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A dynamical system representation of the knowledge process: Projection of physiological dynamics on epistemic spaces

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ABSTRACT

We study the dynamical relation between physiological realizations and epistemic organizations, without selecting a particular philosophical theory of knowledge. A task specification supplies the adequacy criteria represented at the chosen resolution; their recognition by the subject is modeled separately. Physiological states that are indistinguishable under a realization map can have different epistemic continuations, so an autonomous epistemic dynamics is not presumed. A compatibility inequality yields quantitative bounds on persistence near a moving recognized target and on deviation from a selected task-adequate target. A nonlinear constitutive example derived from firing–connectivity balance equations verifies that inequality with explicit constants and compares its bound with numerical trajectories. Additional variational and state–process structures distinguish rate selection, accessibility, and minimum action to different targets.

1. Introduction

Human cognition involves physiological and epistemic dynamics. Neuronal firing, plastic connectivity, and the histories stored by their evolution constrain what a subject can discriminate, retain, combine, predict, and communicate. Epistemic descriptions concern organizations of contents and relations among them, together with the tasks for which those organizations can be judged adequate.

The epistemological question of what qualifies as knowledge and the dynamical question of how an epistemic organization is realized and transformed are distinct. Here, we address the latter without choosing among the theories of knowledge discussed in the current philosophical debate. Thus, state spaces and adequacy data introduced in what follows must be interpreted consistently with the epistemological account adopted in an application. This does not assert that every such account is already expressible by the particular observables and losses below: truth conditions, justification, or relations to an environment may require additional variables, histories, or constraints. The mathematical claims expressed here are conditional on the resulting specification and do not make task success a sufficient definition of knowledge.

We study organizations of epistemic states whose availability under specified perturbations and transformations, including retention and synthesis, can be expressed through persistence conditions. This selects a dynamical question and does not impose persistence as a universal philosophical criterion for knowledge. However, stability and persistence alone do not distinguish knowledge from other organized phenomena. A persistent organization may be false, irrelevant, or incapable of performing the task for which it is used. Thus, the epistemic qualification is supplied here by adequacy relative to specified tasks and standards. Prediction, discrimination, classification, inference, control, communicability, and access to information give different instances of

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such specifications. In the analysis below, tasks and standards are fixed over the time interval considered, while their recognition by the subject may evolve.

Adequacy and its recognition are distinct. Once a family of tasks and its standards has been fixed, adequacy does not depend on the particular subject who judges it. The formation of those tasks and standards may nevertheless be historical. Their recognition by a subject is interpretive and can be altered by psychological condition, individual memory, previous trauma, cultural formation, and historical–social circumstances. These factors may affect which adequate range is perceived, which path is selected, and whether an accessible path of knowledge is recognized as accessible. The knowledge of accessibility is itself an epistemic condition and needs not follow from accessibility alone.

Physiological limitation and interpretive conditioning have different roles [1–3], despite possible interactions. Stable cognitive capacities delimit the processes that can be physiologically realized. Variable psychological and historical–social conditions influence the assessment of their epistemic effects and the selection among the paths that remain available. The distinction permits a task-adequate target to be fixed while the target recognized by the subject moves. It also permits a physiologically economical process to fail to be optimal with respect to the selected epistemic target.

Dynamical approaches to cognition represent neural or cognitive activity by trajectories, invariant sets, and patterns in state spaces [4,5]. Formal epistemology represents changes of belief and epistemic status through logical, probabilistic, or ordinal structures [6–8]. Accounts based on propositions, justification, belief, and truth conditions specify the status of contents without assigning them a physiological dynamics [9–12]. The construction developed here begins with neither an autonomous cognitive dynamics nor a purely formal change of epistemic status. It addresses the compatibility between a physiological evolution, the epistemic organizations it realizes, the persistence and adequacy of those organizations, and their interpretive recognition.

The relation between neural coarse-graining and macroscopic content also occurs in the information closure theory of consciousness [13]. There, a stochastic microscopic process is mapped to a coarse-grained process, and informational closure supplies a criterion for conscious content. On the other side, Kolchinsky and Wolpert [14] define semantic information through correlations causally necessary for the viability of a physical system. Informational closure and viability concern relations different from task-relative adequacy and its recognition, as interpreted here. A system may remain viable without realizing the organization adequate for an assigned epistemic task, and a subject may fail to recognize an adequate organization that is physiologically realizable.

The physiological dynamics is tied to the balance equations of the interactions associated with variations of neuronal firing and connectivity, as derived in [15]; their structure is independent of constitutive specification, according to the conceptual hierarchy at the ground of classical field theories. Indeed, the representation of contact interactions follows from the geometric description adopted for the morphology of the body and from the range of the interactions retained in that description, whereas constitutive choices specify the response only after these structures have been fixed. Constitutive choices, initial data, and boundary data close a particular evolution problem. Here, we do not enter into the discussion of existence of solutions to specific initial and boundary value problems. We just restrict the analysis by a presumption that the physiological/psychological dynamics considered develops in terms of a semiflow. Convex dissipation, variational selection, and the action assigned to a process require further hypotheses.

The structure that we present here is based on the notions of state and related transformations: (1) physiological states contain firing, connectivity, and their histories; (2) epistemic states represent (conceptual) organizations; (3) psychological–historical states (interpreted also as psychological-cultural states) describe the variable conditions under which adequacy is recognized and paths are selected.

Physiological evolution is associated with a path of epistemic states: More generally, processes are intended as operators that act on the extended physiological–interpretive state space and induce epistemic transformations. Their action measures the accumulated cost of following an admissible interpretative process; the residual action required to reach a target is derived by minimization over processes and is therefore a function of the initial state.

Exact reduction to an autonomous epistemic dynamics is not imposed. Physiological states identified at the epistemic level can retain different histories and different continuations. Elimination of unresolved variables commonly preserves such dependence, as in the Mori–Zwanzig representation [16]. Here nonprojectability is part of the modeled relation: the same present epistemic organization can have different futures because of physiological distinctions suppressed by the realization or because of different interpretive conditions.

The fiberwise descent statements below are only factorization criteria. Among them, we discuss the semiconjugacy condition associated with exact lumpability (condition (21)). The persistence estimate uses a scalar dissipative comparison, and the process value has the dynamic-programming structure of a cost to a terminal set [17]. The action inequality is related, with the sign and terminal conventions specified in Section 7, to the dissipativity theory discussed in [18]. Approximate bisimulation instead compares the behavior of two systems through trajectory-matching conditions [19]. The specific contribution of what we propose here is to connect these structures to distinct physiological, adequacy, and recognition data, to identify the separate obstructions to their reduction, and to verify the persistence hypothesis in a constitutive firing–connectivity model. Section 9.3 makes the memory mechanism explicit.

2. Conceptual structure of the model

2.1. Epistemic organization, persistence, and adequacy

An epistemic state is an organization of variables selected to represent contents and relations relevant to assigned tasks. Equality of epistemic states means indistinguishability within that representation; it does not assert identity of their physiological realizations

or completeness of the description. The epistemic state space is consequently determined together with the observables through which prediction, discrimination, classification, inference, control, communication, or access are evaluated.

Persistence refers to the preservation of an epistemic organization along the realized evolution, either exactly or within a controlled error. Its mathematical expressions include invariance, attraction, robustness, and confinement to a moving neighborhood. None of these properties alone assigns epistemic value. Adequacy is introduced through the performance of the organization under the fixed tasks and standards. A selected target inside the adequate region singles out an organization whose maintenance can be studied dynamically, without identifying it with the whole adequate region.

The qualification "absolute" used below is conditional on the selected task specification. It means independence from the individual act of recognition after the tasks, standards, and tolerances have been assigned. It does not mean that the specification is independent of history, social practices, or the purposes for which the tasks were formulated by others or by the subject.

Complete metric spaces enter into play: a physiological state space $(\mathscr{M}, d_{\mathcal{M}})$, the space (X, d_X) of epistemic organization, and $(\mathfrak{P}, d_{\mathfrak{P}})$ the space of psychological, mnemonic, and historical–social conditions relevant to recognition and path selection. The physiological state space $\mathscr{M}$ contains the fields and history variables required by the selected constitutive realization of the balance structure. Extended states are elements of $\mathscr{M} \times \mathfrak{P}$, while epistemic realizations belong to X.

Definition 2.1. Fix $\mathscr{M}$, X, a realization $\pi : \mathscr{M} \to X$, an epistemic task specification in X, its adequate region $\mathcal{A}_*$, and a selected closed target $A_* \subset \mathcal{A}_*$. A curve $\mathfrak{e} : I \to X$ is realized when $\mathfrak{e}(t) = \pi(z(t))$ for an admissible physiological history $z(t) \in \mathscr{M}$. It is r-persistent relative to A_* when

$$\sup_{t \in I} d_X(\mathfrak{e}(t), A_*) \leq r. \tag{1}$$

A history is task-adequate when $\mathfrak{e}(t) \in \mathcal{A}_*$ for every $t \in I$. Let L_* and ε_* be the task loss and tolerance (a real function and a real number, respectively) associated with the specification, as made explicit in Section 3.2. For prescribed $r, \eta \geq 0$, a realized history is (r, η)-target-compatible relative to A_* and the task specification when it is r-persistent relative to A_* and

$$\sup_{t \in I} \big(L_*(\mathfrak{e}(t)) - \varepsilon_*\big)_+ \leq \eta. \tag{2}$$

Exact target compatibility means $(0, 0)$-target compatibility. A task-adequate, r-persistent realized history is $(r, 0)$-target-compatible.

The selected target supplies a quantitative persistence criterion for the results below; it does not exhaust the adequate region. A realized history can be persistent and task-adequate without remaining near the particular subset A_*. Target compatibility with $\eta = 0$ is therefore a sufficient quantified realization of the persistence–adequacy conjunction adopted in the model, rather than a definition of knowledge. In particular, no predicate asserting that a subject knows is identified with target compatibility. Recognition is excluded from the definition of task adequacy so that conformity to a fixed standard can be compared with the subject's assessment of it. This separation does not declare adequacy sufficient for knowledge, nor does it equate target recognition with belief, justification, or reflective access to reasons. Requirements concerning those notions must be represented explicitly when an application adopts an epistemological account that includes them.

2.2. Realization and physiological multiplicity

The passage from physiological to epistemic description requires a selected resolution. A continuous *realization map* $\pi : \mathscr{M} \to X$ is a datum of the model. It induces the equivalence relation

$$z_1 \sim_\pi z_2 \quad \Longleftrightarrow \quad \pi(z_1) = \pi(z_2). \tag{3}$$

Its fibers contain the physiological states indistinguishable at the adopted epistemic resolution. They retain distinctions irrelevant to the current epistemic description but potentially relevant to later evolution.

