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Principal blocks with 5 irreducible characters ☆

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We show that if the principal p-block of a finite group G contains exactly 5 irreducible ordinary characters, then a Sylow p-subgroup of G has order 5, 7 or is isomorphic to one of the non-abelian 2-groups of order 8.

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0. Introduction

The problem of classifying finite groups G depending on the number $k(G)$ of irreducible ordinary characters of G goes back to Burnside [8], and a complete classification up to $k(G) = 14$ has been achieved in [52]. On the other hand, in Modular Representation Theory, very little is known about the analogous number $k(B)$ of irreducible ordinary characters belonging to a Brauer p-block B of G. Brauer's celebrated $k(B)$-conjecture asserts that the number of irreducible characters in a p-block B is bounded from above by the order of a defect group D of B. This conjecture remains open (and even without a reduction theorem to finite simple groups) at the present time. In this paper, we are concerned with the related problem of classifying the defect groups of p-blocks B with a small number $k(B)$, which already appears to be difficult.

It is well-known that $k(B) = 1$ if, and only if, D is trivial. It is also true that $k(B) = 2$ if, and only if, $D \cong \mathsf{C}_2$ ([3]). Külshammer, Navarro, Sambale, and Tiep conjecture that $k(B) = 3$ if, and only if, $D \cong \mathsf{C}_3$ in [33]. They prove that their conjecture holds for a p-block B if that block satisfies the statement of the Alperin-McKay conjecture in [33, Theorem 4.2]. The case where $l(B) = 1$, that is, the p-block B has a unique simple module, was proved in [32]. In general, this conjecture is still open.

More can be said if we restrict ourselves to the case of principal p-blocks. In the following, we denote by $B_0(G)$ (or simply B_0 if the context is clear) the principal p-block of G, so that $D \in \mathrm{Syl}_p(G)$. For example, Belonogov showed that $k(B_0) = 3$ if, and only if, $D \cong \mathsf{C}_3$ in [2]. This paper is not easily available. Recently, Koshitani and Sakurai have provided in [31] an alternative proof of this result and we will rely on their proof throughout this article. Moreover, Koshitani and Sakurai [31] have shown that $k(B_0) = 4$ implies that $D \in \{\mathsf{C}_2 \times \mathsf{C}_2, \mathsf{C}_4, \mathsf{C}_5\}$. In this note, we go one step further and analyze the isomorphism classes of Sylow p-subgroups of groups G for which $B_0(G)$ has exactly 5 irreducible characters.

Theorem A. *Let G be a finite group, p be a prime, and $P \in \mathrm{Syl}_p(G)$. Denote by B_0 the principal p-block of G. If $k(B_0) = 5$, then $P \in \{\mathsf{C}_5, \mathsf{C}_7, \mathsf{D}_8, \mathsf{Q}_8\}$.*

Notice that all the groups described in of Theorem A occur as Sylow p-subgroups of principal p-blocks with 5 irreducible characters. For example, $G = \mathsf{C}_5$ and $p = 5$ yields the case $P = \mathsf{C}_5$, $G = \mathsf{D}_{14}$ and $p = 7$ yields the case $P = \mathsf{C}_7$, $G = \mathsf{D}_8$ and $p = 2$ yields the case $P = \mathsf{D}_8$, and $G = \mathsf{Q}_8$ and $p = 2$ yields the case $P = \mathsf{Q}_8$.

We remark that for general p-blocks B with $k(B) = 5$ and $l(B) = 1$, Chlebowitz and Külshammer proved that $D \in \{\mathsf{C}_5, \mathsf{D}_8, \mathsf{Q}_8\}$ in [15].

Our proof of Theorem A is based on the analysis of the structure of a counterexample G of minimal order. We first prove that such a G must be almost simple in Theorem 2.1. Our methods for proving Theorem 2.1 can also be used to provide independent reductions to simple and, respectively, almost simple groups of the corresponding statements for $k(B_0) = 3$ [31, Theorem 3.1] and $k(B_0) = 4$ [31, Theorem 5.1 and 6.1]. As explained by the authors in [31], their reductions depend on the study of the structure of Sylow p-subgroups of finite groups having exactly 2 G-conjugacy classes of p-elements carried out in [33]. The main difference is that our methods would remove the dependence on that result.

With Theorem 2.1 in place and by using the Classification of Finite Simple Groups, we are left to study the principal p-blocks of finite almost simple groups whose socle is a group of Lie type. This is done in Section 3.

We finish this introduction by mentioned that, as noticed by one of the referees, the statement of Theorem A follows from the principal block case of the Alperin-McKay conjecture. In particular, we can see Theorem A as new evidence in support of this long-standing conjecture.

1. Preliminaries and background results

Our notation for characters and p-blocks follows [42]. For a fixed prime p, we refer to p-blocks just as blocks. In this section we collect some results on blocks that we will use later on, as well as background results on blocks with a small number of irreducible characters.

We start by recalling some facts about *covering blocks*. For a definition and first properties of covering blocks we refer the reader to [42, Chapter 9]. Recall that if N is a normal subgroup of G and B and b are blocks of G and N respectively, then B covers b if there are $\chi \in \mathrm{Irr}(B)$ and $\theta \in \mathrm{Irr}(b)$ such that θ is an irreducible constituent of the restriction χ_N (see [42, Theorem 9.2]). Then it is clear that $B_0(G)$ covers $B_0(N)$.

Lemma 1.1. *Let $N \lhd G$. Then for every $\theta \in \mathrm{Irr}(B_0(N))$, there exists $\chi \in \mathrm{Irr}(B_0(G) \mid \theta)$.*

Proof. This is [42, Theorem 9.4]. □

We will often use the following two facts in Sections 2 and 3.

Lemma 1.2. *Let $N \lhd G$. Suppose that $B \in \mathrm{Bl}(G)$ is the only block covering $b \in \mathrm{Bl}(N)$. Then for every $\theta \in \mathrm{Irr}(b)$, the set $\mathrm{Irr}(G|\theta) \subseteq \mathrm{Irr}(B)$.*

Proof. Let $\chi \in \mathrm{Irr}(G|\theta)$ and let $\mathrm{Bl}(\chi)$ be the p-block of G containing χ. Then $\mathrm{Bl}(\chi)$ covers b (see [42, Theorem 9.2]) and hence $\mathrm{Bl}(\chi) = B$. □

Lemma 1.3. *Let $M \lhd G$ and $P \in \mathrm{Syl}_p(G)$. If $P\mathbf{C}_G(P) \subseteq M$, then $B_0(G)$ is the only block covering $B_0(M)$. In particular, $k(G/M) < k(B_0(G))$ as long as $P > 1$.*

Proof. Write $B_0 = B_0(G)$ and $b_0 = B_0(M)$ to denote the principal p-blocks of G and M respectively. By [42, Theorem 4.14 and Problem 4.2] and Brauer's third main theorem [42, Theorem 6.7], we have that $b_0^G = B_0$. Let B be another block of G covering b_0. By [42, Theorem 9.26], $B \in \mathrm{Bl}(G|P)$. By [42, Theorem 9.19, Lemma 9.20] we have that

B is regular with respect to M and hence $B = b_0^G = B_0$. The last part follows from Lemma 1.2 as $k(b_0) > 1$ if $P > 1$. □

Next we record some observations that will be useful when dealing with quotients. The following can be seen using Clifford theory. A proof may be found, for example, in [47, Lemma 3.2.7].

Lemma 1.4. *Let G be a finite group and $N \lhd G$ such that G/N is cyclic. Let $\chi \in \mathrm{Irr}(G)$. Then the number of irreducible constituents of the restriction χ_N is the number of $\beta \in \mathrm{Irr}(G/N)$ such that $\chi\beta = \chi$.*

The next lemma can be found, for example, as [10, Lemma 17.2].

Lemma 1.5. *Let G be a finite group. Two characters of $G/\mathbf{Z}(G)$ are in the same block if and only if they are in the same block as a character of G.*

If B is a p-block with defect group D, then $\mathrm{Irr}_0(B) = \{\chi \in \mathrm{Irr}(B) \mid \chi(1)_p = |G : D|_p\}$, where n_p denotes the p-part of the integer n. We write $k_0(B) = |\mathrm{Irr}_0(B)|$ for the number of height zero characters in B (if $D > 1$ then it is well-known that $k_0(B) \geq 2$). If $B = B_0(G)$ is the principal p-block of G, then $D \in \mathrm{Syl}_p(G)$ and $\mathrm{Irr}_0(B)$ is the subset of p'-degree characters of B.

We will often use the following results on the number of height zero characters in (principal) blocks for $p \in \{2, 3\}$ in Section 2.

Theorem 1.6. *Let B be a 2-block of G with defect group D.*

(a) *If $2 \mid |D|$ then $2 \mid k_0(B)$.*
(b) *If $4 \mid |D|$ then $4 \mid k_0(B)$.*
(c) *$|D| = 2$ if, and only if, $k_0(B) = 2$.*

Proof. Parts (a) and (b) can be proven following the ideas in the proof of [43, Lemma 2.2] (part (b) is [34, Corollary 1.3]). The "if" implication in part (c) follows from parts (a) and (b). If $|D| = 2$, then the block B is nilpotent and by [6, Theorem 1.2, Corollary 1.4] $k_0(B) = 2$. □

We remark that, although Theorem 1.6 does not rely on the Classification of Finite Simple Groups, the following extension to the case $p = 3$ (more specifically, part (b) in Theorem 1.7 below) does.

