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Abstract

A nonlinear differential equation with inhomogeneous differential operator Φ, which is regularly varying at zero, is considered. The operator Φ can be viewed as an extension of the p-Laplacian operator and arises in many physical problems, as illustrated by several examples. In particular, the existence of global positive bounded solutions on the half-line with the Neumann type boundary conditions is studied by means of an abstract fixed point theorem and certain properties of an associated half-linear equation. The results do not require the explicit form of the inverse operator of Φ, and are completed by an asymptotic analysis of these solutions near infinity.

Keywords: nonlinear differential equation, regularly varying operator, inhomogeneous differential operator, Neumann boundary conditions, positive solutions, fixed point theorem, principal solution
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1. Introduction

Consider the differential equation on the half line $I = [t_0, \infty), t_0 \geq 0$,

$$(a(t)\Phi(x'))' + b(t)F(x) = 0, \tag{1}$$

where:

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(i_1) the operator $\Phi\colon (-\rho,\rho) \to (-\sigma,\sigma)$, $0<\rho,\sigma\le\infty$, is an increasing odd homeomorphism, regularly varying at zero with index $i=p-1>0$, i.e., for any $\lambda>0$

$$\lim_{u\to 0}\frac{\Phi(\lambda u)}{\Phi(u)} = \lambda^{p-1};$$

(i_2) the function F is continuous on $\mathbb{R}$ with $uF(u)>0$ for $u\neq 0$ and

$$\lim_{u\to 0}\frac{F(u)}{\Phi_p(u)} = F_0, \quad 0\le F_0<\infty, \tag{2}$$

where $\Phi_p(u)=|u|^{p-2}u$, $p>1$, is the p-Laplacian operator;

(i_3) the functions a, b are continuous and positive on $[t_0,\infty)$.

In some results throughout the paper, we also need the following additional assumptions

$$J_a = \int_{t_0}^{\infty}\Phi_q\left(\frac{1}{a(t)}\right)dt<\infty, \tag{3}$$

$$\liminf_{t\to\infty}\Phi_q\left(a(t)\right)\int_t^{\infty}\Phi_q\left(\frac{1}{a(s)}\right)ds>0, \tag{4}$$

where q is the Hölder conjugate of p, that is $p^{-1}+q^{-1}=1$.

Functions satisfying (i_1) are usually called *Karamata functions* and have been widely employed in many fields, such as probability theory, number theory, and complex analysis (see, e.g., [4, 24]). More recently, Karamata functions have also been used to study the asymptotic behavior of solutions to second-order differential equations (see, e.g., [17, 20], the monograph [21], and references therein).

A prototype of the homeomorphism Φ is the p-Laplacian operator Φ_p, which is regularly varying at $u=0$ with index $i=p-1$. The operator Φ_p is homogeneous, and $\operatorname{dom}\Phi_p=\operatorname{Im}\Phi_p=\mathbb{R}$. Nevertheless, many physical problems are modeled by equations with differential operators involving regularly varying functions that are inhomogeneous and whose domain or image may be bounded; see Section 2 for more details. Indeed, solutions of (1) are related to radially symmetric solutions of certain classes of elliptic differential equations with differential operators in divergence form, possibly including a weight, i.e., equations of the type

$$\operatorname{div}\Big(g(|w|)\varphi(|\nabla u|)\nabla u\Big)+\Psi(|w|,u)=0, \tag{5}$$

where $w = (w_1, \cdots, w_n) \in \mathbb{R}^n$, $n \geq 2$, $|\cdot|$ denotes the Euclidean norm, g is a positive weight function, and φ is a C^1 function such that $(t\varphi(t))' > 0$. The search for radially symmetric solutions u of (5), outside a ball of radius R centered at the origin, leads to the nonlinear equation on the half-line

$$\left(r^{n-1}g(r)\varphi(|x'|)x'\right)' + r^{n-1}\Psi(r,x) = 0, \quad r \geq R,$$

which is of the type (1) if $\Psi(r,x) = b(r)F(x)$.

As observed in [19], for small values of the variable, the classical acceleration operator approximates the so-called *Minkowski mean curvature operator* $\Phi_M \colon (-1,1) \to \mathbb{R}$ given by

$$\Phi_M(u) = \frac{u}{\sqrt{1-u^2}}.$$

Similarly, for small arguments, the p-Laplacian operator Φ_p can be regarded as an approximation of the generalized relativity operator $\Phi_R \colon (-1,1) \to \mathbb{R}$ defined by

$$\Phi_R(u) = \frac{|u|^{p-2}u}{(1-|u|^p)^{(p-1)/p}}, \quad p > 1,$$

and this analogy is highlighted in the search for periodic solutions; see [19]. This observation may be expressed in terms of Karamata functions by saying that the homeomorphisms Φ_M and Φ_R are regularly varying at zero with indices 1 and $p-1$, respectively, and $\Phi_M(u)/u$, $\Phi_R(u)/(|u|^{p-2}u)$ both tend to 1 as $u \to 0$.

The similarity between particular cases of the homeomorphism Φ and the p-Laplacian Φ_p has also been investigated in other contexts concerning oscillation or nonoscillation; see [8, Theorem 2.1] or [10] and references therein, respectively.

If the function a is not differentiable, then equation (1) may be interpreted as the first order differential system

$$x' = \Phi^*\left(\frac{y}{a(t)}\right), \quad y' = -b(t)F(x),$$

where Φ^* is the inverse of Φ. Therefore, as usual, a *solution* of (1) is a differentiable function x on $[t_0, T_x)$, $T_x > t_0$, such that $a\Phi(x')$ is differentiable and satisfy (1) for $t_0 \leq t < T_x$.

Here we deal with solutions x of (1) that are positive on the entire half-line $I = [t_0, \infty)$. In the following, these solutions are called *global positive solutions*. Note that if b is non-negative with isolated zeros, then the equation with the p-Laplacian

$$(a(t)\Phi_p(z'))' + b(t)F(z) = 0 \tag{6}$$

may have solutions that change sign multiple times in the left neighborhood of some $\bar{t} > t_0$, and therefore these solutions are not continuable to infinity; see [9]. When b is positive on the entire interval I, even if solutions of (1) exist for arbitrarily large t under general assumptions, see, e.g., [25, Appendix A] or [23, Theorem 1.2], their positivity over I is generally nontrivial and may fail for (6) as well as for the half-linear equation

$$(a(t)\Phi_p(z'))' + b(t)\Phi_p(z) = 0. \tag{7}$$

The existence of global positive solutions of (1) is also motivated by the search for radially symmetric solutions outside a ball for elliptic differential equations with operators in divergence form, even in the presence of a weight; see also examples in Section 2.

