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A REDUCTION THEOREM FOR THE GALOIS–MCKAY CONJECTURE

GABRIEL NAVARRO, BRITTA SPÄTH, AND CAROLINA VALLEJO

Abstract. We introduce $\mathcal{H}$-triples and a partial order relation on them, generalizing the theory of ordering character triples developed by Navarro and Späth. This generalization takes into account the action of Galois automorphisms on characters and, together with previous results of Ladisch and Turull, allows us to reduce the Galois–McKay conjecture to a question about simple groups.

Introduction

The origin of the McKay conjecture dates back to a paper of John McKay from 1972 ([McK72]), where it is stated for finite simple groups and for $p = 2$.

Conjecture (The McKay conjecture). *Let G be a finite group, let p be a prime, and let $H = \mathbf{N}_G(P)$ be the normalizer in G of a Sylow p-subgroup P of G. Then*

$$|\mathrm{Irr}_{p'}(G)| = |\mathrm{Irr}_{p'}(H)|,$$

where $\mathrm{Irr}_{p'}(G)$ is the set of irreducible complex characters of G of degree not divisible by p.

In 2007, Martin Isaacs, Gunter Malle and the first-named author reduced the McKay conjecture to a problem about simple groups in [IMN07]. Using this reduction theorem, G. Malle and the second-named author have recently proven that the McKay conjecture holds for all finite groups for $p = 2$ in [MS16].

In 2004, the first-named author predicted that not only the degrees of the complex characters of G and H were related but also their values (see [Nav04]). For a fixed prime p, let $\mathcal{H}$ be the subgroup of $\mathcal{G} = \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$ consisting of the $\sigma \in \mathcal{G}$ for which there exists an integer f such that $\sigma(\xi) = \xi^{p^f}$ for every root of unity ξ of order not divisible by p.

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Conjecture (The Galois–McKay conjecture)**.** *Let G be a finite group, let p be a prime, and let $H = \mathbf{N}_G(P)$ be the normalizer in G of a Sylow p-subgroup P of G. Then the actions of $\mathcal{H}$ on $\mathrm{Irr}_{p'}(G)$ and $\mathrm{Irr}_{p'}(H)$ are permutation isomorphic.*

As a matter of fact, this conjecture is stated only for cyclic subgroups of $\mathcal{H}$ in Conjecture A of [Nav04], but it is suggested in the above more general form at the end of the same paper. The Galois–McKay conjecture as stated above is equivalent to the existence of a McKay bijection preserving fields of values of characters over the field $\mathbb{Q}_p$ of p-adic numbers. Recall that if $\mathbb{Q} \subseteq \mathbb{F}$ is a field extension and χ is a character of a group G, then the field of values $\mathbb{F}(\chi)$ of χ over $\mathbb{F}$ is obtained by adjoining to $\mathbb{F}$ all the values of χ. The conjecture appeared in this latter form in [Tur08a] (also including local Schur indices).

The Galois–McKay conjecture has been proved for p-solvable groups in [Tur08b] and for alternating groups in [Nat09] and [BN18]. It has been established for groups with cyclic Sylow p-subgroups in [Nav04], and for groups of Lie type in defining characteristic in [Ruh17]. For sporadic groups, it can now be easily checked with [GAP].

Also, some of its main consequences have been obtained since its formulation. For instance, in [NTT07] it was proven that, for p odd, $\mathbf{N}_G(P) = P$ if and only if G has no non-trivial p-rational valued irreducible character of p'-degree. More recently, for $p = 2$, it has been proved in [SF18] that $\mathbf{N}_G(P) = P$ if and only if all the odd-degree irreducible characters of G are fixed by $\sigma_0 \in \mathcal{H}$, where σ_0 squares odd roots of unity and fixes 2-power roots of unity. Some other consequences, such as determining the exponent of P/P' from the character table, have been treated recently in [NT19]. In particular, we now know that the character table determines the exponent of the abelianization of a Sylow 2-subgroup thanks to [NT19] and [Mal19]. In all these papers, ad hoc reductions to simple groups have been provided for fixed elements $\sigma \in \mathcal{H}$, and then the classification of finite simple groups has been used to prove the theorems. However, the Galois–McKay conjecture has eluded a general reduction until now. The following is the main result of this paper. We recall that a simple group S is **involved** in G if $S \cong K/N$ for some $N \lhd K \leqslant G$.

Theorem A. *Suppose that G is a finite group, and p is a prime. If all simple groups involved in G satisfy the inductive Galois–McKay condition for p (Definition* 3.1*), then the Galois–McKay conjecture holds for G and p.*

One of the main differences between our reduction theorem and the reduction theorem for the McKay conjecture is that we cannot make use of the general theory of character triples and character triple isomorphisms, since these do not preserve in general fields of values. We remedy this by introducing the notion of $\mathcal{H}$-triples in Section 1. There we also introduce a partial order relation between $\mathcal{H}$-triples that allows us to construct $\mathcal{H}$-equivariant bijections between character sets. The original partial order relation between character triples that we now generalize and whose use is crucial in this work was introduced in [NS14]. Here we mostly refer to the exposition given in [Nav18]. In Section 2 we study how to construct new ordered $\mathcal{H}$-triple pairs from old ones. In Section 3 we give the inductive Galois–McKay condition that we expect all finite simple groups to satisfy. Finally in Section 4, we prove Theorem A relying on key results due to Friedrich Ladisch and Alexandre Turull. We care to remark that our Theorem A does not provide a different proof of the p-solvable case of the Galois–McKay conjecture as our method depends on the study of the character theory over Glauberman correspondents that has been carried out by A. Turull in different papers.

The verification of the inductive Galois–McKay condition for finite simple groups brings up a new challenge, as it requires a vast knowledge of the character values of decorated simple groups and the interplay between Galois action and the action of group automorphisms on characters, a subject that is still not fully understood. Examples of families of simple groups satisfying the inductive Galois–McKay condition will appear in [Spä20].

1. $\mathcal{H}$-triples

Let G be a finite group, let $N \lhd G$, and let $\theta \in \mathrm{Irr}(N)$. We denote by $\mathrm{Irr}(G|\theta)$ the set of $\chi \in \mathrm{Irr}(G)$ such that θ is an irreducible constituent of the restriction χ_N. If θ is G-invariant, then it is said that (G, N, θ) is a **character triple**. The aim of this section is to extend the theory of ordering character triples developed in [NS14] by taking into account the action of Galois automorphisms on characters.

Let $\mathcal{G} = \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$, where $\mathbb{Q}^{\mathrm{ab}}$ is the field generated by all roots of unity in $\mathbb{C}$. Recall that $\mathcal{G}$ is abelian. By Brauer's theorem on splitting fields [Bra45], every character of a finite group can be afforded by a representation with entries in $\mathbb{Q}^{\mathrm{ab}}$. Also, the group $\mathcal{G}$ acts on the irreducible characters of every finite group. If $\sigma \in \mathcal{G}$, and N and $\theta \in \mathrm{Irr}(N)$ are as before, we denote by θ^σ the irreducible character of N given by $\theta^\sigma(n) = \theta(n)^\sigma = \sigma(\theta(n))$ for every $n \in N$.

Let p be a prime which is fixed but arbitrary. Let $\mathcal{H}$ be the subgroup of $\mathcal{G}$ consisting of the $\sigma \in \mathcal{G}$ for which there exists an integer f such that $\sigma(\xi) = \xi^{p^f}$ for every root of unity ξ of order not divisible by p. For every non-negative integer n, the restriction of the automorphisms in $\mathcal{H}$ to $\mathbb{Q}(\xi_n)$ yields a group $\mathcal{H}_n \leqslant \mathrm{Gal}(\mathbb{Q}(\xi_n)/\mathbb{Q})$ which is isomorphic to $\mathrm{Gal}(\mathbb{Q}_p(\xi_n)/\mathbb{Q}_p)$, where ξ_n is a primitive nth root of unity and $\mathbb{Q}_p$ is the field of p-adic numbers.

We denote by $\theta^{\mathcal{H}}$ the $\mathcal{H}$-orbit of θ and by $\mathrm{Irr}(G|\theta^{\mathcal{H}})$ the set of irreducible characters of G which lie over some $\mathcal{H}$-conjugate of θ. This set is

$$\bigcup_{\sigma \in \mathcal{H}} \mathrm{Irr}(G|\theta^\sigma)\,.$$

If $\chi \in \mathrm{Irr}(G|\theta^{\mathcal{H}})$, then we call the natural number $\chi(1)/\theta(1)$ the **character degree ratio** of χ (with respect to $\theta^{\mathcal{H}}$).

We denote by $G_{\theta^{\mathcal{H}}}$ the stabilizer in G of the set $\theta^{\mathcal{H}}$ with respect to the action of G on $\mathrm{Irr}(N)$ by conjugation. Note that $G_{\theta^{\mathcal{H}}} = \{g \in G \mid \theta^g = \theta^\sigma \text{ for some } \sigma \in \mathcal{H}\}$. We write $(G, N, \theta)_{\mathcal{H}}$ if $G_{\theta^{\mathcal{H}}} = G$; in other words, if

$$\{\theta^g \mid g \in G\} \subseteq \{\theta^\sigma \mid \sigma \in \mathcal{H}\}.$$

In this case, we call $(G, N, \theta)_{\mathcal{H}}$ an **$\mathcal{H}$-triple**. Notice that if $(G, N, \theta)_{\mathcal{H}}$ is an $\mathcal{H}$-triple, then $(G, N, \theta^\sigma)_{\mathcal{H}}$ is also an $\mathcal{H}$-triple for every $\sigma \in \mathcal{H}$. Also, note that $(G_{\theta^{\mathcal{H}}}, N, \theta)_{\mathcal{H}}$ is always an $\mathcal{H}$-triple.

Suppose that $(G, N, \theta)_{\mathcal{H}}$ is an $\mathcal{H}$-triple. Let G_θ be the stabilizer of θ in G. If $g \in G$, then there is $\sigma \in \mathcal{H}$ such that $\theta^g = \theta^\sigma$. Therefore $(G_\theta)^g = G_\theta$, and we have that $G_\theta \lhd G$. Furthermore, notice that via $gG_\theta \mapsto \sigma\mathcal{H}_\theta$ we obtain an injective homomorphism $G/G_\theta \to \mathcal{H}/\mathcal{H}_\theta$. We will denote by $\mathcal{H}_{G,\theta}$ the subgroup of $\mathcal{H}$ such that $\mathcal{H}_{G,\theta}/\mathcal{H}_\theta$ is the image under the above monomorphism. We will write just $\mathcal{H}_G$ whenever θ is clear from the context.

We start with the following result on projective representations. For background on these, see Chapter 11 of [Isa06] or Section 10.4 of [Nav18]. It is a version of the main result of [Rey65]. Recall that two (projective) representations $\mathcal{P}$ and $\mathcal{P}_1$ of G are similar if there exists an invertible complex matrix M such that $\mathcal{P}(g) = M^{-1}\mathcal{P}_1(g)M$ for every $g \in G$. Notice that two similar projective representations have the same factor set.

Theorem 1.1. *Suppose that $\mathcal{Q}$ is a projective representation of a finite group G whose factor set α only takes roots of unity values and satisfies $\alpha(1,1) = 1$. Then there is a similar representation $\mathcal{Q}'$ with entries in a finite cyclotomic extension of $\mathbb{Q}$.*

Proof. As in Theorem 5.6 of [Nav18], let Z be a finite subgroup of $\mathbb{C}^\times$ containing all values of α. Define $\widehat{G} = \{(g,z) \mid g \in G,\ z \in Z\}$ with multiplication given by

$$(g_1, z_1)(g_2, z_2) = (g_1g_2, \alpha(g_1, g_2)z_1z_2).$$

The product above is associative as α is a factor set. Since $\alpha(g,1) = \alpha(1,1) = \alpha(1,g) = 1$ for every $g \in G$ (by Lemma 11.5 of [Isa06]), we get that $\alpha(g,g^{-1}) = \alpha(g^{-1},g)$, and that $\hat{G}$ is a finite group. Define $\widehat{\mathcal{Q}}((g,z)) = z\mathcal{Q}(g)$ for every $(g,z) \in \widehat{G}$. Then $\widehat{\mathcal{Q}}$ is an ordinary representation of $\widehat{G}$. By Brauer's theorem (Theorem 10.3 of [Isa06] or [Bra45]), there exists a representation $\widehat{\mathcal{D}}$ of $\widehat{G}$ similar to $\widehat{\mathcal{Q}}$ with matrix entries in some finite cyclotomic extension of $\mathbb{Q}$. Then we easily check that $\mathcal{Q}'(g) = \widehat{\mathcal{D}}(g,1)$ is a projective representation of G similar to $\mathcal{Q}$ and such that the values of $\mathcal{Q}'$ lie in some finite cyclotomic extension of $\mathbb{Q}$. ∎

Given a character triple (G, N, θ), one can find a projective representation $\mathcal{P}$ of G **associated** with θ in the sense of Definition 5.2 of [Nav18]. Since $\mathcal{P}|_N$ is an ordinary representation affording θ, notice that $\alpha(1,1) = 1$.

Corollary 1.2. *If (G, N, θ) is a character triple, then there is a projective representation $\mathcal{P}$ of G associated with θ with entries in $\mathbb{Q}^{\mathrm{ab}}$ and whose factor set only takes roots of unity values. If $\mathcal{P}$ is any such representation, then $\mathcal{P}(g)$ has finite order for every $g \in G$.*

Proof. By Theorem 5.5 of [Nav18], there exists a projective representation $\mathcal{P}'$ of G associated with (G, N, θ) such that the factor set α only takes roots of unity values. Since α is the factor set of $\mathcal{P}'$, we have that $\alpha(1,1) = 1$. By Theorem 1.1, let $\mathcal{P}$ be a similar projective representation of G with values in $\mathbb{Q}^{ab}$. Since $\mathcal{P}$ and $\mathcal{P}'$ have the same factor set, it easily follows that $\mathcal{P}$ is a projective representation associated with θ satisfying the required properties.

Finally, if $\mathcal{P}$ is any such representation, let Z be a finite subgroup of $\mathbb{C}^\times$ containing all values of α. Define $\widehat{G} = \{(g,z) \mid g \in G,\ z \in Z\}$ with multiplication given by

$$(g_1, z_1)(g_2, z_2) = (g_1g_2, \alpha(g_1, g_2)z_1z_2)$$

as in the proof of Theorem 1.1. Then $\widehat{\mathcal{P}}$ defined as $\widehat{\mathcal{P}}(g,z) = z\mathcal{P}(g)$ for every $(g,z) \in \widehat{G}$ is an ordinary representation of $\widehat{G}$. The last statement follows since $\widehat{\mathcal{P}}(g,z)$ has finite order for every $(g,z) \in \widehat{G}$. ∎

Remark 1.3. Recall that if (G, N, θ) is a character triple, then two projective representations $\mathcal{P}$ and $\mathcal{P}_1$ of G are associated with θ if and only if there is a function $\mu\colon G \to \mathbb{C}^\times$ constant on cosets of N, with $\mu(1) = 1$ and a complex invertible matrix M such that $\mathcal{P}_1(g) = \mu(g)M^{-1}\mathcal{P}(g)M$ for all $g \in G$ (this is Lemma 10.10(b) of [Nav18]). Notice that μ is uniquely determined by the pair $(\mathcal{P}, \mathcal{P}_1)$. Indeed, if $\mathcal{P}_1(g) = \mu_1(g)M_1^{-1}\mathcal{P}(g)M_1$ for all $g \in G$ and for some function μ_1 with $\mu_1(n) = 1$ for all $n \in N$, then we will have that $M^{-1}\mathcal{P}(n)M = M_1^{-1}\mathcal{P}(n)M_1$ for all $n \in N$. By Schur's lemma, we have that $M_1 = \lambda M$ for some $\lambda \in \mathbb{C}^\times$. Hence $\mu(g)M^{-1}\mathcal{P}(g)M = \mu_1(g)M^{-1}\mathcal{P}(g)M$ for all $g \in G$, and thus $\mu(g) = \mu_1(g)$ using that $M^{-1}\mathcal{P}(g)M$ is non-zero.

Let $\mathcal{P}$ be a projective representation of G with factor set α. If $f\colon G \to G_1$ is a group isomorphism (we will use exponential notation for images of f), then we define $\mathcal{P}^f(g_1) = \mathcal{P}(g_1^{f^{-1}})$ for $g_1 \in G_1$. This is a projective representation of G_1 with factor set α^f, where $\alpha^f(x^f, y^f) = \alpha(x, y)$ for $x, y \in G$. (See Theorem 10.9 of [Nav18].) If $\sigma \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$ and $\mathcal{P}$ has entries in $\mathbb{Q}^{\mathrm{ab}}$, then $\mathcal{P}^\sigma(g) := \mathcal{P}(g)^\sigma$ (where σ is applied entrywise) defines a projective representation of G with factor set $\alpha^\sigma(x, y) = \alpha(x, y)^\sigma$ for $x, y \in G$. Notice that $\alpha(x, y) \in \mathbb{Q}^{\mathrm{ab}}$ in this case.

In what follows, we shall use that if $N \lhd G$, then $G \times \mathcal{G}$ acts naturally on $\mathrm{Irr}(N)$. Indeed, if $\theta \in \mathrm{Irr}(N)$, $g \in G$ and $\sigma \in \mathcal{G}$, then $\theta^{g\sigma}$ is the irreducible character of N given by $\theta^{g\sigma}(n) = \theta(gng^{-1})^\sigma$ for $n \in N$. (To simplify notation, we often use $g\sigma$ instead of (g, σ).)

Throughout this work, we will use $\sim$ to denote similarity of (projective) representations. In order to avoid too much notation, we also employ $\sim$ to denote matrix similarity. We hope that which meaning is in use should be clear from the context.

