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Further observations on the definition of global hyperbolicity under low regularity

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Abstract

The definitions of global hyperbolicity for closed cone structures and topological preordered spaces are known to coincide. In this work we clarify the connection with definitions of global hyperbolicity proposed in recent literature on Lorentzian length spaces and Lorentzian optimal transport, suggesting possible corrections for the terminology adopted in these works. It is found that in Kunzinger–Sämman’s Lorentzian length spaces the definition of global hyperbolicity coincides with that valid for closed cone structures and, more generally, for topological preordered spaces: the causal relation is a closed order and the causally convex hull operation preserves compactness. In particular, it is independent of the metric, chronological relation or Lorentzian distance.

Keywords: low regularity, causality, Lorentzian length spaces

1. Introduction

Among the many causality properties that can be imposed on a Lorentzian spacetimes, global hyperbolicity is certainly one of the most useful. The PDE evolution of Cauchy data for the Einstein’s equations naturally lead to, potentially extendible, globally hyperbolic spacetimes. The belief that under physically reasonable conditions the globally hyperbolic spacetimes so obtained are inextendible is known as the strong cosmic censorship.

Global hyperbolicity is also the strongest property in the causal ladder of spacetimes, and the evolution and refinement of its definition has reflected the progress of mathematical relativity in the last decades.



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Recently, studies in low regularity Lorentzian geometry, by means of metric geometry, cone structures, length spaces, Lorentzian optimal transport, have led to new investigations and adaptations of this property to more general frameworks.

Some years ago we started introducing elements of the theory of topological ordered spaces as developed by Nachbin [24] in the study of the spacetime causal structure [17]. The idea was to regard the spacetime structure as a topological ordered space (X, \mathcal{T}, J) endowed with a measure μ . As argued in [20], this type of framework could be sufficiently general to correctly describe a quantum spacetime theory, the manifold smoothness being expected to be lost at small length scales/high energies.

Nachbin theory was not sufficiently general though, as it was particularly lacking in connection with non-compact manifolds. In order to extend its range of application we obtained some results that could be applied to locally compact σ -compact spaces and hence to manifolds [18, 19]. In this connection, we proposed a definition of global hyperbolicity that applies to closed ordered spaces [18].

It is the purpose of this work to show that this definition passes the test of time, as it is consistent with all the definitions of global hyperbolicity that have subsequently been proposed. This consistency was proved for closed cone structures in [14, 21], but recent work on Lorentzian length spaces and Lorentzian optimal transport, adopting new terminology, has brought us to reconsider this problem for these type of structures.

We recall that a *topological preordered space* is a triple (X, \mathcal{T}, J) where (X, \mathcal{T}) is a topological space and J is a reflexive and transitive relation (*preorder*). It is called an *order* if it is antisymmetric, in which case the triple is a *topological ordered space*. Several consistency conditions can be imposed between topology and order, in fact, as Nachbin showed [24], there is a beautiful topological theory for these spaces that is analogous to the usual topology. One of the most important conditions is that the preorder J , regarded as a subset of $X \times X$ be closed in the product topology $\mathcal{T} \times \mathcal{T}$, in which case we speak of *closed preordered space* (in some papers in topology this property is referred as *continuity* of J). A weaker property is that of *semi-closedness* (or semi-continuity) namely J is such that the sets $J^+(p) := \{q : (p, q) \in J\}$ and $J^-(p) = \{q : (q, p) \in J\}$ are closed for every p .

Observe that the antisymmetry of J reads $\Delta = J \cap J^{-1}$ where $J^{-1} = \{(x, y) : (y, x) \in J\}$ and $\Delta \subset X \times X$ is the diagonal. Thus for a closed order the diagonal Δ is closed, i.e. the topology \mathcal{T} is Hausdorff.

The definition proposed in [18] was

Definition 1.1. A topological preordered space (X, \mathcal{T}, J) is *globally hyperbolic* if

(\star) J is a closed order and for every compact set K the set $J^+(K) \cap J^-(K)$ is compact.