In this paper, we consider the existence of global positive bounded solutions of (1) for a wide variety of operators Φ satisfying (i_1), for which homogeneity fails. Moreover, their asymptotic behavior at infinity is studied. The obtained results can be applied even when the explicit form of the inverse Φ^* of Φ is unknown. They also illustrate that a certain proximity between the solutions of (1) and those of the half-linear equation (7) persists, and they extend recent ones in [10, 11].

The plan of the paper is as follows. In Section 2, some properties of Karamata functions, together with examples of inhomogeneous operators Φ, are presented. In Section 3, a recent fixed point result ([10, Theorem 2.1]), originating from one by Cecchi, Furi, and Marini, is presented. It concerns maps defined in a Fréchet space via Schauder's quasi-linearization method. This tool involves the inverse Φ^* of Φ, but its explicit form is not required. In Section 3, some properties of the half-linear equation (7) are also recalled. These properties are related to the notion of the principal solution for (7) and play a crucial role in the sequel. In Section 4, the existence of global positive bounded solutions for (1) is established, and this result is applied to some special cases of operators Φ already considered in Section 2. In Section 5, the behavior at infinity of global positive solutions of (1) is studied, and a similarity between solutions of (1) and of (7) is highlighted.

2. Inhomogeneus operator Φ

Since any continuous and increasing solution X of the functional equation $X(u)X(v) = X(uv)$ for any $u, v \in \mathbb{R}$, has the form $X(u) = |u|^r u$ for some $r > -1$, see, e.g., [1], the p-Laplacian Φ_p is the unique map which satisfies the homogeneity property $\Phi_p(uv) = \Phi_p(u)\Phi_p(v)$ for any $u, v \in \mathbb{R}$.

Nevertheless, as claimed, in many physical phenomena also regularly varying inhomogeneous operators Φ arise. We start by recalling some elementary properties of them, by reporting these properties just in case when the function Φ is an odd increasing homeomorphism, even if they can be stated in a more general setting, see, e.g., [4, 24].

Lemma 1. *Let Φ satisfy (i_1) with index $i = p - 1$. The following holds.*

(s_1) *There exists an even map L, such that for $u \neq 0$*

$$\Phi(u) = |u|^{p-2}uL(u) = \Phi_p(u)L(u) \tag{8}$$

and

$$\lim_{u\to 0} \frac{L(\lambda u)}{L(u)} = 1 \quad \text{for any } \lambda > 0. \tag{9}$$

The function L in (9) is a slowly varying function at $u = 0$, i.e., it is regularly varying at zero with index $i = 0$.

(s_2) *The inverse Φ^* of Φ is an odd increasing regularly varying homeomorphism at $v = 0$ with index $i = q - 1$, where q is the Hölder conjugate of p.*

(s_3) *Let Λ be the map such that for $v \neq 0$*

$$\Phi^*(v) = |v|^{q-2}v\ \Lambda(v) = \Phi_q(v)\Lambda(v) \tag{10}$$

and

$$\lim_{u\to 0} \frac{\Lambda(\lambda v)}{\Lambda(v)} = 1 \ \ \text{for any } \lambda > 0.$$

Then Λ is an even slowly varying map, and for every $u \neq 0, v \neq 0$

$$\Lambda(v) = \Big(L(\Phi^*(v))\Big)^{1-q}, \qquad L(u) = \Big(\Lambda(\Phi(u))\Big)^{1-p}, \tag{11}$$

where L is given in (s_1).

(s_4) *The function* L *is continuous and positive for* $u \neq 0$ *and, similarly,* Λ *is also continuous and positive for* $v \neq 0$*. Moreover, if* $\lim_{u\to 0} L(u) = L_0 > 0$*, then* $\lim_{v\to 0} \Lambda(v) = \Lambda_0 = L_0^{1-q} > 0$*.*

(s_5) *If* L *is increasing for* $u > 0$*, then* Λ *is decreasing for* $v > 0$*. Similarly, if* L *is decreasing for* $u > 0$*, then* Λ *is increasing for* $v > 0$*.*

Proof. Claim (s_1). This is a basic property of regularly varying functions. It follows by considering the function L given by $L(u) = \Phi(u)/\Phi_p(u)$, $u \neq 0$ and using (i_1).

Claim (s_2). The assertion is a nontrivial consequence of the Uniform Convergence Theorem, see, e.g., [22, Theorem 2.2] or [24, Theorem 1.1]. A complete proof of claim (s_2) is in [22, Lemma 3.3] or [24, Section 1.5 (v) pages 25–27].

Claim (s_3). We prove the claim by considering the function Φ^* in the half-domain $(0, \sigma)$, that is, for $v > 0$. We have

$$v = \Phi(\Phi^*(v)) = (\Phi^*(v))^{p-1} L(\Phi^*(v)) = \left(v^{q-1}\Lambda(v)\right)^{p-1} L(\Phi^*(v))$$

or, since $(q-1)(p-1) = 1$,

$$v = v\,\left(\Lambda(v)\right)^{p-1} L(\Phi^*(v)),$$

which gives the first equality in (11). From this, setting $v = \Phi(u)$ we get the second equality in (11). Since Φ and Φ^* are odd functions, the assertion is valid also for $v < 0$.

Claim (s_4). It follows from Claims (s_1), (s_2) and (s_3).

Claim (s_5). Since $p > 1$ and Φ^* is increasing, (11) gives the assertion. $\square$

The slowly varying function L given in (8) can be unbounded near zero. Moreover, when the limit $\lim_{u\to 0} L(u)$ exists, it may be zero or positive or infinity. Nevertheless, the positivity of L in a complete neighborhood of zero plays an important role in the developed approach, as illustrated below in Section 4. The following regularly varying operators, for which the homogeneity property fails, have widely been considered in the literature. In these examples, the function L is continuous and positive in a complete neighborhood of zero.

Example 1. The Euclidean mean curvature operator $\Phi_E\colon \mathbb{R} \to (-1,1)$ given by

$$\Phi_E(u) = \frac{u}{\sqrt{1+u^2}},$$

and its generalization $\Phi_C\colon \mathbb{R} \to (-1,1)$ given by

$$\Phi_C(u) = \frac{|u|^{p-2}u}{(1+|u|^p)^{(p-1)/p}}, \quad p > 1,$$

model fluid mechanics problems, in particular capillarity-type phenomena for compressible and incompressible fluids, see, e.g., [3, 6]. Clearly, the operators Φ_E and Φ_C are regularly varying with index $i = 1$ and $i = p-1$, respectively. Moreover, the associated slowly varying functions, defined in (8), are

$$L_E(u) = \Big(1+u^2\Big)^{-1/2}, \quad L_C(u) = \Big(1+|u|^p\Big)^{(1-p)/p},$$

respectively.