Theorem 1.7. *Let B be a principal 3-block of G with defect group D.*

(a) *If $3 \mid |D|$ then $3 \mid k_0(B)$.*
(b) *If $B = B_0(G)$ is the principal 3-block of G, then $|D| = 3$ if, and only if, $k_0(B) = 3$*

Proof. Part (a) is [34, Corollary 1.6]. Part (b) is [43, Theorem C]. $\square$

We summarize below what is known on (principal) blocks with up to 4 irreducible characters. Our proof of Theorem A relies on those facts.

Theorem 1.8. *Let B be a p-block of a finite group G with defect group D.*

(a) $k(B) = 1$ *if, and only if,* $D = 1$.
(b) $k(B) = 2$ *if, and only if,* $D \cong \mathsf{C}_2$.
(c) *If* $B = B_0(G)$ *the principal p-block, then* $k(B) = 3$ *if, and only if,* $D \cong \mathsf{C}_3$.
(d) *If* $B = B_0(G)$ *the principal p-block, then* $k(B) = 4$ *implies* $D \in \{\mathsf{C}_2 \times \mathsf{C}_2, \mathsf{C}_4, \mathsf{C}_5\}$.

Proof. Part (a) is [42, Theorem 3.18]. Part (b) is [3, Theorem A]. The "only if" implication in part (c) follows from [32] (when $l(B) = 1$) and [31] (when $l(B) = 2$). The converse in part (c) follows since $k(B) \leq 3$ by [4, Theorem 1*] and parts (a) and (b). Part (d) is [31, Theorem 1.1] (note that if $l(B_0(G)) = 1$, then G has a normal p-complement K and $k(B_0(G)) = k(B_0(G/K)) = k(D) = 4$). $\square$

2. A reduction to almost simple groups

The aim of this section is to prove the following reduction theorem.

Theorem 2.1. *Let G be a finite group, p a prime and $P \in \mathrm{Syl}_p(G)$. If G is a minimal counterexample to the statement of Theorem A, then G is almost simple. Write $S = \mathrm{soc}(G)$ and $\bar{B}_0 = B_0(G/S)$. Then either p does not divide $|G : S|$ and $S\mathbf{C}_G(P) < G$, or $k(\bar{B}_0) = 4$. Moreover S is a simple group of Lie type not isomorphic to an alternating or sporadic group.*

The following technical result will be useful for handling some cases in the proof of Theorem 2.1.

Lemma 2.2. *Let $M \lhd G$ be such that $P\mathbf{C}_G(P) \subseteq M$, where $P \in \mathrm{Syl}_p(G)$. Write $B_0 = B_0(G)$ and $b_0 = B_0(M)$. If $k(B_0) = 5$, then $k(b_0) \in \{4, 5, 7, 11, 13\}$. Moreover, if $p = 2$ and $M < G$, then $k(b_0) = 7$.*

Proof. By Lemma 1.3, the block B_0 is the only block covering b_0, so that $\mathrm{Irr}(G/M) \subseteq \mathrm{Irr}(B_0)$ and $1 \leq k(G/M) < 5$. If $G/M = 1$, then $k(b_0) = k(B_0) = 5$, and we are done. Hence $1 < k(G/M) < 5$. By [53], $G/M \in \{\mathsf{C}_2, \mathsf{C}_3, \mathsf{S}_3, \mathsf{C}_4, \mathsf{C}_2 \times \mathsf{C}_2, \mathsf{D}_{10}, \mathsf{A}_4\}$. We analyze $k(b_0)$ depending on the isomorphism class of G/M.

Suppose that $G/M \cong \mathsf{C}_2$. Write $\mathrm{Irr}(B_0) = \{1_G, \lambda, \chi, \psi, \xi\}$, where λ lies over 1_M. If some $1_M \neq \theta \in \mathrm{Irr}(b_0)$ is G-invariant, then θ has two extensions, say χ and ψ, both in $\mathrm{Irr}(B_0)$ by Lemma 1.2. Then ξ_M decomposes as a sum of two distinct elements of $\mathrm{Irr}(B_0)$

and $k(b_0) = 4$. Otherwise, every character in $\mathrm{Irr}(B_0) \setminus \mathrm{Irr}(G/M)$ is induced from M and $k(b_0) = 7$.

Suppose that $G/M \cong \mathsf{C}_3$, reasoning as before one can show that every character in $\mathrm{Irr}(B_0)\setminus\mathrm{Irr}(G/M)$ must be induced from an irreducible character of M, hence $k(b_0) = 7$.

Suppose that $G/M \cong \mathsf{S}_3$. Let $M \le K \triangleleft G$ with $|K : M| = 3$. Note that $B_0(K)$ is the only block covering b_0 and B_0 is the only block covering $B_0(K)$ by Lemma 1.3. If some $1_M \neq \theta \in \mathrm{Irr}(b_0)$ is G-invariant, then θ has 3 extensions in $B_0(K)$ and at least one of them, namely ξ, is G-invariant by counting. Hence ξ has two extensions in B_0. That leaves no space for characters over the other two extensions of θ to K. Hence, we may assume no nontrivial character in $\mathrm{Irr}(b_0)$ is G-invariant. Similarly, if some $1_M \neq \theta \in \mathrm{Irr}(b_0)$ is K-invariant, using the Clifford correspondence, we would obtain 3 distinct characters in $\mathrm{Irr}(B_0)$ not containing M in their kernels, absurd. Write $\mathrm{Irr}(B_0) = \{1_G, \lambda, \mu, \chi, \psi\}$, where λ and μ lie over 1_M. Let $\theta \in \mathrm{Irr}(b_0)$ lie under χ or ψ. If $M < G_\theta$, then $|G_\theta : M| = 2$, then θ has 2 extensions $\xi_i \in \mathrm{Irr}(B_0(G_\theta))$ for $i = 1, 2$ and, since B_0 is the only block covering b_0, by the Clifford correspondence $\xi_i^G \in \mathrm{Irr}(B_0)$ for $i = 1, 2$ are distinct so $\{\xi_i^G\} = \{\chi, \psi\}$. In particular, $k(b_0) = 2$. In this case $P \cong \mathsf{C}_2$ which implies $k(B_0) = 2$, a contradiction. Hence $G_\theta = M$, so χ_M and ψ_M decompose as a sum of $|G/M|$ distinct characters, yielding $k(b_0) = 13$.

Otherwise $k(G/M) = 4$, and consequently G/M acts transitively on the set of nontrivial characters of $\mathrm{Irr}(b_0)$, all of which lie under the only $\chi \in \mathrm{Irr}(B_0)$ that does not lie over 1_M. In particular, by Clifford's theorem $k(b_0) = 1 + |G : G_\theta|$, where $1_M \neq \theta \in \mathrm{Irr}(b_0)$. If $G_\theta = G$, then $k(b_0) = 2$ and $P \cong \mathsf{C}_2$ contradicting $k(B_0) = 5$. Reasoning as before, one can check case by case that if $1_M \neq \theta \in \mathrm{Irr}(b_0)$, then $G_\theta = M$ or $|G_\theta : M| = 4$. Hence [53] implies that $k(b_0) = 1 + |G : G_\theta| \in \{4, 5, 11, 13\}$ completing the proof. Note that if $p = 2$ then $|G/M|$ is odd and hence $G/M \cong \mathsf{C}_3$ and $k(b_0) = 7$. □

Our proof of Theorem 2.1 uses the principal block case of a celebrated result of Kessar–Malle, whose proof relies on the Classification of Finite Simple Groups.

Theorem 2.3. *[30] Let B be the principal block of G and let $P \in \mathrm{Syl}_p(G)$. If P is abelian then $k(B) = k_0(B)$.*

We will also make use of Alperin-Dade's theory of isomorphic blocks.

Theorem 2.4. *Suppose that N is a normal subgroup of G, with G/N a p'-group. Let $P \in \mathrm{Syl}_p(G)$ and assume that $G = N\mathbf{C}_G(P)$. Then restriction of characters defines a natural bijection between the irreducible characters of the principals blocks of G and N.*

Proof. The case where G/N is solvable was proved in [1] and the general case in [17]. □

Proof of Theorem 2.1. If x is a p-element of G, then $B_0(\mathbf{C}_G(x))^G$ is defined by [42, Theorem 4.14] and hence, by [42, Theorem 5.12] and Brauer's third main theorem [42, Theorem 6.7], we have that

$$5 = k(B_0) = |\mathbf{Z}(G)|_p l(B_0) + \sum_{i=1}^{k} l(B_0(\mathbf{C}_G(x_i))\,, \tag{1}$$

where $\{x_1, x_2, \dots, x_k\}$ is a complete set of representatives of the conjugacy classes of non-central p-elements of G.

Step 1. We may assume that $\mathbf{O}_{p'}(G) = 1$. This is due to the minimality of G as a counterexample and [42, Theorem 9.9.(c)].

Step 2. We may assume that G is not p-solvable. Otherwise, by Step 1 and [42, Theorem 10.20], G has a unique p-block and hence $k(B_0) = k(G) = 5$. In this case, by [53] we have that $G \in \{\mathsf{C}_5, \mathsf{D}_8, \mathsf{Q}_8, \mathsf{D}_{14}, \mathsf{C}_5 \rtimes \mathsf{C}_4, \mathsf{C}_7 \rtimes \mathsf{C}_3, \mathsf{S}_4\}$ (the prime is clear in each case since G has a unique p-block). Then $P \in \{\mathsf{C}_5, \mathsf{C}_7, \mathsf{D}_8, \mathsf{Q}_8\}$, and G is not a counterexample.

