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Variational Analysis of Sharp Folds and Cusps in Shells: A Generalized Functional Setting with Atomic Measures

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Abstract
We characterize the functional structure of deformations that describe the occurrence of possibly irrecoverable sharp folds and cusps in shell-like bodies, while they avoid self-penetration of matter. The combination of special bounded variation maps with one-dimensional jump set and atomic measures provides a functional environment in which it is possible to represent configurations not otherwise described by special bounded Hessian maps, often used for energy minimization of shells. The introduction of atomic measures provides necessary control over the singular component of the generalized gradient measure, ensuring the robust compactness required by the direct method. In the proposed framework, we prove the existence of minimizers under Dirichlet boundary conditions for a class of energies that are appropriate for an ambient setting foreseeing large strains. Although we treat a traditional topic, the analysis goes beyond the classical format of nonlinear elasticity.

Keywords Continuum mechanics · Calculus of variations · Shells · Energy minimization

Mathematics Subject Classification 74K25 · 49Q20 · 49J45 · 26B25

1 Introduction

To be determined along the deformation of thin shells, possibly irrecoverable sharp folds and cusps require large strains at least locally, namely at least in some of

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their 'small' neighborhoods. The adjective "irrecoverable" refers to permanence after unloading.

We look at shells in terms of Gauss graphs; that is, we consider the "middle surface" and the associated normal vector field. Thus, we do not account for through-the-thickness shear, which we neglect due to the smallness of the thickness considered. So, we refer to two-dimensional bodies with additional structure embedded in three-dimensional space.

We give a functional characterization of pertinent deformations that avoid self-penetration of matter and may describe the formation of cusps and sharp folds, granting compactness.

Two-dimensional models of shells with additional structure, as the one provided by the normal vector field, can be justified in different ways. One is 'top-to-bottom': It means that we collapse a body 'around a surface' evaluating the energy. Plates are a paradigmatic case: Appropriate pertinent descriptions emerge by shrinking a parallelepiped up to a plane. More precisely, consider a domain $\Omega \subset \mathbb{R}^2$ and the 'thick' region $\Omega_h := \Omega \times \big(-h/2,\ h/2\big)$, with $h > 0$ the thickness. Thus, deformations are maps $\tilde{u} : \Omega_h \to \mathbb{R}^3$ assumed to be one-to-one, differentiable, and such that they preserve the local spatial orientation. For the elastic energy, we then write

$$\mathscr{E}_h := \int_{\Omega_h} e\left(Du\left(x\right)\right)\ \mathrm{d}x\ ,$$

with $e\left(\cdot\right)$ the energy density; it is objective in the sense that $e\left(Du\right) = e\left(QDu\right)$ for any $Q \in SO(3)$. (Under large strains, objectivity renders the energy density physically incompatible with a quadratic dependence on Du alone.) A possible program is then to think of plate models as the result of an energy limit by letting h to zero in the sense of De Giorgi's Γ-convergence. A key factor is the scaling, that is, how the ratio $\mathscr{E}_h/h$ behaves. Another essential rôle is the scaling of the applied loads, that is, the ratio between force densities and shell thickness. In-plane and out-of-plane energy components must be evaluated differently. The result for homogeneous materials has been obtained in Friesecke et al. (2006), based on analyses in Friesecke et al. (2003) and Le Dret and Raoult (2000). Precisely: (1) For $(\mathscr{E}_h/h) \sim 1$, the membrane theory emerges. (2) $(\mathscr{E}_h/h) \sim h^2$ gives the energy pertaining to shells that bend without stretching Ω. (3) Condition $(\mathscr{E}_h/h) \sim h^4$ leads to the nonlinear von Kármán's theory. (4) The scaling $(\mathscr{E}_h/h) \sim h^\gamma$, with $\gamma > 4$, yields the linearized version of case (3); when the material is linearly elastic and homogeneous, a biharmonic equation describes equilibrium. (5) Finally, the scaling $(\mathscr{E}_h/h) \sim h^\gamma$, with $2 < \gamma < 4$, implies non-standard theories of plates. For analogous analysis in the heterogeneous case, see (Schmidt 2007); see also (Neff 2010; Saem et al. 2023, 2025).