Example 2. The Minkowski mean curvature operator $\Phi_M\colon (-1,1) \to \mathbb{R}$ given by

$$\Phi_M(u) = \frac{u}{\sqrt{1-u^2}},$$

and its generalization $\Phi_R\colon (-1,1) \to \mathbb{R}$ given by

$$\Phi_R(u) = \frac{|u|^{p-2}u}{(1-|u|^p)^{(p-1)/p}}, \quad p > 1,$$

arise in studying certain extrinsic properties of the mean curvature of hypersurfaces in the relativity theory, see [2, 14, 18, 19]. For this reason, they are also referred to as (generalized) relativistic operators. Moreover, the operator Φ_M occurs also in the theory of nonlinear electromagnetism, where it is referred to as Born–Infeld operator, see [16, page 3]. The operators Φ_M and Φ_R are regularly varying with index $i = 1$ and $i = p - 1$, respectively. Moreover, the associated slowly varying functions, defined in (8), are

$$L_M(u) = \Big(1-u^2\Big)^{-1/2}, \quad L_R(u) = \Big(1-|u|^p\Big)^{(1-p)/p},$$

respectively. Observe that the inverse map of Φ_C is obtained by replacing p with q in the map Φ_R. Similarly, the inverse map of Φ_R is obtained by replacing p with q in Φ_C .

Example 3. The torsional creep operator $\Phi_{TC}\colon \mathbb{R} \to \mathbb{R}$ given by

$$\Phi_{TC}(u) = |u|^{p-2}u + |u|^{n-2}u, \ 1 < p < n,$$

arises in studying the deformation of materials under stress [5]. It is clear that this operator is regularly varying at zero with index $i = p - 1$ and

$$L_{TC}(u) = 1 + |u|^{n-p}. \tag{12}$$

Example 4. The operator $\Phi_{NE}\colon \mathbb{R} \to \mathbb{R}$ given by

$$\Phi_{NE} = (1 + u^2)^p u, \quad p > 1/2,$$

arises in studying nonlinear elasticity phenomena in dynamical problems [15]. The operator Φ_{NE} is regularly varying at zero with index $i = 1$ and

$$L_{NE}(u) = \Big(1 + u^2\Big)^p.$$

Example 5. The operator $\Phi_D\colon \mathbb{R} \to \mathbb{R}$ given by

$$\Phi_D(u) = \begin{cases} \Big(\log(1 + |u|)\Big)^{-1} |u|^{p-2}u, & \text{if } u \neq 0, \\ 0, & \text{if } u = 0, \end{cases}$$

with $p > 2$ occurs in studying the plastic deformation exhibited by certain materials subject to a torsional moment for an extended period of time at high temperature [5]. It is regularly varying at zero with index $i = p - 2$ and

$$L_D(u) = \frac{|u|}{\log(1 + |u|)}, \quad u \neq 0.$$

It is obvious that the operator L_D can be continuously defined at $u = 0$, setting $L_D(0) = 1$.

3. Auxiliary results

In view of (2), define

$$\widetilde{F}(u) = \ F(u)/\Phi_p(u) \ \text{ if } u \neq 0, \ \text{ and } \ \widetilde{F}(0) = F_0. \tag{13}$$

The main tool here used to prove the existence of positive bounded solutions of (1) is the following fixed point result.

Theorem 1. *Let S_0 be a given subset of $C(I,\mathbb{R}^2)$. Suppose that there exists a nonempty closed bounded convex subset $\Omega \subset C(I,\mathbb{R}^2)$ such that*

$$|v(t)| < \sigma\, a(t), \quad \Lambda\left(v(t)/a(t)\right) \neq 0 \ \text{ for all } (u,v) \in \Omega \text{ and } t \in I, \tag{14}$$

and a nonempty closed subset S_1 of $S_0 \cap \Omega$ such that for each $(u,v) \in \Omega$, the half-linear equation

$$\frac{d}{dt}\left(H(t,v(t))\, \Phi_p(y')\right) + b(t)\widetilde{F}(u(t))\, \Phi_p(y) = 0 \tag{15}$$

has a unique solution y_{uv} with $(y_{uv}, y_{uv}^{[1]}) \in S_1$, where

$$H(t,v(t)) := \frac{a(t)}{\Phi_p\Big(\Lambda\left(v(t)/a(t)\right),\Big)} \tag{16}$$

and $y_{uv}^{[1]}$ is the quasiderivative of y_{uv}, that is,

$$y_{uv}^{[1]}(t) = H(t,v(t))\, \Phi_p(y_{uv}'(t)).$$

For each (u,v) denote by $\mathcal{T} : \Omega \to C(I,\mathbb{R}^2)$ the operator defined by $\mathcal{T}(u,v) = (y_{uv}, y_{uv}^{[1]})$. Then $\mathcal{T}$ has a fixed point $(\widehat{x}, \widehat{z}) \in S_1 \subset S_0$ with $\widehat{z}(t) = a(t)\Phi(\widehat{x}'(t))$ and $\widehat{x}$ is a solution of (1).

Proof. The assertion follows from [10, Theorem 2.1] by choosing $F_1(t,u) = b(t)F(u)$ and $G(t,u) = b(t)\widetilde{F}(u)$. For the sake of completeness, we here briefly show how a fixed point of the operator T yields a solution of (1), see also the final part of the proof of Theorem 2.1 in [10], with minor changes. Let $(\widehat{x}, \widehat{z})$ be a fixed point of T. Since $\widehat{x}$ is a solution of (15) with $u = \widehat{x}$, $v = \widehat{z}$ and $\widetilde{F}(\widehat{x}(t))\, \Phi_p(\widehat{x}) = F(\widehat{x}(t))$, we obtain

$$\widehat{z}(t) = H(t,\widehat{z}(t))\, \Phi_p(\widehat{x}\,'(t)) \tag{17}$$

and

$$\frac{d}{dt}\left(H(t,\widehat{z}(t))\Phi_p(\widehat{x}\,'(t))\right) + b(t)F(\widehat{x}(t)) \, = 0.$$

Thus,

$$\frac{d}{dt}\widehat{z}(t) + b(t)F(\widehat{x}(t)) \, = 0. \tag{18}$$

From (10), (16) and (17) we obtain

$$\widehat{x}'(t) = \frac{\Phi_q\left(\widehat{z}(t)\right)}{\Phi_q\left(H(t, \widehat{z}(t)\right)} = \frac{\Phi_q\left(\widehat{z}(t)\right)}{\Phi_q\left(a(t)\right)}\Lambda\Big(\frac{\widehat{z}(t)}{a(t)}\Big) = \Phi^*\left(\frac{\widehat{z}(t)}{a(t)}\right)$$

or $a(t)\Phi(\widehat{x}'(t)) = \widehat{z}(t)$, which, in view of (18), gives that $\widehat{x}$ is a solution of (1). □

Theorem 1 requires that a certain half-linear equation has a unique solution in S_1. To check this assumption, in the following we will use a comparison criterion concerning principal solutions of half-linear equations.