Lemma 1.4. *Suppose that $N \lhd G$, $\theta \in \mathrm{Irr}(N)$, and assume that $\theta^{g\sigma} = \theta$ for some $g \in G$ and $\sigma \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$. Let $\mathcal{P}$ be a projective representation of G_θ associated with θ with values in $\mathbb{Q}^{\mathrm{ab}}$ and factor set α. Then*

$$\mathcal{P}^{g\sigma}(x) = \mathcal{P}(gxg^{-1})^\sigma,$$

where we apply σ entrywise, defines a projective representation of G_θ associated with θ, with factor set $\alpha^{g\sigma}(x, y) = \alpha^g(x, y)^\sigma$. In particular, there is a unique function

$$\mu_{g\sigma}\colon G_\theta \to \mathbb{C}^\times$$

with $\mu_{g\sigma}(1) = 1$, constant on cosets of N such that $\mathcal{P}^{g\sigma} \sim \mu_{g\sigma}\mathcal{P}$.

Proof. Notice that g normalizes G_θ. The rest is straightforward using Remark 1.3. □

We are now ready to define a partial order relation between $\mathcal{H}$-triples.

Definition 1.5. Suppose that $(G, N, \theta)_\mathcal{H}$ and $(H, M, \varphi)_\mathcal{H}$ are $\mathcal{H}$-triples. We write $(G, N, \theta)_\mathcal{H} \geqslant_c (H, M, \varphi)_\mathcal{H}$ if the following conditions hold:

(i) $G = NH$, $N \cap H = M$, $\mathbf{C}_G(N) \subseteq H$.
(ii) $(H \times \mathcal{H})_\theta = (H \times \mathcal{H})_\varphi$. In particular, $H_\theta = H_\varphi$.

(iii) There are projective representations $\mathcal{P}$ of G_θ and $\mathcal{P}'$ of H_φ associated with θ and φ with entries in $\mathbb{Q}^{\mathrm{ab}}$ with factor sets α and α', respectively, such that α and α' take roots of unity values, $\alpha_{H_\theta \times H_\theta} = \alpha'_{H_\theta \times H_\theta}$, and for $c \in \mathbf{C}_G(N)$, the scalar matrices $\mathcal{P}(c)$ and $\mathcal{P}'(c)$ are associated with the same scalar ζ_c.

(iv) For every $a \in (H \times \mathcal{H})_\theta$, the functions μ_a and μ'_a given by Lemma 1.4 agree on H_θ.

In (iii), notice that if $c \in \mathbf{C}_G(N)$, then $c \in H_\theta$, and $\mathcal{P}(c)$ and $\mathcal{P}'(c)$ are scalar matrices by Schur's Lemma (applied to the irreducible representations $\mathcal{P}_N$ and $\mathcal{P}'_M$).

In the situation described above we say that $(\mathcal{P}, \mathcal{P}')$ **gives**

$$(G, N, \theta)_{\mathcal{H}} \geqslant_c (H, M, \varphi)_{\mathcal{H}} .$$

Note that if $(\mathcal{P}, \mathcal{P}')$ gives $(G, N, \theta)_{\mathcal{H}} \geqslant_c (H, M, \varphi)_{\mathcal{H}}$ as above, then $(\mathcal{P}, \mathcal{P}')$ is associated with $(G_\theta, N, \theta) \geqslant_c (H_\varphi, M, \varphi)$ in the sense of Definition 10.14 of [Nav18].

The following technical result will be useful at the end of Section 2.

Lemma 1.6. *Assume that $(\mathcal{P}, \mathcal{P}')$ gives $(G, N, \theta)_{\mathcal{H}} \geqslant_c (H, M, N)_{\mathcal{H}}$. Let $U \leqslant \mathbb{C}^\times$ be the subgroup of roots of unity of $\mathbb{C}$. If $\epsilon\colon G_\theta \to U$ is any map constant on N-cosets and such that $\epsilon(1) = 1$, then $(\epsilon\mathcal{P}, \epsilon_{H_\theta}\mathcal{P}')$ also gives $(G, N, \theta)_{\mathcal{H}} \geqslant_c (H, M, \varphi)_{\mathcal{H}}$.*

Proof. Conditions (i) and (ii) of Definition 1.5 are satisfied. Write $\hat{\mathcal{P}} = \epsilon\mathcal{P}$ and $\hat{\mathcal{P}}' = \epsilon_{H_\theta}\mathcal{P}'$; we will show that $(\hat{\mathcal{P}}, \hat{\mathcal{P}}')$ gives $(G, N, \theta)_{\mathcal{H}} \geqslant_c (H, M, \varphi)_{\mathcal{H}}$.

Note that $\hat{\mathcal{P}}$ is a projective representation with entries in $\mathbb{Q}^{\mathrm{ab}}$ associated with θ. If $\nu\colon G_\theta \to U$ is any arbitrary function, we can define $\delta(\nu)\colon G_\theta \times G_\theta \to U$ by

$$\delta(\nu)(x, y) = \nu(x)\nu(y)\nu(xy)^{-1}$$

so that $\delta(\nu)$ is a factor set. It is routine to check that the factor set of $\widehat{\mathcal{P}}$ is $\beta = \delta(\epsilon)\alpha$. Also, $\hat{\mathcal{P}}'$ is a projective representation with values in $\mathbb{Q}^{\mathrm{ab}}$ associated with φ, and with factor set $\beta' = \delta(\epsilon_{H_\theta})\alpha' = \beta_{H_\theta \times H_\theta}$. For every $c \in \mathbf{C}_G(N)$, the matrices $\hat{\mathcal{P}}'(c)$ and $\hat{\mathcal{P}}'(c)$ correspond to the same scalar $\epsilon(c)\zeta_c$, where $\mathcal{P}(c)$ and $\mathcal{P}'(c)$ correspond to the scalar ζ_c. Hence $(\hat{\mathcal{P}}, \hat{\mathcal{P}}')$ satisfies condition (iii) of Definition 1.5. Whenever $(h, \sigma) \in (H \times \mathcal{H})_\theta$, it is straightforward to check that

$$\hat{\mathcal{P}}^{h\sigma} \sim \hat{\mu}_{h\sigma}\hat{\mathcal{P}} \quad \text{and} \quad (\hat{\mathcal{P}}')^{h\sigma} \sim \hat{\mu}'_{h\sigma}\hat{\mathcal{P}}' ,$$

where $\hat{\mu}_{h\sigma} = \mu_{h\sigma}\frac{\epsilon^{h\sigma}}{\epsilon}$, $\hat{\mu}'_{h\sigma} = \mu'_{h\sigma}(\frac{\epsilon^{h\sigma}}{\epsilon})_{H_\theta}$, and the functions $\mu_{h\sigma}$ and $\mu'_{h\sigma}$ given by Lemma 1.4 satisfy

$$\mathcal{P}^{h\sigma} \sim \mu_{h\sigma}\mathcal{P} \quad \text{and} \quad (\mathcal{P}')^{h\sigma} \sim \mu'_{h\sigma}\mathcal{P}' .$$

In particular $\hat{\mu}'_{h\sigma} = (\hat{\mu}_{h\sigma})_{H_\theta}$, and thus the pair $(\hat{\mathcal{P}}, \hat{\mathcal{P}}')$ satisfies all the conditions in Definition 1.5. ☐

The following technical lemma allows us to show that in order to check (iv) of Definition 1.5 on $(H \times \mathcal{H})_\theta$ it is enough to check the condition on a transversal of H_θ in $(H \times \mathcal{H})_\theta$.

Lemma 1.7. *Suppose that $(G, N, \theta)_{\mathcal{H}}$ is an $\mathcal{H}$-triple. Let $\mathcal{P}$ be a projective representation of G_θ associated with θ with entries in $\mathbb{Q}^{\mathrm{ab}}$ and whose factor set α takes roots of unity values. Then the following hold:*

(a) *Let $g \in G_\theta$. Then $\mathcal{P}^g(y) = \mu_g(y)M\mathcal{P}(y)M^{-1}$ for all $y \in G_\theta$, where*

$$\mu_g(y) = \frac{\alpha(g, g^{-1})}{\alpha(g, yg^{-1})\alpha(y, g^{-1})}$$

and $M = \mathcal{P}(g)$. In particular, μ_g has values in $\mathbb{Q}^{\mathrm{ab}}\backslash\{0\}$.

(b) *Let $(g, \sigma) \in (G \times \mathcal{H})_\theta$ and suppose that we write $g = tx$, where $t \in G_\theta$ and $x \in G$. Then $\theta^{x\sigma} = \theta$, $(G_\theta)^x = G_\theta$ and*

$$\mu_{g\sigma} = \mu_t^{x\sigma}\mu_{x\sigma}\,,$$

where $\mu_t^{x\sigma}(y) = \mu_t(xyx^{-1})^\sigma$ for every $y \in G_\theta$, and the functions μ_t, $\mu_{x\sigma}$ and $\mu_{g\sigma}$ are given by Lemma 1.4.

Proof. Part (a) follows directly from the definitions of $\mathcal{P}^g$, of a projective representation, the uniqueness in Remark 1.3 and Lemma 1.4 applied to $\sigma = 1$. Given $(g, \sigma) \in (G \times \mathcal{H})_\theta$, suppose that we write $g = tx$ for some $t \in G_\theta$ and $x \in G$. Note that $\theta^{x\sigma} = \theta$. In particular, x normalizes G_θ. By Lemma 1.4 we have that $\mathcal{P}^t = \mu_t M^{-1}\mathcal{P}\mathcal{M}$ and $\mathcal{P}^{g\sigma} = \mu_{g\sigma}X^{-1}\mathcal{P}\mathcal{X}$. Notice that since $\mathcal{P}$ has entries in $\mathbb{Q}^{\mathrm{ab}}$, then $M = \mathcal{P}(g)$ has entries in $\mathbb{Q}^{\mathrm{ab}}$. Then for every $y \in G_\theta$ we have that

$$\begin{aligned}\mathcal{P}^{g\sigma}(y) &= \mathcal{P}(gyg^{-1})^\sigma\\ &= \mathcal{P}(txyx^{-1}t^{-1})^\sigma\\ &= (\mathcal{P}^t(xyx^{-1}))^\sigma\\ &\sim \mu_t(xyx^{-1})^\sigma\mathcal{P}(xyx^{-1})^\sigma\\ &= \mu_t^{x\sigma}(y)\mathcal{P}^{x\sigma}(y)\\ &\sim \mu_t^{x\sigma}(y)\mu_{x\sigma}(y)\mathcal{P}(y),\end{aligned}$$

being the conjugating matrix $M^\sigma X$ which does not depend on $y \in G_\theta$. Hence $\mathcal{P}^{g\sigma} \sim \mu_t^{x\sigma}\mu_{x\sigma}\mathcal{P}$ as claimed. □

We will often use the following fact in Section 2.

Lemma 1.8. *Suppose that $(G, N, \theta)_{\mathcal{H}}$ and $(H, M, \varphi)_{\mathcal{H}}$ are $\mathcal{H}$-triples satisfying the conditions* (i) *and* (ii) *in Definition* 1.5*. Write $A = (H \times \mathcal{H})_\theta$. Suppose that $\mathcal{P}$ and $\mathcal{P}'$ are projective representations satisfying* (iii) *from Definition* 1.5*. Then condition* (iv) *of Definition* 1.5 *holds for every $a \in A$ if and only if it holds for a complete set of representatives of H_θ-cosets in A.*

Proof. Note that $H_\theta \lhd A$ since $\theta^{h\sigma} = \theta$ implies that h normalizes H_θ. The direct implication trivially holds. Assume that (iv) of Definition 1.5 holds for a complete set of representatives $\mathbb{T}$ of the H_θ-cosets in A. Given $a \in A$, write $a = hx\sigma$ for $h \in H_\theta$ and $x\sigma \in \mathbb{T}$. By Lemma 1.7(b) we have that

$$\mu_a = \mu_h^{x\sigma}\mu_{x\sigma} \;\text{ and }\; \mu'_a = (\mu'_h)^{x\sigma}\mu'_{x\sigma}.$$

By assumption $\mu'_{x\sigma}$ is the restriction of $\mu_{x\sigma}$. By Lemma 1.7(a) we have that μ_h depends only on the factor set α of $\mathcal{P}$ and μ'_h depends only on the factor set α' of

$\mathcal{P}'$. Since $\mathcal{P}$ and $\mathcal{P}'$ satisfy condition (iii) of Definition 1.5, we have that α' is the restriction of α. Hence μ'_a is the restriction of μ_a as wanted. □

Let $(\mathcal{P}, \mathcal{P}')$ be associated with $(G, N, \theta)_{\mathcal{H}} \geqslant_c (H, M, \varphi)_{\mathcal{H}}$. Since $(\mathcal{P}, \mathcal{P}')$ is associated with $(G_\theta, N, \theta) \geqslant_c (H_\varphi, M, \varphi)$ as in Definition 10.14 of [Nav18], we have defined character bijections via $(\mathcal{P}, \mathcal{P}')$

$$\tau_J \colon \mathrm{Irr}(J|\theta) \to \mathrm{Irr}(J \cap H|\varphi)$$

whenever $N \subseteq J \leqslant G_\theta$ as in Theorem 10.13 of [Nav18]. These bijections preserve ratios of character degrees.

Lemma 1.9. *Suppose that $(\mathcal{P}, \mathcal{P}')$ is associated with $(G, N, \theta)_{\mathcal{H}} \geqslant_c (H, M, \varphi)_{\mathcal{H}}$.*

(a) *For every $N \subseteq J \leqslant G_\theta$, let $\tau_J \colon \mathrm{Irr}(J|\theta) \to \mathrm{Irr}(J \cap H|\varphi)$ be the bijective map defined via $(\mathcal{P}, \mathcal{P}')$. If $(h, \sigma) \in (H \times \mathcal{H})_\theta$, then $J^h \subseteq G_\theta$ and*

$$\tau_{J^h}(\chi^{h\sigma}) = \tau_J(\chi)^{h\sigma}$$

for every $\chi \in \mathrm{Irr}(J|\theta)$.

(b) *Let $\mathcal{H}_G/\mathcal{H}_\theta \leqslant \mathcal{H}/\mathcal{H}_\theta$ be the image of G/G_θ in $\mathcal{H}/\mathcal{H}_\theta$ under the natural monomorphism. If for $\chi \in \mathrm{Irr}(G|\theta)$, we define $\tau(\chi) = (\tau_{G_\theta}(\psi))^H$, where $\psi \in \mathrm{Irr}(G_\theta)$ is the Clifford correspondent of χ lying over θ, then the map $\tau \colon \mathrm{Irr}(G|\theta) \to \mathrm{Irr}(H|\varphi)$ is an $\mathcal{H}_G$-equivariant bijection preserving ratios of character degrees.*

Proof. We have that $(\mathcal{P}, \mathcal{P}')$ is associated with $(G_\theta, N, \theta) \geqslant_c (H_\varphi, M, \varphi)$ in the sense of Definition 10.14 of [Nav18]. For every $N \subseteq J \leqslant G_\theta$, we have defined character bijections

$$\tau_J \colon \mathrm{Irr}(J|\theta) \to \mathrm{Irr}(J \cap H|\varphi).$$

Given $\chi \in \mathrm{Irr}(J|\theta)$, recall that χ is the trace of a representation of the form $\mathcal{Q} \otimes \mathcal{P}_J$, where $\mathcal{Q}$ is an irreducible projective representation of J/N, with factor set $\beta = (\alpha^{-1})_{J \times J}$, that can be chosen with matrix entries in some finite cyclotomic extension of $\mathbb{Q}$ (by Theorem 1.1). Then $\tau_J(\chi)$ is the character afforded by $\mathcal{Q}_{J \cap H} \otimes \mathcal{P}'_{J \cap H}$.

Let $a = (h, \sigma) \in A = (H \times \mathcal{H})_\theta$. Then $\theta^a = \theta$ and also $\varphi^a = \varphi$, as $A = (H \times \mathcal{H})_\varphi$. Hence $\chi^a \in \mathrm{Irr}(J^h|\theta)$. The character χ^a of J^h is afforded by

$$(\mathcal{Q} \otimes \mathcal{P}_J)^a = \mathcal{Q}^a \otimes (\mathcal{P}_J)^a = \mathcal{Q}^a \otimes (\mathcal{P}^a)_{J^h} \sim \mathcal{Q}^a \otimes (\mu_a)_{J^h} \mathcal{P}_{J^h} = (\mu_a)_{J^h} \mathcal{Q}^a \otimes \mathcal{P}_{J^h}\,.$$

This implies that $(\mu_a)_{J^h} \mathcal{Q}^a$ is a projective representation of J^h/N with factor set $(\alpha^{-1})_{J^h \times J^h}$. By definition, we have that $\tau_J(\chi^a)$ is afforded by

$$(\mu_a \mathcal{Q}^a)_{J^h \cap H} \otimes \mathcal{P}'_{J^h \cap H} = (\mathcal{Q}^a)_{J^h \cap H} \otimes (\mu'_a \mathcal{P}')_{J^h \cap H} \sim (\mathcal{Q}_{J \cap H})^a \otimes (\mathcal{P}'_{J \cap H})^a\,.$$

Just notice that $(\mathcal{Q}_{J \cap H})^a \otimes (\mathcal{P}'_{J \cap H})^a = (\mathcal{Q}_{J \cap H} \otimes \mathcal{P}'_{J \cap H})^a$ affords $\tau_J(\chi)^a$.