Continuity of the realization expresses compatibility of the selected epistemic observables with variations of the physiological state. It does not imply that the physiological evolution descends to an autonomous evolution on the epistemic state space. Such descent requires every two representatives of one fiber to have epistemically indistinguishable continuations. Failure of that condition records memory and unresolved physiological structure rather than a defect of the realization.

2.3. Recognition, accessibility, and selection

The subject does not act directly on the task-adequate region as an externally given mathematical entity. Adequacy is approached through a recognized target formed under the current psychological, mnemonic, cultural, and historical–social condition. The recognized target can deviate from the selected adequate target and can vary while the latter remains fixed. A comparison between the two targets measures this deviation without identifying recognition with adequacy.

Transmission among subjects and passage from individual to collective scales can alter both organization and adequacy. A variable $\mathfrak{p}$ retains the historical–social conditions relevant to the interpretation of one subject, without defining a population dynamics or a change-of-scale operator (a theory of propagation across social scales would require state spaces and admissible transformations at each scale and compatibility maps between them).

Existence of an adequate state does not imply its accessibility, which depends on the processes admitted by the subject's physiological state and cognitive capacities. Recognition of accessibility is a further condition: a realizable path may not be perceived

as available, while a path perceived as favorable may not lead to the selected adequate target. This separates the geometry of target sets from the process structure used to reach them.

Processes are therefore introduced as operators that induce transformations of the extended physiological–interpretive state source. Their domains encode which transformations are admissible from which initial states; their prolongation encodes successive transformations. The induced epistemic transformation is obtained only after applying the realization π. It becomes an operator on the epistemic state space precisely when it is constant on every relevant realization fiber.

An action evaluates a process from an initial state and is additive under prolongation. It can contain physiological dissipation, applied actions, interpretive effort, and other constitutively selected costs. The minimum action required to reach an adequate or recognized target is a derived value. Different target families and different process domains can therefore produce different values from the same initial state. This distinction formalizes the possibility that physiological economy, interpreted convenience, and epistemic adequacy select different paths.

3. Physiological, interpretive, and epistemic data

3.1. Physiological balances and evolution

Let $\mathcal{B} \subset \mathbb{R}^3$ be a bounded connected region with surface-like boundary oriented by the normal n to within a finite number of corners and edges, a region occupied by cortical matter. Differentiable fields

$$u : \mathcal{B} \times [0, +\infty) \to \mathbb{R}, \qquad \boldsymbol{\xi} : \mathcal{B} \times [0, +\infty) \to \mathbb{R}^{3\times 3}, \tag{4}$$

describe mean firing and neuronal connectivity. Bulk and contact interactions are associated with the time rate of u and $\boldsymbol{\xi}$, indicated by a superposed dot. With $\mathfrak{b}$ a generic part of $\mathcal{B}$, that is, a nonempty subset of $\mathcal{B}$ with non-null volume and regularity as the one of $\mathcal{B}$, the external power $\mathcal{P}^{ext}_{\mathfrak{b}}$ of interactions over $\mathfrak{b}$ is defined by

$$\mathcal{P}^{ext}_{\mathfrak{b}}(\dot{u}, \dot{\boldsymbol{\xi}}) := \int_{\mathfrak{b}} (\beta \dot{u} + \boldsymbol{\gamma} \cdot \dot{\boldsymbol{\xi}})\, dx + \int_{\partial \mathfrak{b}} (\tau_{u\partial} \dot{u} + \boldsymbol{\tau}_{\xi\partial} \cdot \dot{\boldsymbol{\xi}})\, d\mathcal{H}^2(x)\,,$$

where $d\mathcal{H}^2$ is the two-dimensional Hausdorff measure and the interposed dot indicates the duality pairing from now on. The bulk interactions β and $\boldsymbol{\gamma}$ are, respectively, scalar and tensor functions of space and time. So are, respectively, $\tau_{u\partial}$ and $\boldsymbol{\tau}_{\xi\partial}$; the subscript ∂ means that, in addition to space and time, the contact (first-neighbor) interactions depend on the boundary $\partial\mathfrak{b}$.

With $\psi = \psi(x, \varsigma)$ the Helmholtz free energy density, which depends on the state ς, left unspecified at this level, the pertinent version of the Clausius-Duhem inequality in isothermal setting reads

$$\mathscr{D}(\dot{u}, \dot{\boldsymbol{\xi}}; \mathfrak{b}) := \frac{d}{dt} \int_{\mathfrak{b}} \psi\, dx - \mathcal{P}^{ext}_{\mathfrak{b}}(\dot{u}, \dot{\boldsymbol{\xi}}) \leq 0\,. \tag{5}$$

Changes of observers in the physical space are changes of coordinate frames. Since firing and ψ are densities per unit volume and connectivity is a tensor, generic diffeomorphism-based changes of observer alter their representations because, in general, the volume is altered. The changes of coordinates act on the representations of u and $\boldsymbol{\xi}$ through families of differentiable homomorphisms from $\mathrm{Diff}(\mathbb{R}^3, \mathbb{R}^3)$ into the diffeomorphism groups of the spaces in which these fields take values. The quantities υ and $\boldsymbol{\zeta}$ below are the corresponding infinitesimal generators after pull-back into the frame of the first observer. By assumption they are bounded as functions of space and time. The induced transformations are

$$\dot{u} \longmapsto \dot{u} + \upsilon\,, \qquad \dot{\boldsymbol{\xi}} \longmapsto \dot{\boldsymbol{\xi}} + \boldsymbol{\zeta}\,, \qquad \dot{\psi} \longmapsto \dot{\psi} + \varphi(x, \varsigma, \upsilon, \nabla\upsilon, \boldsymbol{\zeta}, \nabla\boldsymbol{\zeta}, \ldots)\,,$$

where φ is a bounded function and the dots indicate the possible dependence on higher gradients of υ and $\boldsymbol{\zeta}$. When changes of observers in the physical space are of rigid-body type, the scalar densities $\dot{u}$ and $\dot{\psi}$ do not vary, so υ and φ vanish; the tensorial generator $\boldsymbol{\zeta}$ need not vanish (further details are in [15,20,21]). In this setting, the *covariance axiom* [20,21] for (5) reads

$$\mathscr{D}(\upsilon, \boldsymbol{\zeta}; \mathfrak{b}) \leq 0, \tag{6}$$

for any choice of υ, $\boldsymbol{\zeta}$, and $\mathfrak{b}$. The identity to zero in (6) holds only when the changes of observers are isometric.

When the bulk terms are bounded while $\tau_{u\partial}$ and $\boldsymbol{\tau}_{\xi\partial}$ are continuous with respect to the space variable, we find that the contact actions depend on $\partial\mathfrak{b}$ only through the normal n (see proofs in [15,20,21]), that is,

$$\tau_{u\partial}(x, t) = \tau_u(x, t, n) = -\tau_u(x, t, -n)$$

and

$$\boldsymbol{\tau}_{\xi\partial}(x, t) = \boldsymbol{\tau}_{\xi}(x, t, n) = -\boldsymbol{\tau}_{\xi}(x, t, -n)\,.$$

Moreover, we find that τ_u and $\boldsymbol{\tau}_{\xi}$ are linear with respect to n, that is (Cauchy's stress-type theorem),

$$\tau_u(x, t, n) = \mathcal{S}_u(x, t) n(x) \quad \text{and} \quad \boldsymbol{\tau}_{\xi}(x, t, n) = \boldsymbol{\mathcal{S}}_{\xi}(x, t) n(x)\,.$$

We also determine (see details in [20,21]) the existence of internal self-actions z_u and $\boldsymbol{z}_{\xi}$, which we assume to be sums of energetic and dissipative components, the former indicated by a superscript e, the latter by a superscript d; the same assumption can be accepted for $\mathcal{S}_u$ and $\boldsymbol{\mathcal{S}}_{\xi}$.

Thus, the arbitrariness of the induced generators v and $\boldsymbol{\zeta}$ implies (see also [15,20,21]) the local balance equations

$$\begin{aligned} z_u^d &= \beta - z_u^e + \operatorname{div}(S_u^e + S_u^d), \\ \boldsymbol{z}_{\boldsymbol{\xi}}^d &= \boldsymbol{\gamma} - \boldsymbol{z}_{\boldsymbol{\xi}}^e + \operatorname{div}(\boldsymbol{S}_{\boldsymbol{\xi}}^e + \boldsymbol{S}_{\boldsymbol{\xi}}^d). \end{aligned} \tag{7}$$

They require constitutive assignment. Let the state ς be the list $u, \nabla u, \boldsymbol{\xi}, \nabla \boldsymbol{\xi}$. Assume that S_u and $\boldsymbol{S}_{\boldsymbol{\xi}}$ have only energetic-type components. By exploiting once again the Clausius-Duhem inequality, a priori constitutive restrictions follow so that the balance equations reduce to

$$\begin{aligned} a_u(\cdots)\dot{u} &= \beta - \partial_u \psi + \operatorname{div} \partial_{\nabla u} \psi, \\ \widehat{a}_{\boldsymbol{\xi}}(\cdots)\dot{\boldsymbol{\xi}} &= \boldsymbol{\gamma} - \partial_{\boldsymbol{\xi}} \psi + \operatorname{div} \partial_{\nabla \boldsymbol{\xi}} \psi\,, \end{aligned} \tag{8}$$

where a_u and $\widehat{a}_{\boldsymbol{\xi}}$ are positive real functions of the state variables and, possibly, all their derivatives. Choices of ψ, a_u, $\widehat{a}_{\boldsymbol{\xi}}$, and their possible dependence on present values and histories specify a constitutive class and close a particular initial–boundary value problem. Existence and uniqueness depend on those choices and on the prescribed data.

The tensor field $\boldsymbol{\xi}$ describes connectivity at a selected continuum resolution; it is not a representation of every pairwise neuronal connection. Retaining only contact interactions and local gradient dependence is a modeling assumption. Nonlocal kernels, additional (thermodynamic non-equilibrium) internal variables, or history-dependent constitutive choices would define other models and require their own well-posedness and compatibility estimates. No learning or memory mechanism follows from the balance structure without such constitutive assignments.

The balance setting provides a framework for constitutive neural models, as discussed in [15]. Identifying a particular model, including a continuous Hodgkin–Huxley-type structure [22], requires constitutive prescriptions. Other representations could also be admitted. The general analysis is not restricted to one neural model; a specific constitutive example is supplied below (the taxonomy of specific models is very wide; an incomplete list of recent proposals includes [23–29]). What is relevant below is the assumption that the physiological dynamics implies (at least in principle) a semiflow in the considered state space, under conditions assuring well-posedness to the pertinent differential problem. Thus, the definition below applies.

Definition 3.1. Let $(\mathscr{M}, d_{\mathscr{M}})$ be a complete metric space containing the firing field, the connectivity field, and the history variables required by a fixed constitutive selection. An admissible physiological evolution is the physiological component of a continuous semiflow

$$\mathscr{S} : [0, +\infty) \times (\mathscr{M} \times \mathfrak{P}) \longrightarrow \mathscr{M} \times \mathfrak{P}, \tag{9}$$

whose physiological trajectories solve the corresponding initial–boundary value problem in the adopted weak or strong sense.