The half-linear equation (7) has been extensively studied in the last years, especially in virtue of the striking similarity with the corresponding linear equation

$$(a(t)z')' + b(t)z = 0,$$

concerning the Sturmian theory, Kneser-type or Hille-type oscillation/nonoscillation criteria, see, e.g., [12] and references therein. In particular, a solution x of (7) is said to be nonoscillatory if $x(t) \neq 0$ for large t and oscillatory otherwise. It is known, see, e.g, [12, Section 1.2.4], that a solution of (7) is oscillatory if and only if any other solution is oscillatory. Thus, equation (7) is said to be oscillatory if one solution is oscillatory and nonoscillatory otherwise. Moreover, in [13, 23] the notion of principal solution was extended to (7) by the Riccati equation approach and reads as follows.

Definition ([13, 23]). Let (7) be nonoscillatory. A nontrivial solution u of (7) is said to be a *principal solution* if for every nontrivial solution z of (7) such that $z \neq \lambda u$, $\lambda \in \mathbb{R}$, it holds

$$\frac{u'(t)}{u(t)} < \frac{z'(t)}{z(t)} \qquad \text{as } t \to \infty.$$

The set of principal solutions of (7) is nonempty ([13, 23]), and for any $\mu \neq 0$ there exists a unique principal solution u such that $u(0) = \mu$, i.e., principal solutions are defined up to a constant factor. Any nontrivial solution of (7) that is not principal is called a *nonprincipal solution*. A detailed analysis of the main properties of the principal solutions of (7) can be found in [12]. Here, we report some of these properties, which are needed in the sequel.

Jointly with (7), consider the half-linear equation

$$(\widehat{a}(t)\Phi_p(y'))' + \widehat{b}(t)\Phi_p(y) = 0, \tag{19}$$

where $\widehat{a}, \widehat{b}$ are continuous functions for $t \in I$ and

$$\widehat{a}(t) \geq a(t), \quad \widehat{b}(t) \leq b(t) \text{ on } I.$$

Then, (19) is a minorant of (7). The following comparison result holds.

Lemma 2. *Assume (3). If (7) is nonoscillatory and its principal solution z_0 is positive decreasing on I, then the principal solution y_0 of (19) is positive decreasing on I and satisfies the inequality*

$$\frac{y_0^{[1]}(t)}{\Phi_p(y_0(t))} \leq \frac{z_0^{[1]}(t)}{\Phi_p(z_0(t))} \quad \textit{for any } t \in I. \tag{20}$$

Proof. Set

$$\eta_0(t) = \frac{y_0^{[1]}(t)}{\Phi_p(y_0(t))}, \quad \zeta_0(t) = \frac{z_0^{[1]}(t)}{\Phi_p(z_0(t))}.$$

In virtue of [23], see also [12, Section 4.2.2], the functions η_0 and ζ_0 are the so-called *minimal solutions* of the Riccati equations associated to (19) and (7), respectively. Thus, using [12, Theorem 4.2.2] with minor changes, see also [10, Proposition 3.1-(j_2)], for any $t \in I$ we get $\eta_0(t) \leq \zeta_0(t)$ and $y_0(t) > 0$. Hence (20) is verified. Since z_0 is positive decreasing on I, again from (20) we have $y_0'(t) < 0$ and the assertion follows. □

We close the section by recalling a useful property of two solutions of (7). If z, y are solutions of (7), then the function

$$W(z, y)(t) = z(t)y'(t) - z'(t)y(t), \tag{21}$$

is called *the Wronskian of z and y*. Similarly to the linear case, the following holds.

Lemma 3 ([12, Lemma 1.3.1])**.** *Let z and y be two nontrivial solutions of (7) defined on $I = [t_0, \infty)$. Then either $W(z, y) \equiv 0$, and in this case z and y are linearly dependent on I, or $W(z, y) \neq 0$ on I and z and y are linearly independent on I.*

4. Global positive bounded solutions

In this section, we investigate the existence of global positive solutions of (1), that is, solutions of (1) which can be extended to infinity and are positive throughout the entire interval $I = [t_0, \infty)$. The analysis is carried out by applying Theorem 1.

If conditions (3) and (4) hold, in view of (i_2), set

$$M = \max_{u \in [0,1]} \widetilde{F}(u), \quad N = \inf_{t \geq t_0} \; \Phi_q\left(a(t)\right) \int_t^\infty \Phi_q\left(\frac{1}{a(s)}\right) ds, \tag{22}$$

where $\widetilde{F}$ is given in (13). Denote by Φ^* the inverse of the regularly varying function Φ and let Λ be the map such that $\Phi^*(v) = v^{q-1}\, \Lambda(v)$, see Lemma 1. Assume that

$$\Lambda \text{ is positive and continuous on } (-\sigma, \sigma). \tag{23}$$

Fixed $K \in (0, \rho)$, put

$$\gamma = \min_{|v| \leq \Phi(K)} \Lambda(v), \qquad \vartheta = \max_{|v| \leq \Phi(K)} \Lambda(v). \tag{24}$$

The following holds.

Theorem 2. *Assume (3) and (4). Let (23) be satisfied and let M, ϑ be the constants given in (22) and in (24), respectively. Then equation (1) has infinitely many global positive decreasing solutions, provided that the half-linear equation*

$$\left(\vartheta^{1-p} a(t) \Phi_p(z')\right)' + \; M b(t) \Phi_p(z) = 0 \tag{25}$$

is nonoscillatory and its principal solution z_0 is positive decreasing on the whole interval $I = [t_0, \infty)$.

Proof. Let $c > 0$ be a constant satisfying

$$c^{p-1} \leq \min\left\{1, (\gamma N)^{p-1} \Phi(K)\right\}, \tag{26}$$

where N and γ are given in (22) and (24), respectively. Define

$$w_0(t) = \gamma \int_t^\infty \Phi_q\left(\frac{1}{a(s)}\right) ds. \tag{27}$$

Thus

$$w_0(t) \leq \gamma J_a, \tag{28}$$

where J_a is given in (3). Consider the sets S_0 and Ω given by

$$S_0 = \{(u,v) \in C(I,\mathbb{R}^2) : u(t_0) = c,\ v(t_0) = 0\},$$

$$\Omega = \left\{(u,v) \in C(I,\mathbb{R}^2) :\ c\left(\frac{w_0(t)}{\gamma J_a}\right)^{\vartheta/\gamma} \le u(t) \le c,\ 0 \le |v(t)| \le \Phi(K)\, a(t)\right\}. \tag{29}$$

Clearly, Ω is a nonempty bounded convex subset of $C(I,\mathbb{R}^2)$. Observe that for any $(u,v) \in \Omega$, we have $u(t_0) = c$, and condition (14) is satisfied.