We next prove the second statement. If $\sigma \in \mathcal{H}_G$, then let $g \in H$ be such that $\theta^{g\sigma} = \theta$ (there exists such $g \in H$ by the definition of $\mathcal{H}_G$, and using the fact that $G = G_\theta H$). Since $(g, \sigma) \in (H \times \mathcal{H})_\theta = (H \times \mathcal{H})_\varphi$, we have $\varphi^{g\sigma} = \varphi$. Consequently $\chi^\sigma \in \mathrm{Irr}(G|\theta)$ and $\tau(\chi)^\sigma \in \mathrm{Irr}(H|\varphi)$. Since $\psi^{g\sigma}$ is the Clifford correspondent of χ^σ, we have that

$$\tau(\chi^\sigma) = \tau_{G_\theta}(\psi^{g\sigma})^H = (\tau_{G_\theta}(\psi)^{g\sigma})^H = (\tau_{G_\theta}(\psi)^H)^\sigma = \tau(\chi)^\sigma\,,$$

where $\tau_{G_\theta}(\psi^{g\sigma}) = \tau_{G_\theta}(\psi)^{g\sigma}$ by the first part of this proof, so τ is $\mathcal{H}_G$-equivariant. Notice that our map preserves ratios of character degrees because character triple isomorphisms do, and it is a bijection since τ_{G_θ} and the Clifford correspondence are bijections. □

Ordered pairs of $\mathcal{H}$-triples yield $\mathcal{H}$-equivariant bijections between related character sets.

Theorem 1.10. *Suppose that $(G, N, \theta)_{\mathcal{H}} \geqslant_c (H, M, \varphi)_{\mathcal{H}}$. Then there is an $\mathcal{H}$-equivariant bijection $\mathrm{Irr}(G|\theta^{\mathcal{H}}) \to \mathrm{Irr}(H|\varphi^{\mathcal{H}})$ that preserves ratios of character degrees.*

Proof. By the definition of $\mathcal{H}$-character triples, we have that G acts on $\theta^{\mathcal{H}}$. Suppose that $\theta_1, \dots, \theta_s$ are representatives of the G-orbits, where say $\theta_1 = \theta$ and $\theta_i = \theta^{\sigma_i}$. Notice that $G_{\theta_i} = G_\theta$. Also notice that if we set $\varphi_i = \varphi^{\sigma_i}$, then $\varphi_1, \dots, \varphi_s$ are representatives of the H-orbits on $\varphi^{\mathcal{H}}$ (using that $(H \times \mathcal{H})_\theta = (H \times \mathcal{H})_\varphi$). If $(\mathcal{P}, \mathcal{P}')$ is associated with $(G, N, \theta)_{\mathcal{H}} \geqslant_c (H, M, \varphi)_{\mathcal{H}}$, then $(\mathcal{P}^{\sigma_i}, (\mathcal{P}')^{\sigma_i})$ is associated with $(G, N, \theta_i)_{\mathcal{H}} \geqslant_c (H, M, \varphi_i)_{\mathcal{H}}$. We have that

$$\mathrm{Irr}(G|\theta^{\mathcal{H}}) = \dot{\bigcup_i} \mathrm{Irr}(G|\theta_i) \quad \text{and} \quad \mathrm{Irr}(H|\varphi^{\mathcal{H}}) = \dot{\bigcup_i} \mathrm{Irr}(H|\varphi_i).$$

Note that if we write τ_i for the bijection $\mathrm{Irr}(G|\theta_i) \to \mathrm{Irr}(H|\varphi_i)$ given by Lemma 1.9, then $\tau_i \circ \sigma_i = \sigma_i \circ \tau_1$. Let $\mathcal{H}_G$ be as in Lemma 1.9. Recall $\mathcal{H}_G/\mathcal{H}_\theta \leqslant \mathcal{H}/\mathcal{H}_\theta$ is isomorphic to G/G_θ. Note that $\mathcal{H}_G$ is the stabilizer in $\mathcal{H}$ of any of the G-orbits on $\theta^{\mathcal{H}}$ (this is because $\mathcal{H}$ is abelian). Hence $\mathcal{H} = \dot{\bigcup}_i \mathcal{H}_G \sigma_i$. Let $\tau \colon \mathrm{Irr}(G|\theta^{\mathcal{H}}) \to \mathrm{Irr}(H|\varphi^{\mathcal{H}})$ be the bijection defined in the obvious way from the bijections τ_i. Given $\chi \in \mathrm{Irr}(G|\theta_1)$ and $\sigma = \omega\sigma_i \in \mathcal{H}$ with $\omega \in \mathcal{H}_G$, we have that

$$\tau(\chi^\sigma) = \tau_i(\chi^{\omega\sigma_i}) = \tau_i(\chi^{\sigma_i})^\omega = (\tau_1(\chi))^{\sigma_i\omega} = \tau(\chi)^\sigma,$$

where we use that the bijections τ_i are $\mathcal{H}_G$-equivariant and $\tau_i \circ \sigma_i = \sigma_i \circ \tau_1$. It easily follows that τ is $\mathcal{H}$-equivariant. As each τ_i preserves character degree ratios, then so does τ. □

Let $N \lhd G$ and $\theta \in \mathrm{Irr}(N)$ not necessarily satisfying $G_{\theta^{\mathcal{H}}} = G$. By the Clifford correspondence, induction of characters defines a bijection

$$\mathrm{Irr}(X|\theta^\sigma) \to \mathrm{Irr}(G|\theta^\sigma)$$

for every $\sigma \in \mathcal{H}$, whenever $G_\theta \subseteq X \leqslant G$. Hence induction of characters defines an $\mathcal{H}$-equivariant surjective map

$$\mathrm{Irr}(X|\theta^{\mathcal{H}}) \to \mathrm{Irr}(G|\theta^{\mathcal{H}}),$$

which turns out to be injective if and only if $G_{\theta^{\mathcal{H}}} \subseteq X$.

Corollary 1.11. *Let $N \lhd G$ and $H \leqslant G$ be such that $G = NH$. Write $M = N \cap H$. Suppose that $(G_{\theta^{\mathcal{H}}}, N, \theta)_{\mathcal{H}} \geqslant_c (H_{\varphi^{\mathcal{H}}}, M, \varphi)_{\mathcal{H}}$. Then there is an $\mathcal{H}$-equivariant bijection*

$$\mathrm{Irr}(G|\theta^{\mathcal{H}}) \to \mathrm{Irr}(H|\varphi^{\mathcal{H}})$$

that preserves ratios of character degrees.

Proof. Let $\tau \colon \mathrm{Irr}(G_{\theta^{\mathcal{H}}}|\theta^{\mathcal{H}}) \to \mathrm{Irr}(H_{\varphi^{\mathcal{H}}}|\varphi^{\mathcal{H}})$ be the $\mathcal{H}$-equivariant bijection preserving ratios of character degrees given by Theorem 1.10. Define $\hat{\tau} \colon \mathrm{Irr}(G|\theta^{\mathcal{H}}) \to \mathrm{Irr}(H|\varphi^{\mathcal{H}})$ in the following way. For $\chi \in \mathrm{Irr}(G|\theta^{\mathcal{H}})$, let $\psi \in \mathrm{Irr}(G_{\theta^{\mathcal{H}}}|\theta^{\mathcal{H}})$ be such that $\psi^G = \chi$; then $\hat{\tau}(\chi) := \tau(\psi)^H$. The conclusion then follows from the comments preceding this result. □

2. Constructing new $\mathcal{H}$-triples from old ones

The following results show easy ways to construct $\mathcal{H}$-triples from given ones. The proofs are straightforward from the definitions. Note that if $(G,N,\theta)_{\mathcal{H}} \geqslant_c (H,M,\varphi)_{\mathcal{H}}$ and $N \subseteq J \leqslant G$, then

$$(J,N,\theta)_{\mathcal{H}} \geqslant_c (J \cap H, M, \varphi)_{\mathcal{H}}.$$

Lemma 2.1. *Let $(G,N,\theta)_{\mathcal{H}} \geqslant_c (H,M,\varphi)_{\mathcal{H}}$. Suppose that $f\colon G \to \hat{G}$ is a group isomorphism. Then $(\hat{G},\hat{N},\theta^f)_{\mathcal{H}} \geqslant_c (\hat{H},\hat{M},\varphi^f)_{\mathcal{H}}$, where we write $\hat{J} = J^f$ (using exponential notation for images of f) and $\psi^f(x^f) = \psi(x)$ for every $\psi \in \mathrm{Irr}(J)$ and $x \in J$.*

Sometimes it will be easier to apply the following weaker version of the above result.

Lemma 2.2. *Let $N \lhd G$ and $H \leqslant G$ be such that $G = NH$. Write $M = H \cap N$. Suppose that $N \subseteq K \leqslant G$ and $(K,N,\theta)_{\mathcal{H}} \geqslant_c (K \cap H, M, \varphi)_{\mathcal{H}}$. Then for every $h \in H$*

$$(K^h,N,\theta^h)_{\mathcal{H}} \geqslant_c (K^h \cap H, M, \varphi^h)_{\mathcal{H}}.$$

Under the hypotheses of the above lemma, assume that $(\mathcal{P},\mathcal{P}')$ gives $(K,N,\theta)_{\mathcal{H}} \geqslant_c (K \cap H, M, \varphi)_{\mathcal{H}}$ and let $h \in H$. Then we will consider that

$$(K^h,N,\theta^h)_{\mathcal{H}} \geqslant_c (K^h \cap H, M, \varphi^h)_{\mathcal{H}}$$

is given by $(\mathcal{P}^h,(\mathcal{P}')^h)$. Hence, if τ_θ and τ_{θ^h} are the bijections given by Theorem 1.10, we will have that $\tau_{\theta^h}(\chi^h) = \tau_\theta(\chi)^h$ for every $\chi \in \mathrm{Irr}(G|\theta^{\mathcal{H}})$. In particular, if $\hat{\tau}_\theta$ and $\hat{\tau}_{\theta^h}$ are the bijections given by Corollary 1.11, then $\hat{\tau}_\theta = \hat{\tau}_{\theta^h}$.

Lemma 2.3. *Let $(G,N,\theta)_{\mathcal{H}} \geqslant_c (H,M,\varphi)_{\mathcal{H}}$ and $\sigma \in \mathcal{H}$. Then $(G,N,\theta^\sigma)_{\mathcal{H}} \geqslant_c (H,M,\varphi^\sigma)_{\mathcal{H}}$.*

Suppose that $(\mathcal{P},\mathcal{P}')$ gives $(G,N,\theta)_{\mathcal{H}} \geqslant_c (H,M,\varphi)_{\mathcal{H}}$. For every $\sigma \in \mathcal{H}$, we will always consider that $(G,N,\theta^\sigma)_{\mathcal{H}} \geqslant_c (H,M,\varphi^\sigma)_{\mathcal{H}}$ is given by $(\mathcal{P}^\sigma,(\mathcal{P}')^\sigma)$. In this way, if τ_θ and τ_{θ^σ} are the bijections given by Theorem 1.10, then by construction $\tau_{\theta^\sigma}(\chi^\sigma) = \tau_\theta(\chi)^\sigma$ for every $\chi \in \mathrm{Irr}(G|\theta^{\mathcal{H}})$. By Theorem 1.10 τ_θ is $\mathcal{H}$-equivariant, in particular $\tau_{\theta^\sigma} = \tau_\theta$. Hence, by construction the bijections given by Corollary 1.11 are also equal.

Lemma 2.4. *Let $(G,N,\theta)_{\mathcal{H}} \geqslant_c (H,M,\varphi)_{\mathcal{H}}$. Suppose that $L \lhd G$ is contained in $\ker(\theta) \cap \ker(\varphi) \cap \mathbf{C}_G(N)$, and $\mathbf{C}_{G/L}(N/L) = \mathbf{C}_G(N)/L$. Then*

$$(G/L,N/L,\theta)_{\mathcal{H}} \geqslant_c (H/L,M/L,\varphi)_{\mathcal{H}},$$

where θ and φ are considered as characters of N/L and M/L.

Lemma 2.5. *Let $(G_i,N_i,\theta_i)_{\mathcal{H}} \geqslant_c (H_i,M_i,\varphi_i)_{\mathcal{H}}$ for $i = 1,2$. Then*

$$(G_{\theta^{\mathcal{H}}},N,\theta)_{\mathcal{H}} \geqslant_c (H_{\varphi^{\mathcal{H}}},M,\varphi)_{\mathcal{H}},$$

where $G = G_1 \times G_2$, $H = H_1 \times H_2$, $N = N_1 \times N_2$, $M = M_1 \times M_2$, $\theta = \theta_1 \times \theta_2$ and $\varphi = \varphi_1 \times \varphi_2$.

Proof. The group-theoretical conditions are easily checked. Also, it is easy to check that $(H_{\theta^{\mathcal{H}}} \times \mathcal{H})_\theta = (H_{\varphi^{\mathcal{H}}} \times \mathcal{H})_\varphi$. Now we have to construct appropriate projective representations of $(G_{\theta^{\mathcal{H}}})_\theta = G_\theta = (G_1)_{\theta_1} \times (G_2)_{\theta_2}$ and $(H_{\varphi^{\mathcal{H}}})_\varphi = H_\varphi = (H_1)_{\varphi_1} \times (H_2)_{\varphi_2}$. This is done as in Lemma 10.20 of [Nav18]. Checking conditions (ii), (iii) and (iv) of Definition 1.5 is straightforward. $\square$

Denote by S_m the symmetric group acting on m letters. In the next two results we deal with $\mathcal{H}$-triples and wreath products of groups. We follow the notation in Chapter 10 of [Nav18]. If G is a finite group, then G^m will denote the direct product $G \times \cdots \times G$ (m times), and if $\theta \in \mathrm{Irr}(G)$, then in our context $\theta^m = \theta \times \cdots \times \theta \in \mathrm{Irr}(G^m)$. Recall that S_m acts naturally on G^m by

$$(g_1, \ldots, g_m)^\omega = (g_{\omega^{-1}(1)}, \ldots, g_{\omega^{-1}(m)})$$

whenever $g_i \in G$ and $\omega \in \mathsf{S}_m$.

Lemma 2.6. *Let $(G, N, \theta)_\mathcal{H}$ and $(H, M, \varphi)_\mathcal{H}$ be $\mathcal{H}$-triples such that $(G, N, \theta)_\mathcal{H} \geqslant_c (H, M, \varphi)_\mathcal{H}$. For any $m \in \mathbb{Z}_{>0}$*

$$((G_\theta \wr \mathsf{S}_m)\Delta^m G, N^m, \theta^m)_\mathcal{H} \geqslant_c ((H_\theta \wr \mathsf{S}_m)\Delta^m H, M^m, \varphi^m)_\mathcal{H},$$

where $\Delta^m \colon G \to G^m$ denotes the diagonal embedding of G into the direct product G^m.

Proof. We claim that $(G^m)_{(\theta^m)^\mathcal{H}} = (G_\theta)^m \Delta^m G$. Let $(g_1, \ldots, g_m) \in (G^m)_{(\theta^m)^\mathcal{H}}$. Then there exists some $\sigma \in \mathcal{H}$ such that $(\theta^m)^{(g_1,\ldots,g_m)} = (\theta^m)^\sigma$. Hence $\theta^{g_i} = \theta^\sigma$ for each i. In particular, $G_\theta g_i = G_\theta g_j$ for every i and j. Hence, we can write $g_i = x_i g_1$ for some $x_i \in G_\theta$ and for every i. Thus $(g_1, \ldots, g_m) \in (G_\theta)^m \Delta^m G$. The other inclusion is also clear using that given $g \in G$, there is $\sigma \in \mathcal{H}$ such that $\theta^g = \theta^\sigma$, by our hypothesis. This also implies that

$$(G \wr \mathsf{S}_m)_{(\theta^m)^\mathcal{H}} = (G_\theta \wr \mathsf{S}_m)\Delta^m G\,.$$

Similarly, $(H \wr \mathsf{S}_m)_{(\varphi^m)^\mathcal{H}} = (H_\varphi \wr \mathsf{S}_m)\Delta^m H = (H_\theta \wr \mathsf{S}_m)\Delta^m H$ as $H_\theta = H_\varphi$ by hypothesis.

We follow the proof of Theorem 10.21 of [Nav18]. First, we easily check that $(G \wr \mathsf{S}_m)_{\theta^m} = G_\theta \wr \mathsf{S}_m$. Conditions (i) and (ii) of Definition 1.5 for

$$((G_\theta \wr \mathsf{S}_m)\Delta^m G, N^m, \theta^m)_\mathcal{H} \geqslant_c ((H_\theta \wr \mathsf{S}_m)\Delta^m H, M^m, \varphi^m)_\mathcal{H}$$

follow from the above discussion together with the discussions in Theorem 10.21 of [Nav18]. Let $(\mathcal{P}, \mathcal{P}')$ be associated with $(G, N, \theta)_\mathcal{H} \geqslant_c (H, M, \varphi)_\mathcal{H}$. Construct projective representations $\tilde{\mathcal{P}}$ and $\tilde{\mathcal{P}}'$ of $G_\theta \wr \mathsf{S}_m$ and of $H_\theta \wr \mathsf{S}_m$ as in Theorem 10.21 of [Nav18]. Condition (iii) of Definition 1.5 is proven in Theorem 10.21 of [Nav18]. It remains to check condition (iv) of Definition 1.5.

For any $(\gamma, \sigma) \in ((H \wr \mathsf{S}_m) \times \mathcal{H})_{\theta^m}$, we have that $\gamma \in (H_\theta \wr \mathsf{S}_m)\Delta^m H$. We denote by $\tilde{\mu}_{\gamma\sigma}$ and $\tilde{\mu}'_{\gamma\sigma}$ the functions given by Lemma 1.4 with respect to the action of $\gamma\sigma$ on $\tilde{\mathcal{P}}$ and $\tilde{\mathcal{P}}'$. By Lemma 1.8 we only need to check the condition for a transversal of $H_\theta \wr \mathsf{S}_m$ in $((H_\theta \wr \mathsf{S}_m)\Delta^m H \times \mathcal{H})_{\theta^m}$. In particular, it is enough to check the condition for elements (γ, σ) such that $\gamma = (y, \ldots, y) = \Delta^m y$ for some $y \in H$ with $\theta^{y\sigma} = \theta$.

