\{e_1, \dots, e_k\}$  and  $F = \text{span}\{e_{k+1}, \dots, e_n\}$ . Set  $E' = \text{span}\{e_1, \dots, e_{k-1}\}$  and  $F' = \text{span}\{e_{k+2}, \dots, e_n\}$ . In particular, for  $k = n-1$ , we have  $F' = \{0\}$  and  $\text{Conv}_{(0)}(F'; \mathbb{R})$  can be identified with  $\mathbb{R}$ . In order to simplify notation, we write  $\int_{F'} d\Phi_0^0(v_{F'}, x_{F'}) = 1$  for  $v_{F'} \in \text{Conv}_{(0)}(F'; \mathbb{R})$ .

LEMMA 4.14. *Let  $1 \leq i < k$  and  $a \in \mathbb{R} \setminus \{0\}$ . For  $v_{E'} \in \text{Conv}_{(0)}(E'; \mathbb{R})$  and  $v_{F'} \in \text{Conv}_{(0)}(F'; \mathbb{R})$ ,*

$$\begin{aligned} &\int_{F'} \tilde{\zeta}_{0,l}(|(a, x_{F'})|) d\Phi_{l-1}^{n-k-1}(v_{F'}, x_{F'}) \\ &= \int_{E'} \int_{\mathbb{R}} \int_{F'} \zeta_{1,l-1}(|(x_{E'}, a)|, |(t, x_{F'})|) d\Phi_{l-1}^{n-k-1}(v_{F'}, x_{F'}) dt dx_{E'} \end{aligned}$$

and

$$\begin{aligned} &\int_{E'} \int_{\mathbb{R}} \int_{F'} \zeta_{i,l-i}(|(x_{E'}, s)|, |(a, x_{F'})|) d\Phi_{l-i-1}^{n-k-1}(v_{F'}, x_{F'}) ds d\Phi_i^{k-1}(v_{E'}, x_{E'}) \\ &= \int_{E'} \int_{\mathbb{R}} \int_{F'} \zeta_{i+1,l-i-1}(|(x_{E'}, a)|, |(t, x_{F'})|) d\Phi_{l-i-1}^{n-k-1}(v_{F'}, x_{F'}) dt d\Phi_i^{k-1}(v_{E'}, x_{E'}) \end{aligned}$$

where  $l + k - n \leq i < l$ .

*Proof.* Let  $\alpha, \beta \in \mathbb{R}$  and  $t_\alpha, t_\beta \in \mathbb{R} \setminus \{0\}$ . Set

$$v_E(x_E) = v_{E'}(x_{E'}) + \frac{\alpha}{2}|x_k - t_\alpha|$$

for  $v_{E'} \in \text{Conv}_{(0)}(E'; \mathbb{R})$  and

$$v_F(x_F) = \frac{\beta}{2}|x_{k+1} - t_\beta| + v_{F'}(x_{F'})$$

for  $v_{F'} \in \text{Conv}_{(0)}(F'; \mathbb{R})$ . Using Lemma 4.8, (2.6) and (4.11), we obtain

$$\begin{aligned} d\Phi_i^k(v_E, x_E) &= d\Phi_i^{k-1}(v_{E'}, x_{E'}) dx_k + \alpha d\Phi_{i-1}^{k-1}(v_{E'}, x_{E'}) d\delta_{t_\alpha}(x_k), \\ d\Phi_{l-i}^{n-k}(v_F, x_F) &= dx_{k+1} d\Phi_{l-i}^{n-k-1}(v_{F'}, x_{F'}) + \beta d\delta_{t_\beta}(x_{k+1}) d\Phi_{l-i-1}^{n-k-1}(v_{F'}, x_{F'}) \end{aligned}$$

for  $l + k - n \leq i \leq l$ . Thus, by Lemma 4.13,

$$\begin{aligned} &Z(v_E + v_F) \\ &= \int_F \tilde{\zeta}_{0,l}(|x_F|) d\Phi_l^{n-k}(v_F, x_F) \\ &\quad + \sum_{i=1 \vee (l+k-n)}^{k \wedge l} \int_E \int_F \zeta_{i,l-i}(|x_E|, |x_F|) d\Phi_{l-i}^{n-k}(v_F, x_F) d\Phi_i^k(v_E, x_E) \\ &= \int_{F'} \int_{\mathbb{R}} \tilde{\zeta}_{0,l}(|(x_{k+1}, x_{F'})|) dx_{k+1} d\Phi_l^{n-k-1}(v_{F'}, x_{F'}) \\ &\quad + \beta \int_{F'} \tilde{\zeta}_{0,l}(|(t_\beta, x_{F'})|) d\Phi_{l-1}^{n-k-1}(v_{F'}, x_{F'}) \\ &\quad + \sum_{i=1 \vee (l+k-n)}^{k \wedge l} \left( \int_{E'} \int_{\mathbb{R}} \int_{F'} \int_{\mathbb{R}} \zeta_{i,l-i}(|(x_{E'}, s)|, |(t, x_{F'})|) \right. \\ &\qquad\qquad\qquad \left. dt d\Phi_{l-i}^{n-k-1}(v_{F'}, x_{F'}) ds d\Phi_i^{k-1}(v_{E'}, x_{E'}) \right. \\ &\quad + \alpha \int_{E'} \int_{\mathbb{R}} \int_{F'} \zeta_{i,l-i}(|(x_{E'}, t_\alpha)|, |(t, x_{F'})|) d\Phi_{l-i}^{n-k-1}(v_{F'}, x_{F'}) dt d\Phi_{i-1}^{k-1}(v_{E'}, x_{E'}) \\ &\quad + \beta \int_{E'} \int_{\mathbb{R}} \int_{F'} \zeta_{i,l-i}(|(x_{E'}, t)|, |(t_\beta, x_{F'})|) d\Phi_{l-i-1}^{n-k-1}(v_{F'}, x_{F'}) dt d\Phi_i^{k-1}(v_{E'}, x_{E'}) \\ &\quad \left. + \alpha\beta \int_{E'} \int_{F'} \zeta_{i,l-i}(|(x_{E'}, t_\alpha)|, |(t_\beta, x_{F'})|) d\Phi_{l-i-1}^{n-k-1}(v_{F'}, x_{F'}) d\Phi_{i-1}^{k-1}(v_{E'}, x_{E'}) \right). \end{aligned}$$