Else, as before we can shrink to the plane a parallelepiped made of a second-grade elastic material: This means that the energy density is a function of the first and second deformation gradients, namely

$$\mathscr{E}_{h2} := \int_{\Omega_h} e\left(Dy\,(x)\,,\,D^2y\,(x)\right)\,\mathrm{d}x\,.$$

When $(\mathscr{E}_{h2}/h) \sim 1$, the shrinking process leads to a surface with out-of-tangent-plane directors that can rotate independently from each other (Bhattacharya and James 1999) (see also (Anzellotti et al. 1994; Le Dret and Raoult 1996)), a Cosserat surface indeed.

Beyond elasticity, Γ-convergence analyses leading to plate and shell models have also been carried out considering deformations as special bounded Hessian (SBH) maps (Demengel 1984, 1989), which may represent peculiar aspects of elastic–plastic behavior (Percivale 1990; Acerbi and Percivale 1992; Tomarelli et al. 1992; Carriero et al. 1992, 1994; Percivale and Tomarelli 2002). Clearly, such a variational view does not account for plastic flows unless the related minimization problems are inserted in De Giorgi's minimizing movements procedure. Also, resorting to SBH maps does not account for the possible formation of cusps, those we describe here in terms of atomic measures. For example, when looking at cuspidate prismatic shells, concentrated actions may appear, as pointed out from a completely different path in Chinchaladze et al. (2011).

Always in variational terms, we also have a bottom-to-top approach to shell models based on a superposition of atomic layers. The result is still a Cosserat surface but with multiple out-of-tangent-plane directors, one for each atomic layer (Friesecke and James 2000). (An analysis that foresees a growth to infinity of the atoms in each layer is in Schmidt (2008); see also (Schmidt 2006).) If we assume that the n-tuple of directors is defined pointwise to within a permutation, we obtain a scheme that is adequate for the representation of shells or thin films made of solid mixtures, because it accounts for the unknown component mixing (Mariano 2018).

Beyond Γ-convergence, pointwise asymptotic procedures can be adopted to derive plate and shell models (Ciarlet 2001; Ciarlet and Mardare 2019; Steigmann 1999). Another efficient way is to exploit the theory of internal constraint in the $3D$ setting (Podio-Guidugli 1989; Antman and Marlow 1991; Lembo and Podio-Guidugli 2007).

It is also possible to derive a model for the mechanics of shells by looking only at the geometry of surfaces embedded in a $3D$ Euclidean point space, which is Koiter's proposal (Koiter 1966, 1960). Such an approach, however, does not account for the through-the-thickness behavior that is at least roughly incorporated in the Cosserat-surface-based view, which can be adopted *directly*, as Ericksen and Truesdell made first in 1958 (Ericksen and Truesdell 1958), opening a fruitful research path, the one of so-called *direct models* (e.g., (Simo et al. 1988; Simo and Fox 1989; Di Carlo et al. 2001; Steigmann 2013, 2018; Mariano and Mucci 2023b); a wide view on shell models is in the treatises (Antman 1995; Ciarlet 1997; Villaggio 1997)).

The viewpoint adopted here falls within the context of direct models (Ericksen and Truesdell 1958). We consider deformations of the middle shell surface as continuous maps with sufficiently regular derivatives. However, detaching from a pure SBH setting, we consider conical points (cusps) that can emerge in the folding. As already mentioned, to characterize cusps, we consider atomic measures supported by a countable set of points. To our knowledge, existential analyses of minimum problems for shell models that account for the emergence of cusps under large strains do not appear to have been considered so far in the available literature. The normal vector field is