Step 3. We have $1 < l(B_0) < 5$ and $1 \le k \le 3$. If $l(B_0) = 1$ or $P \subseteq \mathbf{Z}(G)$ then G has a normal p-complement (by [42, Corollary 6.13] or by the Schur-Zassenhaus theorem respectively), contradicting Step 2.

Step 4. We may assume that $\mathbf{Z}(G) = 1$. Indeed, by Step 1, $\mathbf{Z}(G)$ is a p-group. If $\mathbf{Z}(G) > 1$, Equation (1) and Step 3 force $|\mathbf{Z}(G)| = 2$, $l(B_0) = 2$, $k = 1$ and $l(B_0(\mathbf{C}_G(x_1))) = 1$. Note that $p = 2$ in this case. Write $N = \mathbf{Z}(G)$ and $\bar{B}_0 = B_0(G/N) \subseteq B_0$. Since N is a nontrivial 2-group, we have that $k(\bar{B}_0) < k(B_0)$. By [42, Theorem 9.10] we have $l(\overline{B}_0) = l(B_0) = 2$.

If $x_1 N \in \mathbf{Z}(G/N)$, since G has a unique conjugacy class of non-central 2-elements, then $P/N \subseteq \mathbf{Z}(G/N)$. In particular, $P \triangleleft G$ contradicting Step 2. Otherwise $x_1 N$ is a non-central p-element of G/N and then, applying [42, Theorem 5.12] to G/N, we have $2 \le |\mathbf{Z}(G/N)|_2 l(\bar{B}_0) < k(\bar{B}_0) \le 4$. Then $k(\bar{B}_0) \in \{3, 4\}$. If $k(\bar{B}_0) = 3$, we have that $p = 3$, a contradiction. If $k(\bar{B}_0) = 4$, since $p = 2$ we get $|P/N| = 4$ by Theorem 1.8(d). Hence $|P| = 8$ and is non-abelian (if P is abelian we get 4 divides 5 by Theorem 1.6 and Theorem 2.3), and G is not a counterexample.

Step 5. G has a unique minimal normal subgroup N.

By Step 1, p divides the order of every minimal normal subgroup of G. Since $k \le 3$ and $\mathbf{Z}(G) = 1$, if G has more than one minimal normal subgroup, then it has exactly two, namely N_1 and N_2, and $k = 3$. Write $N = N_1 \times N_2 \triangleleft G$. Note that G has no p-elements outside N, so $P \subseteq N$ and G/N is a p'-group. Let $1 \ne x_1 \in N_1$ and $1 \ne x_2 \in N_2$, be p-elements. By Equation (1) we know that $l(B_0(\mathbf{C}_G(x_i))) = 1$ hence the groups $\mathbf{C}_G(x_i)$ have normal p-complements by [42, Corollary 6.13]. Since $N_2 \le \mathbf{C}_G(x_1)$ and $N_1 \le \mathbf{C}_G(x_2)$, we see that N_i have normal p-complements too. By Step 1 this forces N_i to be p-groups. Then $P = N \triangleleft G$ and G is p-solvable, contradicting Step 2.

Write $\bar{B}_0 = B_0(G/N)$. Since $\bar{B}_0 \subseteq B_0$ we have that $1 \le k(\bar{B}_0) \le 5$. Moreover $k(\bar{B}_0) < 5$ as otherwise $k(B_0(N)) = 1$ (contradicting Step 1). We have seen that $1 \le k(\bar{B}_0) < 5$.

Step 6. N is semisimple of order divisible by p.

Otherwise N is abelian and by Step 1 we have that N is an elementary abelian p-group. Note that $P/N > 1$ as otherwise G would be p-solvable contradicting Step 2. Hence, in this case, $1 < k(\bar{B}_0) < 5$.

Also by Step 1, we have that $\mathbf{O}_{p'}(\mathbf{C}_G(N)) = 1$, and by Step 4, we have that $\mathbf{C}_G(N) < G$. Suppose that $\mathbf{C}_P(N) = \mathbf{C}_G(N) \cap P = N$. Then $N \in \mathrm{Syl}_p(\mathbf{C}_G(N))$, and $\mathbf{C}_G(N)$ has a normal p-complement, forcing $\mathbf{C}_G(N) = N$. By [42, Theorem 9.21] B_0 is the only block covering $B_0(N)$. In particular, we have that $k(G/N) < 5$. By the classification in [53], G/N is solvable, so is G, a contradiction with Step 2. Hence $N < \mathbf{C}_P(N)$. We study now the different values of $k(\bar{B}_0)$.

If $k(\bar{B}_0) = 2$, then $p = 2$ and $P/N \cong \mathsf{C}_2$ by Theorem 1.8(b). It is well-known that a group with a cyclic Sylow 2-subgroup has a normal 2-complement (see Corollary 5.14 of [29], for instance). Then G/N has a normal p-complement and therefore G is p-solvable contradicting Step 2.

If $k(\bar{B}_0) = 3$, then $p = 3$ and $P/N \cong \mathsf{C}_3$ by Theorem 1.8(c). Then $P = \mathbf{C}_P(N)$, so that $N \subseteq \mathbf{Z}(P)$ and P is abelian. By Theorem 2.3, $k_0(B_0) = k(B_0) = 5$. By Theorem 1.7, 3 divides $k_0(B_0)) = 5$, a contradiction.

If $k(\bar{B}_0) = 4$ then, by Theorem 1.8(d), either $p = 2$ and $|P/N| = 4$ or $p = 5$ and $|P/N| = 5$. Write $M = \mathbf{C}_G(N)$. Recall that $N < M \cap P \leq P$.

- Suppose that $|P/N| = 4$ (so $p = 2$).

If $|M \cap P : N| = 2$, then $|G/M|_2 = |P : M \cap P|_2 = |M : N|_2 = 2$. Hence G/M and M/N have normal p-complements, and G is p-solvable contradicting Step 2.

It remains to consider the situation where $P \subseteq M$. In this case $P\mathbf{C}_G(P) \subseteq M$. By Lemma 2.2, $k(B_0(M)) = 7 \geq |\mathbf{Z}(M)|_p l(B_0(M))$ (using Theorem 5.12 of [42]). In particular, since $N \subseteq \mathbf{Z}(M)$, we have that $|N|$ divides 4. If $|N| = 2$, then $|P| = 8$ contradicting the minimality of G as a counterexample (again P is nonabelian as otherwise Theorem 2.3 and Theorem 1.6 imply 4 divides 5). Otherwise $|N| = 4$ and $l(B_0(M)) = 1$. Then M has a normal 2-complement. By Step 1 this forces $M = P \triangleleft G$, a contradiction with Step 2.

- Suppose that $|P/N| = 5$ (so $p = 5$). Then $P \subseteq \mathbf{C}_G(N) = M$, so that Lemma 2.2 applies to $M \triangleleft G$. By Lemma 2.2 and Theorem 5.12 of [42] $13 \geq k(B_0(M)) \geq |\mathbf{Z}(M)|_p l(B_0(M))$. Since $N \subseteq \mathbf{Z}(M)$, these inequalities force $N \cong \mathsf{C}_5$. Then $G/M \leq \mathrm{Aut}(N) \cong \mathsf{C}_4$. From the proof Lemma 2.2 one can actually conclude that $5 \leq k(B_0(M)) \in \{4, 5, 7\}$, so that $l(B_0(M)) = 1$. Yielding $M = P \triangleleft G$, a contradiction with Step 2.

Step 7. N is (nonabelian) simple of order divisible by p. Moreover either G/N is a p'-group or $k(\bar{B}_0) = 4$.

By Step 6 we have that N is semisimple. Write $N = S_1 \times S_2 \times \cdots \times S_t$, where $S_i \cong S$ and S is a nonabelian simple with $p \mid |S|$.

If $|G/N|$ is not divisible by p, then $P \subseteq N$. Let $M = N\mathbf{C}_G(P) \triangleleft G$ by the Frattini argument. By Theorem 2.4, we have that $k(B_0(M)) = k(B_0(N)) = k(B_0(S))^t$. By Lemma 2.2, the equality $k(B_0(M)) = k(B_0(S))^t$ with $t > 1$ yields a contradiction unless $t = 2$ and $k(B_0(S)) = 2$, which is absurd as S is nonabelian simple.

If $|G/N|$ divisible by p, then $1 < k(\bar{B}_0) < 5$. We again study the different values of $k(\bar{B}_0)$.

Suppose that $k(\bar{B}_0) = 2$. Then $p = 2$ and $|PN/N| = 2$ by Theorem 1.8(b). In particular, G/N has a normal 2-complement X/N. By [42, Corollary 9.6], B_0 is the only block covering $B_0(X)$ and hence $\mathrm{Irr}(G/X) \subseteq \mathrm{Irr}(B_0)$. Let $1_X \neq \tau \in \mathrm{Irr}(B_0(X))$. If τ extends to G then it has two extensions, all in B_0 by Lemma 1.2. Therefore $k(B_0(X)) = 4$. Since $p = 2$ and $P \cap N = P \cap X \in \mathrm{Syl}_p(X)$, by Theorem 1.8(d) we have $|P \cap N| = 4$ and hence $|P| = 8$ (note that P is nonabelian as otherwise Theorem 2.3 and Theorem 1.6 imply 4 divides 5). This contradicts the minimality of G as a counterexample. Hence we may assume that $G_\tau = X$ for all non-trivial $\tau \in \mathrm{Irr}(B_0(X))$ and therefore 2 divides $\chi(1)$ for all $\chi \in \mathrm{Irr}(B_0) \setminus \mathrm{Irr}(G/X)$. Hence $k_0(B_0) = 2$ and Theorem 1.6 implies that $|G|_2 = 2$, a contradiction by Theorem 1.8(b).