Write

$$\mathscr{S}(t)(z, \mathfrak{p}) = \big(S_{\mathfrak{p}}(t) z, \mathfrak{p}_{z,\mathfrak{p}}(t)\big). \tag{10}$$

The semigroup relations hold on $\mathscr{M} \times \mathfrak{P}$. The notation does not assert that either component separately defines a semiflow. If residual time dependence is not incorporated into the extended state, the appropriate object is a two-parameter evolution process; that case is not considered below.

Let $B \subset \mathscr{M}$ be closed. It is exponentially attracting along the physiological component in a positively invariant neighborhood $U \subset \mathscr{M} \times \mathfrak{P}$ if constants $C_B \geq 1$ and $\lambda > 0$ satisfy

$$d_{\mathscr{M}}(S_{\mathfrak{p}}(t) z, B) \leq C_B e^{-\lambda t} d_{\mathscr{M}}(z, B), \tag{11}$$

for every $(z, \mathfrak{p}) \in U$ and $t \geq 0$. No epistemic meaning is assigned to B by (11) alone.

3.2. Absolute epistemic adequacy

Persistence becomes epistemically qualified only after tasks and standards have been assigned. The following specification converts their performance requirements into observables and losses on X.

An absolute epistemic specification is a finite family

$$\mathfrak{E}_* = \{(Y_j, O_j, T_j, \ell_j, w_j)\}_{j=1}^m, \tag{12}$$

where Y_j is a metric space, $O_j : X \to Y_j$ is continuous, $T_j \subset Y_j$ is closed, $\ell_j : Y_j \to [0, +\infty)$ is continuous, $w_j > 0$, and

$$\ell_j(y) = 0 \quad \Longleftrightarrow \quad y \in T_j. \tag{13}$$

Every ℓ_j is a loss associated with the O_j task. The associated *total loss* is

$$L_*(\mathfrak{e}) := \sum_{j=1}^m w_j \ell_j(O_j(\mathfrak{e})). \tag{14}$$

Prediction, discrimination, classification, control, and communicability can be represented among the task observables. Accessibility can be represented by an observable on X only when the corresponding process reachability is constant on the relevant realization fibers; without that invariance it is a property of the physiological–interpretive representative.

Fix $\varepsilon_* \geq 0$ and set

$$\mathcal{A}_* := \{\mathfrak{e} \in X : L_*(\mathfrak{e}) \leq \varepsilon_*\}. \tag{15}$$

A nonempty compact set $A_* \subset \mathcal{A}_*$ represents a *selected target within the region of absolute adequacy*. Membership in A_* implies task adequacy, whereas proximity generally gives only an approximate loss bound. If L_* is K_*-Lipschitz, then $L_*(\mathfrak{e}) \leq \sup_{a\in A_*} L_*(a) + K_* d_X(\mathfrak{e}, A_*)$. Exact adequacy throughout a neighborhood follows when the target has a margin $\sup_{A_*} L_* \leq \varepsilon_* - \sigma$, with $\sigma > 0$, and $K_* d_X(\mathfrak{e}, A_*) \leq \sigma$. Adequate histories far from the selected subset A_* are not covered by a target-confinement conclusion. The specification, the tolerance, and the selected target are independent of the individual physiological trajectory and of the condition $\mathfrak{p}$. Absoluteness has this restricted meaning. The tasks and their standards may themselves have a historical origin.

3.3. Interpretive recognition

As already mentioned, the elements of $\mathfrak{P}$ collect the psychological condition of the subject, individual memory, and the historical–social conditions relevant to the subject's interpretation. The latter are included as contextual state variables, together with the history needed for autonomy; no autonomous dynamics of a social system is asserted. A *recognized loss* is a continuous map

$$\widehat{L} : \mathfrak{P} \times X \longrightarrow [0, +\infty), \qquad (\mathfrak{p}, \mathfrak{e}) \longmapsto \widehat{L}_{\mathfrak{p}}(\mathfrak{e}). \tag{16}$$

Associated with $\widehat{L}$ we consider a *psychology-dependent tolerance* $\widehat{\varepsilon}(\mathfrak{p})$. Given a nonnegative continuous $\widehat{\varepsilon}(\mathfrak{p})$, define $\widehat{A}_{\mathfrak{p}}$ as the set

$$\widehat{A}_{\mathfrak{p}} := \{\mathfrak{e} \in X : \widehat{L}_{\mathfrak{p}}(\mathfrak{e}) \leq \widehat{\varepsilon}(\mathfrak{p})\}. \tag{17}$$

Below, the sets used are assumed nonempty and closed; compactness is imposed when needed.

The variables in $\mathfrak{P}$ are descriptive coordinates for the factors retained in an application, not independent substances or arbitrary labels. Their dynamics and the functions $\widehat{L}$, $\widehat{\varepsilon}$, c, and δ are constitutive data. Consistency relations between physiological and interpretive variable can restrict the admissible invariant subset of $\mathscr{M} \times \mathfrak{P}$. A change of $\pi(z)$ changes the represented epistemic organization; a change of $\mathfrak{p}$ at fixed $\pi(z)$ changes its assessment or the recognized possibilities for action. An application must specify observables for that distinction rather than infer $\mathfrak{p}$ retrospectively from the desired persistence estimate.

Let $\mathfrak{P}_0 \subset \mathfrak{P}$ contain the psychology-driven interpretive states under consideration; they implicitly refer to a given cultural level. Recognition of the selected target is controlled if there are functions $c : \mathfrak{P}_0 \to [1, +\infty)$ and $\delta : \mathfrak{P}_0 \to [0, +\infty)$ that satisfy

$$d_X(\mathfrak{e}, A_*) \leq c(\mathfrak{p}) d_X(\mathfrak{e}, \widehat{A}_{\mathfrak{p}}) + \delta(\mathfrak{p}), \tag{18}$$

for the epistemic states under consideration. The coefficient c measures interpretive distortion relative to the selected target and δ a residual target-recognition defect. Neither quantity measures distance from the whole adequate region $\mathcal{A}_*$.

3.4. Epistemic realization and projectability

Let $\pi : \mathscr{M} \to X$ be the continuous realization fixed above, with the equivalence relation (3). Define

$$\mathfrak{e}_{z,\mathfrak{p}}(t) := \pi(S_{\mathfrak{p}}(t)z). \tag{19}$$

On the extended state space, set

$$\Pi(z, \mathfrak{p}) := \pi(z). \tag{20}$$

Exact projectability requires

$$\Pi(\zeta_1) = \Pi(\zeta_2) \quad \Longrightarrow \quad \Pi(\mathscr{S}(t)\zeta_1) = \Pi(\mathscr{S}(t)\zeta_2), \tag{21}$$

for all $t \geq 0$, where $\zeta_i = (z_i, \mathfrak{p}_i)$. Under (21), the formula

$$T(t)\Pi(\zeta) := \Pi(\mathscr{S}(t)\zeta), \tag{22}$$

defines an induced semigroup on $\Pi(\mathscr{M} \times \mathfrak{P})$. For a continuous semiflow on the chosen topology, continuity of the induced action is an additional requirement; it follows, for example, if $\mathrm{id}_{[0,\infty)} \times \Pi$ is a quotient map. Mere continuity of a surjective Π does not supply that topological condition. Exact projectability is not assumed.

For the fields in (4), finite-rank realization maps include components

$$\pi_\alpha(u, \boldsymbol{\xi}, H) = \int_B \big(p_\alpha(x)u(x) + Q_\alpha(x) \cdot \boldsymbol{\xi}(x)\big)\, dx + \Pi_\alpha(H), \qquad \alpha = 1, \dots, N. \tag{23}$$

The functions p_α, tensors Q_α, and history functionals Π_α specify the physiological quantities entering the epistemic realization.

Each component must be justified by the task observables it is intended to support: weights can specify regional firing averages or connectivity moments, and a history functional can retain a declared temporal statistic. Such a choice is fixed before a compatibility test. The theory supplies neither a unique decoder nor an identification of arbitrary weighted averages with semantic content.

All reduction results concern a fixed realization. A different map can change both the fibers and projectability. There is invariance under an invertible relabeling $\pi' = h \circ \pi$: the fibers and algebraic descent conditions are unchanged. If h is bi-Lipschitz with constants $m, M > 0$, then distances to the transported targets satisfy $m\, d_X(e, A) \leq d_{X'}(h(e), h(A)) \leq M\, d_X(e, A)$, and quantitative bounds must be rescaled accordingly. No robustness under arbitrary changes of observables or loss of additional coordinates is claimed.

4. Closed epistemic dynamics and periodic training

A differential equation written directly on X presupposes that the epistemic variables and their evolution have already been closed. Such a closure can result from exact projectability, from an approximation retaining the unresolved effects through effective terms, or from a phenomenological model posed independently at the epistemic level. These alternatives have the same formal state variable but different relations to the physiological dynamics.

For example, with $X = \mathbb{R}^n$, an epistemic-level model with inertia and periodic training $\bar{h}(t)$ has the form

$$\frac{d}{dt}\varphi\big(\mathfrak{e}(t), \dot{\mathfrak{e}}(t), \lambda\bar{h}(t)\big) = g\big(\mathfrak{e}(t), \dot{\mathfrak{e}}(t)\big) + \lambda f\big(t, \mathfrak{e}(t), \dot{\mathfrak{e}}(t), \lambda\big), \tag{24}$$

where $\bar{h}$ and f are periodic, φ is a generalized momentum, and g contains state- and rate-dependent restoring effects. Under a nonzero topological degree condition and a quantitative restriction on the forcing period, this model admits periodic solutions synchronized with the training for sufficiently small positive λ [30].

The periodic solution supplies a recurrent response of the closed epistemic model. It is persistent relative to the selected target in the sense of Definition 2.1 only when its orbit satisfies the prescribed target-confinement estimate; target compatibility additionally requires the corresponding task-loss bound. Its physiological realization, task performance, and robustness under the admitted transformations remain distinct questions. Periodicity alone does not decide its epistemological status. Eq. (24) also does not determine whether its state variable is the exact projection of a physiological evolution. In the present construction it is induced as an autonomous epistemic dynamics by the physiological evolution only when the fiber condition (21) holds; otherwise it is a phenomenological or approximate closure. Periodic training thus gives recurrent organization; physiological realization, target confinement, and epistemic adequacy remain separate requirements. Inertia in cognitive dynamics can also be considered in other contexts, as in [31].