We check below that, for every $x_i \in H_\theta$ and $\omega \in \mathsf{S}_m$,

$$\tilde{\mu}_{\gamma\sigma}((x_1, \ldots, x_m)\omega) = \prod_{i=1}^m \mu_{y\sigma}(x_i).$$

First let $\mathcal{X}_{\theta(1)}$ be the permutation representation as in Theorem 10.21 of [Nav18]. We use that $\mathcal{P}^{y\sigma} = \mu_{y\sigma} M \mathcal{P} M^{-1}$ and

$$(M \otimes \cdots \otimes M)\mathcal{X}_{\theta(1)}(\omega) = \mathcal{X}_{\theta(1)}(\omega)(M \otimes \cdots \otimes M).$$

Given $(x_1, \dots, x_m)\omega \in H_\theta \wr \mathsf{S}_m$ we have that

$$\begin{aligned}\tilde{\mathcal{P}}^{\gamma\sigma}((x_1,\dots,x_m)\omega) &= \tilde{\mathcal{P}}((x_1^{y^{-1}},\dots,x_m^{y^{-1}})\omega)^\sigma \\ &= (\mathcal{P}^{y\sigma}(x_1)\otimes\cdots\otimes\mathcal{P}^{y\sigma}(x_m))\mathcal{X}_{\theta(1)}(\omega) \\ &\sim \prod_{i=1}^m \mu_{y\sigma}(x_i)\tilde{\mathcal{P}}((x_1,\dots,x_m)\omega);\end{aligned}$$

the conjugating matrix $M\otimes\cdots\otimes M$ does not depend on $(x_1,\dots,x_m)\omega \in H_\theta \wr \mathsf{S}_m$.

Similarly one can check that for every $(x_1,\dots,x_m)\omega \in H_\theta \wr \mathsf{S}_m$

$$(\tilde{\mu'}_{\gamma\sigma})((x_1,\dots,x_m)\omega) = \prod_{i=1}^m (\mu'_{y\sigma})(x_i).$$

Since $\mu'_{y\sigma}$ is the restriction of $\mu_{y\sigma}$ the proof is finished. □

The following is a special feature of $\mathcal{H}$-triples with respect to wreath products.

Theorem 2.7. *Let $(G,N,\theta)_{\mathcal{H}}$ and $(H,M,\varphi)_{\mathcal{H}}$ be $\mathcal{H}$-triples such that $(G,N,\theta)_{\mathcal{H}} \geqslant_c (H,M,\varphi)_{\mathcal{H}}$. Let $k,m \in \mathbb{Z}_{>0}$. Let $\sigma_i \in \mathcal{H}$ for $i=1,\dots,k$. Write $\theta_i = \theta^{\sigma_i}$ and $\varphi_i = \varphi^{\sigma_i}$. Suppose that θ_i and θ_j are not G-conjugate whenever $i \neq j$. Then for $n = mk$*

$$((G\wr\mathsf{S}_n)_{\tilde{\theta}^{\mathcal{H}}}, N^n, \tilde{\theta})_{\mathcal{H}} \geqslant_c ((H\wr\mathsf{S}_n)_{\tilde{\varphi}^{\mathcal{H}}}, M^n, \tilde{\varphi})_{\mathcal{H}},$$

where $\tilde{\theta} = \theta_1^m \times\cdots\times\theta_k^m$ and $\tilde{\varphi} = \varphi_1^m\times\cdots\times\varphi_k^m$.

Proof. The statement makes sense since $G\wr\mathsf{S}_n = N^n(H\wr\mathsf{S}_n)$, $N^n \cap (H\wr\mathsf{S}_n) = M^n$ and $\mathbf{C}_{G\wr\mathsf{S}_n}(N^n) \subseteq \mathbf{C}_G(N)^n \subseteq H^n \subseteq H\wr\mathsf{S}_n$. Moreover, we see next that $(H\wr\mathsf{S}_n\times\mathcal{H})_{\tilde{\theta}} = (H\wr\mathsf{S}_n\times\mathcal{H})_{\tilde{\varphi}}$. Write $\tilde{\theta} = \beta_1\times\cdots\times\beta_n$ and $\tilde{\varphi} = \xi_1\times\cdots\times\xi_n$. We know that each β_i is θ^{τ_i} for some $\tau_i \in \mathcal{H}$ and then $\xi_i = \varphi^{\tau_i}$. Let $a \in (H\wr\mathsf{S}_n\times\mathcal{H})_{\tilde{\theta}}$. Hence $a = (\gamma,\tau)$, where $\gamma = (a_1,\dots,a_n)\omega \in H\wr\mathsf{S}_n$ and $\tau\in\mathcal{H}$. The equality $\tilde{\theta}^a = \tilde{\theta}$ implies that $\beta_{\omega^{-1}(i)}^{a_{\omega^{-1}(i)}\tau} = \beta_i$ for every $i=1,\dots,n$. This is exactly the same as

$$\theta^{a_j\tau_j\tau\tau_{\omega(j)}^{-1}} = \theta$$

for every $j=1,\dots,n$. Write $c_j = a_j\tau_j\tau\tau_{\omega(j)}^{-1} \in (H\times\mathcal{H})_\theta = (H\times\mathcal{H})_\varphi$. Then $\varphi^{c_j} = \varphi$ for every $j=1,\dots,n$ implies $\tilde{\varphi}^a = \tilde{\varphi}$. The above discussion shows that conditions (i) and (ii) of Definition 1.5 are satisfied by the $\mathcal{H}$-triples $((G\wr\mathsf{S}_n)_{\tilde{\theta}^{\mathcal{H}}}, N^n, \tilde{\theta})_{\mathcal{H}}$ and $((H\wr\mathsf{S}_n)_{\tilde{\varphi}^{\mathcal{H}}}, M^n, \tilde{\varphi})_{\mathcal{H}}$.

Next we explain how to construct projective representations giving the relation between the aforementioned $\mathcal{H}$-triples. Let $(\mathcal{P},\mathcal{P}')$ be associated with $(G,N,\theta)_{\mathcal{H}} \geqslant_c (H,M,\varphi)_{\mathcal{H}}$. As in Lemma 2.6 we can construct projective representations $\tilde{\mathcal{P}}$ and $\tilde{\mathcal{P}}'$ associated with

$$((G_\theta\wr\mathsf{S}_m)\Delta^m G, N^m, \psi)_{\mathcal{H}} \geqslant_c ((H_\theta\wr\mathsf{S}_m)\Delta^m H, M^m, \xi)_{\mathcal{H}}, \tag{1}$$

where $\psi = \theta^m$ and $\xi = \varphi^m$. Write $\tilde{\mathcal{P}}_i = (\tilde{\mathcal{P}})^{\sigma_i}$ and $\tilde{\mathcal{P}}'_i = (\tilde{\mathcal{P}}')^{\sigma_i}$ for $i=1,\dots,k$. In particular, each $(\tilde{\mathcal{P}}_i, \tilde{\mathcal{P}}'_i)$ gives

$$(G_\theta\wr\mathsf{S}_m, N^m, \psi_i) \geqslant_c (H_\theta\wr\mathsf{S}_m, M^m, \xi_i),$$

where $\psi_i = \psi^{\sigma_i} = \theta_i^m$ and $\xi_i = \xi^{\sigma_i} = \varphi_i^m$. The pair $(\tilde{\mathcal{P}},\tilde{\mathcal{P}}')$ of tensor product representations

$$\tilde{\mathcal{P}} = \tilde{\mathcal{P}}_1\otimes\cdots\otimes\tilde{\mathcal{P}}_k \quad\text{and}\quad \tilde{\mathcal{P}}' = \tilde{\mathcal{P}}'_1\otimes\cdots\otimes\tilde{\mathcal{P}}'_k$$

gives

$$((G_\theta \wr \mathsf{S}_m)^k, N^n, \tilde{\theta}) \geqslant_c ((H_\theta \wr \mathsf{S}_m)^k, M^n, \tilde{\varphi}).$$

Notice that $(G_\theta \wr \mathsf{S}_m)^k = (G \wr \mathsf{S}_n)_{\tilde{\theta}}$ and $(H_\theta \wr \mathsf{S}_m)^k = (H \wr \mathsf{S}_n)_{\tilde{\varphi}}$. This is because θ_i and θ_j are not G-conjugate whenever $i \neq j$. Notice that we have constructed $\tilde{\mathcal{P}}$ and $\tilde{\mathcal{P}}'$ as in Definition 1.5(iii).

It only remains to check condition (iv) of Definition 1.5. As before write $\tilde{\theta} = \beta_1 \times \cdots \times \beta_n$. Note that $\beta_i = \theta_j$ whenever $i \in \Lambda_j = \{(j-1)m+1, \ldots, jm\}$ for $j = 1, \ldots, k$.

Let $a \in (H \wr \mathsf{S}_n \times \mathcal{H})_{\tilde{\theta}}$. Hence $a = (\gamma, \tau)$, where $\gamma = (a_1, \ldots, a_n)\omega \in (H \wr \mathsf{S}_n)_{\tilde{\theta}^{\mathcal{H}}}$ and $\tau \in \mathcal{H}$. The equality $\tilde{\theta}^a = \tilde{\theta}$ is equivalent to

$$\beta_{\omega^{-1}(i)}^{a_{\omega^{-1}(i)}\tau} = \beta_i$$

for every $i = 1, \ldots, n$. In particular

$$\beta_{\omega^{-1}(i)}^{a_{\omega^{-1}(i)}\tau} = \theta_j \quad \text{whenever} \quad i \in \Lambda_j.$$

Hence $\omega^{-1}(\Lambda_j) = \Lambda_l$ for some $l \in \{1, \ldots, k\}$, and $\theta^{\sigma_l a_i \tau} = \theta_i^{a_i \tau} = \theta_j = \theta^{\sigma_j}$ for every $i \in \Lambda_l$. In particular $\sigma_l \tau \sigma_j^{-1} \in \mathcal{H}_{H,\theta}$. Fix for each $l \in \{1, \ldots, k\}$ an element $c_l \in H$ such that $\theta^{c_l} = \theta^{\sigma_l \tau \sigma_j^{-1}}$. Hence $a_i = a_i' c_l$ for some $a_i' \in H_\theta$ for every $i \in \Lambda_l$ and for $l = 1, \ldots, k$. Write $b_l = \Delta^m c_l$ for each l.

Define $\pi \in \mathsf{S}_n$ by $\pi((l-1)m + i) = (j-1)m + i$ if $\omega(\Lambda_l) = \Lambda_j$ for every $i = 1, \ldots, m$. Hence $\pi(l) = j$ if $\omega(\Lambda_l) = \Lambda_j$, and in this way we can view $\pi \in \mathsf{S}_k$. For $j = 1, \ldots, k$, define $\pi_j \in \mathsf{S}_n$ by $\pi_j|_{\Lambda_j} = \omega\pi^{-1}|_{\Lambda_j}$ and fixing $\{1, \ldots, n\} \backslash \Lambda_j$. By definition $\omega = \pi_1 \cdots \pi_k \pi$.

Then $\gamma = x\gamma'$, where $x = (a_1', \ldots, a_n')\pi_1 \cdots \pi_k \in (H_\theta \wr \mathsf{S}_m)^k$ and $\gamma' = (b_1, \ldots, b_k)\pi \in (H \wr \mathsf{S}_n)_{\tilde{\theta}^{\mathcal{H}}}$ (this is because $(b_1, \ldots, b_k)$ and $\pi_1 \cdots \pi_k$ commute). By Lemma 1.8, in order to verify condition (iv) of Definition 1.5 for γ we may assume that $\gamma = \gamma'$.

With the above assumptions and notation, we have that $\psi^{b_l \sigma_l \tau \sigma_{\pi(l)}^{-1}} = \psi$ for all l. Note that $\gamma^{-1} = (b_{\pi^{-1}(1)}^{-1}, \ldots, b_{\pi^{-1}(k)}^{-1})\pi^{-1}$. Let $(y_1, \ldots, y_k) \in (H_\theta \wr \mathsf{S}_m)^k$. Then we have

$$\begin{aligned}
&(\tilde{\mathcal{P}}_1 \otimes \cdots \otimes \tilde{\mathcal{P}}_k)^{\gamma\tau}(y_1, \ldots, y_k) \\
&\quad = \tilde{\mathcal{P}}^{\sigma_1 \tau} \otimes \cdots \otimes \tilde{\mathcal{P}}^{\sigma_k \tau}(y_{\pi(1)}^{b_1^{-1}}, \ldots, y_{\pi(k)}^{b_k^{-1}}) \\
&\quad = \tilde{\mathcal{P}}^{\sigma_1 \tau b_1}(y_{\pi(1)}) \otimes \cdots \otimes \tilde{\mathcal{P}}^{\sigma_k \tau b_k}(y_{\pi(k)}) \\
&\quad = (\tilde{\mathcal{P}}^{b_1 \sigma_1 \tau \sigma_{\pi(1)}^{-1}}(y_{\pi(1)}))^{\sigma_{\pi(1)}} \otimes \cdots \otimes (\tilde{\mathcal{P}}^{b_k \sigma_k \tau \sigma_{\pi(k)}^{-1}}(y_{\pi(k)}))^{\sigma_{\pi(k)}}.
\end{aligned}$$

Write $\tau_l = \sigma_l \tau \sigma_{\pi(l)}^{-1}$. For each $l \in \{1, \ldots, k\}$, we have that $\psi^{b_l \tau_l} = \psi$. By Lemma 1.4, we have functions $\tilde{\mu}_{b_l \tau_l}$ and invertible matrices M_l such that

$$(\tilde{\mathcal{P}})^{b_l \tau_l} = \tilde{\mu}_{b_l \tau_l} M_l^{-1} \tilde{\mathcal{P}} M_l \,. \tag{2}$$

Write

$$\tilde{\mu}_{\gamma\tau}(y_1, \ldots, y_k) := \prod_{l=1}^{k} \tilde{\mu}_{b_l \tau_l}(y_{\pi(l)})^{\sigma_{\pi(l)}}.$$

Then for $(y_1, \dots, y_k) \in (H_\theta \wr \mathsf{S}_m)^k$ we have

$$\begin{aligned}(\tilde{\mathcal{P}}^{\sigma_1} \otimes \cdots \otimes \tilde{\mathcal{P}}^{\sigma_k})^{\gamma\tau}(y_1, \dots, y_k) \\ \sim \tilde{\mu}_{\gamma\tau}(y_1, \dots, y_k)\tilde{\mathcal{P}}^{\sigma_{\pi(1)}}(y_{\pi(1)}) \otimes \cdots \otimes \tilde{\mathcal{P}}^{\sigma_{\pi(k)}}(y_{\pi(k)}) \\ \sim \tilde{\mu}_{\gamma\tau}(y_1, \dots, y_k)\tilde{\mathcal{P}}^{\sigma_1} \otimes \cdots \otimes \tilde{\mathcal{P}}^{\sigma_k}(y_1, \dots, y_k)\,,\end{aligned}$$

where the first similarity relation is given by $M = M_1 \otimes \cdots \otimes M_k$, and the second similarity relation is obtained by conjugating by the matrix $\mathcal{X}(\pi)$ associated with the action of π on the tensors

$$v_1 \otimes \cdots \otimes v_k \mapsto v_{\pi^{-1}(1)} \otimes \cdots \otimes v_{\pi^{-1}(k)}.$$

Since the matrices M and $\mathcal{X}(\pi)$ do not depend on $(y_1, \dots, y_k) \in (H_\theta \wr \mathsf{S}_m)^k$, we have that $(\tilde{\mathcal{P}}^{\sigma_1} \otimes \cdots \otimes \tilde{\mathcal{P}}^{\sigma_k})^{\gamma\tau} \sim \tilde{\mu}_{\gamma\tau}\tilde{\mathcal{P}}^{\sigma_1} \otimes \cdots \otimes \tilde{\mathcal{P}}^{\sigma_k}$.

We have analogous relations for $\tilde{\mathcal{P}}'_1 \otimes \cdots \otimes \tilde{\mathcal{P}}'_k$ with

$$\tilde{\mu'}_{\gamma\tau}(y_1, \dots, y_k) := \prod_{l=1}^{k} \tilde{\mu'}_{b_l\tau_l}(y_{\pi(l)})^{\sigma_{\pi(l)}}.$$

By equation (1) (in the second paragraph of this proof) each $\tilde{\mu'}_{\tilde{b}_j\tau_j}$ is the restriction of $\tilde{\mu}_{\tilde{b}_j\tau_j}$, and hence the result follows. □

We will need to control the character theory and $\mathcal{H}$-action over some characters of central products. Suppose that K is the product of two subgroups N and Z with $N \lhd K$ and $Z \leqslant \mathbf{C}_K(N)$. Then K is the central product of N and Z. In this case

$$\mathrm{Irr}(K) = \dot{\bigcup_{\nu \in \mathrm{Irr}(Z \cap N)}} \mathrm{Irr}(K|\nu),$$

where $\mathrm{Irr}(K|\nu) = \{\theta \cdot \lambda \mid \theta \in \mathrm{Irr}(N|\nu) \text{ and } \lambda \in \mathrm{Irr}(Z|\nu)\}$. Note that, whenever a group A acts by automorphisms on K, stabilizing N and Z, if $a \in A$ and $\theta \cdot \lambda \in \mathrm{Irr}(K)$, then $(\theta \cdot \lambda)^a = \theta \cdot \lambda$ if and only if $\theta^a = \theta$ and $\lambda^a = \lambda$. The same happens if $A \leqslant \mathcal{G} = \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$.

