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GROUPS WITH VANISHING CLASS SIZE p

MARIAGRAZIA BIANCHI, MARK L. LEWIS, AND EMANUELE PACIFICI

Abstract. Let G be a finite group. A conjugacy class of G is said to be vanishing if there exists an irreducible character of G which takes the value 0 on the elements of this class. In this note, we describe the groups whose vanishing classes all have size p for some prime p.

Throughout this note all groups are finite. The study of the sizes of conjugacy classes of groups goes back to the beginning of character theory: W. Burnside proved that a simple group does not have a conjugacy class of size that is a prime power (larger than 1), and this is a key element in his proof of his p^aq^b-Theorem (see Theorem (3.9) of [8]). N. Itô began the study of the structure of a group in terms of the number of conjugacy class sizes in [10], where he proved that a group having only two conjugacy class sizes must be the direct product of a p-group and an abelian group. K. Ishikawa had a major breakthrough in [12], where he proved that a p-group with only two conjugacy class sizes has nilpotence class at most 3; furthermore, he shows in [11] that a p-group has conjugacy class sizes $\{1, p\}$ if and only if the group is isoclinic to an extra-special p-group. There has followed a great deal of work on this topic. Rather than trying to summarize this work, we refer the reader to the excellent expository paper by A.R. Camina and R.D. Camina [3].

The study of group elements that are zeros for some irreducible character also goes back to the beginnings of character theory. Burnside also proved that if χ is a nonlinear irreducible character of a group G, then there is some element $g \in G$ so that $\chi(g) = 0$ (see Theorem (3.15) of [8]). With this in mind, an element $g \in G$ is said to be *vanishing* if there exists an irreducible character $\chi \in \mathrm{Irr}(G)$ so that $\chi(g) = 0$. Notice that if one element of a conjugacy class is vanishing, then all elements of that class are vanishing. Hence, we say that a conjugacy class is *vanishing* if the elements in the conjugacy class are vanishing.

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Recently, there has been more concentrated study on the vanishing elements and classes in a group. This began with the work of I.M. Isaacs, G. Navarro, and T.R. Wolf in [9], where they proved that if G is solvable, P is a Sylow 2-subgroup of G, and $F(G)$ denotes the Fitting subgroup of G, then every element outside the conjugates of $PF(G)$ is a vanishing element. Furthermore, if G is nilpotent, then they prove that every element outside $Z(G)$ is vanishing and if G is supersolvable, then every element outside $Z(F(G))$ is vanishing. In [6] and [7], the authors attach a graph to orders of the vanishing elements and they obtain results regarding which graphs can arise. We suggest the reader consult [5] for a nice expository paper that presents most of the known results regarding vanishing elements.

Recently, there has begun to be research that studies the sizes of the vanishing classes; although looking at [5], one will see that there are very few results in this subject so far. However, in [4], the third author with S. Dolfi and L. Sanus prove that if a prime p does not divide the size of any vanishing conjugacy class of G, then G has a normal p-complement and abelian Sylow p-subgroups. Also, in [1], vanishing class sizes in nonsolvable groups are studied.

In this paper, we follow the lead of Ishikawa, and we classify the groups where the only vanishing class size is a prime p. The main result of our paper is the following:

Theorem 1. *Let G be a group and let p be a prime. The vanishing classes of G all have size p if and only if one of the following occurs:*

(1) *$G = P \times H$, where P is a p-group whose conjugacy class sizes are 1 or p, and H is an abelian p'-group.*
(2) *$G/Z(G)$ is a Frobenius group whose Frobenius kernel has order p.*

Note that in case (2) of Theorem 1 a Frobenius complement of $G/Z(G)$ is cyclic of order dividing $p-1$.

Proof. Assume first that G is nilpotent: then we claim that all the vanishing conjugacy classes of G have size p if and only if G is as in (1). In fact, under the nilpotency assumption, Theorem B of [9] shows that the vanishing elements of G are precisely the elements in $G \setminus Z(G)$; hence, the set of vanishing conjugacy class sizes of G is actually the whole set of conjugacy class sizes of G, excluding 1. As it is easily seen, the conjugacy class sizes of G are all 1 or p if and only if G has the structure described in (1), so the theorem is proved in this situation. In view of the above discussion, we will henceforth assume that G is not nilpotent.

As the next step we show that, if the structure of G is as in (2), then the vanishing conjugacy class sizes of G are all p. So, let $G/Z(G)$ be a Frobenius group whose Frobenius kernel has prime order p, and let H, K be subgroups of G such that $K/Z(G)$ and $H/Z(G)$ are the Frobenius kernel and a Frobenius complement of $G/Z(G)$, respectively. Note that, as both the Sylow p-subgroup $K/Z(G)$ and the Hall p'-subgroup $H/Z(G)$ of $G/Z(G)$ are cyclic, every Sylow subgroup of $G/Z(G)$ is cyclic. In particular, every irreducible character of $Z(G)$ has an extension to G (see [8, (6.26) and (11.22)]). Now, let χ be in $\mathrm{Irr}(G)$ and let μ be the unique irreducible constituent of $\chi_{Z(G)}$; since μ has an extension $\eta \in \mathrm{Irr}(G)$, by Gallagher's Theorem ([8, (6.17)]) there exists $\psi \in \mathrm{Irr}(G)$ such that $\chi = \eta\psi$ and $\ker(\psi) \supseteq Z(G)$. As η is linear, for any $x \in G$ we have $\chi(x) = 0$ if and only if $\psi(x) = 0$; from this, it is easily seen that x is a vanishing element of G if and only if $xZ(G)$ is a vanishing element of $G/Z(G)$. Finally, an application of [9, Theorem A] yields that every element of $K/Z(G)$ is non-vanishing in $G/Z(G)$; as a consequence, if x is a vanishing element of G, then $xZ(G)$ lies in a Frobenius complement of $G/Z(G)$ (i.e., x lies in a conjugate of H) and, as H is abelian, the class size of x is p, as wanted.