Suppose that $k(\bar{B}_0)) = 3$. Then $p = 3$ and $|PN/N| = 3$ by Theorem 1.8(c). By Theorem 1.7, 3 divides $k_0(B_0) \leq 5$, and hence $3 = k_0(B_0)$. By Theorem 1.7, this implies that $|G|_3 = 3$. As $|PN/N| = 3$, this forces $P \cap N = 1$, a contradiction.

Suppose that $k(\bar{B}_0) = 4$. Then all nontrivial irreducible characters in $\mathrm{Irr}(B_0(N))$ lie under the same irreducible character of $\mathrm{Irr}(B_0)$ and hence are G-conjugate. This forces $t = 1$.

By Steps 5 and 7, the unique minimal normal subgroup N of G is nonabelian simple and $\mathbf{C}_G(N) = 1$. Hence G is almost simple and $N = \mathrm{soc}(G)$. Note that in the case where G/N is a p'-group, we have that $N\mathbf{C}_G(P) < G$ by Theorem 2.4 together with the minimality of G as a counterexample. Otherwise $k(\bar{B}_0) = 4$ by Step 7.

Write $S = \mathrm{soc}(G)$. We have seen that G/S is either a nontrivial p'-group or its principal p-block $\bar{B}_0$ has exactly 4 irreducible characters. Using [22] and the GAP Character Table Library, one can check that if S is a sporadic group or a simple alternating group A_n with $n \leq 6$, then G is not a counterexample to the statement of Theorem A. Suppose that $S \cong \mathsf{A}_n$ is a simple alternating group with $n > 6$. In such cases $|G : S| \leq 2$. In particular $k(\bar{B}_0) \neq 4$ and hence $|G : S| = 2$ as $S < G$. Then $G \cong \mathsf{S}_n$ and p is odd. In this case $P \in \mathrm{Syl}_p(S)$. By [45, Proposition 4.10], we have that $k(B_0(S)) \leq k(B_0) = 5$. By minimality of G as a counterexample, we actually have that $k(B_0(S)) < 5$. By Theorem 1.8, either $p = 3$ and $P \cong \mathsf{C}_3$, which is absurd as $n > 6$, or $p = 5$ and $P \cong \mathsf{C}_5$, contradicting the choice of G as a counterexample. □

3. Simple groups of Lie type

In this section, we prove that finite almost simple groups with socle a group of Lie type do not provide counterexamples to the statement of Theorem A. In view of Theorem 2.1, that will complete the proof of Theorem A.

3.1. Preliminaries

First, we check that the almost simple groups extending several small groups of Lie type satisfy Theorem A.

Proposition 3.1. *Let G be an almost simple group such that its simple socle S is* ${}^2\mathrm{F}_4(2)'$, $\mathrm{F}_4(2)$, $\mathrm{G}_2(3)$, $\mathrm{G}_2(4)$, $\mathrm{PSL}_2(8) \cong {}^2\mathrm{G}_2(3)'$, $\mathrm{B}_3(3)$, ${}^2\mathrm{B}_2(8)$, ${}^2\mathrm{E}_6(2)$, $\mathrm{Sp}_4(4)$, $\mathrm{Sp}_6(2)$, $\mathrm{D}_4(2)$, $\mathrm{PSU}_6(2)$, $\mathrm{PSL}_3(2)$, $\mathrm{PSL}_3(4)$, $\mathrm{PSp}_4(3) \cong \mathrm{PSU}_4(2)$, $\mathrm{PSU}_4(3)$, *or* $\mathrm{PSL}_2(7)$. *Then Theorem A holds for G for all primes p dividing $|G|$. Further, Theorem A holds for G if $p = 2$ and S is one of* $\mathrm{Sp}_4(8)$ *or* $\mathrm{PSU}_5(4)$ *and for the prime $p = 3$ if S is* $\mathrm{D}_4(3)$.

Proof. This can be seen using [22]. $\square$

Now, given an almost simple group G with socle S, with Lemma 1.1 in mind, our strategy for proving Theorem A in most cases will be to demonstrate at least 6 non-$\mathrm{Aut}(S)$-conjugate members of $B_0(S)$. For $N \lhd G$, we write $k_G(B_0(N))$ for the number of distinct G-classes of characters in $B_0(N)$. Note that $k(B_0(G)) \geq k_G(B_0(N))$, using Clifford theory and Lemma 1.1.

We next address some immediate cases that further reduce the situation, when combined with Theorem 2.1.

Lemma 3.2. *Let G be a finite group and p a prime dividing $|G|$ such that $P \in \mathrm{Syl}_p(G)$ is cyclic. Then G is not a counterexample to Theorem A for the prime p.*

Proof. Write $B_0 := B_0(G)$. According to [16, Theorem 5.1.2(ii)], we have

$$k(B_0) = e + \frac{|P| - 1}{e}$$

where $e := |\mathbf{N}_G(P) : \mathbf{C}_G(P)|$. From this we see that if $|P| \geq 8$, then $k(B_0) > 5$, and if $|P| \leq 4$, then $k(B_0) \leq 4$. In either of these cases, G is then not a counterexample to Theorem A. If $|P| \in \{5, 7\}$, then $P \in \{\mathsf{C}_5, \mathsf{C}_7\}$, and hence G is again not a counterexample to Theorem A, completing the claim. $\square$

Lemma 3.3. *Assume that G is almost simple with socle S and that $k(B_0(G/S)) = 4$. If $k_G(B_0(S)) \geq 3$, then G is not a counterexample to Theorem A.*

Proof. Note that by assumption, $B_0(G)$ contains at least 2 characters that are nontrivial on S, using Lemma 1.1. But $k(B_0(G/S)) = 4$ implies there are 4 additional characters in $B_0(G)$ that are trivial on S, since $B_0(G/S) \subseteq B_0(G)$. Hence $k(B_0(G)) \geq 6$. $\square$

3.2. Additional notation

To ease our notation, we will switch to using A for the remainder of the paper to denote an almost simple group. We will let G be a quasisimple group of Lie type with $S = G/\mathbf{Z}(G)$, where $\mathbf{Z}(G)$ is a non-exceptional Schur multiplier for S and G is defined over $\mathbb{F}_q$, where q is a power of the prime q_0. (Note that with Proposition 3.1 and Theorem 2.1, we have already completed the case that S has an exceptional Schur multiplier. See e.g. [25,

Table 6.1.3] for the list of simple groups of Lie type with exceptional Schur multipliers.) Then we may assume that G is the set of fixed points $\mathbf{G}^F$ of a simple, simply connected algebraic group $\mathbf{G}$ defined over $\overline{\mathbb{F}}_q$, under a Steinberg endomorphism F.

Throughout, for $\epsilon \in \{\pm 1\}$ (or simply $\{\pm\}$), the notation $\mathrm{PSL}_n^\epsilon(q)$ will denote $\mathrm{PSL}_n(q)$ for $\epsilon = 1$ and $\mathrm{PSU}_n(q)$ for $\epsilon = -1$. Similarly, $\mathrm{P\Omega}_{2n}^\epsilon(q)$ will denote the simple group $\mathrm{P\Omega}_{2n}(q)$ of type $\mathrm{D}_n(q)$ for $\epsilon = 1$ and its twisted counterpart $\mathrm{P\Omega}_{2n}^-(q)$ of type ${}^2\mathrm{D}_n(q)$ for $\epsilon = -1$. An analogous meaning will be used for related groups such as $\mathrm{GL}_n^\epsilon(q)$, $\mathrm{SL}_n^\epsilon(q)$, $\mathrm{GO}_{2n}^\epsilon(q)$, and $\mathrm{SO}_{2n}^\epsilon(q)$ and for the groups $\mathrm{E}_6^\epsilon(q)$ representing $\mathrm{E}_6(q)$ and ${}^2\mathrm{E}_6(q)$.

3.3. Defining characteristic

In this section, we will assume $q_0 = p$, so that S is defined in the same characteristic as the blocks under consideration. In this case, using [9, Theorem 3.3], we see that there are exactly two p-blocks for S: $B_0(S)$ and a defect-zero block containing the Steinberg character. With this in mind, we have $k(B_0(S)) = k(S) - 1$. Combining with Lemma 1.1, for $S \leq A \leq \mathrm{Aut}(S)$, it suffices to show $\frac{k(S)-1}{|A/S|} \geq 6$ when $P \in \mathrm{Syl}_p(A)$ is not one of $\{\mathsf{C}_5, \mathsf{C}_7, \mathsf{D}_8, \mathsf{Q}_8\}$. Now, note further that $k(S) \geq k(G)/|\mathbf{Z}(G)|$, giving a rough bound

$$k(B_0(A)) \geq k_A(B_0(S)) \geq \frac{k(G) - |\mathbf{Z}(G)|}{|\mathbf{Z}(G)||\mathrm{Out}(S)|.} \tag{2}$$

Further, [12, Theorem 3.7.6] implies that the number of semisimple classes of G is $|(\mathbf{Z}(\mathbf{G})^\circ)^F|q^r$, where r is the rank of G, which forces $k(G) > q^r$ since G also contains non-semisimple classes. Combining this with (2), we see that

$$k(B_0(A)) > \frac{q^r - |\mathbf{Z}(G)|}{|\mathbf{Z}(G)||\mathrm{Out}(S)|}. \tag{3}$$

Proposition 3.4. *Let S be a simple group of Lie type defined in characteristic p such that S is not isomorphic to an alternating group nor one of the groups treated in Proposition 3.1. Then $k(B_0(A)) \geq 6$ for any $S \leq A \leq \mathrm{Aut}(S)$, where $B_0(A)$ is the principal p-block of A.*

Proof. Recall that our assumptions assure $S = G/\mathbf{Z}(G)$, where G is a group of Lie type in characteristic p whose underlying algebraic group is simple and simply connected and $\mathbf{Z}(G)$ is a nonexceptional Schur multiplier for S.