Property (\star) consists of two conditions. The former might also be called *causal simplicity* while the latter reads: the operation of taking the causally convex hull preserves compactness. The weaker condition of *causality* is the request: J is antisymmetric (i.e. an order). Finally, it is possible to introduce an intermediate level between causal simplicity and causality, namely *stable causality* (which we shall not use): the smallest closed preorder containing J is antisymmetric. This proves that portions of the causal ladder for spacetimes [22] pass to the topological preordered space framework.

The property (\star) is important as for locally compact sigma-compact topologies it implies another very important property know as quasi-uniformizability, which essentially establishes the representability of the topological ordered space by continuous isotone functions and the

fact that the space can be Nachbin compactified [18] (we shall not expand on these properties as they will not be used in what follows).

2. Global hyperbolicity on smooth manifolds

Let us briefly recall the improvements on the definition of global hyperbolicity that took place along the years. We start with the regular setting, meaning with this C^2 (or $C^{1,1}$) metrics on smooth manifolds.

The traditional definition, as can be found in the oldest textbooks [1, 13], is *strong causality and compactness of the causal diamonds*: $\forall p, q \in M, J^+(p) \cap J^-(q)$. Bernal and Sánchez proved that *strong causality* could be weakened to *causality* [3], while in [14] we proved that for physically reasonable spacetimes, i.e. with dimension larger than three which are non-compact or non-totally vicious, the condition of causality could be dropped altogether resulting in just *compactness of the causal diamonds*.

Other equivalent definitions were also obtained, for instance *non-total imprisonment and the causal diamonds are relatively compact* [16] (a spacetime is non-total imprisoning if no inextendible causal curve is imprisoned in a compact set). This definition shows clearly that shrinking the cones does not spoil global hyperbolicity (as the family of causal curves get smaller, as do the causal diamonds) and is particularly convenient for proving its stability under C^0 perturbations of the cones [2, 21, 25]. It also sets a balance between compact sets (and hence open sets) and causality, as the more the compact sets the harder for causal curves to be non-imprisoned, but the easier for causal diamonds to be relatively compact.

The first work in a low regularity setting was due to Fathi and Siconolfi [9, 10] who studied C^0 cone distributions. Their definition of global hyperbolicity was essentially the traditional one, but for strong causality that was strengthened to stable causality. Chruściel and Grant [8] studied systematically the C^0 Lorentzian metric theory, pointing out that the equalities $J^\pm(p) = \bar{I}^\pm(p)$ (no causal bubbles), $I \circ J \cup J \circ I \subset I$ (push up), do not hold at this level of regularity. The two pathologies were in fact one and the same as was later show in [21, theorem 2.8], see also [11, theorem 2.12]. The C^0 Lorentzian geometry approach was also developed by Sbierski [26] in his study on the C^0 inextendibility of Schwarzschild spacetime. Sämann studied specifically global hyperbolicity and its many equivalent definitions in the same C^0 Lorentzian framework [25].

A more general point of view was taken by Bernard and Suhr, who studied closed cone structures [4], thus reconsidering Fathi and Siconolfi's cone distribution approach. Subsequently, we explored quite in deep causality in closed cone structures and non-regular Lorentz–Finsler spaces [21].

We recall that a *cone structure* on a smooth manifold is a multivalued map $x \mapsto C_x$, where $C_x \subset T_x M \setminus 0$, is a closed sharp convex non-empty cone. It is a *closed cone structure* if $C = \cup_x C_x$ is a closed subbundle of the slit tangent bundle $TM \setminus 0$ (this is the terminology in [21], Bernard and Suhr would call it *regular* closed cone structure). This is essentially an upper semi-continuity condition on the cone distribution [21, proposition 2.3]. It turns out that most of causality theory and the very causal ladder of spacetimes makes sense for closed cone structures [21, theorem 2.47]. However, in order to make sense of the chronological relation and some Lipschitzness condition on achronal hypersurfaces, one needs to work with a slightly more specialized object, namely a *proper cone structure* which is a closed cone structure which is proper, i.e. such $\text{Int}C \subset TM \setminus 0$ has non-empty fiber at each point of M . The C^0 metric Lorentzian framework is contained in the proper cone structure theory and hence in the theory of closed cone structures.