It remains to prove that, if G is not nilpotent and every vanishing conjugacy class of G has size p, then $G/Z(G)$ is a Frobenius group whose Frobenius kernel has order p. By Theorem A of [4], under our assumptions, G has a normal q-complement and abelian Sylow q-subgroups for every prime q other than p. Therefore G has a normal Sylow p-subgroup P, and a Hall p'-subgroup H of G is nilpotent because it has a normal q-complement for every prime q dividing its order; but H is in fact abelian, as all its Sylow subgroups are abelian. Observe also that, since H is abelian and $\mathrm{Irr}(P/\Phi(P))$ can be viewed as a completely reducible H-module, there exists $\lambda \in \mathrm{Irr}(P)$ (such that $\ker(\lambda) \supseteq \Phi(P)$) whose inertia subgroup in G is $PC_H(P)$. So, if $\beta \in \mathrm{Irr}(PC_H(P))$ is such that λ is a constituent of β_P, Clifford Correspondence yields that β^G is in $\mathrm{Irr}(G)$; moreover, since $PC_H(P)$ is a normal subgroup of G, β^G vanishes outside $PC_H(P)$ and therefore every element in $G \setminus PC_H(P)$ is vanishing in G. By our assumptions, we get $|G : C_G(x)| = p$ for every $x \in G \setminus PC_H(P)$.

As G is not nilpotent, we have $C_H(P) \neq H$ and so we can choose h in $H \setminus C_H(P)$. Now, assume that $y \in P$ commutes with h. Since $C_H(P)$ is the unique Hall p'-subgroup of $PC_H(P)$, our choice of h ensures that both h and hy do not lie in $PC_H(P)$, hence we get

$$|G : C_G(h)| = p = |G : C_G(hy)|$$

by the paragraph above. But, as h and y are commuting elements of coprime orders, we get $C_G(hy) = C_G(h) \cap C_G(y)$, whence $C_G(h) = C_G(hy)$. Moreover, as H is abelian, we have $H \leq C_G(h) = C_G(hy) \leq C_G(y)$. We conclude that $C_P(H) = C_P(h)$ has index p in P, thus $C_P(H) \trianglelefteq G$.

Next, we work to show that $C_P(H)$ is central in G. Let z be in $C_P(H)$, and choose h in $H \setminus C_H(P)$: by the paragraph above we have $C_G(h) \leq C_G(z)$. Thus, we deduce that $C_P(H) = C_P(h)$ centralizes z and, as this holds for every $z \in C_P(H)$, we see that $C_P(H)$ is abelian. In particular, the centralizer in G of $C_P(H)$ contains the subgroup $C_P(H)H$, whose index in G equals $|P : C_P(H)| = p$; on the other hand $C_G(C_P(H))$ is normal in G, and if it coincides with $C_P(H)H$ (which clearly has H as a characteristic subgroup) then H is normal in G, a contradiction. As a consequence, the whole G centralizes $C_P(H)$.

We are now in a position to finish the proof. Set $Z = C_P(H)C_H(P)$; in view of the above paragraph (and of the fact that H is abelian) we have $Z \leq Z(G)$. As PZ/Z is a normal subgroup of G/Z having order p, and which is acted on faithfully by HZ/Z, we have that $G/Z = PH/Z$ is a Frobenius group whose Frobenius kernel is PZ/Z, and $Z = Z(G)$. Conclusion (2) holds, and our proof is complete. □

We believe there is some hope that Theorem 1 can be extended to groups whose only vanishing class size is a prime power p^n. In particular, we would not be surprised if it were true that, under this hypothesis, G is either the direct product of a p-group with class sizes 1 or p^n and an abelian p'-group, or $G/Z(G)$ is a Frobenius group whose Frobenius kernel has order p^n and whose Frobenius complements are abelian.

However, the converse of this is not true. As an example, let $G = \mathrm{SL}(2, 3)$: then $G/Z(G)$ is a a Frobenius group whose Frobenius kernel has order 4, but the vanishing class sizes of G are 4 and 6. Restricting our attention to Frobenius groups, it is not difficult to see that any Frobenius group where all the vanishing elements lie outside the Frobenius kernel and a Frobenius complement is cyclic will have only one vanishing class size, which is the order of the Frobenius kernel. We do not know of any examples of Frobenius groups whose Frobenius kernel is a p-group for some prime p and a Frobenius complement is cyclic, where there are vanishing elements in the Frobenius kernel (by Theorem A of [9], there are no such examples if the Frobenius kernel is abelian). We believe that proving the existence or nonexistence of such groups will likely involve results similar to those in [2] and [13].

We already mentioned the result by Itô which establishes that, if a group G has only two conjugacy class sizes, then the unique size larger than 1 is a prime power; as a concluding remark, we note that this does not translate to vanishing class sizes. In fact, it is not difficult to find examples of Frobenius groups, where the Frobenius kernel is not of prime-power order, the Frobenius complements are cyclic, and all the vanishing elements lie outside of the Frobenius kernel: these give examples of groups with only one vanishing class size which is not a prime power.

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