First, [35] contains the explicit values of $k(G)$ for G of exceptional type ${}^2\mathrm{B}_2(q)$, ${}^2\mathrm{G}_2(q)$, $\mathrm{G}_2(q)$, ${}^2\mathrm{F}_4(q)$, $\mathrm{F}_4(q)$, ${}^3\mathrm{D}_4(q)$, ${}^2\mathrm{E}_6(q)$, $\mathrm{E}_6(q)$, $\mathrm{E}_7(q)$, and $\mathrm{E}_8(q)$. Using the bound in (2), together with this information and the knowledge of $|\mathbf{Z}(G)|$ and $|\mathrm{Out}(S)|$ in each case, we see that the statement holds for these groups.

We may therefore assume that G is of classical type. Let $q = p^a$. Table 1 gives upper bounds on $|\mathbf{Z}(G)|$ and $|\mathrm{Out}(S)|$ in each case. We see using these bounds and (3) that $k(B_0(A)) \geq 6$ except possibly in the cases $\mathrm{PSp}_4(5)$, $\mathrm{Sp}_8(2)$, $\mathrm{P\Omega}_8^-(3)$, $\mathrm{PSU}_5(2)$, or $\mathrm{PSL}_n^\epsilon(q)$ with $n \leq 4$. (Recall that S is not as in Proposition 3.1.) It can be readily checked in GAP

Table 1
Bounds for $|\mathbf{Z}(G)|$ and $|\mathrm{Aut}(S)|$.

S	Upper Bound on $\lvert\mathbf{Z}(G)\rvert$	Upper Bound on $\lvert\mathrm{Out}(S)\rvert$
$\mathrm{PSL}_n^\epsilon(q)$	$p^a - \epsilon$	$2(p^a - \epsilon)a$
$\mathrm{P}\Omega_{2n}^\epsilon(q)$ with $n \geq 4$, q odd, and $(n, \epsilon) \neq (4, +1)$	4	$8a$
$\mathrm{P}\Omega_{2n}^\epsilon(q)$ with $n \geq 4$, $p = 2$, and $(n, \epsilon) \neq (4, +1)$	1	$2a$
$\mathrm{P}\Omega_8(q)$, q odd	4	$24a$
$\mathrm{P}\Omega_8(q)$, $p = 2$	1	$6a$
$\mathrm{PSp}_{2n}(q)$ or $\mathrm{P}\Omega_{2n+1}(q)$ with $n \geq 2$ and $(n, p) \neq (2, 2)$	2	$2a$
$\mathrm{Sp}_4(2^a)$	2	$4a$

that $k(B_0(S)) \geq 6$ if S is one of the first four in this list. Further, using the character tables in [14], we see that, since the cases with exceptional Schur multiplier are omitted here, the simple groups $\mathrm{PSL}_3^\epsilon(q)$ and $\mathrm{PSL}_4^\epsilon(q)$ under consideration have at least 6 distinct character degrees other than the Steinberg character, showing that $k_A(B_0(S)) \geq 6$ in this case when combined with Lemma 1.5 and recalling that $k(B_0(S)) = k(S) - 1$.

So, we now assume $S = \mathrm{PSL}_2(q)$. Then $\mathrm{Out}(S)$ is generated by the action of $\mathrm{PGL}_2(q)$ and the field automorphisms. The character tables for $\mathrm{PSL}_2(q)$, $\mathrm{SL}_2(q)$, and $\mathrm{PGL}_2(q)$ are well-known. First let q be odd and write $q \equiv \eta \pmod 4$ with $\eta \in \{\pm 1\}$. Then we have two characters of degree $\frac{q+\eta}{2}$ that are switched by the action of $\mathrm{PGL}_2(q)$ and fixed by the field automorphisms. For $\eta = 1$, there are $\frac{q-5}{4}$ characters of degree $q+1$ and $\frac{q-1}{4}$ characters of degree $q-1$. For $\eta = -1$, there are $\frac{q-3}{4}$ each of characters of degree $q-1$ and $q+1$. These characters are fixed by the action of $\mathrm{PGL}_2(q)$ but permuted by the field automorphisms. Recall that $q = p^a$, so the size of $\mathrm{Out}(S)$ is $2a$. Then in the case that p is odd, this yields $k_{\mathrm{Aut}(S)}(B_0(S)) \geq 2 + \frac{q-3}{2a} = 2 + \frac{p^a-3}{2a}$. This is larger than 5 for $q \geq 11$. Then by our assumption that S is not solvable, alternating, or as in Proposition 3.1, we are done in this case.

Finally, assume $p = 2$. Then S has $\frac{q}{2} - 1$ irreducible characters of degree $q+1$ and $\frac{q}{2}$ irreducible characters of degree $q-1$. Here $\mathrm{Out}(S)$ is cyclic of size a, where $q = 2^a$. This yields $k_{\mathrm{Aut}(S)}(B_0(S)) \geq 1 + \frac{2^a-1}{a}$, which is larger than 5 for $a \geq 5$. Further, the statement can be checked using GAP when $S = \mathrm{PSL}_2(16)$, completing the proof. □

3.4. Non-defining characteristic

From now on, let q be a power of a prime q_0 different from p and let $d_p(q)$ be the order of q modulo p if p is odd, and let $d_2(q)$ be the order of q modulo 4.

In this situation, it will often be useful to consider a certain collection of characters of a simple group S of Lie type known as *unipotent characters*. A block containing a unipotent character is called a unipotent block, and in particular the principal block is one such unipotent block.

If G is a group of Lie type such that the underlying algebraic group is simply connected and such that $S = G/\mathbf{Z}(G)$, and $\widetilde{G}$ is a group of Lie type such that the underlying algebraic group has a connected center and $G \lhd \widetilde{G}$ via a regular embedding as in [10, Section 15.1], then the work of Lusztig [36] (see also [24, 2.3.14 and 2.3.15]) shows that the unipotent characters of $\widetilde{G}$ are irreducible on restriction to G and trivial on $\mathbf{Z}(\widetilde{G})$ (and hence on $\mathbf{Z}(G)$). Therefore, unipotent characters in $B_0(\widetilde{G})$ may also be viewed as unipotent characters of S lying in $B_0(S)$ by applying Lemma 1.5. Further, Lusztig has determined the stabilizers in $\mathrm{Aut}(S)$ of the unipotent characters, which is also summarized in [39, Theorem 2.5]. In particular, the unipotent characters are stabilized by $\mathrm{Aut}(S)$ except in the following cases: $\mathrm{B}_2(2^{2n+1})$, $\mathrm{G}_2(3^{2n+1})$, and $\mathrm{F}_4(2^{2n+1})$, in which cases the exceptional graph automorphism permutes certain pairs of unipotent characters; $\mathrm{D}_4(q)$, in which case the graph automorphism of order three permutes two specific triples of unipotent characters, and $\mathrm{D}_n(q)$ with even $n \geq 4$, in which case the graph automorphism of order 2 interchanges the pairs of unipotent characters labeled by so-called degenerate symbols.

The set $\mathrm{Irr}(\widetilde{G})$ can be partitioned into so-called Lusztig series $\mathcal{E}(\widetilde{G}, s)$, which are indexed by $\widetilde{G}^*$-classes of semisimple elements s in a group $\widetilde{G}^*$, known as the dual group of $\widetilde{G}$. The elements of $\mathcal{E}(\widetilde{G}, s)$ are further in bijection with unipotent characters of $\mathbf{C}_{\widetilde{G}^*}(s)$, with $1_{\mathbf{C}_{\widetilde{G}^*}(s)}$ corresponding to the so-called *semisimple* character χ_s. We record the following, which is a specific case of [10, Theorem 9.12].

Lemma 3.5. *With the notation above, the characters lying in the unipotent p-blocks of $\widetilde{G}$ are exactly those characters in Lusztig series $\mathcal{E}(\widetilde{G}, t)$ indexed by semisimple p-elements t in $\widetilde{G}^*$. Further, $B_0(\widetilde{G})$ is the unique unipotent block of $\widetilde{G}$ if and only if all unipotent characters lie in $B_0(\widetilde{G})$.*

We also record the following observation, which has been useful in many contexts for determining elements of $\mathrm{Irr}_{p'}(B_0)$.