Here we have split the integral over  $E$  with respect to  $d\Phi_i^k(v_E, x_E)$  and the integral over  $F$  with respect to  $d\Phi_{l-i}^{n-k}(v_F, x_F)$  into multiple integrals. We are allowed to do so since, for given  $i$ , each of the integrals so obtained is homogeneous of different degrees in  $v_{E'}$  and  $v_{F'}$ . In particular, each of the integrals in the last expression is well-defined and finite. Since  $Z$  is rotation invariant and this expression only depends on the absolute values of  $t_\alpha$  and  $t_\beta$ , respectively, we may exchange  $(\alpha, t_\alpha)$  and  $(\beta, t_\beta)$

while preserving equality. Making this exchange, setting  $t_\alpha = t_\beta$ , comparing with the original expression and considering those parts which are  $i$ -homogeneous in  $v_{E'}$  and  $(l - i - 1)$ -homogeneous in  $v_{F'}$  gives

$$\begin{aligned} & \beta \int_{F'} \tilde{\zeta}_{0,l}(|(t_\alpha, x_{F'})|) d\Phi_{l-1}^{n-k-1}(v_{F'}, x_{F'}) \\ & \quad + \alpha \int_{E'} \int_{\mathbb{R}} \int_{F'} \zeta_{1,l-1}(|(x_{E'}, t_\alpha)|, |(t, x_{F'})|) d\Phi_{l-1}^{n-k-1}(v_{F'}, x_{F'}) dt dx_{E'} \\ & = \alpha \int_{F'} \tilde{\zeta}_{0,l}(|(t_\alpha, x_{F'})|) d\Phi_{l-1}^{n-k-1}(v_{F'}, x_{F'}) \\ & \quad + \beta \int_{E'} \int_{\mathbb{R}} \int_{F'} \zeta_{1,l-1}(|(x_{E'}, t_\alpha)|, |(t, x_{F'})|) d\Phi_{l-1}^{n-k-1}(v_{F'}, x_{F'}) dt dx_{E'} \end{aligned}$$

for  $i = 0$  and

$$\begin{aligned} & \beta \int_{E'} \int_{\mathbb{R}} \int_{F'} \zeta_{i,l-i}(|(x_{E'}, t)|, |(t_\alpha, x_{F'})|) d\Phi_{l-i-1}^{n-k-1}(v_{F'}, x_{F'}) dt d\Phi_i^{k-1}(v_{E'}, x_{E'}) \\ & \quad + \alpha \int_{E'} \int_{\mathbb{R}} \int_{F'} \zeta_{i+1,l-i-1}(|(x_{E'}, t_\alpha)|, |(t, x_{F'})|) d\Phi_{l-i-1}^{n-k-1}(v_{F'}, x_{F'}) dt d\Phi_i^{k-1}(v_{E'}, x_{E'}) \\ & = \alpha \int_{E'} \int_{\mathbb{R}} \int_{F'} \zeta_{i,l-i}(|(x_{E'}, t)|, |(t_\alpha, x_{F'})|) d\Phi_{l-i-1}^{n-k-1}(v_{F'}, x_{F'}) dt d\Phi_i^{k-1}(v_{E'}, x_{E'}) \\ & \quad + \beta \int_{E'} \int_{\mathbb{R}} \int_{F'} \zeta_{i+1,l-i-1}(|(x_{E'}, t_\alpha)|, |(t, x_{F'})|) d\Phi_{l-i-1}^{n-k-1}(v_{F'}, x_{F'}) dt d\Phi_i^{k-1}(v_{E'}, x_{E'}) \end{aligned}$$

for  $i \geq 1$ . The desired equality now follows after rearranging, using the fact that  $\alpha$  and  $\beta$  are arbitrary, and setting  $a = t_\alpha$ .  $\square$

LEMMA 4.15. *For  $1 \leq i < k$  with  $l + k - n \leq i < l$ , there exists  $\zeta_i \in C_b((0, \infty))$  such that*

$$\zeta_{i,l-i}(a, b) = \zeta_i(\sqrt{a^2 + b^2})$$

for every  $a, b > 0$ , and by continuity this extends to all  $a, b \geq 0$  with  $(a, b) \neq (0, 0)$ . Moreover, if  $k = n - 1$ , then  $\zeta_{l-1} \in D_l^n$ .