For a proper cone structure all the definitions of global hyperbolicity traditionally developed for smooth spacetimes have a straightforward analog and remain equivalent among each other [21]. The relation of global hyperbolicity with causal simplicity and other causality properties also does not change. We might say that there are no surprises for proper cone structures, and no need to adjust the definitions [14, 21].

For closed cone structures one has to be careful. The generalization of the traditional definition which makes use of causal diamonds does not work in the sense that it is not equivalent to other desirable properties, such as the existence of a Cauchy hypersurface [21, example 2.6]. Nevertheless, the following properties are equivalent [4] [21, theorems 2.39, 2.45]

- (\star) J is a closed order and for every compact set K , $J^+(K) \cap J^-(K)$ is compact.
- ($*$) Causality and for every compact sets K_1 and K_2 , the ‘causal emerald’ $J^+(K_1) \cap J^-(K_2)$ is compact.
- (2) Non-imprisonment and the causally convex hulls of relatively compact sets are relatively compact.
- (3) Existence of a Cauchy hypersurface.
- (4) Existence of a Cauchy time function.

and hence any of them can provide the correct definition of global hyperbolicity in this framework. Observe that (\star) coincides with the definition 1.1 we gave in the context of topological ordered spaces, while property ($*$) was introduced by Bernard and Suhr in [4]. In the recent work [14] we established that for closed cone structures (\star) and ($*$) can be improved as follows (in the smooth setting this result had been already obtained in [2]).

- (1) J is an order (causality) and for every compact set K , $J^+(K) \cap J^-(K)$ is compact.

3. Global hyperbolicity for topological ordered spaces

The most general setting for studying causality is that of topological preordered spaces. Any framework for spacetime, including those not using a notion of smooth manifold, such as that of Lorentzian length spaces, can be seen as an instance of this general framework (sometimes with some caveats, see below).

In the introduction we provided definitions for causality, stable causality, causal simplicity and global hyperbolicity, i.e. property (\star).

We observe that in (\star) the replacement of the compact set K with two sets K_1, K_2 , does not change the property as we have

Proposition 3.1. *For a closed preordered space the properties*

- (i) *for every compact subset K , $J^+(K) \cap J^-(K)$ is compact,*
- (ii) *for any two compact subsets K_1, K_2 , $J^+(K_1) \cap J^-(K_2)$ is compact,*

are equivalent.

Proof. One direction is obvious setting $K = K_1 = K_2$. For the other direction, it is well known that in a closed preordered space if K is compact, $J^\pm(K)$, is closed [24, p. 44] [19, proposition 2.2]. Thus $J^+(K_1) \cap J^-(K_2)$ is a closed subset of the compact set $J^+(K) \cap J^-(K)$, where $K = K_1 \cup K_2$, hence compact. \square

Proposition 3.2. *Let (M, \mathcal{F}, J) be a topological preordered space. Assume that the topology is Hausdorff and ‘first countable or locally compact’. The property*

(†) for every compact sets K_1, K_2 , $J^+(K_1) \cap J^-(K_2)$ is compact,

implies

(‡) J is closed in the product topology.

We recall that under the Hausdorff condition if every point admits a compact neighborhood then every point admits a basis of compact neighborhoods. Thus there is not ambiguity in what we mean by local compactness. The proof is similar to [21, theorem 2.38].