5. Persistence of recognized and selected targets

Fix $R > 0$ and define the set

$$\mathcal{T}_R := \{(\mathfrak{p}, \mathfrak{e}) \in \mathfrak{P}_0 \times X \,:\, d_X(\mathfrak{e}, \widehat{A}_{\mathfrak{p}}) \le R\}. \tag{25}$$

Assume that $(\mathfrak{p}, \mathfrak{e}) \mapsto d_X(\mathfrak{e}, \widehat{A}_{\mathfrak{p}})$ is continuous on the relevant region. This is an additional regularity requirement on the moving target, not a consequence of continuity of a loss defining its sublevel sets. Local Hausdorff continuity of nonempty compact target sets is one sufficient condition. Let W be nonnegative and continuous on a neighborhood of $\mathcal{T}_R$, with constants $a_1, a_2 > 0$ such that

$$a_1 d_X(\mathfrak{e}, \widehat{A}_{\mathfrak{p}})^2 \le W(\mathfrak{p}, \mathfrak{e}) \le a_2 d_X(\mathfrak{e}, \widehat{A}_{\mathfrak{p}})^2 \tag{26}$$

on $\mathcal{T}_R$. The comparison functional is thus assigned relative to the recognized family.

Assume that the interpretive component remains in $\mathfrak{P}_0$ for the initial states considered. For $\zeta = (z, \mathfrak{p}) \in U$, write

$$q_\zeta(t) := \sqrt{W(\mathfrak{p}_{z,\mathfrak{p}}(t), \mathfrak{e}_{z,\mathfrak{p}}(t))}. \tag{27}$$

The following conditions apply on intervals on which this expression is defined.

Definition 5.1. The evolution, realization, recognized family, and fixed W are locally compatible if q_ζ is locally absolutely continuous and constants $\mu > 0$, $\kappa_0, \kappa_1 \ge 0$, together with a nonnegative $g_\zeta \in L^1_{\mathrm{loc}}([0, \infty))$, satisfy

$$\dot{q}_\zeta(t) \le -\mu q_\zeta(t) + \kappa_0 + \kappa_1 d_{\mathscr{M}}(S_{\mathfrak{p}}(t)z, B) + g_\zeta(t) \tag{28}$$

for almost every time in $\mathcal{T}_R$. The control g_ζ must be specified or bounded from the interpretive dynamics and initial data without using the unknown comparison error q_ζ or its derivative to define it.

The subscript ζ allows dependence on the full initial state. Autonomy of $\mathfrak{p}$ is not required. For example, if $\dot{\mathfrak{p}} = F(z, \mathfrak{p})$, a bound on F over an independently established invariant region can provide a control envelope. For $W(\mathfrak{p}, e) = \|e - a(\mathfrak{p})\|^2$, the target-motion contribution is bounded by $\|Da(\mathfrak{p})\|\,\|F(z, \mathfrak{p})\|$. Other mismatches must be estimated from the projected dynamics. An inequality obtained only by setting g equal to the observed residual of (28) has no independent predictive content. Define

$$J_\zeta(t) := \int_0^t e^{-\mu(t-s)} g_\zeta(s)\, ds, \qquad J^*_\zeta := \sup_{t \ge 0} J_\zeta(t). \tag{29}$$

Global integrability of g_ζ is not required; bounded inputs, for example, have bounded J_ζ.

Theorem 5.2. *Assume that $\mathscr{S}$ is an admissible evolution, B satisfies (11), and local compatibility holds. Let $\zeta_0 = (z_0, \mathfrak{p}_0) \in U$ with $(\mathfrak{p}_0, \pi(z_0)) \in \mathcal{T}_R$, and set*

$$q_0 := \sqrt{W(\mathfrak{p}_0, \pi(z_0))}, \qquad d_0 := d_{\mathscr{M}}(z_0, B). \tag{30}$$

If $J^*_{\zeta_0} < \infty$ and

$$q_0 + \frac{\kappa_0}{\mu} + \frac{\kappa_1 C_B d_0}{\max\{\mu, \lambda\}} + J^*_{\zeta_0} < \sqrt{a_1} R, \tag{31}$$

the trajectory remains in $\mathcal{T}_R$ *and*

$$d_X(\mathfrak{e}_{z_0,\mathfrak{p}_0}(t), \widehat{A}_{\mathfrak{p}_{z_0,\mathfrak{p}_0}(t)}) \le \frac{1}{\sqrt{a_1}} \left(e^{-\mu t} q_0 + \frac{\kappa_0}{\mu}(1 - e^{-\mu t}) + \kappa_1 C_B d_0 I_{\mu,\lambda}(t) + J_{\zeta_0}(t) \right), \tag{32}$$

where

$$I_{\mu,\lambda}(t) := \begin{cases} (e^{-\lambda t} - e^{-\mu t})/(\mu - \lambda), & \mu \neq \lambda, \\ t e^{-\mu t}, & \mu = \lambda. \end{cases} \tag{33}$$

If (18) *holds along the trajectory, then*

$$d_X(\mathfrak{e}_{z_0,\mathfrak{p}_0}(t), A_*) \le c(\mathfrak{p}_{z_0,\mathfrak{p}_0}(t)) d_X(\mathfrak{e}_{z_0,\mathfrak{p}_0}(t), \widehat{A}_{\mathfrak{p}_{z_0,\mathfrak{p}_0}(t)}) + \delta(\mathfrak{p}_{z_0,\mathfrak{p}_0}(t)). \tag{34}$$

Proof. Before a possible exit, compatibility and physiological attraction give

$$\dot{q}_{\zeta_0}(t) \le -\mu q_{\zeta_0}(t) + \kappa_0 + \kappa_1 C_B e^{-\lambda t} d_0 + g_{\zeta_0}(t). \tag{35}$$

Multiplying by $e^{\mu t}$ and integrating yields

$$q_{\zeta_0}(t) \le e^{-\mu t} q_0 + \frac{\kappa_0}{\mu}(1 - e^{-\mu t}) + \kappa_1 C_B d_0 I_{\mu,\lambda}(t) + J_{\zeta_0}(t). \tag{36}$$

The convolution representation gives

$$0 \le I_{\mu,\lambda}(t) \le \frac{1}{\max\{\mu, \lambda\}}. \tag{37}$$

Indeed, bounding either exponential in its defining integral by one yields both $1/\mu$ and $1/\lambda$. The initial point is strictly inside the tube by (26) and (31). Continuity of the target-distance function along the continuous extended trajectory ensures that any exit crosses the boundary at distance R. There, (26) gives $W \ge a_1 R^2$, hence $q_{\zeta_0} = \sqrt{W} \ge \sqrt{a_1} R$, contradicting (36)–(37) and the strict smallness condition. Thus, the threshold in (31) is in the scale of $\sqrt{W}$; $a_1 R^2$ is the corresponding threshold for W, not for its square root. Dividing (36) by $\sqrt{a_1}$ proves (32); 34 is the recognition comparison. □

Corollary 5.3. *Under the hypotheses of Theorem* 5.2*, write* $\mathfrak{p}(t) = \mathfrak{p}_{z_0,\mathfrak{p}_0}(t)$ *and* $\mathfrak{e}(t) = \mathfrak{e}_{z_0,\mathfrak{p}_0}(t)$*. Suppose that*

$$\limsup_{t\to\infty} J_{\zeta_0}(t) \le J_\infty, \quad \sup_{t\ge 0} c(\mathfrak{p}(t)) \le c_*, \quad \limsup_{t\to\infty} \delta(\mathfrak{p}(t)) \le \delta_*, \tag{38}$$

with finite nonnegative J_∞, δ_**. Then,*

$$\limsup_{t\to\infty} d_X(\mathfrak{e}(t), \widehat{A}_{\mathfrak{p}(t)}) \le \frac{1}{\sqrt{a_1}} \left(\frac{\kappa_0}{\mu} + J_\infty \right), \qquad \limsup_{t\to\infty} d_X(\mathfrak{e}(t), A_*) \le \frac{c_*}{\sqrt{a_1}} \left(\frac{\kappa_0}{\mu} + J_\infty \right) + \delta_*. \tag{39}$$

If L_* *is* K_**-Lipschitz on a region containing the trajectory and* A_**, we have*

$$\limsup_{t\to\infty} L_*(\mathfrak{e}(t)) \le \varepsilon_* + K_* \left[\frac{c_*}{\sqrt{a_1}} \left(\frac{\kappa_0}{\mu} + J_\infty \right) + \delta_* \right]. \tag{40}$$

In particular, $J_\infty = 0$ *recovers the decaying-control conclusion. For an input with finite essential upper limit* g_* *as* $t \to \infty$*, one may take* $J_\infty = g_*/\mu$*; the recognized-target bound then becomes* $(\kappa_0 + g_*)/(\mu\sqrt{a_1})$*.*

Proof. The transients in (32) decay for fixed $\mu, \lambda > 0$, giving the two distance bounds. For $a \in A_*$, the loss estimate follows from $L_*(\mathfrak{e}) \le \varepsilon_* + K_* d_X(\mathfrak{e}, a)$, followed by an infimum and upper limit. To control J, split its integral at T such that $g_{\zeta_0}(s) \le g_* + \epsilon$ for almost every $s \ge T$. The initial part tends to zero and the remaining part is at most $(g_* + \epsilon)/\mu$. Let $\epsilon \downarrow 0$. □

Proposition 5.4. *Assume that* X *and* $\mathfrak{P}$ *are differentiable Banach manifolds,* W *is continuously differentiable near* $\mathcal{T}_R$*, and the extended trajectory and* q_ζ *are locally absolutely continuous. If*

$$\frac{d}{dt} W(\mathfrak{p}(t), \mathfrak{e}(t)) \le -2\mu W(\mathfrak{p}(t), \mathfrak{e}(t)) + 2\sqrt{W(\mathfrak{p}(t), \mathfrak{e}(t))} \big(\kappa_0 + \kappa_1 d_{\mathscr{M}}(z(t), B) + g_\zeta(t) \big) \tag{41}$$

almost everywhere, then (28) *holds. Local absolute continuity of* q *holds, for example, for a squared norm distance to a locally Lipschitz target center along locally absolutely continuous paths.*

Proof. For $W > 0$, use the chain rule. At almost every point of the zero set of a nonnegative absolutely continuous q, its derivative is zero, so the required inequality also holds there. Equivalently, for $q_\epsilon = \sqrt{W + \epsilon}$,

$$\dot{q}_\epsilon \le -\mu q_\epsilon + \mu\sqrt{\epsilon} + \kappa_0 + \kappa_1 d_{\mathscr{M}}(z(t), B) + g_\zeta(t), \tag{42}$$

and the integrated inequality passes to the limit. □

5.1. Relaxation rates and the range of the estimates

The estimates require positive rates for the specified attraction and comparison problems. Attraction to B constrains distance to that set, not every motion inside it. The framework therefore does not require all physiological degrees of freedom to relax to a single equilibrium. It also does not prove a positive rate in a critical regime. For fixed $\mu > 0$, the limit $\lambda \downarrow 0$ gives $I_{\mu,\lambda}(t) \to (1 - e^{-\mu t})/\mu$: physiological transients cease to decay uniformly in that limit, although the finite-time bound need not diverge. If $\mu \downarrow 0$, residual and persistent-input bounds proportional to $1/\mu$ can become uninformative. System-size-dependent rates must be retained explicitly; no uniform thermodynamic-limit or zero-rate conclusion follows. Algebraic or scale-dependent relaxation would require a different attraction hypothesis and its corresponding convolution estimate.