*Proof.* Since  $i \geq 1$ , it follows from Lemma 4.9 and Lemma 4.14 that

$$\int_{\mathbb{R}} \int_{F'} \left( \zeta_{i+1,l-i-1}(|(a, b)|, |(t, x_{F'})|) - \zeta_{i,l-i}(|(a, t)|, |(b, x_{F'})|) \right) d\Phi_{l-i-1}^{n-k-1}(v_{F'}, x_{F'}) dt = 0$$

for every  $a, b > 0$  and  $v_{F'} \in \text{Conv}_{(0)}(F'; \mathbb{R})$ . Writing the same equation again but this time replacing  $b$  by  $\varepsilon$  with  $0 < \varepsilon < b$  and  $a$  by  $\sqrt{a^2 + b^2 - \varepsilon^2}$ , we obtain

$$\begin{aligned} & \int_{\mathbb{R}} \int_{F'} \left( \zeta_{i+1,l-i-1}(|(a, b)|, |(t, x_{F'})|) \right. \\ & \quad \left. - \zeta_{i,l-i}(|(\sqrt{a^2 + b^2 - \varepsilon^2}, t)|, |(\varepsilon, x_{F'})|) \right) d\Phi_{l-i-1}^{n-k-1}(v_{F'}, x_{F'}) dt = 0 \end{aligned}$$

for every  $a > 0$  and  $b > \varepsilon > 0$ . By subtracting the last two equations, we obtain

$$\int_{\mathbb{R}} \int_{F'} \left( \zeta_{i,l-i}(|(\sqrt{a^2 + b^2 - \varepsilon^2}, t)|, |(\varepsilon, x_{F'})|) \right. \\ \left. - \zeta_{i,l-i}(|(a, t)|, |(b, x_{F'})|) \right) d\Phi_{l-i-1}^{n-k-1}(v_{F'}, x_{F'}) dt = 0$$

for every  $a > 0$  and  $b > \varepsilon > 0$ . By the properties of  $\zeta_{i,l-i}$  we may apply Lemma 2.14 for fixed  $b > \varepsilon > 0$  and obtain

$$\int_{F'} \left( \zeta_{i,l-i}(\sqrt{a^2 + b^2 - \varepsilon^2}, |(\varepsilon, x_{F'})|) - \zeta_{i,l-i}(a, |(b, x_{F'})|) \right) d\Phi_{l-i-1}^{n-k-1}(v_{F'}, x_{F'}) = 0$$

for every  $a > 0$ . Similarly to before, we write the last equation again, but this time, we replace  $a$  by  $\delta$  with  $0 < \delta < a$ . Furthermore, we may replace  $b$  by  $\sqrt{a^2 + b^2 - \delta^2}$  since by our choice of  $\delta$  we have  $\sqrt{a^2 + b^2 - \delta^2} > b > \varepsilon$ . Thus, we obtain

$$\int_{F'} \left( \zeta_{i,l-i}(\sqrt{a^2 + b^2 - \varepsilon^2}, |(\varepsilon, x_{F'})|) \right. \\ \left. - \zeta_{i,l-i}(\delta, |(\sqrt{a^2 + b^2 - \delta^2}, x_{F'})|) \right) d\Phi_{l-i-1}^{n-k-1}(v_{F'}, x_{F'}) = 0$$

for every  $a > \delta > 0$  and  $b > \varepsilon > 0$ . Subtracting the last two equations yields

$$\int_{F'} \left( \zeta_{i,l-i}(\delta, |(\sqrt{a^2 + b^2 - \delta^2}, x_{F'})|) \right. \\ \left. - \zeta_{i,l-i}(a, |(b, x_{F'})|) \right) d\Phi_{l-i-1}^{n-k-1}(v_{F'}, x_{F'}) = 0 \quad (4.21)$$

for every  $a > \delta > 0$  and  $b > \varepsilon > 0$ . Since this expression does not depend on  $\varepsilon$  anymore and  $\varepsilon > 0$  was arbitrary, we conclude that (4.21) holds for every  $b > 0$ .

We now claim that

$$\zeta_{i,l-i}(\sqrt{a^2 + b^2 - \delta^2}, \delta) = \zeta_{i,l-i}(a, b) \quad (4.22)$$

for every  $a > \delta > 0$  and  $b > 0$ . In case  $l - i - 1 > 0$ , this is a direct consequence of Lemma 4.9. In case  $l - i - 1 = 0$  and  $k = n - 1$ , we have  $F' = \{0\}$  and  $\int_{F'} d\Phi_{l-i-1}^{n-k-1}(v_{F'}, x_{F'}) = 1$  and the claim is immediate. In the remaining case  $l - i - 1 = 0$  and  $k < n - 1$ , we want to emphasize that

$$d\Phi_{l-i-1}^{n-k-1}(v_{F'}, x_{F'}) = dx_{F'}$$

and therefore, (4.22) follows from Lemma 2.14 after switching to spherical coordinates in (4.21) and considering that  $b > 0$  is arbitrary.

Note that the right side of (4.22) does not depend on the choice of  $\delta < a$ . Hence, there exists a function  $\zeta_i : (0, \infty) \rightarrow \mathbb{R}$  such that

$$\zeta_{i,l-i}(a, b) = \lim_{\delta \rightarrow 0^+} \zeta_{i,l-i}(\sqrt{a^2 + b^2 - \delta^2}, \delta) =: \zeta_i(\sqrt{a^2 + b^2}) \quad (4.23)$$

for every  $a, b > 0$ . Furthermore, it follows from the properties of  $\zeta_{i,l-i}$  that  $\zeta_i$  is continuous with bounded support. Moreover, for fixed  $a > 0$  we obtain

$$\lim_{b \rightarrow 0^+} \zeta_{i,l-i}(a, b) = \lim_{b \rightarrow 0^+} \zeta_i(\sqrt{a^2 + b^2}) = \zeta_i(\sqrt{a^2/2 + a^2/2}) = \zeta_{i,l-i}(a/\sqrt{2}, a/\sqrt{2}).$$

Thus, (4.23) continuously extends to all  $a, b \geq 0$  with  $(a, b) \neq (0, 0)$ .