Proof. Let us give the proof in the ‘first countable’ case. First we prove that J is semi-closed. Let $q \in \overline{J^+(p)}$ then there is a sequence $q_k \rightarrow q$, $p \leq q_k$. The set $K = \{q, q_1, q_2, \dots\}$ is compact, thus as $\{p\}$ is compact, $B = J^+(p) \cap J^-(K)$ is compact hence closed. But $q_k \in B$, thus $q \in B$ which implies $p \leq q$. By the arbitrariness of q , $J^+(\overline{p}) = J^+(p)$. The fact that $J^-(p)$ is closed is proved analogously.

Now, let $(p, q) \in \overline{J}$, then we can find $(p_k, q_k) \in J$, $(p_k, q_k) \rightarrow (p, q)$. Let us consider the compact sets $K_n^p = \{p, p_n, p_{n+1}, \dots\}$, $K_n^q = \{q, q_n, q_{n+1}, \dots\}$, then $J^+(K_n^p) \cap J^-(K_n^q)$ is non-empty (as it contains p_k and q_k for every $k \geq n$) and compact. By the finite intersection property, there is $r \in M$ such that $r \in J^+(K_n^p) \cap J^-(K_n^q)$ for every n . By the semi-closure of J , $p \leq r \leq q$, thus $p \leq q$.

In the locally compact case the proof is analogous, just let K be a compact neighborhood in the first part, and let K_α^p, K_β^q be generic compact neighborhoods of p and q in the second part. \square

We are ready to prove that the equivalence between (\star) and $(*)$, already proved in the context of closed ordered spaces [21, theorem 2.39], actually holds in general for topological preordered spaces (in that result the Hausdorff property for the topology was contained in the manifold condition on M).

Theorem 3.3. *Let (M, \mathcal{T}, J) be a topological preordered space such that the topology is ‘first countable or locally compact’. The following properties are equivalent*

- (\star) J is a closed order and for every compact set K , $J^+(K) \cap J^-(K)$ is compact,
- ($\tilde{\star}$) J is antisymmetric, the topology is Hausdorff and for every compact sets K_1, K_2 , $J^+(K_1) \cap J^-(K_2)$ is compact.

Proof. The direction $(\star) \Rightarrow (\tilde{\star})$ is proposition 3.1, noting that, as previously mentioned, a closed ordered space has Hausdorff topology. The direction $(\tilde{\star}) \Rightarrow (\star)$ is proposition 3.2. \square

Example 3.4. Let (M, \mathcal{T}, J) be a topological ordered space. Consider the properties

- (†) for every compact set K , $J^+(K) \cap J^-(K)$ is closed,
- (‡) J is closed in the product topology.

Is it true that the former implies the latter? The other direction is a well known as the closedness of J implies the closedness of $J^\pm(K)$, see [24, p. 44].

We know that the result holds true in the smooth Lorentzian setting [2, lemma 2.1] and, more generally, for closed cone structures [14]. However, it does not pass to topological ordered spaces, not even under good properties for the topology (e.g. metrizable).

We can give the following minimal counterexample. Let $M = \{p, q, q_1, q_2, \dots\}$, $M \subset \mathbb{R}$, where $q = 0$, $q_n = 1/n$, $p = -1$, and where the topology is the induced topology. Moreover,

define \leq as follows: set $p \leq q_k \not\leq p$ for every k , $p, q_k \not\leq q \not\leq p, q_k$, for every k , and $q_j \not\leq q_k$ for $j \neq k$. Observe that $q_k \rightarrow q$. In this example every compact set is such that $J^+(K) \cap J^-(K) = K$ which is compact hence closed. Clearly, J is not closed (not even semiclosed).