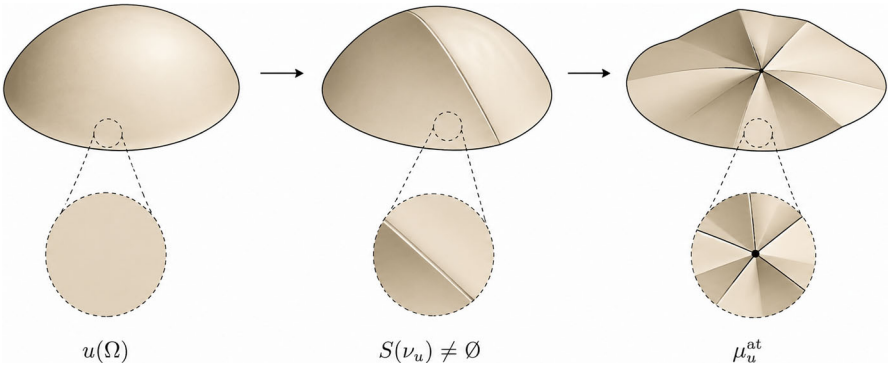

Fig. 1 Typical geometries considered in the paper: from smooth shell configurations to sharp folds and cusp-like singularities associated with concentrated curvature measures

considered to be given by a special function of bounded variation (SBV). So, at every point along a sharp fold, we only consider the two normal vectors that limit the normal cone of a fold.

The energy class we look at is surface polyconvex; also, it involves the gradient of the normal, considered as an SBV map with jump set endowed with peculiar line energy; the possible presence of an external potential is considered to cover cases in which there is activation of shape morphing. For this energy class, we prove the existence of minimizers under Dirichlet-type boundary conditions, intended in the sense of traces, in the functional setting that we propose.

The conceptual path that we follow is summarized in Fig. 1: from smooth deformation, to those with nonzero singular one-dimensional set, to the introduction of atomic measures.

Shell singularities emerge in dimensional reduction and energy scaling in terms of Γ-convergence. Cusps are seen as results of metric incompatibility associated with embedding, scaling energy and relaxation (Lewicka 2020; Lewicka and Müller 2011). These scaling approaches to shells are linked with the hierarchy analysis proposed in Friesecke et al. (2003).

Always in the scaling perspective, a question tackled in Müller and Olbermann (2014); Olbermann (2016, 2017, 2018) concerns the amount of energy that we need to pay in order to get a cusp. We tackle another question: What's the functional structure that is necessary to represent cusps, maintaining compactness?

We adopt tools from geometric measure theory. On one side, it is a technical convenience: It allows us to avoid cancellations, control limits of the Gauss map and, at the same time, folds and cusps, obtaining compactness and eventually using the Federer–Fleming theorem. However, there is something more at the interplay of analytical and mechanical aspects. These tools allow us to select a subspace of $W^{1,p}$, with $p > 1$, of maps that avoid the non-required presence of holes, as the very concept of elasticity imposes. (Otherwise we were referring to elastic-brittle materials.) These maps have the minimal analytical regularity needed to formulate the mechanical setting. These aspects point out that the choice of a functional class has per se constitutive character.

Notions from geometric measure theory are necessary tools for analysis; we recall them in Sect. 2. Then, we first analyze smooth deformations in Sect. 3, clarifying how we impose a condition that avoids self-penetration of matter and how our approach prevents a possible cancelation in the Gauss graph when two distant shell parts come into contact with opposite orientation. Then, in Sect. 4 we prove summability properties of not necessarily smooth deformations that form cusps but are convex in a neighborhood of the cusps themselves; the analysis, however, is not reduced to such deformations, although they have a special status. In Sect. 5, we describe possibly irrecoverable folds in terms of vector-valued measures concentrated on 1-dimensional sets where jumps of the unit normal to the deformed shell may appear. The representation of cusps in terms of atomic measures supported by a countable point set is discussed in Sect. 6. An energy functional that accounts for the above-mentioned geometries and the energy weight of each of the corresponding features is introduced in Sect. 7; for it we prove the existence of minimizers.

2 Background Material

Basic tools of geometric measure theory and the theory of smooth surface Gauss graphs are recalled below for the sake of reader's convenience.