In the case  $k = n - 1$  and  $i = l - 1$ , we have  $\zeta_{l-1,1}(s, \cdot) \in D_1^1$  for every  $s > 0$ , which follows from Lemma 4.13 and is also evident from (4.23). By Lemma 4.13, we have  $\int_F \zeta_{l-1,1}(\cdot, |x_F|) d\Phi_1^1(v_F, x_F) \in D_{l-1}^{n-1}$  for every  $v_F \in \text{Conv}(F; \mathbb{R})$ . Thus, choosing  $v_F(x_F) = \frac{1}{2} \sum_{i=k+1}^n |x_i|$ , we obtain by (4.11) that

$$\zeta_{l-1}(s) = \zeta_{l-1,1}(s, 0) = \int_F \zeta_{l-1,1}(s, |x_F|) d\Phi_1^1(v_F, x_F)$$

for every  $s > 0$ . Hence  $\zeta_{l-1} \in D_{l-1}^{n-1} = D_l^n$ .  $\square$

LEMMA 4.16. *For  $\lceil \frac{n}{2} \rceil \leq k \leq n - 1$ , there exists  $\zeta_{(k)} \in C_b((0, \infty))$  such that*

$$\begin{aligned} Z(v_E + v_F) &= \int_F \int_E \zeta_{(k)}(\sqrt{|x_E|^2 + |x_F|^2}) d\Phi_0^k(v_E, x_E) d\Phi_l^{n-k}(v_F, x_F) \\ &\quad + \sum_{i=1 \vee (l+k-n)}^{k \wedge l} \int_E \int_F \zeta_{(k)}(\sqrt{|x_E|^2 + |x_F|^2}) d\Phi_{l-i}^{n-k}(v_F, x_F) d\Phi_i^k(v_E, x_E) \end{aligned}$$

for every  $v_E \in \text{Conv}_{(0)}(E; \mathbb{R})$  and  $v_F \in \text{Conv}_{(0)}(F; \mathbb{R})$ . Moreover,  $\zeta_{(n-1)} \in D_l^n$ .

*Proof.* By Lemma 4.13 combined with Lemma 4.15, we have

$$\begin{aligned} &Z(v_E + v_F) \\ &= \int_F \tilde{\zeta}_{0,l}(|x_F|) d\Phi_l^{n-k}(v_F, x_F) \\ &\quad + \sum_{i=1 \vee (l+k-n)}^{(k \wedge l) - 1} \int_E \int_F \zeta_i(\sqrt{|x_E|^2 + |x_F|^2}) d\Phi_{l-i}^{n-k}(v_F, x_F) d\Phi_i^k(v_E, x_E) \\ &\quad + \int_E \int_F \zeta_{k \wedge l, l - (k \wedge l)}(|x_E|, |x_F|) d\Phi_{l - (k \wedge l)}^{n-k}(v_F, x_F) d\Phi_{k \wedge l}^k(v_E, x_E) \end{aligned} \tag{4.24}$$

for every  $v_E \in \text{Conv}_{(0)}(E; \mathbb{R})$  and  $v_F \in \text{Conv}_{(0)}(F; \mathbb{R})$ . Here,  $\zeta_i \in C_b((0, \infty))$  satisfies

$$\zeta_{i,l-i}(a, b) = \zeta_i(\sqrt{a^2 + b^2}) \tag{4.25}$$

for every  $a, b \geq 0$  with  $(a, b) \neq (0, 0)$  and for every  $1 \vee (l + k - n) \leq i \leq (k \wedge l) - 1$ . In addition, if  $k = n - 1$ , we have  $\zeta_{l-1} \in D_l^n$ . We need to show that there exists  $\zeta_{(k)} \in C_b((0, \infty))$  such that

$$\zeta_{(k)} = \zeta_{1 \vee (l+k-n)} = \cdots = \zeta_{(k \wedge l) - 1} \tag{4.26}$$

and that  $\zeta_{(n-1)} \in D_l^n$ . Furthermore, we will show that, in case it does not vanish, we can rewrite the first integral on the right side in (4.24) using  $\zeta_{(k)}$ . Moreover, for  $l \leq k$ , we will also rewrite the last integral in (4.24) using  $\zeta_{(k)}$  and, for  $k < l$ , we will directly show that

$$\zeta_{k \wedge l, l - (k \wedge l)}(a, b) = \zeta_{k, l - k}(a, b) = \zeta_{(k)}(\sqrt{a^2 + b^2}) \quad (4.27)$$

for every  $a, b \geq 0$  with  $(a, b) \neq (0, 0)$ .

We will first prove (4.26) and (4.27). Observe that in the special cases  $k = 2$ ,  $k = n - 1$ ,  $l = 2$ , or  $l = n - 1$ , there is only one summand in the sum in (4.24) and therefore (4.26) is trivially true. Also note that for  $k = n - 1$  this summand has index  $i = l - 1$  and  $\zeta_{l-1} \in D_l^n$ , which shows that  $\zeta_{(n-1)} \in D_l^n$ .