Lemma 3.6. *Let $G = \mathbf{G}^F$ be a group of Lie type defined over $\mathbb{F}_q$ such that the underlying algebraic group $\mathbf{G}$ is simple and simply connected and such that G is not of Suzuki or Ree type. Let $p \nmid q$ be a prime such that $d := d_p(q)$ is a regular number for G in the sense of Springer [51]. Then $B_0(G)$ is the unique block of G containing unipotent characters with p'-degree.*

Proof. By [38, Corollary 6.6], any unipotent character of p'-degree lies in a d-Harish-Chandra series indexed by (L, λ) where L is the centralizer $\mathbf{C}_G(\mathbf{S})$ of a Sylow d-torus $\mathbf{S}$ of $\mathbf{G}$. Further, $\mathbf{C}_{\mathbf{G}}(\mathbf{S})$ is a torus since d is regular (see [50, Definition 2.5]), and hence is further a maximal torus since $\mathbf{C}_{\mathbf{G}}(\mathbf{S})$ necessarily contains a maximal torus. Hence there is a unique such series. By [19, Theorem A], all members of this series lie in the same block of G, which is therefore the principal block $B_0(G)$. $\square$

We begin by considering the simple exceptional groups of Lie type, by which we mean ${}^2\mathrm{B}_2(2^{2n+1})$, $\mathrm{G}_2(q)$, ${}^2\mathrm{G}_2(3^{2n+1})$, $\mathrm{F}_4(q)$, ${}^2\mathrm{F}_4(2^{2n+1})$, ${}^3\mathrm{D}_4(q)$, $\mathrm{E}_6^{\pm}(q)$, $\mathrm{E}_7(q)$, and $\mathrm{E}_8(q)$. In the cases of Suzuki and Ree groups, we only consider $n \geq 1$, as the Tits group ${}^2\mathrm{F}_4(2)'$ and the group ${}^2\mathrm{G}_2(3)' \cong \mathrm{A}_1(8)$ are dealt with in Proposition 3.1.

Lemma 3.7. *Let S be an exceptional group of Lie type defined over $\mathbb{F}_q$ as above, and let p be a prime not diving q. Let A be an almost simple group with simple socle S. Then A is not a minimal counterexample to Theorem A.*

Proof. For $S = {}^2\mathrm{B}_2(2^{2n+1})$ or $S = {}^2\mathrm{G}_2(3^{2n+1})$, the Sylow p-subgroups are cyclic if $p \geq 5$, by applying [25, Theorem 4.10.2] for example. (And note that $3 \nmid |S|$ in the first case.) When $p \geq 5$, we further see that $k_A(B_0(S)) \geq 3$ since by [7], $B_0(S)$ contains 3 characters of distinct degrees for ${}^2\mathrm{B}_2(2^{2n+1})$, and by [26], there are at least 3 characters in $B_0(S)$ in the case ${}^2\mathrm{G}_2(3^{2n+1})$, of which at least 2 are unipotent. Then we are done in this case by applying Theorem 2.1 and Lemmas 3.2 and 3.3. For $p = 2$ and $S = {}^2\mathrm{G}_2(3^{2n+1})$, there are at least 6 characters of distinct degree in $B_0(S)$ by [54], forcing $k_A(B_0(S)) \geq 6$. For $S = {}^2\mathrm{F}_4(2^{2n+1})$ and $p \geq 3$, we see using [37] that there are at least 6 unipotent characters in $B_0(S)$, so that again $k_A(B_0(S)) \geq 6$.

Hence we may assume that S is not a Suzuki or Ree group. We claim that we may further assume that $d_p(q)$ is a regular number, in the sense of Springer [51]. If $d_p(q)$ is a non-regular number, we see from [5, Table 2] combined with [19, Theorem A] that there are at least 6 unipotent characters in $B_0(S)$, with the possible exception of $\mathrm{E}_6(q)$ when $d_p(q) = 5$ or ${}^2\mathrm{E}_6(q)$ when $d_p(q) = 10$, in which cases [5, Table 2] instead yields at least 5 unipotent characters in $B_0(S)$. Recalling that these unipotent characters are invariant under $\mathrm{Aut}(S)$, this gives $k_A(B_0(S)) \geq 6$, respectively 5. In the cases $\mathrm{E}_6(q)$ when $d_p(q) = 5$ or ${}^2\mathrm{E}_6(q)$ when $d_p(q) = 10$, a Sylow p-subgroup of S is cyclic, and hence A is not a minimal counterexample to Theorem A by again applying Theorem 2.1 with Lemmas 3.2 and 3.3.

We now assume $d := d_p(q)$ is a regular number. Let G be a group of Lie type of simply connected type such that $G/\mathbf{Z}(G) = S$. By Lemma 3.6, the principal block $B_0(G)$ is the unique block of G containing unipotent characters of p'-degree. Then from the discussion above, it would suffice to show there exist at least 6 unipotent characters of p' degree for G (hence S) that are not conjugate under $\mathrm{Aut}(S)$. Recall that in the cases under consideration, the unipotent characters are $\mathrm{Aut}(S)$-invariant except for the case of $\mathrm{G}_2(3^{2m+1})$ and $\mathrm{F}_4(2^{2m+1})$.

For G of type $\mathrm{E}_6^{\pm}(q)$, $\mathrm{E}_7(q)$, or $\mathrm{E}_8(q)$, we see from the list of unipotent character degrees in [12, Section 13.9] that there are at least 6 p'-degree unipotent characters for each regular d, so we are done in these cases.

For $\mathrm{F}_4(q)$, again using the list in [12, Section 13.9], we see for $d > 2$ or $p > 3$ that there are at least 6 p'-degree unipotent characters with distinct degrees. Further, there are at least 6 p'-degree unipotent characters for $p = 2, 3$. In the case $p = 3$ and q is an odd power of 2, we have the trivial character, the Steinberg character, and at least four pairs of $3'$-degree unipotent characters, where the pairs are permuted by the graph automorphism (see [39, Theorem 2.5]) but left invariant by all other members of $\mathrm{Aut}(S)$.

This still yields at least 6 $\mathrm{Aut}(S)$-orbits of unipotent characters of p'-degree in $B_0(S)$, so that $k_A(B_0(S)) \geq 6$.

For $S = \mathrm{G}_2(q)$, [28,27] show that there are at least 6 characters in $B_0(S)$ for $p = 2$ and $p = 3$, respectively, with distinct degrees. For $d_p(q) \geq 3$, we see using [12, Section 13.9] that there are again at least 6 p'-degree unipotent characters, which are further not permuted by the exceptional graph automorphism if q is an odd power of 3. For $d_p(q) \in \{1, 2\}$ and $p > 3$, there are six p'-degree unipotent characters. In the case that q is an odd power of 3, two of these characters are interchanged by the exceptional graph automorphism. Hence in the latter case, there are 5 different $\mathrm{Aut}(S)$-classes of unipotent characters in $B_0(S)$. But there are also non-unipotent characters in $B_0(S)$ (using e.g. [49]), yielding $k_{\mathrm{Aut}(S)}(B_0(S)) \geq 6$.

Finally, if $S = {}^3\mathrm{D}_4(q)$, [18] shows that there are at least 6 characters in $B_0(S)$ with distinct degrees for $p = 2, 3$. If $d \in \{3, 6\}$ or $d \in \{1, 2\}$ with $p > 3$, [12, Section 13.9] shows that there are at least 6 p'-degree unipotent characters. Hence we see $k_{\mathrm{Aut}\,S}(B_0(S)) \geq 6$, except possibly for $d = 12$. If $d = 12$, there are four unipotent characters of p'-degree, and hence $k_{\mathrm{Aut}(S)}(B_0(S)) \geq 4$. But in this case, a Sylow p-subgroup is cyclic, so we are again done by applying Theorem 2.1 and Lemmas 3.2 and 3.3. $\square$

In the context of Theorem 2.1, we have now further reduced ourselves to the case that S is a finite classical group defined in a characteristic different from p. In the remaining sections, we address these cases.

3.4.1. Linear and unitary groups with p odd

Let $S = \mathrm{PSL}_n^\epsilon(q)$ and write $G = \mathrm{SL}_n^\epsilon(q)$ and $\widetilde{G} = \mathrm{GL}_n^\epsilon(q)$, where q is a power of some prime. Let p be an odd prime not dividing q and let $\widetilde{B}_0$ be the principal p-block of $\widetilde{G}$. Recall from before that unipotent characters in $\widetilde{B}_0$ may also be viewed as characters of $B_0(S)$ and that they are invariant under $\mathrm{Aut}(S)$. In most cases, we aim to show that $\widetilde{B}_0$ contains at least 6 unipotent characters, which will force $k(B_0(A)) \geq k_A(B_0(S)) \geq 6$ for any almost simple group A with simple socle S.

Let e be the order of q modulo p if $\epsilon = 1$ and of q^2 modulo p if $\epsilon = -1$, and let e' be as follows:

$$e' := \begin{cases} e & \text{if } \epsilon = 1 \\ e & \text{if } \epsilon = -1 \text{ and } p \mid q^e - (-1)^e \\ 2e & \text{if } \epsilon = -1 \text{ and } p \mid q^e + (-1)^e. \end{cases}$$

Write $n = me' + r$ with $0 \leq r < e'$. The unipotent characters of $\widetilde{G}$ are indexed by partitions of n, and two such characters lie in the same block exactly when the corresponding partitions have the same e'-core, by [20]. Identifying the trivial character with the partition (n), a unipotent character is then contained in $\widetilde{B}_0$ if and only if the corresponding partition has e'-core (r).