Theorem 2.8. *Let $(G, N, \theta)_\mathcal{H}$ and $(H, M, \varphi)_\mathcal{H}$ be $\mathcal{H}$-triples such that*

$$(G, N, \theta)_\mathcal{H} \geqslant_c (H, M, \varphi)_\mathcal{H}\,.$$

Suppose that $Z \lhd G$ is abelian and satisfies $Z \subseteq \mathbf{C}_G(N)$. Let $\nu \in \mathrm{Irr}(Z \cap N)$ be under θ and $\lambda \in \mathrm{Irr}(Z|\nu)$. Then

$$(G_{(\theta\cdot\lambda)^\mathcal{H}}, NZ, \theta \cdot \lambda)_\mathcal{H} \geqslant_c (H_{(\varphi\cdot\lambda)^\mathcal{H}}, MZ, \varphi \cdot \lambda)_\mathcal{H}.$$

In particular, there exists an $\mathcal{H}$-equivariant bijection

$$\tau_\theta \colon \mathrm{Irr}(G_{(\theta\cdot\lambda)^\mathcal{H}}|(\theta \cdot \lambda)^\mathcal{H}) \to \mathrm{Irr}(H_{(\varphi\cdot\lambda)^\mathcal{H}}|(\varphi \cdot \lambda)^\mathcal{H})$$

that preserves ratios of character degrees.

Proof. Note that $Z \subseteq \mathbf{C}_G(N) \subseteq H$. Hence $Z \cap N = Z \cap M$. Since $(G, N, \theta)_\mathcal{H} \geqslant_c (H, M, \varphi)_\mathcal{H}$, we have that φ lies over ν.

Note that $(\theta \cdot \lambda)^{g\sigma} = \theta \cdot \lambda$ if and only if $\theta^{g\sigma} = \theta$ and $\lambda^{g\sigma} = \lambda$ for $g \in G$ and $\sigma \in \mathcal{H}$. Hence, $G_{(\theta\cdot\lambda)^\mathcal{H}} \cap H = H_{(\varphi\cdot\lambda)^\mathcal{H}}$, and Definition 1.5(i) is easily checked. Since $(H \times \mathcal{H})_\theta = (H \times \mathcal{H})_\varphi$, we have that $(H_{(\varphi\cdot\lambda)^\mathcal{H}} \times \mathcal{H})_\theta = (H_{(\varphi\cdot\lambda)^\mathcal{H}} \times \mathcal{H})_\varphi$, so Definition 1.5(ii) holds.

Since $G_{(\theta\cdot\lambda)} = G_\theta \cap G_\lambda \subseteq G_\theta$ and $H_{\theta\cdot\lambda} = G_{\theta\cdot\lambda} = H_{\varphi\cdot\lambda}$, if $(\mathcal{P}, \mathcal{P}')$ gives

$$(G, N, \theta)_{\mathcal{H}} \geqslant_c (H, M, \varphi)_{\mathcal{H}},$$

then $(\mathcal{P}_{G_{\theta\cdot\lambda}}, \mathcal{P}'_{H_{\theta\cdot\lambda}})$ gives

$$(G_{(\theta\cdot\lambda)^{\mathcal{H}}}, N, \theta)_{\mathcal{H}} \geqslant_c (H_{(\theta\cdot\lambda)^{\mathcal{H}}}, M, \varphi)_{\mathcal{H}}.$$

To ease the notation we assume $G = G_{(\theta\cdot\lambda)^{\mathcal{H}}}$ and write $\mathcal{P}$ for $\mathcal{P}_{G_{\theta\cdot\lambda}}$ and $\mathcal{P}'$ for $\mathcal{P}'_{H_{\theta\cdot\lambda}}$. In particular, $(G, NZ, \theta\cdot\lambda)_{\mathcal{H}}$ and $(H, MZ, \varphi\cdot\lambda)_{\mathcal{H}}$ are $\mathcal{H}$-triples.

We next show how to construct a pair of projective representations giving

$$(G, NZ, \theta\cdot\lambda)_{\mathcal{H}} \geqslant_c (H, MZ, \varphi\cdot\lambda)_{\mathcal{H}}$$

from $(\mathcal{P}, \mathcal{P}')$.

Let $\widehat{\mathcal{P}}$ be a projective representation of $G_{\theta\cdot\lambda}$ associated with $\theta\cdot\lambda$ with entries in $\mathbb{Q}^{\mathrm{ab}}$, as in Corollary 1.2. In particular, $\widehat{\mathcal{P}}$ is associated with $(G_{\theta\cdot\lambda}, N, \theta)$ and by Remark 1.3, $\widehat{\mathcal{P}} \sim \epsilon\mathcal{P}$, where $\epsilon\colon G_\theta \to \mathbb{C}^\times$ is constant on N-cosets and satisfies $\epsilon(1) = 1$. The values of ϵ are roots of unity since $\mathcal{P}(g)$ and $\widehat{\mathcal{P}}(g)$ have finite order for every $g \in G_{\theta\cdot\lambda}$. As in the proof of Lemma 1.7, define

$$\delta(\epsilon)(x, y) = \epsilon(x)\epsilon(y)\epsilon(xy)^{-1}$$

for every $x, y \in G_{\theta\cdot\lambda}$. Then $\hat{\alpha} = \delta(\epsilon)\alpha$. In particular, the values of $\delta(\epsilon)$ are roots of unity. Define $\mathcal{Q} = \epsilon\mathcal{P}$ and $\mathcal{Q}' = \epsilon_{H_{\varphi\cdot\lambda}}\mathcal{P}'$. The proof of Lemma 1.6 applies and the pair $(\mathcal{Q}, \mathcal{Q}')$ gives $(G, N, \theta)_{\mathcal{H}} \geqslant_c (H, M, \varphi)_{\mathcal{H}}$. Note that $\mathcal{Q}$ is associated with $\theta\cdot\lambda$ because $\mathcal{Q} \sim \widehat{\mathcal{P}}$. One can check that $\mathcal{Q}'$ is associated with $\varphi\cdot\lambda$. In particular, $(\mathcal{Q}, \mathcal{Q}')$ gives

$$(G, NZ, \theta\cdot\lambda)_{\mathcal{H}} \geqslant_c (H, MZ, \varphi\cdot\lambda)_{\mathcal{H}}\,.$$

To finish the proof apply Theorem 1.10. □

The following is an $\mathcal{H}$-triple version of the most important result concerning the application of the theory of centrally isomorphic character triples to reduction theorems: the ordering of two $\mathcal{H}$-triples *only* depends on the automorphisms of the normal subgroup defined via conjugation by the overgroup; see the *butterfly* theorem (Theorem 5.3 in [Spä14]).

Theorem 2.9. *Let $(G, N, \theta)_{\mathcal{H}}$ and $(H, M, \varphi)_{\mathcal{H}}$ be $\mathcal{H}$-triples such that $(G, N, \theta)_{\mathcal{H}} \geqslant_c (H, M, \varphi)_{\mathcal{H}}$. Let $\epsilon\colon G \to \mathrm{Aut}(N)$ be the group homomorphism defined by conjugation by G. Suppose that $N \lhd \hat{G}$ and $\hat{\epsilon}(\hat{G}) = \epsilon(G)$, where $\hat{\epsilon}\colon \hat{G} \to \mathrm{Aut}(N)$ is given by conjugation by $\hat{G}$. Let $N\mathbf{C}_{\hat{G}}(N) \subseteq \hat{H} \leqslant \hat{G}$ be such that $\hat{\epsilon}(\hat{H}) = \epsilon(H)$. Then*

$$(\hat{G}, N, \theta)_{\mathcal{H}} \geqslant_c (\hat{H}, M, \varphi)_{\mathcal{H}}.$$

Proof. Note that $\mathbf{C}_G(N) \subseteq H$ and $\mathbf{C}_{\hat{G}}(N) \subseteq \hat{H}$, so the group theory conditions in Definition 1.5(i) are satisfied by Theorem 10.18 of [Nav18].

Recall that the map $\bar{\epsilon}\colon G/\mathbf{C}_G(N) \to \hat{G}/\mathbf{C}_{\hat{G}}(N)$ given by $\bar{\epsilon}(\mathbf{C}_G(N)x) = \mathbf{C}_{\hat{G}}(N)y$ whenever $\epsilon(x) = \hat{\epsilon}(y)$ defines a group isomorphism. Let $x \in G$ and $y \in \hat{G}$. If $\epsilon(x) = \hat{\epsilon}(y)$, notice that $\theta^x = \theta^\sigma$ for some $\sigma \in \mathcal{H}$ if and only if $\theta^y = \theta^\sigma$, so condition (ii) of Definition 1.5 also holds.

Following Theorem 10.18 of [Nav18] and given a tranversal $\mathbb{T}$ of $M\mathbf{C}_G(N)$ in H_φ with $1 \in \mathbb{T}$, we can define a transversal $\widehat{\mathbb{T}}$ of $M\mathbf{C}_{\hat{G}}(N)$ in $\hat{H}_\varphi$ with $\hat{1} = 1$.

Suppose that $(\mathcal{P}, \mathcal{P}')$ gives $(G, N, \theta)_{\mathcal{H}} \geqslant_c (H, M, \varphi)_{\mathcal{H}}$ and let $\lambda\colon \mathbf{C}_G(N) \to \mathbb{Q}^{\mathrm{ab}}\backslash\{0\}$ be given by the scalar associated with $\mathcal{P}$ and $\mathcal{P}'$ for every $c \in \mathbf{C}_G(N)$.

Choose a function $\hat{\lambda}\colon \mathbf{C}_{\hat{G}}(N) \to \mathbb{Q}^{\mathrm{ab}}\backslash\{0\}$ semimultiplicative with respect to $\mathbf{Z}(N)$ as in Theorem 10.18 of [Nav18]. Also following Theorem 10.18 of [Nav18], we can construct projective representations $\hat{\mathcal{P}}$ of $\hat{G}_\theta$ and $\hat{\mathcal{P}}'$ of $\hat{H}_\varphi$ with respect to $\widehat{\mathbb{T}}$, $\hat{\lambda}$ and $\mathcal{P}$ and $\mathcal{P}'$, respectively. Then $\hat{\mathcal{P}}$ is associated with θ, $\hat{\mathcal{P}}'$ is associated with φ, and they satisfy Definition 1.5(iii).

It remains to check Definition 1.5(iv) for $(\hat{\mathcal{P}}, \hat{\mathcal{P}}')$. We have that H acts *by conjugation* on the transversal $\mathbb{T}$ with $t \mapsto t\cdot h$ if and only if $(t\cdot h)^{-1}t^h = m_h c_h \in M\mathbf{C}_G(N)$ for $h \in H$. Similarly $\hat{H}$ acts on $\widehat{\mathbb{T}}$. In fact, from the definition of $\widehat{\mathbb{T}}$ it follows that if $\epsilon(h) = \hat{\epsilon}(\hat{h})$ for $h \in H$, $\hat{h} \in \hat{H}$, then

$$\hat{t}\cdot\hat{h} = \widehat{t\cdot h}$$

and $(\widehat{t\cdot h})^{-1}\hat{t}^{\hat{h}} = m_h \hat{c}_{\hat{h}} \in M\mathbf{C}_{G_1}(N)$ for every $t \in \mathbb{T}$.

Let $(\hat{h}, \sigma) \in (\hat{H} \times \mathcal{H})_\theta$ so that $\theta^{h^{-1}\sigma} = \theta$ and $\mathcal{P}^{h^{-1}\sigma} \sim \mu_{h^{-1}\sigma}\mathcal{P}$ with conjugating matrix X. Let $t \in \mathbb{T}$, $m \in M$ and $\hat{c} \in \mathbf{C}_{\hat{G}}(N)$. Then

$$\begin{aligned}
\hat{\mathcal{P}}^{\hat{h}^{-1}\sigma}(\hat{t}m\hat{c}) &= \hat{\mathcal{P}}(\hat{t}^{\hat{h}}m^h(\hat{c})^{\hat{h}})^\sigma\\
&= \hat{\mathcal{P}}(\widehat{t\cdot h}m_h\hat{c}_{\hat{h}}m^h(\hat{c})^{\hat{h}})^\sigma\\
&= \mathcal{P}(t\cdot h)^\sigma\mathcal{P}(m_hm^h)^\sigma\hat{\lambda}(\hat{c}_{\hat{h}}(\hat{c})^{\hat{h}})^\sigma\\
&= \mathcal{P}(t^hm_h^{-1}c_h^{-1}m_hm^h)^\sigma\lambda(\hat{c}_{\hat{h}}(\hat{c})^{\hat{h}})^\sigma\\
&= \mathcal{P}((tm)^h)^\sigma\lambda(c_h^{-1})^\sigma\hat{\lambda}(\hat{c}_{\hat{h}}(\hat{c})^{\hat{h}})^\sigma\\
&= \mathcal{P}^{h^{-1}\sigma}(tm)\lambda(c_h^{-1})^\sigma\hat{\lambda}(\hat{c}_{\hat{h}}(\hat{c})^{\hat{h}})^\sigma\\
&\sim \mu_{h^{-1}\sigma}(tm)\lambda(c_h^{-1})^\sigma\hat{\lambda}(\hat{c}_{\hat{h}}(\hat{c})^{\hat{h}})^\sigma\mathcal{P}(tm)\hat{\lambda}(\hat{c})\hat{\lambda}(\hat{c})^{-1}\\
&= \mu_{h^{-1}\sigma}(tm)\lambda(c_h^{-1})^\sigma\hat{\lambda}(\hat{c}_{\hat{h}}(\hat{c})^{\hat{h}})^\sigma\lambda(\hat{c})^{-1}\hat{\mathcal{P}}(\hat{t}mc')\\
&= \hat{\mu}_{\hat{h}^{-1}\sigma}(\hat{t}m\hat{c})\hat{\mathcal{P}}(\hat{t}m\hat{c}),
\end{aligned}$$

where $\hat{\mu}_{\hat{h}^{-1}\sigma}(\hat{t}m\hat{c}) = \mu_{h^{-1}\sigma}(tm)\lambda(c_h^{-1})^\sigma\hat{\lambda}(\hat{c}_{\hat{h}}(\hat{c})^{\hat{h}})^\sigma\lambda(\hat{c})^{-1}$ and the conjugating matrix X does not depend on $\hat{t}m\hat{c}$. Hence $\hat{\mathcal{P}}^{\hat{h}^{-1}\sigma} \sim \hat{\mu}_{\hat{h}^{-1}\sigma}\hat{\mathcal{P}}$.

Similarly

$$(\hat{\mathcal{P}}')^{\hat{h}^{-1}\sigma}(\hat{t}m\hat{c}) \sim \hat{\mu}'_{\hat{h}^{-1}\sigma}(\hat{t}m\hat{c})\hat{\mathcal{P}}'(\hat{t}m\hat{c}),$$

where $\hat{\mu}'_{\hat{h}^{-1}\sigma}(\hat{t}m\hat{c}) = \mu'_{h^{-1}\sigma}(tm)\lambda(c_h^{-1})^\sigma\hat{\lambda}(\hat{c}_{\hat{h}}(\hat{c})^{\hat{h}})^\sigma\lambda(\hat{c})^{-1}$, so that $\hat{\mu}_{\hat{h}^{-1}\sigma}$ and $\hat{\mu}'_{\hat{h}^{-1}\sigma}$ agree on $\hat{H}_\varphi$ provided that $\mu_{h^{-1}\sigma}$ and $\mu'_{h^{-1}\sigma}$ agree on H_φ. □

3. The inductive Galois–McKay condition

We refer the reader to Appendix B of [Nav18] for a compendium of the definitions and results on the theory of universal covering groups that are specifically needed in our context.

We can now define the **inductive Galois–McKay condition** on finite non-abelian simple groups.

Definition 3.1. Let S be a finite non-abelian simple group, with p dividing $|S|$. Let X be a universal covering group of S, $R \in \mathrm{Syl}_p(X)$ and $\Gamma = \mathrm{Aut}(X)_R$. We say that S satisfies the inductive Galois–McKay condition for p if there exist some

Γ-stable proper subgroup N of X with $\mathbf{N}_X(R) \subseteq N$ and some $\Gamma \times \mathcal{H}$-equivariant bijection

$$\Omega\colon \mathrm{Irr}_{p'}(X) \to \mathrm{Irr}_{p'}(N)\,,$$

such that for every $\theta \in \mathrm{Irr}_{p'}(X)$ we have

$$(X \rtimes \Gamma_{\theta^{\mathcal{H}}}, X, \theta)_{\mathcal{H}} \geqslant_c (N \rtimes \Gamma_{\theta^{\mathcal{H}}}, N, \Omega(\theta))_{\mathcal{H}}.$$

We recall that for a quasisimple group X (a perfect group whose quotient by its center is simple) and any $n \in \mathbb{Z}_{>0}$,

$$\mathrm{Aut}(X^n) = \mathrm{Aut}(X) \wr \mathsf{S}_n\,.$$

Moreover, whenever $R \leqslant X$

$$\mathrm{Aut}(X^n)_{R^n} = \mathrm{Aut}(X)_R \wr \mathsf{S}_n\,.$$

These results appear as Lemma 10.24 of [Nav18], for example.