Next, we use Lemma 4.9, Lemma 4.14, and (4.25) to obtain

$$\int_{\mathbb{R}} \int_{F'} \left( \zeta_{i+1, l-i-1}(a, |(t, x_{F'})|) - \zeta_i(|(a, t, x_{F'})|) \right) d\Phi_{l-i-1}^{n-k-1}(v_{F'}, x_{F'}) dt = 0 \quad (4.28)$$

for every  $a > 0$ ,  $v_{F'} \in \text{Conv}_{(0)}(F'; \mathbb{R})$  and  $1 \vee (l + k - n) \leq i \leq (k \wedge l) - 1$ . If we have, in addition,  $k < n - 1$  and  $i < l - 1$ , we may choose  $v_{F'}(x_{F'}) = \frac{1}{2}(|x_{k+2} - \bar{x}_{k+2}| + \dots + |x_{k+l-i} - \bar{x}_{k+l-i}|)$  with arbitrary  $\bar{x}_{k+2}, \dots, \bar{x}_{k+l-i} \in \mathbb{R} \setminus \{0\}$  and apply (4.12) to (4.28) (or, when  $i = l + k - n$ , we directly use (4.11)). This gives

$$\begin{aligned} & \int_{\mathbb{R}} \int_{\mathbb{R}^{n+i-k-l}} \left( \zeta_{i+1, l-i-1}(a, |(t, b, x_{k+l-i+1}, \dots, x_n)|) \right. \\ & \quad \left. - \zeta_i(|(a, t, b, x_{k+l-i+1}, \dots, x_n)|) \right) dx_{k+l-i+1} \cdots dx_n dt = 0 \end{aligned}$$

for every  $a, b > 0$ . Hence, by Lemma 2.14,

$$\zeta_{i+1, l-i-1}(a, b) = \zeta_i(\sqrt{a^2 + b^2}) \quad (4.29)$$

for every  $a, b > 0$  and whenever  $k < n - 1$ ,  $1 \leq i \leq k - 1$ , and  $l + k - n \leq i < l - 1$ . Similarly, as in the proof of Lemma 4.15, this extends to  $a, b \geq 0$  with  $(a, b) \neq (0, 0)$ . Note, that if in addition  $i < k - 1$ , we have  $i + 1 < k \wedge l$ , and thus it follows from (4.25) that also  $\zeta_{i+1, l-i-1}(a, b) = \zeta_{i+1}(\sqrt{a^2 + b^2})$  for every  $a, b \geq 0$  with  $(a, b) \neq (0, 0)$ . Combining this with (4.29) gives

$$\begin{aligned} \zeta_{1 \vee (l+k-n), l - (1 \vee (l+k-n))}(a, b) &= \zeta_{1 \vee (l+k-n)}(\sqrt{a^2 + b^2}) \\ &= \zeta_{(1 \vee (l+k-n))+1, l - (1 \vee (l+k-n)) - 1}(a, b) \\ &\quad \vdots \\ &= \zeta_{(k-1) \wedge (l-2)}(\sqrt{a^2 + b^2}) \\ &= \zeta_{k \wedge (l-1), l - (k \wedge (l-1))}(a, b), \end{aligned}$$

where the first equation follows from (4.25) and the last equation is (4.29) for the case  $i = (k-1) \wedge (l-2)$ . Thus, there exists  $\zeta_{(k)} \in C_b((0, \infty))$  such that

$$\zeta_{(k)}(\sqrt{a^2 + b^2}) = \zeta_{i, l-i}(a, b)$$

for every  $a, b \geq 0$  with  $(a, b) \neq (0, 0)$  and every  $1 \vee (l+k-n) \leq i \leq k \wedge (l-1)$  with  $k < n-1$ . In particular, together with (4.25), this completes the proof of (4.26) and it also shows (4.27), where we use that  $k < l$  can only occur for  $k < n-1$ .

For the case  $l \leq k$ , let us now rewrite the last integral in (4.24), which concerns the integrand  $\zeta_{k \wedge l, l - (k \wedge l)} = \zeta_{l, 0}$ . Since  $l-1 \geq 1 \vee (l+k-n)$ , we can choose  $i = l-1$  in (4.28), which together with (4.26) shows

$$\begin{aligned} 0 &= \int_{\mathbb{R}} \int_{F'} \left( \zeta_{l, 0}(a, |(t, x_{F'})|) - \zeta_{l-1}(|(a, t, x_{F'})|) \right) d\Phi_0^{n-k-1}(v_{F'}, x_{F'}) dt \\ &= \int_F \left( \zeta_{l, 0}(a, |x_F|) - \zeta_{(k)}(|(a, x_F)|) \right) d\Phi_0^{n-k}(v_F, x_F) \end{aligned}$$

for every  $a > 0$ ,  $v_{F'} \in \text{Conv}_{(0)}(F'; \mathbb{R})$  and  $v_F \in \text{Conv}_{(0)}(F; \mathbb{R})$ , where for  $k = n-1$  we have used the convention  $\int_{F'} d\Phi_0^0(v_{F'}, x_{F'}) = 1$ . In particular, this implies that

$$\begin{aligned} &\int_E \int_F \zeta_{l, 0}(|x_E|, |x_F|) d\Phi_0^{n-k}(v_F, x_F) d\Phi_l^k(v_E, x_E) \\ &= \int_E \int_F \zeta_{(k)}(\sqrt{|x_E|^2 + |x_F|^2}) d\Phi_0^{n-k}(v_F, x_F) d\Phi_l^k(v_E, x_E) \end{aligned}$$

for every  $v_E \in \text{Conv}_{(0)}(E; \mathbb{R})$  and  $v_F \in \text{Conv}_{(0)}(F; \mathbb{R})$ , which we wanted to show.