Choose $S_1 = S_0 \cap \Omega = \{(u,v) \in \Omega : v(t_0) = 0\}$, and let us prove that for any $(u,v) \in \Omega$ the solution y_{uv} of (15), satisfying

$$y_{uv}(t_0) = c, \quad y'_{uv}(t_0) = 0, \tag{30}$$

is the unique solution of (15) such that $(y_{uv}, y_{uv}^{[1]}) \in S_1$.

From (22) and (24) we get

$$\frac{a(t)}{\vartheta^{p-1}} \le H(t, v(t)) \le \frac{a(t)}{\gamma^{p-1}}, \quad b(t)\widetilde{F}(u(t)) \le Mb(t). \tag{31}$$

Thus, equation (15) is a minorant of (25), and so it is nonoscillatory. Moreover, from Lemma 2, the principal solution η_{uv} of (15), with $\eta_{uv}(t_0) = c$, is positive decreasing on I. We claim that

$$y_{uv}(t) > \eta_{uv}(t) \ \text{ for } t > t_0. \tag{32}$$

Indeed, from the definition of principal solution, it holds

$$0 = y'_{uv}(t_0) > \eta'_{uv}(t_0) \ \text{ and } y_{uv}(t_0) = \eta_{uv}(t_0) = c,$$

and therefore there exists a right neighboorhod of t_0, say $I_\varepsilon = (t_0, t_0 + \varepsilon)$, such that $y_{uv}(t) > \eta_{uv}(t)$ for $t \in I_\varepsilon$. Set $t_\varepsilon = t_0 + \varepsilon$ and, by contradiction, suppose that $y_{uv}(t_\varepsilon) = \eta_{uv}(t_\varepsilon)$. Clearly, we have $y'_{uv}(t_\varepsilon) < \eta'_{uv}(t_\varepsilon)$. Consider the Wronskian $W(\eta_{uv}, y_{uv})(t)$. We have

$$W(\eta_{uv}, y_{uv})(t_\varepsilon) = \eta_{uv}(t_\varepsilon)\,(y'_{uv}(t_\varepsilon) - \eta'_{uv}(t_\varepsilon)) < 0.$$

As the principal solution η_{uv} is decreasing at $t = t_0$, $W(\eta_{uv}, y_{uv})(t_0) = -c\eta'_{uv}(t_0) > 0$, that is, a contradiction with Lemma 3. So (32) is valid.

Thus, the solution y_{uv} is positive on the half-line I. Since $\widetilde{F}(u(t)) > 0$ for any $(u, v) \in \Omega$, integrating (15) we have for $t \in I$

$$y_{uv}^{[1]}(t) = -\int_{t_0}^{t} b(s)\widetilde{F}(u(s))\,\Phi_p(y_{uv}(s))\;ds < 0$$

and y_{uv} is decreasing on I. Moreover, y_{uv} is a nonprincipal solution of (15) and so, from the definition of the principal solution, we obtain for $t \in I$

$$\frac{\eta_{uv}^{[1]}(t)}{\Phi_p(\eta_{uv}(t))} < \frac{y_{uv}^{[1]}(t)}{\Phi_p(y_{uv}(t))}. \tag{33}$$

The function w_0, given in (27), is the principal solution of the half-linear equation

$$(\gamma^{1-p}a(t)\;\Phi_p(w'))' = 0, \tag{34}$$

see, e.g., [12, Example 4.2.1]. In view of (31), the equation (34) is a minorant of (15). From Lemma 2 and (33) we obtain for any $t \geq t_0$

$$\frac{w_0^{[1]}(t)}{\Phi_p(w_0(t))} \leq \frac{\eta_{uv}^{[1]}(t)}{\Phi_p(\eta_{uv}(t))} < \frac{y_{uv}^{[1]}(t)}{\Phi_p(y_{uv}(t))}. \tag{35}$$

Since $y_{uv}^{[1]}$ is negative, using (31), we get

$$\frac{\vartheta}{\gamma}\frac{w_0'(t)}{w_0(t)} < \frac{y_{uv}'(t)}{y_{uv}(t)}.$$

Integrating this inequality on $[t_{0,}t]$ we have for $t \geq t_0$

$$c\left(\frac{w_0(t)}{\gamma J_a}\right)^{\vartheta/\gamma} < y_{uv}(t) \leq c. \tag{36}$$

Moreover, since $w_0^{[1]}(t) = -1$, from (35) we obtain $\left|y_{uv}^{[1]}(t)\right| \leq \Phi_p\left(y_{uv}(t)/w_0(t)\right)$ or, using (27) and (36),

$$\left|y_{uv}^{[1]}(t)\right| \leq \left(\frac{c}{\gamma}\right)^{p-1}\Phi_p\left(\left(\int_t^\infty \Phi_q\left(1/a(s)\right)ds\right)^{-1}\right). \tag{37}$$

Since

$$\Phi_p\left(\int_t^\infty \Phi_q\left(\frac{1}{a(s)}\right)ds\right) = \Phi_p\left(\Phi_q\left(a(t)\right)\int_t^\infty \Phi_q\left(\frac{1}{a(s)}\right)ds\right)\frac{1}{a(t)},$$

from (22), (26) and (37) we get

$$\left|y_{uv}^{[1]}(t)\right| \le a(t)\left(\frac{c}{\gamma N}\right)^{p-1} \le \Phi(K)\,a(t).$$

Thus, from (30) and (36) we obtain

$$(y_{uv}, y_{uv}^{[1]}) \in S_1. \tag{38}$$

Clearly, y_{uv} is the unique solution of (15) satisfying (38). Hence, denoting by $\mathcal{T} : \Omega \to C(I, \mathbb{R}^2)$ the operator $\mathcal{T}(u, v) = (y_{uv}, y_{uv}^{[1]})$ and applying Theorem 1, the assertion follows. $\square$

Remark 1. The positive global solutions of (1) obtained in Theorem 2, start with derivative zero at $t = t_0$. It is not difficult to extend this result, by proving, with minor changes, the existence of positive global solutions x of (1) such that $x'(t_0) > 0$.

Theorem 2 requires the existence of a nonoscillatory half-linear equation (25), having its principal solution positive decreasing on the whole interval I. Moreover, equation (25) depends on the function Λ and therefore on the inverse Φ^* of Φ. When the explicit form of Φ^* is unknown, as it happens for instance in Example 3, then we can overcome this difficulty in the following way.

Corollary 1. *Let (3) and (4) be satisfied. Assume either*

(i_1) the function L, given in (8), is positive increasing for $0 \le u < \varrho$, and the half-linear equation

$$(L(0)a(t)\Phi_p(z'))' + \ Mb(t)\Phi_p(z) = 0$$

is nonoscillatory with principal solution z_0 positive decreasing on the whole interval I;

or

(i_2) the function L, given in (8), is positive decreasing for $u \in [0, K]$, $K < \varrho$, and the half-linear equation

$$(L(K)a(t)\Phi_p(z'))' + \ Mb(t)\Phi_p(z) = 0$$

is nonoscillatory with principal solution z_0 positive decreasing on the whole interval I.