Theorem 3.2. *Suppose that S satisfies the inductive Galois–McKay condition for p, and X, R, Γ, N and Ω are as above. Then for any $n \in \mathbb{Z}_{>0}$ the map*

$$\tilde{\Omega}\colon \mathrm{Irr}_{p'}(X^n) \to \mathrm{Irr}_{p'}(N^n)$$

given by $\tilde{\Omega}(\tilde{\theta}) = \tilde{\varphi}$, where $\tilde{\theta} = \theta_1 \times \cdots \times \theta_n$ and $\tilde{\varphi} = \varphi_1 \times \cdots \times \varphi_n$ with $\Omega(\theta_i) = \varphi_i$, is a bijection. Write $\tilde{\Gamma} = \Gamma \wr \mathsf{S}_n$. Then $\tilde{\Omega}$ is $\tilde{\Gamma} \times \mathcal{H}$-equivariant and for every $\tilde{\theta} \in \mathrm{Irr}_{p'}(X^n)$

$$(X^n \rtimes \tilde{\Gamma}_{\tilde{\theta}^{\mathcal{H}}}, X^n, \tilde{\theta})_{\mathcal{H}} \geqslant_c (N^n \rtimes \tilde{\Gamma}_{\tilde{\theta}^{\mathcal{H}}}, N^n, \tilde{\varphi})_{\mathcal{H}}.$$

Proof. Notice that $\tilde{\Gamma} = \mathrm{Aut}(X^n)_{R^n}$. The fact that $\tilde{\Omega}$ is a $\tilde{\Gamma} \times \mathcal{H}$-equivariant bijection follows straightforwardly from the definitions. Also notice that $X^n \rtimes \tilde{\Gamma} = X^n(N^n \rtimes \tilde{\Gamma})$ and $X^n \cap (N^n \rtimes \tilde{\Gamma}) = N^n$.

Given $\tilde{\theta} \in \mathrm{Irr}_{p'}(X^n)$, we prove the statement on $\mathcal{H}$-triples in a series of steps concerning assumptions on $\tilde{\theta}$. Note that, by applying Lemma 2.2 we can replace $\tilde{\theta}$ by any $\tilde{\Gamma}$-conjugate of $\tilde{\theta}$.

If $\tilde{\theta} = \theta_1 \times \cdots \times \theta_n$, then we write $\mathcal{H}_{\Gamma_i}$ for the subgroup of $\mathcal{H}$ such that $\mathcal{H}_{\Gamma_i}/\mathcal{H}_{\theta_i}$ is the image of the natural monomorphism $\Gamma_{\theta_i^{\mathcal{H}}}/\Gamma_{\theta_i} \to \mathcal{H}/\mathcal{H}_{\theta_i}$. We refer to the θ_i as factors of $\tilde{\theta}$.

Step 1. We may assume that all factors of $\tilde{\theta}$ are $\mathcal{H}$-conjugate and any two Γ-conjugate (in particular $\mathcal{H}_{\Gamma_i}$-conjugate) factors are equal.

By Lemma 2.2 and after conjugating by an element of Γ^n, we may assume that any two factors of $\tilde{\theta}$ are either equal or not Γ-conjugate. In particular, and after maybe conjugation by an element of S_n, we can write $\tilde{\theta} = \times_{j=1}^{\ell} \tilde{\theta}_j$, where all the factors of $\tilde{\theta}_j$ lie in $\theta_j^{\mathcal{H}}$. Notice that any two factors of $\tilde{\theta}_j$ are either equal or not $\mathcal{H}_{\Gamma_j}$-conjugate. Hence $\tilde{\Gamma}_{\tilde{\theta}^{\mathcal{H}}} = \times_{j=1}^{\ell} \Gamma_{\tilde{\theta}_j^{\mathcal{H}}}$ and $\Gamma_{\tilde{\theta}} = \times_{j=1}^{\ell} \Gamma_{\tilde{\theta}_j}$. By Lemma 2.5 we may assume $\ell = 1$, that is, all the factors of $\tilde{\theta}$ lie in the same $\mathcal{H}$-orbit.

Step 2. We may assume that $\tilde{\theta} = \theta_1^m \times \cdots \times \theta_k^m$ for some m, where $\theta_i = \theta^{\sigma_i}$ for some $\sigma_i \in \mathcal{H}$ and $\mathcal{H}_\Gamma \sigma_i \neq \mathcal{H}_\Gamma \sigma_j$ whenever $i \neq j$. Here $\mathcal{H}_\Gamma = \mathcal{H}_{\Gamma_{\theta^{\mathcal{H}}}, \theta}$ as defined above.

By Step 1 and after conjugating $\tilde{\theta}$ by an element of S_n, we can write $\tilde{\theta} = (\theta_1)^{n_1} \times \cdots \times (\theta_k)^{n_k}$, where $\theta_i = \theta^{\sigma_i}$ for some $\sigma_i \in \mathcal{H}$ and two different σ_i define different $\mathcal{H}_\Gamma$-cosets. We work to show that any element $\gamma \in \tilde{\Gamma}_{\tilde{\theta}^{\mathcal{H}}}$ permutes θ_i and θ_j if and only if $n_i = n_j$. If we can show that then, after conjugating $\tilde{\theta}$ by an element of S_n, we may decompose $\tilde{\theta}$ as a direct product of characters $\tilde{\psi} = \psi_1^s \times \cdots \times \psi_r^s$ which do not have any factor in common (pairwise). In particular, $\tilde{\Gamma}_{\tilde{\theta}^{\mathcal{H}}}$ decomposes as a direct sum of $(\Gamma \wr \mathsf{S}_{sr})_{\tilde{\psi}^{\mathcal{H}}}$. By Lemma 2.5 the claim of the step would follow.

Given $\tilde{\xi} = \xi_1 \times \cdots \times \xi_n \in \mathrm{Irr}(X^n)$ such that all ξ_i lie in one $\mathcal{H}$-orbit, we can associate to $\tilde{\xi}$ the multiset $[\tilde{\xi}]$ of the $\mathcal{H}_{\Gamma,\xi_1}$-orbits of the factors, namely $[\tilde{\xi}] = [\xi_1^{\mathcal{H}_{\Gamma,\xi_1}}, \ldots, \xi_n^{\mathcal{H}_{\Gamma,\xi_1}}]$, where $\mathcal{H}_{\Gamma,\xi_1} = \mathcal{H}_{\Gamma_{\xi_1^{\mathcal{H}}},\xi_1}$ Note that for every $\gamma = (a_1, \ldots, a_n)\omega \in \tilde{\Gamma}$ we have that $[\xi^\gamma] = [(\xi_1^{a_1})^{\mathcal{H}_{\Gamma,\xi_1}}, \ldots, (\xi_n^{a_n})^{\mathcal{H}_{\Gamma,\xi_1}}]$.

Write $\tilde{\theta} = \beta_1 \times \cdots \times \beta_n$, so that each $\beta_i = \theta^{\tau_i}$ for some $\tau_i \in \mathcal{H}$. Note that $\mathcal{H}_\Gamma \tau_i = \mathcal{H}_\Gamma \tau_j$ if and only if $\beta_i = \beta_j$. Let $\gamma = (a_1, \ldots, a_n)\omega \in \tilde{\Gamma}$, with $a_i \in \Gamma$ and $\omega \in \mathsf{S}_n$. Then the multiplicity of $(\beta_i^{a_i})^{\mathcal{H}_\Gamma}$ in $[\tilde{\theta}^\gamma]$ is the same as the multiplicity of $\beta_i^{\mathcal{H}_\Gamma}$ in $[\tilde{\theta}]$, which is the same as the number of factors equal to β_i in $\tilde{\theta}$. This is because $\beta_j^{a_j}$ lies in the $\mathcal{H}_\Gamma$-orbit of $\beta_i^{a_i}$ if and only if $\mathcal{H}_\Gamma \tau_i = \mathcal{H}_\Gamma \tau_j$ (using that Γ-conjugate factors of $\tilde{\theta}$ are equal by Step 1). The latter happens if and only if $\beta_i = \beta_j$.

For $(\gamma, \tau) \in (\tilde{\Gamma} \times \mathcal{H})_{\tilde{\theta}}$, we have $\gamma = (a_1, \ldots, a_n)\omega \in \tilde{\Gamma}_{\tilde{\theta}^{\mathcal{H}}}$ and $[\tilde{\theta}^{\gamma\tau}] = [\tilde{\theta}]$. By the above paragraph the multiplicity of $(\beta_i^{a_i\tau})^{\mathcal{H}_\Gamma}$ in $[\tilde{\theta}] = [\tilde{\theta}^{\gamma\tau}]$ equals the multiplicity of $\beta_i^{\mathcal{H}_\Gamma}$ in $[\tilde{\theta}]$, that is, the number of factors equal to β_i in $\tilde{\theta}$. On the other hand $\tilde{\theta}^{\gamma\tau} = \tilde{\theta}$ if and only if $\beta_j = \beta_i^{a_i\tau}$ whenever $\omega(i) = j$. Putting these two facts together we see that if $\omega(i) = j$, then the number of factors equal to β_j in $\tilde{\theta}$ is the same as the number of factors equal to β_i in $\tilde{\theta}$, as wanted.

Final step. By Step 2 we have that $\tilde{\theta} = \theta_1^m \times \cdots \times \theta_k^m$, where $\theta_i = \theta^{\sigma_i}$ for some $\sigma_i \in \mathcal{H}$ and σ_i and σ_j define distinct $\mathcal{H}_\Gamma$-cosets whenever $i \neq j$. The result then follows by applying Theorem 2.7 with $X \lhd G = X \rtimes \Gamma_\theta$. Note that the condition that σ_i and σ_j define distinct $\mathcal{H}_\Gamma$-cosets whenever $i \neq j$ is equivalent to saying that no θ_i is $X \rtimes \Gamma_{\theta^{\mathcal{H}}}$-conjugate to θ_j if $i \neq j$. □

Theorem 3.3. *Suppose that $K \lhd G$, where K is perfect and $K/\mathbf{Z}(K)$ is isomorphic to a direct product of copies of a non-abelian simple group S. Let $Q \in \mathrm{Syl}_p(K)$. Assume that S satisfies the inductive Galois–McKay condition for p. Then there exists an $\mathbf{N}_G(Q)$-invariant subgroup $\mathbf{N}_K(Q) \subseteq M < K$ and an $\mathbf{N}_G(Q) \times \mathcal{H}$-equivariant bijection*

$$\Omega\colon \mathrm{Irr}_{p'}(K) \to \mathrm{Irr}_{p'}(M)\,,$$

such that for every $\theta \in \mathrm{Irr}_{p'}(K)$, we have

$$(G_{\theta^{\mathcal{H}}}, K, \theta)_{\mathcal{H}} \geqslant_c (\mathbf{N}_G(M)_{\theta^{\mathcal{H}}}, M, \Omega(\theta))_{\mathcal{H}}\,.$$

Proof. First note that the $\mathcal{H}$-triples relations make sense. By the Frattini argument $G = K\mathbf{N}_G(Q)$. Since $\mathbf{N}_G(Q) \subseteq \mathbf{N}_G(M)$ by the assumptions on M, then $G = K\mathbf{N}_G(M)$. Also by the Frattini argument $\mathbf{N}_G(M) = M\mathbf{N}_G(Q)$. Hence $\mathbf{N}_K(M) = K \cap \mathbf{N}_G(M) = M\mathbf{N}_K(Q) = M$.

Notice that if the theorem is true for Q, then it is true for Q^k for any $k \in K$. This is because $\Omega_k(\theta) := \Omega(\theta)^k$ would be $\mathbf{N}_G(Q)^k \times \mathcal{H}$-equivariant and by using Lemma 2.2. Hence we may choose any Sylow p-subgroup of K.

Let X be the universal covering of S, and let $\pi\colon X^n \to K$ be a covering of K with $Z = \ker(\pi) \subseteq \mathbf{Z}(X^n)$. Since S satisfies the inductive Galois–McKay condition for p, we have R, N and Ω given by Definition 3.1. We prove the result with respect to $\pi(R^n) = Q \in \mathrm{Syl}_p(K)$. Write $M = \pi(N^n) \supseteq \mathbf{N}_K(Q)$.

The idea is to prove the theorem with respect to $K \lhd \hat{G} = K \rtimes \mathrm{Aut}(K)_Q$ and to use Theorem 2.9 to relate G and $\hat{G}$ via their conjugation homomorphisms into $\mathrm{Aut}(K)$.

We mimic the proof of Theorem 10.25 in [Nav18]; see there for more details. Write $\Gamma = \mathrm{Aut}(X)_R$ as in Definition 3.1 and $\tilde{\Gamma} = \Gamma \wr \mathsf{S}_n$. By Theorem 3.2 we have a $\tilde{\Gamma} \times \mathcal{H}$-equivariant bijection

$$\tilde{\Omega}\colon \mathrm{Irr}_{p'}(X^n) \to \mathrm{Irr}_{p'}(N^n)$$

such that for every $\tilde{\theta} \in \mathrm{Irr}_{p'}(X^n)$

$$(X^n \rtimes \tilde{\Gamma}_{\tilde{\theta}^{\mathcal{H}}}, X^n, \tilde{\theta})_{\mathcal{H}} \geqslant_c (N^n \rtimes \tilde{\Gamma}_{\tilde{\theta}^{\mathcal{H}}}, N^n, \tilde{\varphi})_{\mathcal{H}}.$$

Since these are, in particular, central isomorphisms and $Z \subseteq \mathbf{Z}(X^n)$, then $\tilde{\theta}$ lies over 1_Z if and only if $\tilde{\varphi}$ lies over 1_Z. Let $B = \tilde{\Gamma}_Z \leqslant \tilde{\Gamma}$. In particular, $\tilde{\Omega}$ yields a bijection that we denote by $\tilde{\Omega}$ again,

$$\tilde{\Omega}\colon \mathrm{Irr}_{p'}(X^n/Z) \to \mathrm{Irr}_{p'}(N^n/Z),$$

which is $B \times \mathcal{H}$-equivariant. By Lemma 2.4

$$(X^n/Z \rtimes B_{\tilde{\theta}^{\mathcal{H}}}, X^n, \tilde{\theta})_{\mathcal{H}} \geqslant_c (N^n/Z \rtimes B_{\tilde{\theta}^{\mathcal{H}}}, N^n, \tilde{\varphi})_{\mathcal{H}},$$

for every $\tilde{\theta} \in \mathrm{Irr}_{p'}(X^n/Z)$ and $\tilde{\varphi} = \tilde{\Omega}(\theta)$.

Note that $X^n/Z \times B \cong \hat{G}$ via π and under this isomorphism $N^n/Z \rtimes B$ corresponds to $M \rtimes \mathrm{Aut}(K)_Q$. Hence we have proven that there exists an $\mathrm{Aut}(K)_Q \times \mathcal{H}$-equivariant bijection

$$\Omega\colon \mathrm{Irr}_{p'}(K) \to \mathrm{Irr}_{p'}(M).$$

Moreover, by Lemma 2.1

$$(K \rtimes (\mathrm{Aut}(K)_Q)_{\tilde{\theta}^{\mathcal{H}}}, K, \theta)_{\mathcal{H}} \geqslant_c (M \rtimes (\mathrm{Aut}(K)_Q)_{\tilde{\theta}^{\mathcal{H}}}, M, \Omega(\theta))_{\mathcal{H}},$$

whenever $\theta \in \mathrm{Irr}_{p'}(K)$.

To finish the proof apply Theorem 2.9 as in the end of the proof of Theorem 10.25 of [Nav18]. Let $\epsilon\colon G \to \mathrm{Aut}(K)$ and $\hat{\epsilon}\colon \hat{G} \to \mathrm{Aut}(K)$ be the corresponding conjugation homomorphisms. Let $\theta \in \mathrm{Irr}_{p'}(K)$, and let $V = \epsilon(G_{\theta^{\mathcal{H}}})$. The same arguments as in the proof of Theorem 10.25 of [Nav18] show that if $\hat{V} := \hat{\epsilon}^{-1}(V)$, then $\epsilon^{-1}(\epsilon(\mathbf{N}_{\hat{V}}(M))) = \mathbf{N}_{G_{\theta^{\mathcal{H}}}}(M) = \mathbf{N}_G(M)_{\theta^{\mathcal{H}}}$. ☐

The above result will be key in the reduction theorem carried out in Section 4. Below we write the exact form in which it will be later applied, in which K (in Theorem 3.3) need no longer be perfect.