It remains to rewrite the first integral on the right side in (4.24), which only does not vanish when  $l \leq n-k$ . Since by our assumptions  $l \geq 2$  and  $k \geq 2$ , it follows from (4.26) that  $\zeta_1 = \zeta_{(k)}$ . Thus, by Lemma 4.14 and (4.25),

$$\begin{aligned} &\int_{E'} \int_{\mathbb{R}} \int_{F'} \zeta_{(k)}(|(x_{E'}, a, t, x_{F'})|) d\Phi_{l-1}^{n-k-1}(v_{F'}, x_{F'}) dt dx_{E'} \\ &= \int_{F'} \tilde{\zeta}_{0, l}(|(a, x_{F'})|) d\Phi_{l-1}^{n-k-1}(v_{F'}, x_{F'}) \end{aligned}$$

for every  $a > 0$  and  $v_{F'} \in \text{Conv}_{(0)}(F'; \mathbb{R})$ . If we set  $v_{F'}(x_{F'}) = \frac{1}{2}(|x_{k+2} - \bar{x}_{k+2}| + \dots + |x_{k+l} - \bar{x}_{k+l}|)$  with arbitrary  $\bar{x}_{k+2}, \dots, \bar{x}_{k+l} \in \mathbb{R} \setminus \{0\}$  and consider (4.12), this becomes

$$\begin{aligned} &\int_{E'} \int_{\mathbb{R}} \int_{\mathbb{R}^{n-k-l}} \zeta_{(k)}(|(x_{E'}, a, t, b, x_{k+l+1}, \dots, x_n)|) dx_{k+l+1} \dots dx_n dt dx_{E'} \\ &= \int_{\mathbb{R}^{n-k-l}} \tilde{\zeta}_{0, l}(|(a, b, x_{k+l+1}, \dots, x_n)|) dx_{k+l+1} \dots dx_n \end{aligned}$$

for every  $a, b > 0$ . Hence, by renaming the integration variables on the left side and rearranging, we obtain

$$\int_{\mathbb{R}^{n-k-l}} \left( \int_{\mathbb{R}} \int_{E'} \zeta_{(k)}(|(x_{E'}, a, t, b, x_{k+l+1}, \dots, x_n)|) dx_{E'} dt - \tilde{\zeta}_{0,l}(|(a, b, x_{k+l+1}, \dots, x_n)|) \right) dx_{k+l+1} \cdots dx_n = 0$$

for every  $a, b > 0$ . Thus, by Lemma 2.14 if  $l < n - k$  and trivially if  $l = n - k$ , we have

$$\tilde{\zeta}_{0,l}(a) = \int_{\mathbb{R}} \int_{E'} \zeta_{(k)}(|(x_{E'}, t, a)|) dx_{E'} dt = \int_E \zeta_{(k)}(|(x_E, a)|) d\Phi_0^k(v_E, x_E)$$

for every  $a > 0$  and  $v_E \in \text{Conv}_{(0)}(E; \mathbb{R})$ . In particular, this implies that

$$\int_F \tilde{\zeta}_{0,l}(|x_F|) d\Phi_l^{n-k}(v_F, x_F) = \int_F \int_E \zeta_{(k)}(\sqrt{|x_E|^2 + |x_F|^2}) d\Phi_0^k(v_E, x_E) d\Phi_l^{n-k}(v_F, x_F)$$

for every  $v_E \in \text{Conv}_{(0)}(E; \mathbb{R})$  and  $v_F \in \text{Conv}_{(0)}(F; \mathbb{R})$ .  $\square$

We can now complete the proof of Proposition 4.11. Let

$$v(x_1, \dots, x_n) = \frac{1}{2} (|x_1 - \bar{x}_1| + \cdots + |x_l - \bar{x}_l|)$$

with  $\bar{x}_1, \dots, \bar{x}_l \in \mathbb{R} \setminus \{0\}$  and  $a = \sqrt{\bar{x}_1^2 + \cdots + \bar{x}_l^2} > 0$ . Note that for  $\lceil \frac{n}{2} \rceil \leq k \leq n - 1$ , we can write  $v$  in the form  $v = v_E + v_F$ . Furthermore, by (4.13),  $d\Phi_i^k(v_E, x_E) = 0$  if  $i > l$  and  $d\Phi_j^{n-k}(v_F, x_F) = 0$  if  $j > (l - k) \vee 0$ . Thus, for any  $k$ , it follows from Lemma 4.16, (4.11) and (4.12) that

$$\begin{aligned} Z(v) &= \int_{\mathbb{R}^{n-l}} \zeta_{(\lceil \frac{n}{2} \rceil)}(|(a, x_{l+1}, \dots, x_n)|) dx_{l+1} \cdots dx_n \\ &= \cdots \\ &= \int_{\mathbb{R}^{n-l}} \zeta_{(n-1)}(|(a, x_{l+1}, \dots, x_n)|) dx_{l+1} \cdots dx_n. \end{aligned}$$