6. Variational selection and epistemic rates

The balance equations and the assigned constitutive class determine the physiological evolution without necessarily implying a minimum principle. For constitutive classes possessing a convex dissipation potential, an additional coupled rate selection can be imposed. The admissible rates and their energy–dissipation cost are physiological, whereas the evaluation of their instantaneous epistemic effect depends on both the physiological realization and the current psychological, mnemonic, cultural, and historical–social condition. The minimization is performed in the physiological tangent space. Its contraction to the epistemic tangent space is derived by applying the differential of the realization map.

Suppose that $\mathscr{M}$, $\mathfrak{P}$, and X are differentiable Banach manifolds and that π is continuously differentiable. At $z \in \mathscr{M}$, let

$$C_{\mathrm{phys}}(z) \subset T_z\mathscr{M} \tag{43}$$

be a nonempty closed convex set. It represents physiological capacities and does not depend on the current interpretive evaluation. Let

$$\Psi_z : T_z\mathscr{M} \longrightarrow [0, +\infty] \tag{44}$$

be proper, convex, and lower semicontinuous, with $\Psi_z(0) = 0$, and assume that $C_{\mathrm{phys}}(z)$ intersects its effective domain.

Let $\Phi : \mathscr{M} \to \mathbb{R}$ be differentiable. Assume that $f(z, \mathfrak{p}) \in T_z^*\mathscr{M}$ contains applied and history-dependent interactions. Set

$$\begin{aligned} \Theta_z(v) &:= \Psi_z(v) + I_{C_{\mathrm{phys}}(z)}(v), \\ \mathcal{R}_{z,\mathfrak{p}}(v) &:= \Theta_z(v) + \langle D\Phi(z) - f(z, \mathfrak{p}), v\rangle. \end{aligned} \tag{45}$$

The existence of Ψ_z is an additional property. It does not follow from (8) or from the dissipation inequality.

Assume in this section that the comparison functional W is differentiable in its epistemic argument and set

$$\widehat{U}(\mathfrak{p}, \mathfrak{e}) := \frac{1}{2} W(\mathfrak{p}, \mathfrak{e}). \tag{46}$$

The composition $\widehat{U}(\mathfrak{p}, \pi(z))$ depends on the physiological state through its epistemic realization and on the interpretive condition through $\mathfrak{p}$. For fixed $\mathfrak{p}$, a physiological rate $v \in T_z\mathscr{M}$ produces the epistemic rate $D\pi(z)v$, and the corresponding instantaneous variation of the recognized-target comparison is

$$\left\langle D_{\mathfrak{e}}\widehat{U}(\mathfrak{p}, \pi(z)), D\pi(z)v \right\rangle. \tag{47}$$

The condition $\mathfrak{p}$ is held fixed in this instantaneous minimization; its evolution along the extended semiflow is accounted for separately by the control g_ζ in the persistence estimate. Define

$$\mathcal{J}_{z,\mathfrak{p}}(v) := \mathcal{R}_{z,\mathfrak{p}}(v) + \left\langle D_{\mathfrak{e}}\widehat{U}(\mathfrak{p}, \pi(z)), D\pi(z)v \right\rangle. \tag{48}$$

Within this constitutive setting, a coupled variational realization satisfies

$$\dot{z}(t) \in \operatorname*{argmin}_{v \in T_{z(t)}\mathscr{M}} \mathcal{J}_{z(t),\mathfrak{p}(t)}(v) \tag{49}$$

for almost every time along its sufficiently regular trajectories. Condition (49) is an additional coupling hypothesis: it combines the physiological energy–dissipation cost with the change in recognized adequacy induced by a physiological rate. It links rate selection to the same extended evolution used in the persistence analysis; it does not identify the recognized target with a mechanical energy.

To compare this rule with (8), take unconstrained rates and a quadratic dissipation $\Psi_z(v) = \frac{1}{2}\langle A(z)v, v\rangle$, with a positive operator $A(z)$, and put $h(z, \mathfrak{p}) = D\pi(z)^* D_{\mathfrak{e}}\widehat{U}(\mathfrak{p}, \pi(z))$. The balance law with generalized applied force f_{bal} reads $A(z)\dot{z} = f_{\mathrm{bal}} - D\Phi(z)$, whereas (49) gives $A(z)\dot{z} = f - D\Phi(z) - h$. They coincide when $f_{\mathrm{bal}} = f - h$; if the same applied force is used in both, coincidence requires $h = 0$ in this unconstrained setting. With constraints, the corresponding normal-cone inclusions must agree.

Consequently, the variational rule is either a constitutive representation of feedback already included in the assigned interactions, or an explicitly additional coupled dynamics for which the semiflow and persistence assumptions must be checked again. It is not imposed on every admissible evolution. The feedback depends on the presently recognized target, including erroneous recognition, rather than on future attainment of the adequate target. Its physiological mechanism remains a constitutive hypothesis; the formal minimization does not supply that mechanism or derive it from the second law. The example in Section 9 verifies the persistence theorem directly from the balance law and does not additionally impose this feedback.

The accessible epistemic rates and the contracted physiological functional are

$$\begin{aligned} \mathcal{V}_z &:= D\pi(z)\mathcal{C}_{\text{phys}}(z), \\ \mathcal{D}_{z,\mathfrak{p}}(w) &:= \inf_{\substack{v\in\mathcal{C}_{\text{phys}}(z) \\ D\pi(z)v=w}} \mathcal{R}_{z,\mathfrak{p}}(v), \qquad w\in\mathcal{V}_z. \end{aligned} \tag{50}$$

Proposition 6.1. *If $v_{z,\mathfrak{p}}$ minimizes $\mathcal{J}_{z,\mathfrak{p}}$, then*

$$D\pi(z)v_{z,\mathfrak{p}} \in \operatorname*{argmin}_{w\in\mathcal{V}_z} \left\{ \mathcal{D}_{z,\mathfrak{p}}(w) + \left\langle D_{\mathfrak{e}}\widehat{U}(\mathfrak{p},\pi(z)), w \right\rangle \right\}. \tag{51}$$

The infimum defining $\mathcal{D}_{z,\mathfrak{p}}$ need not be attained for every w.

Proof. Let $m = \mathcal{J}_{z,\mathfrak{p}}(v_{z,\mathfrak{p}})$ and $w_{z,\mathfrak{p}} = D\pi(z)v_{z,\mathfrak{p}}$. Every admissible v satisfies $\mathcal{J}_{z,\mathfrak{p}}(v) \geq m$. Taking the infimum on each fiber of $D\pi(z)$ gives a value not smaller than m. Since $v_{z,\mathfrak{p}}$ belongs to the fiber over $w_{z,\mathfrak{p}}$, equality holds on that fiber. □

The force

$$\zeta_{z,\mathfrak{p}} := f(z,\mathfrak{p}) - D\Phi(z) - D\pi(z)^* D_{\mathfrak{e}}\widehat{U}(\mathfrak{p},\pi(z)), \tag{52}$$

where the asterisk indicates formal adjoint, satisfies

$$\zeta_{z,\mathfrak{p}} \in \partial\Theta_z(v_{z,\mathfrak{p}}). \tag{53}$$

Fenchel's equality therefore gives

$$\Theta_z(v_{z,\mathfrak{p}}) + \Theta_z^*(\zeta_{z,\mathfrak{p}}) = \langle \zeta_{z,\mathfrak{p}}, v_{z,\mathfrak{p}} \rangle. \tag{54}$$

Quadratic viscous, positively homogeneous rate-independent, and nonsmooth anisotropic dissipation are included; related constitutive pseudo-potential structures are discussed in [32].

For $\zeta = (z,\mathfrak{p})$, define

$$\mathcal{K}_\zeta(w) := \mathcal{D}_{z,\mathfrak{p}}(w) + \left\langle D_{\mathfrak{e}}\widehat{U}(\mathfrak{p},\pi(z)), w \right\rangle. \tag{55}$$

Proposition 6.2. *There are sets $\mathcal{V}_{\mathfrak{e}} \subset T_{\mathfrak{e}}X$ and functionals $\mathcal{K}_X(\mathfrak{e},\cdot)$ such that*

$$\mathcal{V}_z = \mathcal{V}_{\pi(z)}, \qquad \mathcal{K}_{(z,\mathfrak{p})} = \mathcal{K}_X(\pi(z),\cdot) \tag{56}$$

if and only if, whenever $\pi(z_1) = \pi(z_2)$,

$$\mathcal{V}_{z_1} = \mathcal{V}_{z_2}, \qquad \mathcal{K}_{(z_1,\mathfrak{p}_1)} = \mathcal{K}_{(z_2,\mathfrak{p}_2)} \tag{57}$$

for all admissible $\mathfrak{p}_1, \mathfrak{p}_2$.

Proof. Factorization implies (57). Conversely, that condition makes the assignments in (56) independent of the physiological–interpretive representative. □

This criterion removes $\mathfrak{p}$ as well as unresolved physiology. It is deliberately stronger than reduction at a retained interpretive state. Dependence on $\mathfrak{p}$ need not vanish in each summand of $\mathcal{K}$; compensating contributions can make the total independent of it. In general it is more useful to retain $\mathfrak{p}$ and define $\Pi_{\text{ret}}(z,\mathfrak{p}) = (\pi(z),\mathfrak{p})$. Then, the accessible rates and their selection functional factor as $\mathcal{V}_{\mathfrak{e}}$ and $\mathcal{K}_{X\times\mathfrak{P}}(\mathfrak{e},\mathfrak{p},\cdot)$ if and only if $\mathcal{V}_{z_1} = \mathcal{V}_{z_2}$ and $\mathcal{K}_{(z_1,\mathfrak{p})} = \mathcal{K}_{(z_2,\mathfrak{p})}$ whenever $\pi(z_1) = \pi(z_2)$, for each common admissible $\mathfrak{p}$. The proof is the same factorization argument. A full reduced evolution on $X\times\mathfrak{P}$ additionally requires equality of the future interpretive components; the rate criterion alone does not establish that evolution.

7. Process accessibility and action values for recognized and selected targets

Accessibility is a relation generated by processes. The existence of a target in X does not imply that it can be reached from a given physiological–interpretive state, and two states with the same epistemic realization need not admit the same processes. A state–process description records the admissible initial states, the transformations induced by processes, and their possible prolongations. The structure follows the state–process–action architecture used in axiomatic thermodynamics [33], here on the extended state space associated with physiological realization and interpretive recognition.