Then equation (1) has infinitely many global positive decreasing solutions.

Proof. We use properties of regularly varying function Φ, which are stated in Lemma 1. Assume (i_1). The function Λ given in (10) is even and positive decreasing on $[0, \sigma)$. Hence, assumption (23) in Theorem 2 is verified. Moreover, from (11) we have

$$\Lambda(\Phi(u)) = \frac{1}{\Phi_q(L(u))} \qquad \text{for } L(u) \neq 0,$$

thus

$$\vartheta = \max_{|v| \leq \Phi(K)} \Lambda(v) = \Lambda(0) = \Lambda(\Phi(0)) = \frac{1}{\Phi_q(L(0))},$$

that is $\vartheta^{1-p} = L(0)$. Hence, the assertion follows from Theorem 2. The case (i_2) follows in a similar way, using the fact that Λ is, in this case, positive increasing and $\Lambda(\Phi(K)) = 1/\Phi_q(L(K))$, which implies $\vartheta^{1-p} = L(K)$. □

Theorem 2 requires the existence of a suitable nonoscillatory half-linear equation with the principal solution positive decreasing on I. This assumption can be verified via a comparison criterion with a half-linear equation, whose principal solution is known and has the desired properties. An example in this direction can be obtained using the half-linear Euler differential equation, as the following result illustrates.

Corollary 2. *Let (3) and (4) be satisfied. Let (23) be satisfied, and let ϑ, M be the constants given in (24) and in (22), respectively. Then equation (1) has infinitely many global positive decreasing solutions, provided that for any $t \geq t_0 > 0$*

$$a(t) \geq \vartheta^{p-1} t^p, \quad b(t) \leq (p^p M)^{-1}. \tag{39}$$

Proof. Consider the half-linear differential equation

$$\left(t^p \Phi_p(\zeta')\right)' + p^{-p} \Phi_p(\zeta) = 0, \qquad t \geq t_0 > 0, \tag{40}$$

which satisfies condition (4). This equation originates from the half-linear Euler differential equation

$$\left(\Phi_q(x')\right)' + \left(\frac{q-1}{q}\right)^q t^{-q} \Phi_q(x) = 0, \qquad t \geq t_0 > 0. \tag{41}$$

where q is the Hölder conjugate of p. Indeed, the change of variable $\zeta = \Phi_q(x')$ transforms (41) into (40). For this reason (40) is called *half-linear*

Euler reciprocal differential equation. It is known that (41) is nonoscillatory. Moreover, the function $x_0(t) = t^{(q-1)/q}$ is the principal solution of (41), see [12, Section 1.4.2.]. Using [12, Theorem 4.2.4], the function ζ_0 given by

$$\zeta_0(t) = \left(\frac{1}{t}\right)^{1/p}$$

is the principal solution of (40). In view of (39), equation (40) is a majorant of (25) and so, applying Lemma 2 equation (25) is nonoscillatory and its principal solution z_0, with $z_0(t_0) > 0$, is positive decreasing on the whole interval $I = [t_0, \infty)$. Thus, the assertion follows from Theorem 2. $\square$

Clearly, other criteria can be obtained by using as majorant of (25), instead of the half-linear Euler reciprocal differential equation, any half-linear equation whose principal solution is positive decreasing on I.

As an application of the previous results, consider the torsional creep operator $\Phi_{TC}\colon \mathbb{R} \to \mathbb{R}$ from Example 3, i.e.,

$$\Phi_{TC}(u) = |u|^{p-2}u + |u|^{n-2}u, \ 1 < p < n.$$

As claimed, in this case the explicit form of the inverse map of Φ_{TC} is unknown. In this case, we can apply Corollary 1. The associated slowly varying function to Φ_{TC} is given in (12), namely

$$L_{TC}(u) = 1 + |u|^{n-p}.$$

Then L is positive increasing and $L(0) = 1$. Since $\Phi_{TC}(u)$ is regularly varying with index $i = p - 1$ near zero, applying Corollaries 1 and 2 we obtain the following.

Example 6. Equation

$$(a(t)\Phi_{TC}(x'))' + b(t)F(x) = 0,$$

where $a(t) \geq t^p$, and $b(t) \leq (p^p M)^{-1}$ for $t \geq t_0 > 0$, has infinitely many global positive decreasing solutions.

Analogously, similar results can be formulated for equations involving the operators in Examples 1, 2, 4–6.

5. Asymptotic behavior

In this section, we study the asymptotic properties of global positive solutions for (1) under the assumption (3) . To this aim we use some asymptotic properties of principal and nonprincipal solutions for the half-linear equation (7). Jointly with the integral J_a given in (3), also the following integrals will play an important role.

$$\begin{aligned} J_b &= \int_{t_0}^{\infty} b(s)\ ds, \\ J_1 &= \int_{t_0}^{\infty} \Phi_q\left(\frac{1}{a(t)}\int_{t_0}^{t} b(s)\ ds\right) dt, \\ Y_1 &= \int_{t_0}^{\infty} b(t)\Phi_p\Big(\int_{t}^{\infty} \Phi_q\left(\frac{1}{a(s)}\right) ds\Big) dt. \end{aligned}$$

More precisely, in the sequel the following cases will be considered:

$$J_a < \infty, J_b = \infty, J_1 = \infty; \tag{42}$$

$$J_a < \infty, J_b = \infty, J_1 < \infty, Y_1 < \infty; \tag{43}$$

$$J_a < \infty, J_b < \infty. \tag{44}$$

Observe that if the case (44) occurs, then $J_1 < \infty, Y_1 < \infty$. The following results for the half-linear equation (7) hold.

Proposition 1. *[7, Theorems 2, 6, 7 and Corollary 1].*

(s_1) If (7) is nonoscillatory and (42) is valid, then any eventually positive solution of (7) tends to zero as $t \to \infty$.

(s_2) If either (43) or (44) is valid, then (7) is nonoscillatory. In addition, in both cases the principal solution of (7) tends to zero and its quasiderivative has a finite nonzero limit as $t \to \infty$. Further, any nonprincipal solution of (7) has a finite nonzero limit as $t \to \infty$.

Proof. The argument is based on some consequence of results in [7].

Claim (s_1). From [7, Lemma 1 and Theorem 4 (i_2)] the set $\mathbb{S}$ of nonoscillatory solutions of (7) is

$$\mathbb{S} = \mathbb{M}^-_{0,\ell} \cup \mathbb{M}^-_{0,\infty}$$

where $\mathbb{M}^- = \{z \text{ solution of } (7) : \exists t_x \geq t_0 : z(t)z'(t) < 0 \text{ for } t > t_z\}$, and

$$\begin{aligned} \mathbb{M}^-_{0,\ell} &= \{z \in \mathbb{M}^- : \lim_{t\to\infty} z(t) = 0, \lim_{t\to\infty} z^{[1]}(t) = d_z,\ 0 < |d_z| < \infty\}, \\ \mathbb{M}^-_{0,\infty} &= \{z \in \mathbb{M}^- : \lim_{t\to\infty} z(t) = 0, \lim_{t\to\infty} |z^{[1]}(t)| = \infty\}. \end{aligned}$$

Therefore the assertion immediately follows.