Now, by [41, Theorem 1.9], $\widetilde{B}_0$ has the same block-theoretic invariants as the principal p-block $B_{me'}$ of $\mathrm{GL}^{\epsilon}_{me'}(q)$. Here, unipotent characters of $\mathrm{GL}^{\epsilon}_{me'}(q)$ lie in $B_{me'}$ exactly when they have trivial e'-core. Further, there is a bijection between partitions of me' with trivial e'-core and partitions of n with e'-core equal to (r). That is, there is a bijection between unipotent characters in $\widetilde{B}_0$ and unipotent characters in $B_{me'}$. By [5, Theorem 3.2], the number of unipotent characters in $\widetilde{B}_0$ is further given by the number of irreducible characters of the relative Weyl group of a Sylow e-torus of $\widetilde{G}$. We also remark that a Sylow p-subgroup is cyclic in the case $m = 1$.

Lemma 3.8. *Let A be an almost simple group with socle $S = \mathrm{PSL}^{\epsilon}_n(q)$, where q is a power of a prime q_0. Let $p \neq q_0$ be an odd prime. With the notation above, A is not a counterexample to Theorem A as long as $me' \geq 5$.*

Proof. Given the discussion above, we know that if $B_{me'}$ contains at least 6 unipotent characters, then $k_A(B_0(S)) \geq 6$. But by taking partitions of the form $(1^c, me' - c)$ for $0 \leq c \leq me'$, we see that $B_{me'}$ contains at least 6 unipotent characters as long as $me' \geq 6$.

Now, assume $me' = 5$. Then either $e' = 1$ or $e' = 5 = me'$. In the first case, all unipotent characters lie in $B_{me'}$, so $B_{me'}$ again contains at least 6 unipotent characters. If $e' = me' = 5$, then $B_{me'}$ contains 5 unipotent characters, and hence $B_0(S)$ contains at least 5 characters that are A-invariant. Further, in this case $\mathrm{GL}^{\epsilon}_{me'}(q)$ has a cyclic Sylow p-subgroup, and hence so does $\widetilde{G}$. The argument in Lemma 3.2 shows that $\widetilde{B}_0$ contains at least one more character, which is necessarily not A-conjugate to the 5 unipotent characters. Now, by Lemma 3.5, the non-unipotent characters in $\widetilde{B}_0$ must be in Lusztig series $\mathcal{E}(\widetilde{G}, t)$ with $t \in \widetilde{G}^* \cong \widetilde{G}$ a nontrivial p-element. Note that since $n \geq 5$, we have $|\mathbf{Z}(\widetilde{G}^*)| = |\widetilde{G}^*/\mathbf{O}^{q_0'}(\widetilde{G}^*)| = |\widetilde{G}^*/[\widetilde{G}^*, \widetilde{G}^*]| = |\widetilde{G}/G|$, and since $e' = 5$ and hence $p \nmid (q - \epsilon)$, we know this number is not divisible by p. Then using [48, Lemma 2.6], we have every member of such a series $\mathcal{E}(\widetilde{G}, t)$ is trivial on the center and cannot lie above a unipotent character of G. This forces a sixth member of $B_0(S)$ that is not A-conjugate to the unipotent characters. □

Now we are ready to complete the case that $S = \mathrm{PSL}^{\epsilon}_n(q)$ with $p \nmid q$ an odd prime.

Proposition 3.9. *Let A be an almost simple group with socle $S = \mathrm{PSL}^{\epsilon}_n(q)$ where $n \geq 2$ and q is a power of a prime, and let p be an odd prime not dividing q. Then A is not a minimal counterexample to Theorem A.*

Proof. By Theorem 2.1, we may assume that either p does not divide $|A : S|$ and $S\mathbf{C}_A(P) < A$, or that $k(B_0(A/S)) = 4$. Keeping the notation from above and applying Lemma 3.8, we may also assume $me' \leq 4$. Let P be a Sylow p-subgroup of S.

First, let $me' = 4$. Then $B_{me'}$ contains at least 4 unipotent characters, taking the forms $(1^c, me' - c)$ as before. In particular, $k_A(B_0(S)) \geq 4$, and we may assume $p \nmid [A :$

S], by Lemma 3.3. If $e' = 1$ or 2, we get one additional unipotent character in $B_{me'}$, corresponding to $(2, 2)$. Hence in these cases, it suffices to note as before that there is at least one non-unipotent character in $B_{me'}$ (and hence in $\widetilde{B}_0$), which we may choose to further be trivial on the center and lie above a non-unipotent character in $B_0(S)$ by choosing a character in $\mathcal{E}(\widetilde{G}, t)$ where $t \in [\widetilde{G}^*, \widetilde{G}^*] \cong G$ is a non-central p-element. This leaves the case $me' = 4 = e'$, in which case we must have $p \geq 5$ and P is cyclic. Hence, we are done by Lemma 3.2.

Now, if $me' = 3$, note that there are three unipotent characters in $B_{me'}$, and hence in $B_0(S)$. If $me' = 2$, there are two unipotent characters in $B_{me'}$, though we may find a third character in $B_0(S)$ by taking $t \in [\widetilde{G}^*, \widetilde{G}^*]$ to have eigenvalues $\{a, a^{-1}\}$ with $|a| = p$ and arguing as before. In any case, we see $k_A(B_0(S)) \geq 3$, so we may again assume $p \nmid [A : S]$. Then by Lemma 3.2, we may assume $n = 3$ and $p \mid (q - \epsilon)$, since in the other cases for $me' \leq 3$, we have P is cyclic. Note that in the case $n = 3$ and $p \mid (q - \epsilon)$, all three unipotent characters lie in the principal block of $B_0(\widetilde{G})$.

If $p = 3$, the semisimple character χ_t of $\widetilde{G}$ indexed by a semisimple element with eigenvalues $\{a, a^{-1}, 1\}$ with $|a| = 3$ is trivial on $\mathbf{Z}(\widetilde{G})$ by [48, Lemma 2.6] since $t \in [\widetilde{G}^*, \widetilde{G}^*]$ and lies in $B_0(\widetilde{G})$ by Lemma 3.5. Further, [48, Lemma 2.6] and Lemma 1.4 also imply that since t is $\widetilde{G}^*$-conjugate to tz where $z \in \mathbf{Z}(\widetilde{G})$ has order 3, we have $\chi_{tz} = \chi_t$ restricts to the sum of 3 irreducible characters in G, which must lie in $B_0(G)$ (and hence $B_0(S)$). This yields $k(B_0(S)) \geq 6$. But since $3 \nmid [A : S]$, we further have that at least one of these three characters must be invariant under A, and hence $k_A(B_0(S)) \geq 5$. Now, these characters (and those above them in A) have height zero, and the same is true for the unipotent characters. Hence it follows from Theorem 1.7 that in fact $k(B_0(A)) \geq 6$.

Finally, assume that $n = 3$, $p \mid (q - \epsilon)$, and $p > 3$. Recall that $B_0(\widetilde{G})$ is the unique unipotent block of $\widetilde{G}$. Let $t_1, t_2 \in [\widetilde{G}^*, \widetilde{G}^*]$ be such that t_1 has eigenvalues $\{a, a^{-1}, 1\}$ and t_2 has eigenvalues $\{a, a, a^{-2}\}$, where $|a| = p$. Then the characters in the series $\mathcal{E}(\widetilde{G}, t_i)$ for $i = 1, 2$ are trivial on $\mathbf{Z}(\widetilde{G})$ and lie in $B_0(\widetilde{G})$ by [48, Lemma 2.6] and Lemma 3.5. Further, note that t_i cannot be $\mathrm{Aut}(S)$-conjugate to $t_j z$ for any $1 \neq z \in Z(\widetilde{G}^*)$ or $j \in \{1, 2\}$, and hence the characters in $\mathcal{E}(\widetilde{G}, t_1)$ are not $\mathrm{Aut}(S)$-conjugate to those in $\mathcal{E}(\widetilde{G}, t_2)$ and restrict irreducibly to G by Lemma 1.4. Hence we see that the characters in these series may be viewed as members of $B_0(S)$, yielding $k_A(B_0(S)) \geq 5$ when combined with the unipotent characters in the block. Finally, since $\mathbf{C}_{\widetilde{G}^*}(t_2) \cong X_1 \times X_2$ with $X_1 \in \{\mathrm{GL}_2^{\pm}(q)\}$ and $X_2 \in \{\mathrm{GL}_1^{\pm}(q)\}$, it follows that there are two members of $\mathcal{E}(\widetilde{G}, t_2)$ with distinct degrees, so $k_A(B_0(S)) \geq 6$. □

3.4.2. Remaining classical groups with p odd

We set some notation to be used throughout this section.

Let q be a power of some prime and let S be a simple group $\mathrm{B}_n(q)$ with $n \geq 3$, $\mathrm{C}_n(q)$ with $n \geq 2$, or $\mathrm{D}_n(q)$ or ${}^2\mathrm{D}_n(q)$ with $n \geq 4$. Let $p \nmid q$ be an odd prime and let $e := d_p(q)/\gcd(2, d_p(q))$ where $d_p(q)$ is the order of q modulo p. Write $n = me + r$, where $0 \leq r < e$ is the remainder when n is divided by r.