2.1 Special Functions of Bounded Variation

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, and let $\mathscr{L}^n$ and $\mathscr{H}^k$ be Lebesgue and Hausdorff measure, respectively. We denote by $BV(\Omega, \mathbb{R}^N)$ the class of summable functions $v : \Omega \to \mathbb{R}^N$ whose distributional derivative Dv is an $\mathbb{R}^{N\times n}$-valued finite Borel measure. Denoting by $S(v)$ the complement of the set of Lebesgue's points of v, a mutually singular decomposition holds,

$$Dv = \nabla v \, \mathscr{L}^n \llcorner \Omega + D^C v + D^J v \,, \tag{2.1}$$

where $\nabla v \in L^1(\Omega, \mathbb{R}^{N\times n})$ is the approximate gradient function, $D^C v$ is the Cantor component that satisfies $|D^C v|(B) = 0$ for every Borel set $B \subset \Omega$ such that $\mathscr{H}^{n-1}(B) < \infty$, and the jump component is given by

$$D^J v = [v] \otimes \mathbf{n} \, \mathscr{H}^{n-1} \llcorner S(v) \,, \tag{2.2}$$

where the jump set $S(v)$ is countably $(n-1)$-rectifiable, $\mathbf{n}$ is a unit normal to $S(v)$, and $[v](x) := v^+(x) - v^-(x) \in \mathbb{R}^N$ is the jump of v at $\mathscr{H}^{n-1}$-almost every (shortly a.e.) point $x \in S(v)$, where $v^{\pm}(x)$ are the one-sided limits of v at x with respect to the normal $\mathbf{n}(x)$. Then, the total variation $|Dv|$ of Dv is given by

$$|Dv|(\Omega) = \int_\Omega |\nabla v| \, \mathrm{d}\mathscr{L}^n + |D^C v|(\Omega) + \int_{S(v)} |[v](x)| \, \mathrm{d}\mathscr{H}^{n-1}(x) \,. \tag{2.3}$$

If, in particular, $D^C \nu = 0$, the map ν is called a special function of bounded variation, and we write $\nu \in SBV(\Omega, \mathbb{R}^N)$, see (Ambrosio et al. 2000).

The case considered deals with $n = 2$, $N = 3$, while ν is identified with the unit normal to a sufficiently smooth parametric surface, so that $\nu : \Omega \to \mathbb{S}^2$, where $\mathbb{S}^2$ is the Gauss sphere $\mathbb{S}^2 := \{z \in \mathbb{R}^3 : |z| = 1\}$. For $X = C^2$, $W^{1,p}$, BV, or SBV, we thus denote

$$X(\Omega, \mathbb{S}^2) := \{\nu \in X(\Omega, \mathbb{R}^3) : |\nu(x)| = 1 \text{ for } \mathscr{L}^2\text{-a.e. } x \in \Omega\}.$$

2.2 Rectifiable Currents

For $U \subset \mathbb{R}^m$ an open set, and $k = 0, \dots, m$, we write $\mathscr{D}_k(U)$ for the dual to the space $\mathscr{D}^k(U)$ of compactly supported smooth k-forms in U. Elements of $\mathscr{D}_k(U)$ are called *currents*. Notions of mass and boundary current are associated with them. Precisely, the *mass* $\mathbf{M}(T)$ of a current T in $\mathscr{D}_k(U)$ is defined to be

$$\mathbf{M}(T) := \sup\{\langle T, \omega\rangle \mid \omega \in \mathscr{D}^k(U),\ \|\omega\| \le 1\},$$

while (when $k \ge 1$) the *boundary* of T is the current ∂T in $\mathscr{D}_{k-1}(U)$ given by

$$\langle \partial T, \eta\rangle := \langle T, \mathrm{d}\eta\rangle, \qquad \forall \eta \in \mathscr{D}^{k-1}(U),$$

where $\mathrm{d}\eta$ is the exterior differential of η. The *weak convergence* $T_h \rightharpoonup T$ in the sense of currents in $\mathscr{D}_k(U)$ is defined through the formula

$$\lim_{h\to\infty} \langle T_h, \omega\rangle = \langle T, \omega\rangle, \qquad \forall\, \omega \in \mathscr{D}^k(U).$$