Claim (s_2). Using the notation from [7], assumption (43) for (7) reads as the case (C_4^-). Analogously, assumption (44) reads as the case (C_0). Thus, from [7, Theorem 2-(i_2)] equation (7) is nonoscillatory and $\mathbb{M}_{0,\ell}^-$ coincides with the set of principal solutions. It remains to shows that any nonprincipal solution of (7) has a finite nonzero limit as $t \to \infty$. If (43) holds, then from [7, Lemma 1 and Theorem 7] the set $\mathbb{S}$ of nonoscillatory solutions of (7) is

$$\mathbb{S} = \mathbb{M}_{0,\ell}^- \cup \mathbb{M}_{\ell,\infty}^-$$

where

$$\mathbb{M}_{\ell,\infty}^- = \{z \in \mathbb{M}^- : \lim_{t\to\infty} z(t) = c_z, \lim_{t\to\infty} |z^{[1]}(t)| = \infty,\ 0 < |c_z| < \infty\}.$$

Since the set of principal solutions is $\mathbb{M}_{0,\ell}^-$, any nonprincipal solution belongs to $\mathbb{M}_{\ell,\infty}^-$ and the assertion follows. Finally, assume (44) and denote by u and z two solutions of (7). Let u be a principal solution and z a nonprincipal one. Without loss of generality, suppose that u is eventually positive. Thus $u^{[1]}$ is eventually negative. Consider the function

$$E_W(z,u)(t) = z^{[1]}(t)\Phi(u(t)) - u^{[1]}(t)\Phi(z(t)).$$

The function E_W can be viewed as an extension of Wronskian W given in (21), as it reduces to the classical Wronskian if $p = 2$. Since all the solutions of (7), together with their quasiderivatives, are bounded, see, e.g., [7, Theorem 4 (i_3)], and $u \in \mathbb{M}_{0,\ell}^-$, we have $\lim_{t\to\infty} z^{[1]}(t)\Phi(u(t)) = 0$. From [7, Corollary 2], the function E_W has a finite nonzero limit, and hence we get $\lim_{t\to\infty} u^{[1]}(t)\Phi(z(t)) \neq 0$, which yields the assertion. $\square$

Using Proposition 1-(s_1) and Theorem 2 we obtain the following.

Corollary 3. *Let (3), (4), and (23) be satisfied. Assume that the half-linear equation (25) is nonoscillatory and its principal solution z_0 is positive decreasing on the whole interval $I = [t_0, \infty)$. Then (1) has infinitely many global positive decreasing solutions x which tend to zero as $t \to \infty$, provided that (42) holds. In addition, any such solution x has the lower bound for any $t \in I$*

$$x(t) \geq x(t_0)\,(J_a)^{-\vartheta/\gamma} \left(\int_t^\infty \Phi_q \left(\frac{1}{a(s)} \right) ds \right)^{\vartheta/\gamma}. \tag{45}$$

Proof. The first assertion follows from Proposition 1-(s_1) and Theorem 2. For proving (45), note that, in view of the proof of Theorem 2, the solution x belongs to the set Ω given by (29) with $c = x(t_0)$. Thus, (45) follows. $\square$

A closer examination of the proof of Theorem 2 shows that the global positive decreasing to zero solutions of (1), which are obtained via Corollary 3, are nonprincipal solutions of the half-linear equation (15). When (43) or (44) holds, i.e., $J_1 < \infty$, the existence of global positive decreasing to zero solutions of (1) can be obtained by considering principal solutions of the same half-linear equation and using again Proposition 1. More precisely, the following holds.

Theorem 3. *Let (3), (4), and (23) be satisfied. Assume that the principal solution z_0 of (25) is positive decreasing on the whole interval $I = [t_0, \infty)$. If either (43) or (44) is valid, then (1) has infinitely many global positive decreasing solutions x satisfying $\lim_{t\to\infty} x(t) = 0$.*

Proof. In the first part, the argument is similar to the one given in the proof of Theorem 2, and therefore most of the calculations are omitted.

Without loss of generality, assume $z_0(t_0) = 1$. By Proposition 1 we have $0 < z_0(t) \le 1, z_0'(t) < 0$ on I and $\lim_{t\to\infty} z_0(t) = 0$. Let c be a positive constant satisfying (26), and consider the function w_0 given by (27). Since w_0 is the principal solution of the half-linear equation (34) which is a minorant of (25), using Lemma 2 and integrating on $[t_0, t]$ we obtain

$$\left(\frac{w_0(t)}{\gamma A}\right)^{\vartheta/\gamma} \le z_0^{\gamma/\vartheta}(t).$$

Consider the sets

$$\begin{aligned}
\Omega = \Bigg\{(u,v) \in C(I,\mathbb{R}^2) : c\left(\frac{w_0(t)}{\gamma A}\right)^{\vartheta/\gamma} \le u(t) \le (z_0(t))^{\gamma/\vartheta},\\
-\Phi(K)\,a(t) \le v(t) \le 0\Bigg\},\\
S_0 = \Big\{(u,v) \in C(I,\mathbb{R}^2) : u(t_0) = c, u(t) \ge 0, v(t) \le 0, \lim_{t\to\infty} u(t) = 0\Big\},
\end{aligned}$$

and put $S_1 = \Omega \cap S_0$. Hence $S_1 = \{(u,v) \in \Omega : u(t_0) = c\}$.

For any $(u,v) \in \Omega$, in the proof of Theorem 2 we have considered the nonprincipal solution y_{uv} of (15) satisfying the initial conditions (30). Now, instead of y_{uv}, consider the principal solution η_{uv} of (15), with $\eta_{uv}(t_0) = c$. From Lemma 2, we have for any $t \in I$

$$\frac{\eta_{uv}^{[1]}(t)}{\Phi_p(\eta_{uv}(t))} \leq \frac{z_0^{[1]}(t)}{\Phi_p(z_0(t))} \,. \tag{46}$$

Since $H_v(t) \leq \gamma^{1-p} a(t)$ and $\eta'_{uv}(t) < 0$, integrating (46) we get for $t \in I$

$$\eta_{uv}(t) \leq c(z_0(t))^{\gamma/\vartheta} \leq (z_0(t))^{\gamma/\vartheta}.$$