Let $\mathscr{D} \subset \mathscr{M}\times\mathfrak{P}$ be an admissibility region and $\mathscr{P}$ a class of processes. A process $P \in \mathscr{P}$ has a nonempty domain $\mathscr{D}_P \subset \mathscr{D}$ and induces a state transformation

$$q_P : \mathscr{D}_P \longrightarrow \mathscr{D}. \tag{58}$$

Its range is $\mathscr{R}_P := q_P(\mathscr{D}_P)$. Since the states contain $\mathfrak{p}$, the domain and the transformation induced by P retain the dependence on the physiological and interpretive conditions.

The prolongation P_2P_1 is defined when $\mathscr{R}_{P_1} \cap \mathscr{D}_{P_2} \neq \varnothing$. Its domain and state transformation are

$$\begin{aligned} \mathscr{D}_{P_2P_1} &:= \{\zeta \in \mathscr{D}_{P_1} \,:\, q_{P_1}\zeta \in \mathscr{D}_{P_2}\}, \\ q_{P_2P_1}\zeta &:= q_{P_2}(q_{P_1}\zeta). \end{aligned} \tag{59}$$

The class $\mathscr{P}$ is closed under every prolongation with nonempty domain and contains a null process I satisfying $\mathscr{D}_I = \mathscr{D}$ and $q_I\zeta = \zeta$. Whenever stopping at an intermediate state is used, the class is also assumed closed under the corresponding temporal restrictions of a process.

The states and epistemic organizations reachable from $\zeta \in \mathscr{D}$ are, respectively,

$$\begin{aligned} \mathrm{Reach}(\zeta) &:= \{q_P\zeta \,:\, P \in \mathscr{P},\ \zeta \in \mathscr{D}_P\}, \\ \mathrm{Reach}_X(\zeta) &:= \Pi(\mathrm{Reach}(\zeta)). \end{aligned} \tag{60}$$

A target $\mathcal{G} \subset \mathscr{D}$ is accessible from ζ when $\mathrm{Reach}(\zeta) \cap \mathcal{G} \neq \varnothing$. This property depends on the actual physiological–interpretive representative and need not be constant on a fiber of Π.

The epistemic transformation induced by P is

$$\mathscr{T}_P := \Pi \circ q_P \,:\, \mathscr{D}_P \longrightarrow X. \tag{61}$$

It generally depends on the full initial state, including its interpretive component.

Proposition 7.1. *There exists a map $\mathscr{T}_P^X \,:\, \Pi(\mathscr{D}_P) \to X$ satisfying*

$$\mathscr{T}_P = \mathscr{T}_P^X \circ \Pi \quad \text{on } \mathscr{D}_P, \tag{62}$$

if and only if

$$\Pi(\zeta_1) = \Pi(\zeta_2) \quad \Longrightarrow \quad \Pi(q_P\zeta_1) = \Pi(q_P\zeta_2), \tag{63}$$

for all $\zeta_1, \zeta_2 \in \mathscr{D}_P$.

Proof. Factorization implies (63). Conversely, that condition makes $\mathscr{T}_P^X(\Pi(\zeta)) := \Pi(q_P\zeta)$ independent of the representative. □

An action is defined on process–initial-state pairs. It assigns to every $P \in \mathscr{P}$ a function

$$a(P, \cdot) \,:\, \mathscr{D}_P \longrightarrow \mathbb{R}, \tag{64}$$

which is continuous with respect to the initial state, satisfies $a(I;\zeta) = 0$, and is additive under prolongation:

$$a(P_2P_1, \zeta) = a(P_1, \zeta) + a(P_2, q_{P_1}\zeta), \qquad \zeta \in \mathscr{D}_{P_2P_1}. \tag{65}$$

For a regular process, let $s \mapsto \zeta_P(s;\zeta) = (z_P(s;\zeta), \mathfrak{p}_P(s;\zeta))$ be the absolutely continuous trajectory generated from $\zeta \in \mathscr{D}_P$, with $\zeta_P(0;\zeta) = \zeta$, $\zeta_P(\tau_P;\zeta) = q_P\zeta$, and $\dot{z}_P \in C_{\mathrm{phys}}(z_P)$ almost everywhere. Let $\Gamma_{z,\mathfrak{p}} \,:\, T_{\mathfrak{p}}\mathfrak{P} \to [0, +\infty]$ be a proper lower semicontinuous cost for the interpretive rate and set

$$\mathscr{L}(z, \mathfrak{p}, v, \eta) := \mathcal{J}_{z,\mathfrak{p}}(v) + \Gamma_{z,\mathfrak{p}}(\eta). \tag{66}$$

On the regular finite-action subclass, the integrand below is measurable and integrable along every trajectory, and the action admits the representation

$$a(P, \zeta) = \int_0^{\tau_P} \mathscr{L}\big(z_P(s,\zeta), \mathfrak{p}_P(s,\zeta), \dot{z}_P(s,\zeta), \dot{\mathfrak{p}}_P(s,\zeta)\big)\, ds. \tag{67}$$

Thus, the abstract action retains the state–process structure, while its regular representation uses the local selection functional from (48) and the interpretive-rate cost.

A state function appears after the process class and the action have been fixed. For a target $\mathcal{G} \subset \mathscr{D}$, assume that every state under consideration admits a terminating process of finite action and that, for each such initial state ζ, the actions of its terminating processes are bounded from below by a finite number. Processes stop at their first entry into $\mathcal{G}$, and only the null process, with zero action, is admitted from a state already in $\mathcal{G}$. Define

$$V_{\mathcal{G}}(\zeta) := \begin{cases} \inf\limits_{\substack{P \in \mathscr{P},\ \zeta \in \mathscr{D}_P \\ q_P\zeta \in \mathcal{G}}} a(P, \zeta), & \zeta \notin \mathcal{G}, \\ 0, & \zeta \in \mathcal{G}. \end{cases} \tag{68}$$

Definition 7.2. A function $U \,:\, \mathscr{D} \to \mathbb{R}$ is a lower action potential referred to $\mathcal{G}$ if $U = 0$ on $\mathcal{G}$ and

$$U(\zeta) - U(q_P\zeta) \le a(P, \zeta) \tag{69}$$

for every $P \in \mathscr{P}$ and $\zeta \in \mathscr{D}_P$, with the process stopped no later than its first entry into $\mathcal{G}$.

Proposition 7.3. *The function $V_{\mathcal{G}}$ is a lower action potential referred to $\mathcal{G}$. Every lower action potential U referred to $\mathcal{G}$ satisfies*

$$U(\zeta) \le V_{\mathcal{G}}(\zeta) \qquad \text{for every } \zeta \in \mathscr{D}. \tag{70}$$

Proof. Let P take ζ to $\zeta_P := q_P\zeta \notin \mathcal{G}$. For every terminating continuation Q with $\zeta_P \in \mathscr{D}_Q$, additivity gives

$$V_{\mathcal{G}}(\zeta) \le a(P,\zeta) + a(Q,\zeta_P). \tag{71}$$

The infimum over Q proves $V_{\mathcal{G}}(\zeta) - V_{\mathcal{G}}(\zeta_P) \le a(P;\zeta)$. If P terminates in $\mathcal{G}$, the same inequality follows from the definition of the value. If U is another lower action potential and P terminates in $\mathcal{G}$, then $U(\zeta) \le a(P;\zeta)$. Taking the infimum over terminating processes proves (70). □

With $S = -U$, the lower-action inequality takes the storage form $S(q_P\zeta) - S(\zeta) \le a(P,\zeta)$, with action as accumulated supply. This identifies its relation to dissipativity [18], but no nonnegative storage is assumed here. Moreover, the terminal-set convention and the direction of optimization differ from a required-supply construction based on a prescribed initial reference. Proposition 7.3 is a dynamic-programming argument for a cost to a terminal set [17], not a new general variational theorem. It compares the selected and recognized targets under the same representative-dependent process structure. Since the running action can be signed, finite reachability and the lower bound on terminating actions are substantive assumptions, not consequences of dissipation positivity.

The selected and recognized targets are

$$\begin{aligned} \mathcal{G}_* &:= \{(z,\mathfrak{p}) \in \mathscr{D} : \pi(z) \in A_*\}, \\ \widehat{\mathcal{G}} &:= \{(z,\mathfrak{p}) \in \mathscr{D} : \pi(z) \in \widehat{A}_{\mathfrak{p}}\}. \end{aligned} \tag{72}$$

The values $V_{\mathcal{G}_*}$ and $V_{\widehat{\mathcal{G}}}$ need not be ordered. Their targets differ, and the processes evaluated as favorable under the current interpretation need not minimize the action required to reach the selected target A_*, even though $A_* \subset \mathcal{A}_*$. The first value quantifies the least action among processes that terminate at the selected task-adequate organization. The second quantifies the least action among processes that terminate at an organization currently recognized as adequate. Their discrepancy contains both the displacement of the target under interpretation and the dependence of admissible processes and action on the full physiological–interpretive state. It is therefore not reducible to the distance between A_* and $\widehat{A}_{\mathfrak{p}}$.

Corollary 7.4. *There exists $V_X : \Pi(\mathscr{D}) \to \mathbb{R}$ satisfying*

$$V_{\mathcal{G}} = V_X \circ \Pi, \tag{73}$$

if and only if $V_{\mathcal{G}}$ is constant on every fiber of Π in $\mathscr{D}$.

Proof. Factorization implies fiberwise constancy. Conversely, constancy makes $V_X(\mathfrak{e}) := V_{\mathcal{G}}(z,\mathfrak{p})$, with $\pi(z) = \mathfrak{e}$, independent of the representative. □

Infimizing a value over an entire fiber of Π would select the most favorable physiological–interpretive representative among states with the same current epistemic realization. It would not give the value associated with the actual subject state.

The physical content of these descent criteria lies in the distinctions they retain or remove. Equal current observables need not imply equal accessible rates, equal reachable organizations, or equal costs for the actual subject. Constancy of a value on fibers can occur without the separate descent of all processes and costs; conversely, a process transformation factoring on its domain does not by itself make admissibility independent of the representative. No equivalence among the separate descent properties is asserted.