Let H be the corresponding symplectic or special orthogonal group $\mathrm{SO}_{2n+1}(q)$, $\mathrm{Sp}_{2n}(q)$, or $\mathrm{SO}^{\epsilon}_{2n}(q)$. For the cases of special orthogonal groups with q odd, let $\Omega \leq H$ be the unique subgroup of index 2, and otherwise let $\Omega := H$, so that $\Omega/\mathbf{Z}(\Omega) = S$. Further, let $\overline{H}$ be the group $\mathrm{GO}^{\epsilon}_{2n}(q)$ in the case of type $\mathrm{D}_n, {}^2\mathrm{D}_n$, and otherwise let $\overline{H} := H$.

Lemma 3.10. *Let A be an almost simple group with socle S of type $\mathrm{B}_n(q)$ with $n \geq 3$, $\mathrm{C}_n(q)$ with $n \geq 2$, or $\mathrm{D}_n(q)$ or ${}^2\mathrm{D}_n(q)$ with $n \geq 4$, where q is a power of a prime q_0. Let $p \neq q_0$ be an odd prime. Then A is not a counterexample to Theorem A as long as $me \geq 3$.*

Proof. Keep the notation from above. Using the main Theorem of [11] to argue as in [40, Discussions before Props 5.4 and 5.5], the number of unipotent characters in $B_0(\overline{H})$ is $k(2e, m)$, where the number $k(2e, m)$ may be computed as in [44, Lemma 1].

Now, similar to before, results of Lusztig give that the unipotent characters of H restrict irreducibly to Ω and are trivial on $\mathbf{Z}(\Omega)$ (see [24, 2.3.14 and 2.3.15]), and hence can be viewed as irreducible characters of S. Recall that for $S \neq \mathrm{D}_4(q)$ nor $\mathrm{PSp}_4(2^{2a+1})$, the only automorphisms of S that do not fix the unipotent characters occur in the case of type D_n. In the latter case, the graph automorphism, induced by the action of $\overline{H}$ on H, interchanges the pairs of unipotent characters of H parameterized by so-called degenerate symbols. Further, unipotent characters of $\overline{H}$ are defined as the characters lying above unipotent characters of H. Then we see in this case that the number of $\overline{H}$-conjugacy classes of unipotent characters in $B_0(H)$ (and hence in $B_0(S)$) is at least $k(2e, m)/2$.

Therefore, if $S \neq \mathrm{D}_4(q)$ nor $\mathrm{PSp}_4(2^{2a+1})$, $B_0(S)$ contains $k(2e, m)/2$ non-$\mathrm{Aut}(S)$-conjugate unipotent characters, and hence $k_A(B_0(S)) \geq k(2e, m)/2$ for any $S \leq A \leq \mathrm{Aut}(S)$. Using [44, Lemma 1], we may calculate that $k(2e, m) \geq 6$ unless $em = 2$, and hence by Lemmas 3.2 and 3.3, we may assume P is not cyclic. Then it follows that $m \geq 2$, in which case we have $k(2e, m) \geq 10$ when $em \geq 3$. Hence we see that $B_0(S)$ contains at least 5 non-$\mathrm{Aut}(S)$-conjugate unipotent characters. However, $B_0(S)$ necessarily contains a non-unipotent character, which cannot be $\mathrm{Aut}(S)$-conjugate to any unipotent character. (Indeed, in this case p is a so-called good prime for H, so we may use [23, Theorem A] to say that $k(B_0(S))$ is larger than the number of unipotent characters in the block, as they form a basic set for the irreducible Brauer characters.)

For $S = \mathrm{D}_4(q)$, the above is still true, except possibly if the unipotent characters permuted by the exceptional graph automorphism of order 3 lie in the principal block. There are 2 orbits of size three of such characters. Using the theory of e-cores and e-cocores of [21], we see that this only happens when $e = 1$ or 2, in which case we may again use [44, Lemma 1] to see that $k(2e, m) \geq 14$, so there are at least $\frac{14-6}{2} + 2 = 6$ non-$\mathrm{Aut}(S)$-conjugate unipotent characters in $B_0(S)$. □

Proposition 3.11. *Let A be an almost simple group with socle S of type $\mathrm{B}_n(q)$ with $n \geq 3$, $\mathrm{C}_n(q)$ with $n \geq 2$, or $\mathrm{D}_n(q)$ or ${}^2\mathrm{D}_n(q)$ with $n \geq 4$, where q is a power of a prime.*

Let p be an odd prime not dividing q. Then A is not a minimal counterexample to Theorem A.

Proof. Note that by Lemma 3.10, we may assume that $em = 2$, and hence $n = 2$ or $n = 3$, so S is type B or C. If $e = 2$, then a Sylow p-subgroup of S, H, or Ω is cyclic. Further, in this case the number of unipotent characters in $B_0(S)$ is 4. Note that for $\mathrm{PSp}_4(2^{2a+1})$, the graph automorphism interchanges two unipotent characters, but in any case we still have $k_A(B_0(S)) \geq 3$ for any $S \leq A \leq \mathrm{Aut}(S)$. Then using Lemmas 3.2 and 3.3 and Theorem 2.1, we see A is not a minimal counterexample for any $S \leq A \leq \mathrm{Aut}(S)$. If $e = 1$, we have $n = 2$ and $S = \mathrm{PSp}_4(q)$. In this case, we see from [56–58] that $k_{\mathrm{Aut}(S)}(B_0(S)) \geq 6$, completing the proof. □

3.4.3. Classical groups with $p = 2$

Lemma 3.12. *Let q be a power of an odd prime q_0 and let $p = 2$. Let A be an almost simple group with simple socle $S = \mathrm{PSL}_n^\epsilon(q)$ with $n \geq 3$, $\mathrm{PSp}_{2n}(q)$ with $n \geq 2$, $\mathrm{P\Omega}_{2n+1}(q)$ with $n \geq 3$ or $\mathrm{P\Omega}_{2n}^\epsilon(q)$ with $n \geq 4$. Then A is not a counterexample to Theorem A.*

Proof. The group $\mathrm{B}_3(3)$ and $\mathrm{PSU}_4(3)$ are dealt with in Proposition 3.1, so our conditions on n and q mean that we may assume S does not have an exceptional Schur multiplier. Then let $\mathbf{G}$ be a simple algebraic group over $\overline{\mathbb{F}}_q$ of simply connected type such that $G = \mathbf{G}^F$ satisfies $G/\mathbf{Z}(G) \cong S$. Then by [10, Theorem 21.14], every unipotent character of G lies in the principal 2-block $B_0(G)$ of G, and the non-unipotent characters in $B_0(G)$ are exactly those characters lying in Lusztig series $\mathcal{E}(G, s)$ with s a 2-element of G^*. Since unipotent characters of G are trivial on the center, we see that $k_A(B_0(S)) \geq 6$ just by considering unipotent characters and also taking into consideration [39, Theorem 2.5], except possibly in the case $\mathrm{PSp}_4(q)$, $\mathrm{PSL}_3^\epsilon(q)$, or $\mathrm{PSL}_4^\epsilon(q)$. In the case of $\mathrm{PSp}_4(q)$, the results of [55] show that there are at least six characters of distinct degree in the principal block of $\mathrm{Sp}_4(q)$ that are trivial on the center, which forces $k_A(B_0(S)) \geq 6$. In the case $S = \mathrm{PSL}_4^\epsilon(q)$, S has five unipotent characters, which are $\mathrm{Aut}(S)$-invariant using [39, Theorem 2.5], and hence the result is obtained by considering a character of G in a series indexed by any 2-element of $\mathbf{O}^{q_0'}(G^*) = [G^*, G^*]$, which will be trivial on the center using [48, Proposition 2.6(iii)]. Finally, in the case $S = \mathrm{PSL}_3^\epsilon(q)$, we may argue as in the last paragraph of Proposition 3.9, but taking $|a| = 4$ in place of $|a| = p$. □

Lemma 3.13. *Let q be a power of an odd prime and let $p = 2$. Let A be an almost simple group with simple socle $S = \mathrm{PSL}_2(q)$. Then A is not a minimal counterexample to Theorem A.*

Proof. In this case, $B_0(S)$ contains two unipotent characters. Taking $t \in [\widetilde{G}^*, \widetilde{G}^*]$ to have eigenvalues $\{a, a^{-1}\}$ with $|a| = 4$, the character χ_t restricts to the sum of two nonunipotent irreducible characters of G trivial on the center. We therefore have $k_A(B_0(S)) \geq 3$, so we may assume $[A : S]$ is odd by applying Lemma 3.3 and Theorem 2.1. Then a Sylow 2-subgroup of A is Dihedral or Klein-four. If $|P| = 2^n \geq 8$, then $k(B_0(A)) = 2^{n-2} + 3$ by [46, Theorem 8.1], which is larger than 5 for $n > 3$. If $|P| = 8$, then P is D_8 and A is not a counterexample. If $|P| = 4$, then $(q^2 - 1)_2 = 8$, and every semisimple 2-element in $[\widetilde{G}^*, \widetilde{G}^*]$ is $\widetilde{G}^*$-conjugate to t above. This means that the only characters of $B_0(S)$ are the four discussed at the beginning of the proof, which are A-invariant. Further, we have $P = \mathsf{C}_2 \times \mathsf{C}_2$. Since $2 \nmid [A : S]$, we know A/S is cyclic generated by field automorphisms. We then see, using the construction in [13], that a generating field automorphism centralizes the Sylow 2-subgroup of G, modulo $\mathbf{Z}(G)$. That is, a generator of A/S centralizes P, contradicting the assumption from Theorem 2.1 that $S\mathbf{C}_A(P) \neq A$. $\square$

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