Hence, by Lemma 2.14,  $\zeta_{(\lceil \frac{n}{2} \rceil)} \equiv \cdots \equiv \zeta_{(n-1)} =: \zeta$  and  $\zeta \in D_l^n$ . Thus, by Lemma 4.16,

$$\begin{aligned} Z(v_E + v_F) &= \int_F \int_E \zeta(|(x_E, x_F)|) d\Phi_0^k(v_E, x_E) d\Phi_l^{n-k}(v_F, x_F) \\ &\quad + \sum_{i=1 \vee (l+k-n)}^{k \wedge l} \int_E \int_F \zeta(|(x_E, x_F)|) d\Phi_{l-i}^{n-k}(v_F, x_F) d\Phi_i^k(v_E, x_E) \end{aligned} \tag{4.30}$$

for every  $v_E \in \text{Conv}_{(0)}(E; \mathbb{R})$  and  $v_F \in \text{Conv}_{(0)}(F; \mathbb{R})$  and every  $\lceil \frac{n}{2} \rceil \leq k \leq n - 1$ . Since Hessian measures are non-negative and since  $v_E + v_F \in \text{Conv}_{(0)}(\mathbb{R}^n; \mathbb{R})$  for



every  $v_E \in \text{Conv}_{(0)}(E; \mathbb{R})$  and  $v_F \in \text{Conv}_{(0)}(F; \mathbb{R})$ , it follows from Lemma 3.1 and Lemma 4.8 that

$$\begin{aligned} & \int_F \int_E \left| \zeta(|(x_E, x_F)|) \right| d\Phi_0^k(v_E, x_E) d\Phi_l^{n-k}(v_F, x_F) \\ & \leq \int_{\mathbb{R}^n} \left| \zeta(|x|) \right| d\Phi_l^n(v_E + v_F, x) < +\infty \end{aligned}$$

for every  $v_E \in \text{Conv}_{(0)}(E; \mathbb{R})$  and  $v_F \in \text{Conv}_{(0)}(F; \mathbb{R})$ . Furthermore, since  $\zeta \in D_l^n$  is measurable and has bounded support and since Hessian measures are locally finite, it now follows from the Fubini-Tonelli theorem that we may exchange the order of integration of the first term on the right side in (4.30). Together with Lemma 4.8 we now obtain

$$\begin{aligned} Z(v_E + v_F) &= \sum_{i=0 \vee (l+k-n)}^{k \wedge l} \int_E \int_F \zeta(|(x_E, x_F)|) d\Phi_{l-i}^{n-k}(v_F, x_F) d\Phi_i^k(v_E, x_E) \\ &= \int_{\mathbb{R}^n} \zeta(|x|) d\Phi_l^n(v_E + v_F, x) \end{aligned}$$

for every  $v_E \in \text{Conv}_{(0)}(E; \mathbb{R})$  and  $v_F \in \text{Conv}_{(0)}(F; \mathbb{R})$  and every  $\lceil \frac{n}{2} \rceil \leq k \leq n-1$ , which completes the proof.

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## References

- [Ale99] ALESKER, S.: Continuous rotation invariant valuations on convex sets. *Ann. Math. (2)* **149**, 977–1005 (1999).

- [Ale01] ALESKER, S.: Description of translation invariant valuations on convex sets with solution of P. McMullen’s conjecture. *Geom. Funct. Anal.* **11**, 244–272 (2001).
- [Ale19] ALESKER, S.: Valuations on convex functions and convex sets and Monge–Ampère operators. *Adv. Geom.* **19**, 313–322 (2019).
- [AF14] ALESKER, S., FAIFMAN, D.: Convex valuations invariant under the Lorentz group. *J. Differ. Geom.* **98**, 183–236 (2014).
- [BGW13] BARYSHNIKOV, Y., GHRIST, R., WRIGHT, M.: Hadwiger’s theorem for definable functions. *Adv. Math.* **245**, 573–586 (2013).
- [BF17] BERNIG, A., FAIFMAN, D.: Valuation theory of indefinite orthogonal groups. *J. Funct. Anal.* **273**, 2167–2247 (2017).
- [BF11] BERNIG, A., FU, J.H.G.: Hermitian integral geometry. *Ann. Math. (2)* **173**, 907–945 (2011).
- [B+20] BÖRÖCZKY, K.J., LUTWAK, E., YANG, D., ZHANG, G.: The Gauss image problem. *Commun. Pure Appl. Math.* **73**, 1406–1452 (2020).
- [B+08] BRANDOLINI, B., NITSCH, C., SALANI, P., TROMBETTI, C.: Serrin-type overdetermined problems: an alternative proof. *Arch. Ration. Mech. Anal.* **190**, 267–280 (2008).
- [CY76] CHENG, S.T., YAU, S.T.: On the regularity of the solution of the  $n$ -dimensional Minkowski problem. *Commun. Pure Appl. Math.* **29**, 495–516 (1976).
- [CH00] COLESANTI, A., HUG, D.: Steiner type formulae and weighted measures of singularities for semi-convex functions. *Trans. Am. Math. Soc.* **352**, 3239–3263 (2000).
- [CLM17] COLESANTI, A., LUDWIG, M., MUSSNIG, F.: Minkowski valuations on convex functions. *Calc. Var. Partial Differ. Equ.* **56**, Article ID 162 (2017).
- [CLM19] COLESANTI, A., LUDWIG, M., MUSSNIG, F.: Valuations on convex functions. *Int. Math. Res. Not.*, 2384–2410 (2019).
- [CLM201] COLESANTI, A., LUDWIG, M., MUSSNIG, F.: A homogeneous decomposition theorem for valuations on convex functions. *J. Funct. Anal.* **279**, Article ID 108573 (2020a).
- [CLM202] COLESANTI, A., LUDWIG, M., MUSSNIG, F.: Hessian valuations. *Indiana Univ. Math. J.* **69**, 1275–1315 (2020b).
- [CLM22] COLESANTI, A., LUDWIG, M., MUSSNIG, F.: The Hadwiger theorem on convex functions, III: Steiner formulas and mixed Monge–Ampère measures. *Calc. Var. Partial Differ. Equ.* **61**, Article ID 181 (2022).
- [CLM23] COLESANTI, A., LUDWIG, M., MUSSNIG, F.: The Hadwiger theorem on convex functions, IV: The Klain approach. *Adv. Math.* **413**, Article ID 108832 (2023).
- [CLM] COLESANTI, A., LUDWIG, M., MUSSNIG, F.: The Hadwiger theorem on convex functions, II: Cauchy–Kubota formulas. *Am. J. Math.* (in press).
- [C+20] COLESANTI, A., PAGNINI, D., TRADACETE, P., VILLANUEVA, I.: A class of invariant valuations on  $\text{Lip}(S^{n-1})$ . *Adv. Math.* **366**, Article ID 107069 (2020).
- [CS10] COLESANTI, A., SAORÍN GÓMEZ, E.: Functional inequalities derived from the Brunn–Minkowski inequalities for quermassintegrals. *J. Convex Anal.* **17**, 35–49 (2010).
- [Hab12] HABERL, C.: Minkowski valuations intertwining with the special linear group. *J. Eur. Math. Soc.* **14**, 1565–1597 (2012).
- [HP14] HABERL, C., PARAPATITS, L.: The centro-affine Hadwiger theorem. *J. Am. Math. Soc.* **27**, 685–705 (2014).
- [Had57] HADWIGER, H.: *Vorlesungen über Inhalt, Oberfläche und Isoperimetrie*. Springer, Berlin (1957).
- [H+16] HUANG, Y., LUTWAK, E., YANG, D., ZHANG, G.: Geometric measures in the dual Brunn–Minkowski theory and their associated Minkowski problems. *Acta Math.* **216**, 325–388 (2016).