8. A non-projectable coupled realization

Let E, F, X, and $\mathfrak{P}$ be Hilbert spaces, with X and $\mathfrak{P}$ finite dimensional. Let $L : E \to X$, $C : F \to E$, $G : \mathfrak{P} \to E$, and $D : \mathfrak{P} \to X$ be bounded. Fix $r_* \in E$ and write $\mathfrak{e}_* := Lr_*$. Consider

$$\begin{aligned} \dot r &= -K_E(r - r_*) + Cy + G\mathfrak{p}, \\ \dot y &= -\Lambda y, \\ \dot{\mathfrak{p}} &= -H\mathfrak{p}. \end{aligned} \tag{74}$$

Assume that $K_E : E \to E$ is bounded, $-\Lambda$ and $-H$ generate exponentially stable semigroups, and a bounded self-adjoint operator $K_X : X \to X$ satisfies

$$\begin{aligned} \langle K_X v, v\rangle_X &\ge \mu \|v\|_X^2, \\ LK_E &= K_X L, \\ LG + DH &= K_X D. \end{aligned} \tag{75}$$

Let

$$\|e^{-t\Lambda} y\|_F \le C_B e^{-\lambda t}\|y\|_F, \qquad \|e^{-tH}\mathfrak{p}\|_{\mathfrak{P}} \le C_H e^{-\omega t}\|\mathfrak{p}\|_{\mathfrak{P}}. \tag{76}$$

Set

$$\mathscr{M} := E \times F, \qquad \pi(r, y) := Lr, \qquad B := E \times \{0\}. \tag{77}$$

Equip $\mathscr{M}$ with the Hilbert product norm. Then, $d_{\mathscr{M}}((r, y), B) = \|y\|_F$. The selected and recognized targets are

$$A_* := \{\mathfrak{e}_*\}, \qquad \widehat{A}_{\mathfrak{p}} := \{\mathfrak{e}_* + D\mathfrak{p}\}. \tag{78}$$

They satisfy

$$d_X(\mathfrak{e}, A_*) \le d_X(\mathfrak{e}, \widehat{A}_{\mathfrak{p}}) + \|D\mathfrak{p}\|_X. \tag{79}$$

Thus, $c = 1$ and $\delta(\mathfrak{p}) = \|D\mathfrak{p}\|_X$.

Let

$$h(t) := Lr(t) - \mathfrak{e}_* - D\mathfrak{p}(t). \tag{80}$$

Relations (74) and (75) give

$$\dot{h} = -K_X h + LCy. \tag{81}$$

For $h \neq 0$,

$$\frac{d}{dt}\|h\|_X \le -\mu\|h\|_X + \|LC\|_{\mathcal{L}(F,X)}\|y\|_F. \tag{82}$$

The regularization in Proposition 5.4 covers $h = 0$. With

$$W(\mathfrak{p}, \mathfrak{e}) := \|\mathfrak{e} - \mathfrak{e}_* - D\mathfrak{p}\|_X^2, \tag{83}$$

compatibility holds globally with

$$\kappa_0 = 0, \qquad \kappa_1 = \|LC\|_{\mathcal{L}(F,X)}, \qquad g_\zeta = 0, \qquad a_1 = a_2 = 1. \tag{84}$$

Consequently,

$$\|h(t)\|_X \le e^{-\mu t}\|h(0)\|_X + \|LC\|_{\mathcal{L}(F,X)}C_B\|y(0)\|_F I_{\mu,\lambda}(t), \tag{85}$$

and

$$\|Lr(t) - \mathfrak{e}_*\|_X \le \|h(t)\|_X + \|D\|_{\mathcal{L}(\mathfrak{P},X)}C_H e^{-\omega t}\|\mathfrak{p}(0)\|_{\mathfrak{P}}. \tag{86}$$

Here $\mathfrak{p} = 0$ represents the calibrated interpretive condition, and $\mathfrak{p}$ its deviation from calibration. The epistemic realization converges to the selected target while the recognized target approaches it through the decay of that deviation.

The realization is not projectable when $LC \neq 0$ or $LG \neq 0$. Equal current values of Lr can be accompanied by different unresolved physiological states or different interpretive conditions and therefore by different projected velocities.

This example illustrates nonprojectability and target tracking. It supplies no task-loss specification and does not, by itself, establish acquisition of information or improved task performance.

9. A nonlinear firing and connectivity model

9.1. Balance law and invariant field class

This example fixes the constitutive functions, an observable, and a calibration task. All quantities are dimensionless. Let $\mathcal{B} = (0, 1)^3$, impose homogeneous Neumann boundary conditions, choose a fixed tensor E with $\|E\|_F = 1$, and write $v = u - u_b$, where u_b is a baseline firing level. With $\alpha, b, c, d_u, d_\xi > 0$, $\chi > 0$, and $\alpha c > \chi^2$, assign

$$\psi = \frac{\alpha}{2}v^2 + b\log\cosh v + \frac{c}{2}\|\xi\|_F^2 - \chi v(E\cdot\xi) + \frac{d_u}{2}|\nabla u|^2 + \frac{d_\xi}{2}\|\nabla\xi\|_F^2, \qquad a_u = \widehat{a}_\xi = 1, \quad \beta = \beta_0, \quad \gamma = 0. \tag{87}$$

Eq. (8) becomes

$$\dot{v} = d_u\Delta v - \alpha v - b\tanh v + \chi(E\cdot\xi) + \beta_0, \qquad \dot{\xi} = d_\xi\Delta\xi - c\xi + \chi v E. \tag{88}$$

The spatially homogeneous class $v(x, t) = v(t)$, $\xi(x, t) = y(t)E$ is invariant, so its evolution, with an assigned interpretive calibration dynamics, is exactly

$$\dot{v} = -\alpha v - b\tanh v + \chi y + \beta_0, \qquad \dot{y} = -cy + \chi v, \qquad \dot{p} = -\nu p, \quad \nu > 0. \tag{89}$$

Here $p \in \mathbb{R}$ denotes the scalar interpretive coordinate $\mathfrak{p}$. This is a constitutive example with a saturating firing nonlinearity, not a fitted cortical model. The scalar y is a signed connectivity component, not an entire adjacency matrix. The estimates and simulations below concern this invariant homogeneous class; they do not assert the same scalar closure for arbitrary inhomogeneous solutions of (88).

On this class take $\mathscr{M} = \mathbb{R}^2$, with its Euclidean metric. The right-hand side is globally Lipschitz, giving a global continuous semiflow. Let $\bar{v}$ be the unique solution of

$$(\alpha - \chi^2/c)\bar{v} + b\tanh\bar{v} = \beta_0, \qquad \bar{y} = \chi\bar{v}/c, \qquad B = \{(\bar{v}, \bar{y})\}. \tag{90}$$

Strict monotonicity and unbounded range give existence and uniqueness. The Hessian of $\alpha v^2/2 + b\log\cosh v + cy^2/2 - \chi vy - \beta_0 v$ is bounded below by λI, where

$$\lambda = \frac{\alpha + c - \sqrt{(\alpha - c)^2 + 4\chi^2}}{2} > 0. \tag{91}$$

Taking the inner product of the difference from equilibrium with its velocity gives $\frac{1}{2}\frac{d}{dt}\|(v, y) - (\bar{v}, \bar{y})\|^2 \le -\lambda\|(v, y) - (\bar{v}, \bar{y})\|^2$. Thus (11) holds with $C_B = 1$. For the numerical parameters below, $[-1, 1]^2$ is positively invariant because the vector field points inward on each side. Choosing $u_b = 2$ keeps firing positive on that invariant region.

9.2. Task, realization, and explicit compatibility

Take $X = \mathbb{R}$ and

$$\pi(u, \xi) = \int_{\mathcal{B}} (u - u_b)\, dx = v, \quad Y_1 = \mathbb{R}, \quad O_1(e) = e, \quad T_1 = \{0\}, \quad \ell_1(e) = e^2, \quad w_1 = 1. \tag{92}$$

The observable is a regional firing deviation, and the assigned task is calibration to zero deviation. The constant offset in π can equivalently be put in O_1, yielding exactly the finite-rank form (23) with $p_1(x) = 1$, $Q_1 = 0$, and no history functional. Set $\varepsilon_* = 10^{-2}$, so $\mathcal{A}_* = [-0.1, 0.1]$, and select $A_* = \{0\}$. These choices define an illustrative task, not a semantic criterion for knowledge. Put

$$\widehat{L}_p(e) = (e - p)^2, \quad \widehat{\varepsilon}(p) = 0, \quad \widehat{A}_p = \{p\}, \quad W(p, e) = (e - p)^2. \tag{93}$$

Then, $a_1 = a_2 = 1$, $c(p) = 1$, and $\delta(p) = |p|$.

Define $h = v - p$, $q = |h|$, $F(s) = \alpha s + b\tanh s$, and $d(t) = \|(v, y) - (\bar{v}, \bar{y})\|$. Directly from the constitutive evolution,

$$\dot{h} = -[F(v) - F(p)] + \chi(y - \bar{y}) + (\beta_0 + \chi\bar{y}) - [F(p) + \dot{p}]. \tag{94}$$

Since $F' \ge \alpha$, $h[F(v) - F(p)] \ge \alpha h^2$. Also, $p(t) = p_0 e^{-\nu t}$ and $|F(p) + \dot{p}| \le (|\alpha - \nu| + b)|p_0|e^{-\nu t}$. Multiplication of (94) by 2 h proves (41), globally, with the explicit choices

$$\mu = \alpha, \quad \kappa_0 = |\beta_0 + \chi\bar{y}|, \quad \kappa_1 = \chi, \quad g_{\zeta_0}(t) = G_0 e^{-\nu t}, \quad G_0 = (|\alpha - \nu| + b)|p_0|. \tag{95}$$

These constants come from the assigned energy, interactions, realization, and recognition dynamics; they are not fitted to the comparison error. Both κ_0 and g are nonzero in the numerical case. The theorem gives

$$q(t) \le \mathcal{B}_q(t) := e^{-\alpha t}|v_0 - p_0| + \frac{\kappa_0}{\alpha}(1 - e^{-\alpha t}) + \chi d_0 I_{\alpha,\lambda}(t) + G_0 I_{\alpha,\nu}(t), \qquad |v(t)| \le \mathcal{B}_q(t) + |p_0|e^{-\nu t}. \tag{96}$$

In particular, $\limsup|v(t)| \le \kappa_0/\alpha$. This can imply eventual task adequacy when the bound is strictly below 0.1, although proximity to a target without such a margin does not generally imply exact adequacy.

9.3. Nonprojectability and unresolved memory

Two states with the same v and p but different y have projected velocities differing by $\chi\Delta y$. Hence the dynamics is not projectable even when the interpretive coordinate is retained. Solving the connectivity equation and substituting gives the exact identity

$$\dot{v}(t) = -\alpha v(t) - b\tanh v(t) + \beta_0 + \chi e^{-ct} y_0 + \chi^2 \int_0^t e^{-c(t-s)} v(s)\, ds. \tag{97}$$

Thus eliminating an unresolved variable leaves a memory kernel and a term depending on unresolved initial data. This explicit calculation exhibits the mechanism motivating the Mori–Zwanzig comparison [16]; it does not identify the chosen realization with a statistical projection operator or derive a stochastic closure. Equal present epistemic observables alone do not fix either term.

9.4. Numerical illustration and sharpness

Use $\alpha = 2$, $b = 0.5$, $c = 1.5$, $\chi = 0.4$, $\beta_0 = 0.05$, $\nu = 0.6$, with $d_u = d_\xi = 1$ in the parent field model. Choose $v_0 = 0.6$, $p_0 = 0.2$, and compare $y_0 = 0.5$ with $y_0 = -0.5$. The constants are

$$\bar{v} \simeq 0.0208920, \ \bar{y} \simeq 0.00557120, \ \lambda \simeq 1.278301, \ \kappa_0 \simeq 0.05222848, \ G_0 = 0.38.$$

For $R = 1$, the left-hand side of (31), using $J^* \le G_0/\max\{\alpha, \nu\}$, is at most 0.768407 and 0.769863, respectively. Thus invariance follows from the above results, independently of the numerical test.