As a preliminary result for the next section, it will be useful to recall that non-total imprisonment in the smooth setting is defined as follows: there are no inextendible causal curves imprisoned in a compact set [22]. The definition remains valid for closed cone structures [21, definition 2.10]. For them it is also true that a continuous causal curve is inextendible if and only if it has infinite h -arc length where h is a complete Riemannian metric [21, corollary 2.1]. Finally, for this structure an easy application of the limit curve lemma [21, lemma 2.1] gives a result which is also valid in the smooth setting and in the C^0 Lorentzian theory [25] [22, theorem 4.39]

Proposition 3.5. *A closed cone structure (M, C) is non-total imprisoning if and only if for an auxiliary (and hence for every) Riemannian metric h on M and every compact set K there is a constant $c(K) > 0$ such that all the continuous causal curves with image in K have h -arc length smaller than $c(K)$.*

3.1. Lorentzian length spaces

Let us discuss Lorentzian (pre-)length spaces in the version by Kunzinger and Sämman [15]. We shall not recall all definitions, referring to [15] for details. A pre-length space (M, ρ, I, J, d) is a quintuple given by a Kronheimer and Penrose's causal space (M, I, J) , a metric ρ , and a lower semi-continuous Lorentzian distance d , satisfying some compatibility conditions.

Proposition 3.5 has an analog in the theory by Kunzinger and Sämman. The two properties

- (α) there is no inextendible causal curve imprisoned in compact set, and
- (β) the causal curves with image in a compact set have bounded length, (the length is that induced from ρ)

are equivalent for locally causally closed ρ -compatible Lorentzian pre-length space [15, lemma 3.12, corollary 3.15] hence for Lorentzian length spaces.

They defined non-total imprisonment in their framework as property (β), thus it depends on ρ . Actually, the distance ρ also appears in (α) as it is present in the Lipschitz condition that they impose on their causal curves [15, definition 2.18]. The distance ρ seems to be essential for their definitions. We observe that on manifolds and over compact subsets distances associated to Riemannian metrics are all Lipschitz equivalent, however, their Lorentzian length spaces M are not manifolds.

This situation clearly complicates causality in the Kunzinger–Sämman theory as global hyperbolicity is obtained from (β) by adding additional conditions. It seems that causality not only depends on the causal relation J and on the topology, but also on the metric ρ .

They defined global hyperbolicity through the property¹

¹ It was claimed in [12] that ‘non-total imprisonment’ can be weakened to ‘causality’ since the proof of [15, theorem 3.26(v)] would not use ‘non-total imprisonment’. Actually, this is not correct as it uses it in applying [15, theorem 3.14] and the uniform bound on lengths implied by non-total imprisonment [15, definition 2.35]. The fact that causality and compactness of causal diamonds implies (β) is not proved in [12], where the authors keep using their definition 3.1 of global hyperbolicity and hence property (β) in their arguments. Still their claim that the assumption in global hyperbolicity can be weakened to causality is correct under minimal assumptions on the Lorentzian length space, see our last corollary 3.8.

(a) property (β) and for every $p, q \in M$, the ‘causal diamonds’ $J^+(p) \cap J^-(q)$ are compact,

Subsequently, in their paper on optimal transport over smooth Lorentzian manifolds, Mondino and Suhr [23] adopted the same definition but realized that in order to develop their theory they needed a stronger property

(b) property (β) and for every compact subsets K_1, K_2 the ‘causal emeralds’ $J^+(K_1) \cap J^-(K_2)$ are compact.

which they called ‘ \mathcal{K} -global hyperbolicity’.

This terminology was adopted in Cavalletti and Modino work on optimal transport over Lorentzian length spaces [7], and in posterior works using the same framework [5, 6].

We have the following result (this is stronger than [7, lemma 1.5]).

Proposition 3.6. *For a Lorentzian pre-length space such that every point admits a timelike curve passing through it (e.g. localizable ones and hence Lorentzian length spaces [15]), properties (a) and (b) actually coincide.*

Actually, the proof does not use the non-total imprisoning property (β) , nor causality. It is worth recalling that localizable Lorentzian pre-length spaces are ρ -compatible [15, definition 3.18].