Corollary 3.4. *Suppose that $K \lhd G$, where $K/\mathbf{Z}(K)$ is isomorphic to a direct product of copies of a non-abelian simple group S, and let $Q \in \mathrm{Syl}_p(K)$. Assume that S satisfies the inductive Galois–McKay condition for p. If $(G, \mathbf{Z}(K), \nu)_{\mathcal{H}}$ is an $\mathcal{H}$-triple, then there exist an $\mathbf{N}_G(Q)$-invariant subgroup $\mathbf{N}_K(Q) \subseteq M < K$ and an $\mathbf{N}_G(Q) \times \mathcal{H}$-equivariant bijection*

$$\Omega\colon \mathrm{Irr}_{p'}(K|\nu^{\mathcal{H}}) \to \mathrm{Irr}_{p'}(M|\nu^{\mathcal{H}}),$$

and for every $\theta \in \mathrm{Irr}_{p'}(K|\nu^{\mathcal{H}})$

$$(G_{\theta^{\mathcal{H}}}, K, \theta)_{\mathcal{H}} \geqslant_c (H_{\varphi^{\mathcal{H}}}, M, \varphi)_{\mathcal{H}} ,$$

where $\varphi = \Omega(\theta)$ *and* $H = M\mathbf{N}_G(Q)$*. In particular, there are character-degree-ratio preserving* $\mathcal{H}$*-equivariant bijections*

$$\hat{\tau}_\theta \colon \mathrm{Irr}(G|\theta^{\mathcal{H}}) \to \mathrm{Irr}(H|\varphi^{\mathcal{H}}) ,$$

satisfying $\hat{\tau}_{\theta^h} = \hat{\tau}_\theta$ *and* $\hat{\tau}_{\theta^\sigma} = \hat{\tau}_\theta$ *for every* $h \in H$ *and* $\sigma \in \mathcal{H}$*.*

Proof. Write $K_1 = K'$ and $Z = \mathbf{Z}(K)$. Hence $K = K_1 Z$ is the central product of K_1 and Z, and K_1 is perfect. Also $Q_1 := Q \cap K_1 \in \mathrm{Syl}_p(K_1)$. Let M_1 and Ω_1 be given by Theorem 3.3 applied with respect to $K_1 \lhd G$. Note that $Z \cap K_1 = Z \cap M_1$. Let $\nu_1 = \nu_{Z \cap K_1} \in \mathrm{Irr}(Z \cap K_1)$, $\theta_1 \in \mathrm{Irr}_{p'}(K_1|\nu_1^\sigma)$ and $\varphi_1 = \Omega_1(\theta_1)$. By Theorem 3.3

$$(G_{\theta_1^{\mathcal{H}}}, K_1, \theta_1)_{\mathcal{H}} \geqslant_c (\mathbf{N}_G(M_1)_{\theta_1^{\mathcal{H}}}, M_1, \varphi_1)_{\mathcal{H}}.$$

In particular, this implies that φ_1 lies over ν_1^σ and thus Ω_1 maps $\mathrm{Irr}_{p'}(K_1|\nu_1^{\mathcal{H}})$ onto $\mathrm{Irr}_{p'}(M_1|\nu_1^{\mathcal{H}})$. Write $M = M_1 Z$, which is the central product of M_1 and Z. Clearly M is $\mathbf{N}_G(Q)$-invariant as $\mathbf{N}_G(Q) \subseteq \mathbf{N}_G(Q_1)$. The desired bijection Ω can be obtained via Ω_1 using the dot product of characters as follows (we refer the reader to the discussion preceding Theorem 2.8 for more details). Let $\theta \in \mathrm{Irr}_{p'}(K|\nu^{\mathcal{H}})$ lie over ν^σ. Then $\theta = \theta_1 \cdot \nu^\sigma$ for some $\theta_1 \in \mathrm{Irr}_{p'}(K_1|\nu_1^\sigma)$. Define $\Omega(\theta) := \Omega_1(\theta_1) \cdot \nu^\sigma = \varphi_1 \cdot \nu^\sigma \in \mathrm{Irr}(M|\nu^\sigma)$. Hence

$$\Omega \colon \mathrm{Irr}_{p'}(K|\nu^{\mathcal{H}}) \to \mathrm{Irr}_{p'}(M|\nu^{\mathcal{H}})$$

is an $\mathbf{N}_G(Q) \times \mathcal{H}$-equivariant bijection with the desired properties. Write $\varphi = \Omega(\theta)$ and recall

$$(G_{\theta_1^{\mathcal{H}}}, K_1, \theta_1)_{\mathcal{H}} \geqslant_c (\mathbf{N}_G(M_1)_{\theta_1^{\mathcal{H}}}, M_1, \varphi_1)_{\mathcal{H}} .$$

Note that $\mathbf{N}_G(M_1) = H = M\mathbf{N}_G(Q)$, by the Frattini argument, as $Q \subseteq M \lhd \mathbf{N}_G(M_1) = M_1 \mathbf{N}_G(Q_1)$. By Theorem 2.8

$$(G_{\theta^{\mathcal{H}}}, K, \theta)_{\mathcal{H}} \geqslant_c (H_{\varphi^{\mathcal{H}}}, M, \varphi)_{\mathcal{H}} .$$

For each θ, denote by $\hat{\tau}_\theta$ the bijection provided by Corollary 1.11. In particular,

$$\hat{\tau}_\theta \colon \mathrm{Irr}(G|\theta^{\mathcal{H}}) \to \mathrm{Irr}(H|\varphi^{\mathcal{H}})$$

is $\mathcal{H}$-equivariant and preserves ratios of character degrees. The claims on $\hat{\tau}_{\theta^h}$ and $\hat{\tau}_{\theta^\sigma}$ in the final part of the statement follow from the comments after Lemma 2.2 and Lemma 2.3. □

4. The reduction

The following key result is due to F. Ladisch. It is based on work by A. Turull. This is an $\mathcal{H}$-triple version of the well-known fact that a character triple (G, N, θ) can be replaced by an isomorphic one (G_1, N_1, θ_1) with $N_1 \subseteq \mathbf{Z}(G_1)$, in such a way that the character properties of G over θ are *the same* as the character properties of G_1 over θ_1. If we wish to control fields of values of characters above θ, this is no longer true. Still we can somehow replace the original $\mathcal{H}$-triple by another one with convenient properties.

Theorem 4.1 (Ladisch). *Suppose that $(G, Z, \lambda)_{\mathcal{H}}$ is an $\mathcal{H}$-triple. Then there exists another $\mathcal{H}$-triple $(G_1, Z_1, \lambda_1)_{\mathcal{H}}$ such that:*

(a) *There is an onto homomorphism $\kappa_1 \colon G_1 \to G/Z$ with kernel Z_1.*

(b) *For every $Z \subseteq X \leqslant G$, there is an $\mathcal{H}$-equivariant bijection $\psi \mapsto \psi_1$ from $\mathrm{Irr}(X|\lambda^{\mathcal{H}}) \to \mathrm{Irr}(X_1|\lambda_1^{\mathcal{H}})$, where $\kappa_1(X_1)/Z = X/Z$, preserving ratios of character degrees; more precisely, if ψ and ψ_1 correspond under the above bijection, then $\psi(1)/\lambda(1) = \psi_1(1)/\lambda_1(1)$. Furthermore, if $g_1 \in G_1$, $g = \kappa(g_1)$ and $\psi \in \mathrm{Irr}(X|\lambda^{\mathcal{H}})$, then*

$$(\psi^g)_1 = (\psi_1)^{g_1}\,.$$

In particular, $(G_1)_{\lambda_1}$ is mapped to G_λ via κ_1.

(c) *There is a normal cyclic subgroup C of G_1 with $C \subseteq Z_1$, and a faithful $\nu \in \mathrm{Irr}(C)$ such that $\nu^{Z_1} = \lambda_1 \in \mathrm{Irr}(Z_1)$.*

(d) *$(G_1, C, \nu)_{\mathcal{H}}$ is an $\mathcal{H}$-triple.*

(e) *If $U = (G_1)_{\lambda_1}$ and $V = (G_1)_\nu$, then $U = Z_1V$ and $C = Z_1 \cap V$. Also $V = \mathbf{C}_{G_1}(C)$ and $C \subseteq \mathbf{Z}(V)$.*

Proof. Let $n = |G|$, and let $\mathcal{H}_n \leqslant \mathrm{Gal}(\mathbb{Q}(\xi_n)/\mathbb{Q})$, where ξ_n is a primitive nth root of unity as in Section 1. Notice that $\mathcal{H}_n$ acts on the characters of any subgroup (or quotient) of G. Let $\mathbb{F} = \mathbb{Q}(\xi_n)^{\mathcal{H}_n}$, so that $\mathcal{H}_n = \mathrm{Gal}(\mathbb{Q}(\xi_n)/\mathbb{F})$. The fact that $(G, Z, \lambda)_{\mathcal{H}}$ is an $\mathcal{H}$-triple means that λ is semi-invariant in G over $\mathbb{F}$ in the sense of [Lad16] (see p. 47, second paragraph). Apply Theorem A and Corollary B of [Lad16]. The conjugation part in (b), which we shall later need, is not mentioned in Corollary B of [Lad16], but in Theorem 7.12(7) of [Tur09]. □

What follows is essentially a deep result by A. Turull concerning Clifford theory and action of $\mathcal{H}$ over Glauberman correspondents.

Theorem 4.2 (Turull). *Suppose that G is a finite p-solvable group. Suppose that K is a normal p'-subgroup of G, and that Q is a p-subgroup such that $KQ \lhd G$. Let $D = \mathbf{C}_K(Q)$. Let C be a normal subgroup of G such that $C \subseteq \mathbf{Z}(KQ)$. Let $\nu \in \mathrm{Irr}(C)$ and assume that $(G, C, \nu)_{\mathcal{H}}$ is an $\mathcal{H}$-triple. Let $\Delta = \mathrm{Irr}_{p'}(KQ|\nu^{\mathcal{H}})$, and let $\Delta' = \mathrm{Irr}_{p'}(DQ|\nu^{\mathcal{H}})$. Then there is an $\mathcal{H}$-equivariant bijection*

$$f \colon \bigcup_{\tau \in \Delta} \mathrm{Irr}(G|\tau) \to \bigcup_{\tau' \in \Delta'} \mathrm{Irr}(\mathbf{N}_G(Q)|\tau')\,.$$

Proof. Write $C = C_{p'} \times C_p$ and $\nu = \nu_{p'} \times \nu_p$, where $\nu_{p'} \in \mathrm{Irr}(C_{p'})$ and $\nu_{p'} \in \mathrm{Irr}(C_p)$. Notice that $C_{p'} \subseteq D$ and $C_p \subseteq Q$ by hypothesis.

Write $\mathrm{Irr}_Q(K)$ for the Q-invariant irreducible characters of K, and for each $\theta \in \mathrm{Irr}_Q(K)$, let $\hat\theta \in \mathrm{Irr}(D)$ be its Glauberman correspondent.

By Theorem 3.2 of [Tur13], for each $\theta \in \mathrm{Irr}_Q(K)$ there is an $\mathcal{H}$-equivariant bijection

$$f_\theta \colon \mathrm{Irr}(G|\theta^{\mathcal{H}}) \cap \mathrm{Irr}(G|\nu_p^{\mathcal{H}}) \to \mathrm{Irr}(\mathbf{N}_G(Q)|\hat\theta^{\mathcal{H}}) \cap \mathrm{Irr}(\mathbf{N}_G(Q)|\nu_p^{\mathcal{H}})$$

satisfying a list of conditions. We can take $f_{\theta^\sigma} = f_\theta$ for every $\sigma \in \mathcal{H}$ and $f_{\theta^x} = f_\theta$ for every $x \in \mathbf{N}_G(Q)$.

We define now $f(\chi)$ for $\chi \in \bigcup_{\tau \in \Delta} \mathrm{Irr}(G|\tau)$. Suppose that χ lies over some $\tau \in \Delta$. We have that $\tau_C = \tau(1)\nu^\sigma$ for some $\sigma \in \mathcal{H}$, by using that $C \subseteq \mathbf{Z}(KQ)$. Also, $\tau_K = \theta \in \mathrm{Irr}_Q(K)$ since $\tau(1)$ has p'-degree and $|KQ : K|$ is a power of p. Note that θ lies over $\nu_{p'}^\sigma$. Define $f(\chi) := f_\theta(\chi) \in \mathrm{Irr}(\mathbf{N}_G(Q)|\hat\theta^{\mathcal{H}}) \cap \mathrm{Irr}(\mathbf{N}_G(Q)|\nu_p^{\mathcal{H}})$.

Note that f is well-defined. First, any other constituent of χ_{KQ} is τ^x for some $x \in \mathbf{N}_G(Q)$ and $f_{\theta^x} = f_\theta$. By Theorem 3.2(1) of [Tur13] $f_\theta(\chi)$ lies over $\hat{\theta}^\sigma$, hence over $\nu_{p'}^\sigma$. In order to see that f is well-defined we need that $f_\theta(\chi)$ lies also over ν_p^σ and this in principle is not guaranteed by Theorem 3.2 of [Tur13] but by Theorem 10.1 of [Tur17]. Hence $f_\theta(\chi)$ lies over ν^σ, and consequently over some $\tau' \in \Delta'$.

The map f is clearly surjective as every element in $\bigcup_{\tau' \in \Delta'} \mathrm{Irr}(\mathbf{N}_G(Q)|\tau')$ lies in $\mathrm{Irr}(\mathbf{N}_G(Q)|\mu) \cap \mathrm{Irr}(\mathbf{N}_G(Q)|\nu_p^\sigma)$ for some $\sigma \in \mathcal{H}$ and $\mu \in \mathrm{Irr}(D|\nu_{p'}^{\mathcal{H}})$. Again using that $f_{\theta^x} = f_\theta$ for every $x \in \mathbf{N}_G(Q)$ and every $\theta \in \mathrm{Irr}_Q(K)$ one can check that f is injective.

Finally f is $\mathcal{H}$-equivariant as every f_θ is $\mathcal{H}$-equivariant. $\square$

In contrast to the proof of the reduction theorem for the McKay conjecture in [IMN07], we cannot work with characters of p'-degree. We have to work instead with characters of relative p'-degree. Those are defined as follows. For $N \lhd G$ and $\theta \in \mathrm{Irr}(N)$, we say that $\chi \in \mathrm{Irr}(G|\theta)$ has **relative p'-degree** with respect to N (or to θ) if the ratio $\chi(1)/\theta(1)$ is not divisible by p. We denote by $\mathrm{Irr}_{p'}^{\mathrm{rel}}(G|\theta)$ the set of irreducible relative p'-degree characters with respect to N. The following are easy properties of relative p'-degree characters.

Lemma 4.3. *Suppose that $N \lhd G$ and $\theta \in \mathrm{Irr}(N)$. Let $P \in \mathrm{Syl}_p(G)$.*

(a) *If $\chi \in \mathrm{Irr}_{p'}^{\mathrm{rel}}(G|\theta)$, then χ_N has some P-invariant irreducible constituent and any two of them are $\mathbf{N}_G(P)$-conjugate. These P-invariant constituents extend to NP.*

(b) *Suppose that $N \subseteq M \lhd G$, and let $\chi \in \mathrm{Irr}(G|\eta)$, where $\eta \in \mathrm{Irr}(M|\theta)$. Then $\chi \in \mathrm{Irr}_{p'}^{\mathrm{rel}}(G|\theta)$ if and only if $\chi \in \mathrm{Irr}_{p'}^{\mathrm{rel}}(G|\eta)$ and $\eta \in \mathrm{Irr}_{p'}^{\mathrm{rel}}(M|\theta)$.*

Proof. We can write $\chi_{PN} = a_1\delta_1 + \cdots + a_k\delta_k$, where $\delta_i \in \mathrm{Irr}(PN)$ lies over some G-conjugate of θ. In particular, $\delta_i(1)/\theta(1)$ is an integer. Since $\chi(1)/\theta(1)$ is not divisible by p, it follows that there is some i such that p does not divide $\delta_i(1)/\theta(1)$. Since this number divides $|PN : N|$, it follows that $(\delta_i)_N = \eta \in \mathrm{Irr}(N)$ is P-invariant. Suppose that η^g is another P-invariant irreducible constituent. Then $P, P^g \subseteq G_{\eta^g}$, and by Sylow theory, we have that η^g and η are $\mathbf{N}_G(P)$-conjugate. In particular, $\eta^g = \eta^h$ for some $h \in \mathbf{N}_G(P)$, then η^h also extends to $(PN)^h = PN$. The second part easily follows using that $\chi(1)/\theta(1) = (\chi(1)/\eta(1))(\eta(1)/\theta(1))$. $\square$

Let $Z \lhd G$ and $\lambda \in \mathrm{Irr}(Z)$. We will denote by $\mathrm{Irr}_{p'}^{\mathrm{rel}}(G|\lambda^{\mathcal{H}})$ the subset of relative p'-degree characters of $\mathrm{Irr}(G|\lambda^{\mathcal{H}})$ (with respect to Z). Recall that whenever $X \leqslant G$ contains $G_{\lambda^{\mathcal{H}}}$, the induction of characters defines an $\mathcal{H}$-equivariant bijection

$$\mathrm{Irr}(X|\lambda^{\mathcal{H}}) \to \mathrm{Irr}(G|\lambda^{\mathcal{H}}) .$$

We are finally ready to prove the main result of this note.

Theorem 4.4. *Let $Z \lhd G$, $P \in \mathrm{Syl}_p(G)$ and $\lambda \in \mathrm{Irr}(Z)$ be P-invariant. Write $H = \mathbf{N}_G(P)Z$. Assume that every simple group involved in G/Z satisfies the inductive Galois–McKay condition for p. Then there exists an $\mathcal{H}$-equivariant bijection between $\mathrm{Irr}_{p'}^{\mathrm{rel}}(G|\lambda^{\mathcal{H}})$ and $\mathrm{Irr}_{p'}^{\mathrm{rel}}(H|\lambda^{\mathcal{H}})$.*

Proof. We argue by induction on $|G : Z|$. We also may assume that $H < G$; otherwise the statement trivially holds.

Step 1. We may assume that $G = G_{\lambda^{\mathcal{H}}}$. In particular $G_\lambda \lhd G$, $G = G_\lambda H$ and $Z < G_\lambda$.

Induction of characters defines $\mathcal{H}$-equivariant bijections $\mathrm{Irr}(G_{\lambda^{\mathcal{H}}}|\lambda^{\mathcal{H}}) \to \mathrm{Irr}(G|\lambda^{\mathcal{H}})$ and $\mathrm{Irr}(H_{\lambda^{\mathcal{H}}}|\lambda^{\mathcal{H}}) \to \mathrm{Irr}(H|\lambda^{\mathcal{H}})$. Since $P \subseteq G_\lambda$, relative p'-degree characters are mapped onto relative p'-degree characters. Hence we may assume $G = G_{\lambda^{\mathcal{H}}}$. In particular $(G, Z, \lambda)_{\mathcal{H}}$ and $(H, Z, \lambda)_{\mathcal{H}}$ are $\mathcal{H}$-triples, and $G_\lambda \lhd G$. Using the Frattini argument we have that $G = G_\lambda H$. If $G_\lambda = Z$, then $G = H$ contradicts our first assumption.