The system was integrated on $[0, 20]$ with the DOP853 method, relative tolerance 10^{-10}, absolute tolerance 10^{-12}, and output interval 0.02. Repeating with tolerances 10^{-12} and 10^{-14} changed sampled state components by less than 3.0×10^{-10}. Numerical

Table 1
Actual recognized-target distance and analytical bound. Both trajectories have the same initial epistemic observable and recognized target.

y_0	$q(1)$	$\mathcal{B}_q(1)$	$q(20)$	$\mathcal{B}_q(20)$
0.5	0.006290	0.249369	0.020891	0.026116
−0.5	0.064545	0.249947	0.020891	0.026116

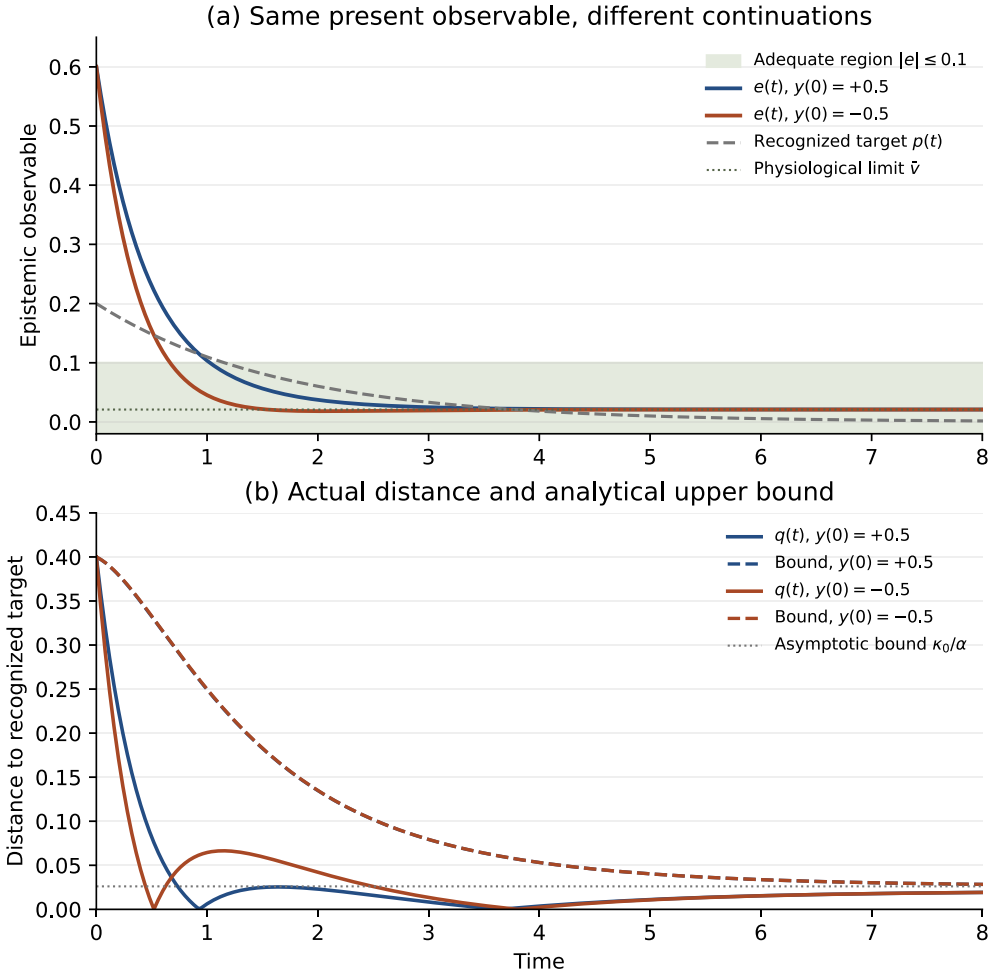

Fig. 1. Constitutive example. (a) Projected trajectories, moving recognized target, and fixed adequate region. Different unresolved initial connectivity produces different continuations. (b) Actual distance $q = |\upsilon - p|$ and the explicit bound (96). The nearly coincident dashed bounds remain above both distances. Curves are shown up to time 8; integration and checks extend to time 20.

results are represented in Fig. 1. These numerical checks complement the analytical proof; they do not establish inequalities between sampling times (see Table 1).

The bound is not sharp in this example. Triangle inequalities remove sign cancellations, and $|y - \bar{y}|$ is bounded by the full physiological distance. The limiting upper bound is $\kappa_0/\alpha \simeq 0.0261142$, whereas the actual limiting distance is $\bar{v} \simeq 0.0208920$. In fact, $\kappa_0 = \alpha\bar{v} + b\tanh\bar{v}$, so the asymptotic gap is exactly $b\tanh\bar{v}/\alpha > 0$. The comparison inequality itself can be saturated by an associated scalar comparison equation, but no sharpness is claimed for a constitutive trajectory. The task loss starts at 0.36 and converges to $\bar{v}^2 \simeq 4.3648 \times 10^{-4} < \varepsilon_*$; this illustrates eventual task adequacy for a prescribed calibration task, not acquisition of information or a demonstrated learning mechanism.

10. Relation with the physiological balance equations

For a fixed constitutive choices to be inserted in the balance equations (8), compatibility can be tested through the weak balances. Suppose that the selected solution has the required time regularity and that the history functionals are differentiable along its

trajectories. For π of the form (23),

$$\frac{d}{dt}\pi_\alpha(u,\boldsymbol{\xi},H) = \langle u_t, p_\alpha\rangle + \langle \boldsymbol{\xi}_t, Q_\alpha\rangle + \frac{d}{dt}\Pi_\alpha(H). \tag{98}$$

Substitution from the balance equations, followed by the integrations by parts associated with the contact actions, expresses (98) in terms of bulk and boundary actions, and state variables.

After W, the interpretive evolution, and the constitutive functions have been specified, the left-hand side of (41) contains

$$D_{\mathfrak{e}}W(\mathfrak{p},\pi(z))[D\pi(z)\dot{z}] + D_{\mathfrak{p}}W(\mathfrak{p},\pi(z))[\dot{\mathfrak{p}}]. \tag{99}$$

The first term is evaluated through the weak physiological balances. The second measures the motion of the recognized target and is estimated by the assigned control g_ζ. Positivity and coercivity of selected dissipative constitutive terms, together with bounds for firing–connectivity, boundary, history, and interpretive couplings, may yield (41).

The balance structure alone implies none of these estimates. Likewise, the existence of a convex dissipation potential and the selection rule (49) require additional constitutive hypotheses.

10.1. Relation with an empirical interpretation

The matter in the previous sections is an axiomatic analysis without empirical validation. A testable application would have to fix the task and its standards, specify measured or reconstructed physiological variables, assign the observation map and a model for recognition, and estimate constitutive parameters and input bounds independently of the trajectory used to test the persistence prediction. A distance to a target has empirical meaning only through that declared observation space and its metric. The present example uses a scalar response deviation and squared calibration error, not an identified neural code for semantic knowledge.

For such a fixed application, a measured violation of the predicted bound beyond stated measurement and parameter errors would reject at least one of the instantiated hypotheses or its parameter estimates. The abstract existence of some suitable map, target, or comparison functional is not a falsifiable empirical prediction. Neither simulation nor a retrospective choice of those objects substitutes for calibration and related corroboration of the theory. The local tensor representation, finite task family, and exclusion of evolving task standards remain explicit limits of the present treatment.

11. Concluding remarks

What we have discussed in this paper describes relations between a physiological dynamics and the epistemic organizations represented at a selected resolution. It does not choose among philosophical theories of knowledge or identify realization, persistence, and task adequacy with a sufficient definition of knowledge. Those notions specify the dynamical and evaluative questions studied here. In particular, confinement near a selected target is a sufficient quantitative route to controlled task loss under stated assumptions; it is not necessary for every adequate history and is not an epistemological necessity theorem. Recognition and accessibility are separate because the existence of a physiologically realizable path does not determine whether the subject identifies or selects it.

The task specification, adequate region, and selected target remain fixed once their tasks and standards have been assigned. This conditional independence from the subject does not remove their possible historical origin. Psychological, mnemonic, cultural, and historical–social conditions determine a moving recognized target and affect selection among physiologically accessible paths. The recognition comparison quantifies the relation between the recognized and selected targets without identifying the selected target with the entire adequate region.

Persistence in the recognized moving tube follows from a compatibility estimate containing physiological transients, residual incompatibility, and interpretive variation bounded independently of the comparison error and its rate, with initial-state dependence stated explicitly. The estimate transfers to the selected target through the recognition comparison, and proximity to that target controls excess task loss; exact adequacy additionally requires a sufficient margin below the tolerance. A closed epistemic model with periodic training supplies recurrent responses after an epistemic dynamics has been postulated or obtained by reduction. Periodicity by itself implies neither target confinement nor adequacy.

The physiological rate set and its energy–dissipation cost express stable cognitive capacities within the assigned constitutive class. Through $\widehat{U}(\mathfrak{p},\pi(z))$, the recognized-target comparison depends on both the physiological realization and the current interpretive condition. The coupled variational hypothesis evaluates the change in that comparison induced by each admissible physiological rate. Infimal contraction preserves the resulting selection among the epistemic rates accessible from the actual physiological state; it does not introduce an independent minimization principle on the epistemic space. The local functional enters the action density only for the regular process subclass on which an integral representation is available.

Processes act on states and determine state transformations. Their domains determine accessibility, their transformations determine continuation, and their prolongation supplies the additivity law for action. Residual values for recognized and selected targets are obtained only after these structures have been fixed. They need not agree because the targets, admissible process families, and action costs retain physiological and interpretive information that is absent from the current epistemic state. Projected dynamics, contracted rate selection, process transformations, and action values become autonomous on the epistemic space only under their separate fiberwise invariance conditions.

The nonlinear example above makes the compatibility hypothesis checkable, with nonzero residual and interpretive controls. Its numerical trajectories satisfy the proved bound and also show its conservativeness. Eliminating the connectivity variable produces an

explicit memory term. These results illustrate what the framework can quantify for a fixed constitutive choice, without establishing learning, a unique epistemic realization, or empirical validity.

The realization map π has been kept independent of $\mathfrak{p}$. An alternative formulation could use a family $\pi_{\mathfrak{p}} : \mathscr{M} \to X$, allowing the interpretive condition to affect not only recognition and selection but also the epistemic realization of a physiological state. Such a choice changes the equivalence relation, the projectability condition, the moving-tube estimate, and every descent criterion; it defines a different construction.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

No data was used for the research described in the article.

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