Proof. The direction (b) \Rightarrow (a) is clear. For the direction (a) \Rightarrow (b) one first proves that J is closed in the usual way [22, theorem 4.12]. Then the proof is word by word that given in [14, proposition 2.3] or [21, proposition 2.21]. \square

Let us denote with \mathcal{T} the topology of the pre-length space, that is, that induced by ρ . We recall that a causally path connected Lorentzian pre-length space has the following property [15, lemma 3.3]: causality (defined by the antisymmetry of J) holds if and only if there are no closed causal curves.

Theorem 3.7. *For a Lorentzian pre-length space which is causally path connected and ρ -compatible (hence for Lorentzian length spaces) the following properties are equivalent*

- (M, \mathcal{T}, J) satisfies (\star) (i.e. global hyperbolicity)
- property (b) (i.e. so called \mathcal{K} -global hyperbolicity).

In particular, the second property does not depend on ρ (or \mathbb{I} , or \mathbb{d}) just on its induced topology.

Proof. Assume (\star) . By theorem 3.3 for every compact sets K_1, K_2 , $J^+(K_1) \cap J^-(K_2)$ is compact. Suppose that (β) does not hold then we can find a compact set C and causal curves $\gamma_k : [0, L_k] \rightarrow C$ parametrized with ρ -arc length that are thus 1-Lipschitz and so equi-Lipschitz. By the Arzelá-Ascoli theorem a subsequence converges uniformly on compact subsets to a 1-Lipschitz curve $\gamma : [0, \infty) \rightarrow C$. Since J is closed (hence locally causally closed), γ is causal. By ρ -compatibility γ is inextendible (the proof goes as in the last paragraph of the proof of [15, theorem 3.14]). The inextendible curve $\gamma : [0, \infty) \rightarrow C$ imprisoned in a compact set C cannot accumulate on just one point [15, lemma 3.12], thus there must be sequences $s_k, t_k \rightarrow \infty$ such that $\lim_k \gamma(s_k) = p$, $\lim_k \gamma(t_k) = q$, $p, q \in C$, $p \neq q$. Passing to subsequences if necessary, we can assume that $s_k < t_k < s_{k+1}$, thus $\gamma(s_k) \leq \gamma(t_k) \leq \gamma(s_{k+1})$, and taking the limit $(p, q), (q, p) \in \bar{J}$. But by (\star) J is closed, thus antisymmetry is violated, a contradiction.

Conversely, assume (b), then by (β) there are no closed causal curves (as their image is compact) hence J is antisymmetric. By theorem 3.3 (\star) follows. \square

The following result shows that under minimal conditions on the Lorentzian length space we can rescue a certain claim on the equivalence of two definitions of global hyperbolicity stated in [12, section 3].

Corollary 3.8. *For a Lorentzian pre-length space which is causally path connected, ρ -compatible (e.g. a Lorentzian length space) and such that through each point passes a timelike curve, the properties (global hyperbolicity) (a), (b), and (\star) , are all equivalent, and they are also equivalent to: causality and the causal diamonds are compact.*

Proof. The first statement follows from proposition 3.6 and theorem 3.7. The last property is clearly implied by (\star) . For the converse, assume causality and that the causal diamonds are compact. First J is closed by the usual argument [22, theorem 4.12], hence J is a closed order, and again by the argument in [14, proposition 2.3] (\star) holds. \square

4. Conclusions

We recalled the definition of global hyperbolicity for topological ordered spaces, the equivalent formulations for closed/proper cone structures, and explored some variations in the broader framework of topological preordered spaces (theorem 3.3).

When it comes to Lorentzian (pre-)length spaces a la Kunzinger–Sämman (M, ρ, I, J, d), in my opinion, the terminology for property (b) introduced in the literature (\mathcal{K} -global hyperbolicity) should be rectified, as (b) could be simply called ‘global hyperbolicity’ (and (a) something like weak global hyperbolicity), as Lorentzian length spaces are special types of topological ordered spaces. As shown in theorem 3.7, property (b) coincides with the definition of global hyperbolicity previously introduced for these more general type of structures and as such it is, quite interestingly, independent of ρ, I and d .

Data availability statement

No new data were created or analysed in this study.

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