Step 2. We may assume that G has a normal cyclic subgroup C contained in Z and a faithful character $\nu \in \mathrm{Irr}(C)$ such that $\nu^Z = \lambda$ and $(G, C, \nu)_{\mathcal{H}}$ is an $\mathcal{H}$-character triple. In particular, $G_\lambda = G_\nu Z$ and $G_\nu \cap Z = C$. Moreover $G_\nu \lhd G$ and $C \subseteq \mathbf{Z}(G_\nu)$.

By Theorem 4.1 there exist an $\mathcal{H}$-character triple $(G_1, Z_1, \lambda_1)_{\mathcal{H}}$, a group epimorphism $\kappa_1 \colon G_1 \to G/Z$ with kernel Z_1 and $\mathcal{H}$-equivariant character bijections $\mathrm{Irr}_{p'}(G|\lambda^{\mathcal{H}}) \to \mathrm{Irr}_{p'}(G_1|\lambda_1^{\mathcal{H}})$ and $\mathrm{Irr}_{p'}(H|\lambda^{\mathcal{H}}) \to \mathrm{Irr}_{p'}(H_1|\lambda_1^{\mathcal{H}})$, where $\kappa_1(H_1) = H$ with $Z_1 \subseteq H_1$. These bijections also commute with group conjugation as in Theorem 4.1(b). Let $Z_1 \subseteq (PZ)_1 \leqslant (G_1)_{\lambda_1}$ be such that $\kappa_1((PZ)_1) = PZ$ and let $P_1 \in \mathrm{Syl}_p((PZ)_1)$. Then $P_1 \in \mathrm{Syl}_p(G_1)$, $(PZ)_1 = P_1 Z_1$ and also $H_1 = \mathbf{N}_{G_1}(P_1)Z_1$ by the Frattini argument. By Theorem 4.1, all the requirements of the claim are satisfied in G_1. Since $|G : Z| = |G_1 : Z_1|$, it is no loss if we work in G_1 instead of in G.

Step 3. If $H \subseteq X < G$, then there exists an $\mathcal{H}$-equivariant bijection between $\mathrm{Irr}^{\mathrm{rel}}_{p'}(X|\lambda^{\mathcal{H}})$ and $\mathrm{Irr}^{\mathrm{rel}}_{p'}(H|\lambda^{\mathcal{H}})$.

This follows by induction since $|X : Z| < |G : Z|$.

Step 4. Let L/Z be a chief factor of G with $L \subseteq G_\lambda$. Then $G = LH$. In other words, $LP \lhd G$.

Recall that $H = \mathbf{N}_G(P)Z$. Define Θ_0 to be a complete set of representatives of the orbits of $\mathbf{N}_G(P) \times \mathcal{H}$ on the P-invariant characters in $\mathrm{Irr}^{\mathrm{rel}}_{p'}(L|\lambda^{\mathcal{H}})$. By Lemma 4.3 every relative p'-degree character of G with respect to Z lies over a unique $\mathbf{N}_G(P)$-orbit of P-invariant characters of relative p'-degree of L with respect to Z. Since $(G, Z, \lambda)_{\mathcal{H}}$ is an $\mathcal{H}$-triple, one can easily check that

$$\mathrm{Irr}^{\mathrm{rel}}_{p'}(G|\lambda^{\mathcal{H}}) = \dot{\bigcup_{\theta \in \Theta_0}} \mathrm{Irr}^{\mathrm{rel}}_{p'}(G|\theta^{\mathcal{H}})$$

is a disjoint union. Similarly

$$\mathrm{Irr}^{\mathrm{rel}}_{p'}(LH|\lambda^{\mathcal{H}}) = \dot{\bigcup_{\theta \in \Theta_0}} \mathrm{Irr}^{\mathrm{rel}}_{p'}(LH|\theta^{\mathcal{H}}).$$

Since $Z < L$, then $|G : L| < |G : Z|$, and by induction we have $\mathcal{H}$-equivariant bijections $\mathrm{Irr}^{\mathrm{rel}}_{p'}(G|\theta^{\mathcal{H}}) \to \mathrm{Irr}^{\mathrm{rel}}_{p'}(LH|\theta^{\mathcal{H}})$ whenever $\theta \in \Theta_0$. This defines an $\mathcal{H}$-equivariant bijection

$$\mathrm{Irr}^{\mathrm{rel}}_{p'}(G|\lambda^{\mathcal{H}}) \to \mathrm{Irr}^{\mathrm{rel}}_{p'}(LH|\lambda^{\mathcal{H}}).$$

If $LH < G$, then by Step 3, we are done. The latter claim of the step follows immediately since $\mathbf{N}_{G/L}(PL/L) = \mathbf{N}_G(P)L/L$ by the Frattini argument.

Step 5. We may assume L/Z is not a p-group.

Notice that, by Step 4, $G = LH$, where $H = \mathbf{N}_G(P)Z$. If L/Z is a p-group, then $H = G$, contradicting a previous assumption.

Step 6. Write $P_\nu = P \cap G_\nu$ and $K = L \cap G_\nu$. Then $H \subseteq \mathbf{N}_G(P_\nu) < G$, $KP_\nu \lhd G$ and $PZ = P_\nu Z$.

Since $(G, C, \nu)_{\mathcal{H}}$ is an $\mathcal{H}$-triple, recall that $G_\nu \lhd G$. Then $K = L \cap G_\nu \lhd G$. Note that $(LP)_\nu = KP_\nu$. Using that ZP/Z is a Sylow p-subgroup of G_λ/Z, that P_ν is a Sylow p-subgroup of G_ν, and that $G_\lambda = G_\nu Z$ with $G_\nu \cap Z = C$, we conclude that $ZP_\nu = ZP$. By Dedekind's lemma $ZP \cap G_\nu = CP_\nu$. A similar argument can be used to show that $LP \cap G_\nu = KP_\nu$. (Note that $L = KZ$ with $K \cap Z = C$ and work with $Q = P_\nu \cap K \in \mathrm{Syl}_p(K)$.)

Recall that $LP \lhd G$, and hence $KP_\nu = LP \cap G_\nu \lhd G$. Also $CP_\nu = C_{p'} \times P_\nu$, since C is central in G_ν. Since Z normalizes CP_ν, it follows that Z normalizes P_ν. Notice that if $P_\nu \lhd G$, then $ZP \lhd G$ (because $ZP_\nu = ZP$), a contradiction. Hence $H \subseteq \mathbf{N}_G(P_\nu) < G$.

Step 7. Let $Y \leqslant G_\nu$ be such that $KY, LY \lhd G$. Then

$$\mathrm{Irr}^{\mathrm{rel}}_{p'}(G|\lambda^{\mathcal{H}}) = \bigcup_{\theta \in \Delta_Y} \mathrm{Irr}^{\mathrm{rel}}_{p'}(G|\theta^{LY}) = \bigcup_{\theta \in \Delta_Y} \mathrm{Irr}^{\mathrm{rel}}_{p'}(G|\theta)\,,$$

where $\Delta_Y = \mathrm{Irr}_{p'}(KY|\nu^{\mathcal{H}})$. Whenever $H \subseteq X \leqslant G$ and $Y \subseteq X$, we also have that

$$\mathrm{Irr}^{\mathrm{rel}}_{p'}(X|\lambda^{\mathcal{H}}) = \bigcup_{\theta' \in \Delta'_Y} \mathrm{Irr}^{\mathrm{rel}}_{p'}(X|\theta^{LY \cap X}) = \bigcup_{\theta' \in \Delta'_Y} \mathrm{Irr}^{\mathrm{rel}}_{p'}(X|\theta)\,,$$

where $\Delta'_Y = \mathrm{Irr}_{p'}(KY \cap X|\nu^{\mathcal{H}})$.

First note that $LY = (KY)Z$ and $KY \cap Z = C$. Moreover $(LY)_{\nu^\sigma} = KY$, whenever $\sigma \in \mathcal{H}$. Note that induction of characters defines a bijection $\mathrm{Irr}(KY|\nu^\sigma) \to \mathrm{Irr}(LY|\lambda^\sigma)$ for every $\sigma \in \mathcal{H}$. Let $\chi \in \mathrm{Irr}^{\mathrm{rel}}_{p'}(G|\lambda^{\mathcal{H}})$. Let $\mu \in \mathrm{Irr}(LY)$ be under χ and over λ^σ for some $\sigma \in \mathcal{H}$. By Lemma 4.3(a) $\mu \in \mathrm{Irr}^{\mathrm{rel}}_{p'}(LY|\lambda^\sigma)$ and $\chi(1)/\mu(1)$ is a p'-number. Let $\theta \in \mathrm{Irr}(KY)$ be under μ and over ν^σ. Then $\theta^{LY} = \mu$. Also $\mu(1)/\lambda(1) = \theta(1)/\nu(1) = \theta(1)$ is a p'-number. The other inclusion is shown similarly. If $H \subseteq X \leqslant G$ and $Y \subseteq X$, we can use the same argument to show that

$$\mathrm{Irr}^{\mathrm{rel}}_{p'}(X|\lambda^{\mathcal{H}}) = \bigcup_{\theta' \in \Delta'_Y} \mathrm{Irr}^{\mathrm{rel}}_{p'}(X|(\theta')^{LY \cap X}) = \bigcup_{\theta' \in \Delta'_Y} \mathrm{Irr}^{\mathrm{rel}}_{p'}(X|\theta')$$

where $\Delta'_Y = \mathrm{Irr}_{p'}((LY \cap X)_\nu|\nu^{\mathcal{H}})$. Notice that in X we still have $X = X_\lambda H$, $X_\lambda = X_\nu Z$ and the rest of the conditions with respect to the normal subgroups $LY \cap X = (L \cap X)Y$ and $KY \cap X = (K \cap X)Y$. We have intentionally omitted the dependence of Δ'_Y on X in the notation, but this shall not lead to confusion as the subgroup X will be clear when applied in this step.

Step 8. We may assume L/Z is not a p'-group.

If L/Z is a p'-group, we see that G is p-solvable and K/C is a p'-group. Since $C \subseteq \mathbf{Z}(K)$, write $K = C_p \times K_{p'}$, where $K_{p'}$ is a normal p-complement of K. Write $P_\nu = P \cap G_\nu$ and $X = \mathbf{N}_G(P_\nu)$. By Step 6, $H \subseteq X < G$. By Step 7, taking $Y = P_\nu$ we have

$$\mathrm{Irr}^{\mathrm{rel}}_{p'}(G|\lambda^{\mathcal{H}}) = \bigcup_{\theta \in \Delta} \mathrm{Irr}^{\mathrm{rel}}_{p'}(G|\theta) = \bigcup_{\theta \in \Delta} \mathrm{Irr}(G|\theta),$$

where $\Delta = \Delta_Y$ and we are using that $LP_\nu = LP$ and $|G : LP|$ is a p'-number. Also by Step 7, since $Y = P_\nu \subseteq X$ we have

$$\mathrm{Irr}^{\mathrm{rel}}_{p'}(X|\lambda^{\mathcal{H}}) = \bigcup_{\theta' \in \Delta'} \mathrm{Irr}(X|\theta')\,,$$

where $\Delta' = \Delta'_Y$. Just note that $(LY \cap X)_\nu = KP_\nu \cap \mathbf{N}_{G_\nu}(P_\nu) = \mathbf{N}_K(P_\nu)P_\nu = DP_\nu$, where $D = \mathbf{C}_{K_{p'}}(P_\nu) = \mathbf{N}_{K_{p'}}(P_\nu)$. Hence $\Delta = \mathrm{Irr}_{p'}(K_{p'}P_\nu|\nu^{\mathcal{H}})$ and $\Delta' = \mathrm{Irr}_{p'}(DP_\nu|\nu^{\mathcal{H}})$. By Theorem 4.2, there is an $\mathcal{H}$-equivariant bijection

$$f\colon \bigcup_{\theta\in\Delta} \mathrm{Irr}(G|\theta) \to \bigcup_{\theta'\in\Delta'} \mathrm{Irr}(X|\theta')\,.$$

By applying Step 3 we are done in this case.

Final step. By Step 5 and Step 8, we may assume that L/Z is a direct product of non-abelian simple groups of order divisible by p isomorphic to some S. Since $K/C \cong L/Z$ and $C \subseteq \mathbf{Z}(K)$ by Step 2, we have $C = \mathbf{Z}(K)$. Let $Q = P \cap K \in \mathrm{Syl}_p(K)$. Thus $\mathbf{N}_G(P) \subseteq \mathbf{N}_G(Q)$. Furthermore, since K/C and Z/C are normal subgroups of G/C with $K \cap Z = C$, we have that Z normalizes $CQ = C_{p'} \times Q$. Thus Z normalizes Q. (Note that $\mathbf{N}_G(Q) < G$ because $\mathbf{N}_K(Q) < K$.) Hence $|\mathbf{N}_G(Q) : Z| < |G : Z|$.

Since S satisfies the inductive Galois–McKay condition, by Corollary 3.4 there exist an $\mathbf{N}_G(Q)$-stable subgroup $\mathbf{N}_K(Q) \subseteq M < K$ and an $\mathbf{N}_G(Q) \times \mathcal{H}$-equivariant bijection

$$\Omega\colon \mathrm{Irr}_{p'}(K|\nu^{\mathcal{H}}) \to \mathrm{Irr}_{p'}(M|\nu^{\mathcal{H}})$$

such that, for every $\theta \in \mathrm{Irr}_{p'}(K|\nu^{\mathcal{H}})$, there is a character-degree-ratio preserving $\mathcal{H}$-equivariant bijection

$$\hat{\tau}_\theta\colon \mathrm{Irr}(G|\theta^{\mathcal{H}}) \to \mathrm{Irr}(M\mathbf{N}_G(Q)|\varphi^{\mathcal{H}})\,,$$

where $\varphi = \Omega(\theta)$. Write $U = M\mathbf{N}_G(Q)$ and notice that $H \subseteq U < G$ and $U \cap K = M$. Moreover, $\hat{\tau}_{\theta^u} = \hat{\tau}_\theta$ and $\hat{\tau}_{\theta^\sigma} = \hat{\tau}_\theta$ for every $\theta \in \mathrm{Irr}_{p'}(K)$, $u \in U$ and $\sigma \in \mathcal{H}$.

By Step 3, there is an $\mathcal{H}$-equivariant bijection

$$\mathrm{Irr}^{\mathrm{rel}}_{p'}(U|\lambda^{\mathcal{H}}) \to \mathrm{Irr}^{\mathrm{rel}}_{p'}(H|\lambda^{\mathcal{H}})\,.$$

Hence, we only need to construct an $\mathcal{H}$-equivariant bijection

$$F\colon \mathrm{Irr}^{\mathrm{rel}}_{p'}(G|\lambda^{\mathcal{H}}) \to \mathrm{Irr}^{\mathrm{rel}}_{p'}(U|\lambda^{\mathcal{H}})\,.$$

By Step 7 we have

$$\mathrm{Irr}^{\mathrm{rel}}_{p'}(G|\lambda^{\mathcal{H}}) = \bigcup_{\theta\in\Delta} \mathrm{Irr}^{\mathrm{rel}}_{p'}(G|\theta) \quad \text{and} \quad \mathrm{Irr}^{\mathrm{rel}}_{p'}(U|\lambda^{\mathcal{H}}) = \bigcup_{\theta'\in\Delta'} \mathrm{Irr}^{\mathrm{rel}}_{p'}(U|\theta')\,,$$

where $\Delta = \mathrm{Irr}_{p'}(K|\nu^{\mathcal{H}})$ and $\Delta' = \mathrm{Irr}_{p'}(M|\nu^{\mathcal{H}})$. We can define F as follows. If $\chi \in \mathrm{Irr}^{\mathrm{rel}}_{p'}(G|\lambda^{\mathcal{H}})$, then $\chi \in \mathrm{Irr}^{\mathrm{rel}}_{p'}(G|\theta)$ for some $\theta \in \mathrm{Irr}_{p'}(K|\nu^\sigma)$ and $\sigma \in \mathcal{H}$. Such θ is determined up to U-conjugacy. Define $F(\chi) = \hat{\tau}_\theta(\chi) \in \mathrm{Irr}(U|\varphi)$, where $\varphi = \Omega(\theta) \in \mathrm{Irr}_{p'}(M|\nu^\sigma)$. (This latter fact follows from the fact that χ and $\hat{\tau}_\theta(\chi)$ lie over the same character of $\mathbf{Z}(K) = C$ by the $\mathcal{H}$-triples relations in Corollary 3.4.)

Now, F is well-defined since $\hat{\tau}_\theta = \hat{\tau}_{\theta^u}$ for every $u \in U$. Suppose that $\chi, \chi' \in \mathrm{Irr}^{\mathrm{rel}}_{p'}(G|\lambda^{\mathcal{H}})$ have the same image ξ under F. If $\theta, \theta' \in \mathrm{Irr}(K)$ lie under χ and χ', respectively, then they must be $\mathbf{N}_G(Q)$-conjugate because Ω is $\mathbf{N}_G(Q)$-equivariant and $\Omega(\theta)$ and $\Omega(\theta')$ lie under ξ. Hence injectivity also follows from the fact that $\hat{\tau}_\theta = \hat{\tau}_{\theta^u}$ for every $u \in U$. F is clearly surjective, and the proof is finished. □

Theorem A follows from Theorem 4.4 by taking $Z = 1$.

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Departament de Matemàtiques, Universitat de València, 46100 Burjassot, València, Spain

Email address: gabriel.navarro@uv.es

BU Wuppertal, Gaussstrasse 20, 42119 Wuppertal, Germany

Email address: bspaeth@uni-wuppertal.de

Departamento de Matemáticas, Universidad Autónoma de Madrid, Campus de Cantoblanco, 28049 Madrid, Spain

Email address: carolina.vallejo@uam.es