Thus, using similar arguments to the ones in the proof of Theorem 2, with minor changes, we obtain

$$(\eta_{uv}, \eta_{uv}^{[1]}) \in S_1.$$

It remains to show that for any $(u, v) \in \Omega$, the solution η_{uv} is the unique solution of (15) such that $(\eta_{uv}, \eta_{uv}^{[1]}) \in S_1$. Now, the argument is different to the one used in the proof of Theorem 2 and it is based on Proposition 1. By contradiction, let $\widehat{\eta}$ be another solution of (15), $(\widehat{\eta}, \widehat{\eta}^{[1]}) \in S_1$ and $\widehat{\eta} \neq \eta_{uv}$. For the sake of simplicity, the dependence of $\widehat{\eta}$ on the variable (u, v) is omitted. Clearly, $\widehat{\eta}$ is a nonprincipal solution of (15). Since $(\widehat{\eta}, \widehat{\eta}^{[1]}) \in S_1$, we have $0 \leq \widehat{\eta}(t) \leq (z_0(t))^{\gamma/\vartheta}$. Hence $\lim_{t\to\infty} \widehat{\eta}(t) = 0$. First, assume (44). Since $u(t) \leq 1$ for all $t \geq t_0$ by the definition of the set Ω, in view of (22) we have

$$\int_{t_0}^{\infty} b(s)\widetilde{F}(u(s))ds \leq M J_b < \infty. \tag{47}$$

Moreover, in view of (31) we have also

$$\int_{t_0}^{\infty} \Phi_q\left(\frac{1}{H_v(t)}\right) dt \leq \vartheta J_a < \infty. \tag{48}$$

Thus, applying Proposition 1 to equation (15), we get

$$\lim_{t\to\infty} \widehat{\eta}(t) \neq 0, \tag{49}$$

that is a contradiction. Now, assume (43). If (47) holds, then, reasoning as above, we obtain a contradiction. On the other hand, if

$$\int_{t_0}^{\infty} b(s)\widetilde{F}(u(s))ds = \infty,$$

using (22), (31) and $J_1 < \infty$, we have

$$\int_{t_0}^{\infty} \Phi_q \left(\frac{1}{H_v(t)} \int_{t_0}^{t} b(s)\widetilde{F}(u(s))ds \right) dt \le \vartheta M^{q-1} J_1 < \infty, \tag{50}$$

and, in a similar way, from $Y_1 < \infty$ we get

$$\int_{t_0}^{\infty} b(t)\widetilde{F}(u(t))\Phi_p\Big(\int_{t}^{\infty} \Phi_q \left(\frac{1}{H_v(s)} \right) ds\Big)dt \le M\vartheta^{p-1} Y_1 < \infty. \tag{51}$$

Thus, using (48), (50), (51) and applying again Proposition 1 to equation (15), we obtain, also in this case, (49), which is a contradiction. Then in both cases (43) and (44), the solution η_{uv} is the unique solution of (15) such that $(\eta_{uv}, \eta_{uv}^{[1]}) \in S_1$.

Hence, let $\mathcal{T} \colon \Omega \to C(I, \mathbb{R}^2)$ be the operator which maps $(u, v) \in \Omega$ into $(\eta_{uv}, \eta_{uv}^{[1]})$. In view of Theorem 1**,** the operator $\mathcal{T}$ has a fixed point $(\widehat{x}, \widehat{x}^{[1]}) \in S_1$, where $\widehat{x}$ is a solution of (1) and $\widehat{x}^{[1]}(t) = a(t)\Phi(\widehat{x}'(t))$. From this, the assertion follows. □

Summarizing, we get the following.

Corollary 4. *Let (3), (4), and (23) be satisfied. Assume that the half-linear equation (25) is nonoscillatory and its principal solution z_0 is positive decreasing on the whole interval $I = [t_0, \infty)$. Then (1) has infinitely many global positive decreasing solutions which tend to zero as $t \to \infty$, provided that either (42), or (43), or (44) is valid. In addition, in cases (43) and (44), equation (1) has also infinitely many global positive decreasing solutions with a finite nonzero limit as $t \to \infty$.*

Proof. If (42) holds, the assertion follows from Corollary 3. In case (43) or (44) are valid, the first assertion follows from Theorem 3. Finally, the existence of solutions of (1) with a finite nonzero limit as $t \to \infty$ follows from Proposition 1 and Theorem 2. □

Remark 2. As Corollary 4 illustrates, when the half-linear equation (25) is nonoscillatory and its principal solution is positive decreasing on I, then equation (1) has infinitely many global positive solutions which decay to zero provided that the integrals J_1 and Y_1 satisfy either $J_1 = \infty$ or $J_1 < \infty, Y_1 < \infty$. In the remaining case

$$J_1 < \infty, Y_1 = \infty \tag{52}$$

the question is open. The case (52) is, roughly speaking, a pathological case for the half-linear equation (25). Indeed, this case can occur only when $1 < p < 2$, see [7, Lemma 2], and so it is impossible in the linear case. Moreover, when (52) holds, then (25) is nonoscillatory and its principal solution decays to zero, but the asymptotic behavior of nonprincipal solutions is not fully known. In particular, it is an open problem if, in this case, any nonprincipal solution has a finite nonzero limit as $t \to \infty$. Consequently, as far as we know, a statement similar to (s_2) in Proposition 1 is not proved, and this fact makes impossible to use an analougous argument to the one given in the final part of the proof of Theorem 3.

When the case (43) or (44) holds, then the precise asymptotic decay to zero of the global positive decreasing solutions to (1), obtained via Corollary 4, can be given. Indeed, the following holds.

Corollary 5. *Let (3), (4), and (23) be satisfied. Assume that the half-linear equation (25) is nonoscillatory and its principal solution z_0 is positive decreasing on the whole interval $I = [t_0, \infty)$. If either (43) or (44) is valid, then (1) has global positive decreasing solutions $\widehat{x}$ such that the limit*

$$\lim_{t\to\infty} \frac{\widehat{x}(t)}{\int_t^\infty \Phi_q\Big(1/a(s)\Big)ds} \tag{53}$$

is finite and different from zero.

Proof. In virtue of Corollary 4, it is sufficient to prove (53). From the proof of Theorem 3, the solution $\widehat{x}$ is also the principal solution η_{uv} of the half-linear equation (15) for $u = \widehat{x}$ and $v = \widehat{x}^{[1]} = a(t)\Phi(\widehat{x}'(t))$.

Assume (44). Reasoning as in the proof of Theorem 3 we obtain (47) and (48). Thus, applying Proposition 1 (s_2) we get

$$\lim_{t\to\infty} \widehat{x}^{[1]}(t) = \lim_{t\to\infty} \eta_{uv}^{[1]}(t) = -\delta, \quad 0 < \delta < \infty. \tag{54}$$

In a similar way, if (43) holds, using Proposition 1 (s_2), the same argument as the one given in the proof of Theorem 3 yields (54). Using (54) and l'Hôpital rule, we obtain that the limit (53) is finite and different from zero. $\square$

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