- [Kla95] KLAIN, D.A.: A short proof of Hadwiger's characterization theorem. *Mathematika* **42**, 329–339 (1995).
- [KR97] KLAIN, D.A., ROTA, G.-C.: Introduction to Geometric Probability. Cambridge University Press, Cambridge (1997).
- [Kno21] KNOERR, J.: The support of dually epi-translation invariant valuations on convex functions. *J. Funct. Anal.* **281**, Article ID 109059 (2021).
- [Kno24] KNOERR, J.: Smooth valuations on convex functions. *J. Differ. Geom.* **126**, 801–835 (2024).
- [LM17] LI, J., MA, D.: Laplace transforms and valuations. *J. Funct. Anal.* **272**, 738–758 (2017).
- [LS07] LONGINETTI, M., SALANI, P.: On the Hessian matrix and Minkowski addition of quasi-convex functions. *J. Math. Pures Appl. (9)* **88**, 276–292 (2007).
- [Lud03] LUDWIG, M.: Ellipsoids and matrix valued valuations. *Duke Math. J.* **119**, 159–188 (2003).
- [Lud10] LUDWIG, M.: Minkowski areas and valuations. *J. Differ. Geom.* **86**, 133–161 (2010).
- [Lud11] LUDWIG, M.: Fisher information and matrix-valued valuations. *Adv. Math.* **226**, 2700–2711 (2011).
- [Lud12] LUDWIG, M.: Valuations on Sobolev spaces. *Am. J. Math.* **134**, 827–842 (2012).
- [LR99] LUDWIG, M., REITZNER, M.: A characterization of affine surface area. *Adv. Math.* **147**, 138–172 (1999).
- [LR10] LUDWIG, M., REITZNER, M.: A classification of  $SL(n)$  invariant valuations. *Ann. Math. (2)* **172**, 1219–1267 (2010).
- [Mus19] MUSSNIG, F.: Volume, polar volume and Euler characteristic for convex functions. *Adv. Math.* **344**, 340–373 (2019).
- [Mus21] MUSSNIG, F.:  $SL(n)$  invariant valuations on super-coercive convex functions. *Can. J. Math.* **73**, 108–130 (2021).
- [PW13] PARAPATITS, L., WANNERER, T.: On the inverse Klain map. *Duke Math. J.* **162**, 1895–1922 (2013).
- [Rei73] REILLY, R.C.: On the Hessian of a function and the curvatures of its graph. *Mich. Math. J.* **20**, 373–383 (1973).
- [Roc97] ROCKAFELLAR, R.T.: Convex Analysis, Princeton Landmarks in Mathematics. Princeton University Press, Princeton (1997). Reprint of the 1970 original, Princeton Paperbacks.
- [RW98] ROCKAFELLAR, R.T., WETS, R.J.-B.: Variational Analysis. Grundlehren der Mathematischen Wissenschaften, vol. 317. Springer, Berlin (1998).
- [Sch14] SCHNEIDER, R.: Convex Bodies: The Brunn-Minkowski Theory, Second expanded edn. Encyclopedia of Mathematics and Its Applications, vol. 151. Cambridge University Press, Cambridge (2014).
- [TW97] TRUDINGER, N.S., WANG, X.-J.: Hessian measures. I. *Topol. Methods Nonlinear Anal.* **10**, 225–239 (1997).
- [TW99] TRUDINGER, N.S., WANG, X.-J.: Hessian measures. II. *Ann. Math. (2)* **150**, 579–604 (1999).
- [Tsa10] TSANG, A.: Valuations on  $L^p$  spaces. *Int. Math. Res. Not.* **20**, 3993–4023 (2010).

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