If $T_h \rightharpoonup T$, by lower semicontinuity we also have

$$\mathbf{M}(T) \le \liminf_{h\to\infty} \mathbf{M}(T_h).$$

For $k \ge 1$, a k-current T with finite mass is called *rectifiable* if

$$\langle T, \omega\rangle = \int_{\mathscr{N}} \theta\, \langle \omega, \xi\rangle\, \mathrm{d}\mathscr{H}^k, \qquad \forall\, \omega \in \mathscr{D}^k(U),$$

where $\mathscr{N}$ is a countably k-rectifiable set in U, $\xi : \mathscr{N} \to \bigwedge^k \mathbb{R}^m$ is a $\mathscr{H}^k \llcorner \mathscr{N}$-measurable function such that $\xi(w)$ is a *simple* unit k-vector orienting the approximate tangent space to $\mathscr{N}$ at $\mathscr{H}^k$-a.e. $w \in \mathscr{N}$, and $\theta : \mathscr{N} \to [0, +\infty)$ is an $\mathscr{H}^k \llcorner \mathscr{N}$-summable and nonnegative function, called a *multiplicity*. Thus, for T a rectifiable current, we have

$$\mathbf{M}(T) = \int_{\mathscr{M}} \theta\, \mathrm{d}\mathscr{H}^k < \infty,$$

and the short-hand notation $T = [\![\mathscr{N}, \xi, \theta]\!]$ is commonly adopted. The *size* of T is defined by

$$\mathbf{S}(T) := \mathscr{H}^k\{w \in \mathscr{N} \mid \theta(w) > 0\}.$$

When θ is integer-valued and T is rectifiable, we say that T is an *integer rectifiable* (in short *i.m.*) current, a class indicated by $\mathscr{R}_k(U)$ (Giaquinta et al. 1998a).

More generally, T is called a *size bounded* current if $T = [\![\mathscr{N}, \xi, \theta]\!]$ is rectifiable and with finite size, $\mathbf{S}(T) < \infty$. We indicate by $\mathscr{S}_k(U)$ the pertinent class. Any current T in $\mathscr{R}_k(U)$ satisfies $\mathbf{S}(T) \le \mathbf{M}(T)$, and therefore, it belongs to $\mathscr{S}_k(U)$.

2.3 Basic Examples

If $\mathscr{N}$ is a smooth, oriented, and compact Riemannian k-manifold embedded in U, a current $[\![\mathscr{N}]\!]$ in $\mathscr{R}_k(U)$ is naturally associated with $\mathscr{N}$ by setting

$$\langle [\![\mathscr{N}]\!], \omega \rangle := \int_{\mathscr{N}} \omega, \quad \omega \in \mathscr{D}^k(U).$$

In that case, we have $\mathbf{M}([\![\mathscr{N}]\!]) = \mathscr{H}^k(\mathscr{N}) < \infty$.

Let $\Omega \subset \mathbb{R}^2$ be a bounded open connected set and $u \in C^2(\overline{\Omega}, \mathbb{R}^3)$. With $U = \Omega \times \mathbb{R}^3$, and $\mathscr{G}_u$ the naturally oriented graph of u, the *graph current* $G_u := [\![\mathscr{G}_u]\!]$ belongs to $\mathscr{R}_2(\Omega \times \mathbb{R}^3)$, and by the area formula we get

$$\mathbf{M}(G_u) = \int_\Omega \sqrt{1 + |\nabla u|^2 + |\partial_1 u \times \partial_2 u|^2}\, \mathrm{d}\mathscr{L}^2 < \infty \tag{2.4}$$

where $\partial_1 u$ and $\partial_2 u$ are the column vectors of the gradient matrix ∇u. Notice that $|\partial_1 u \times \partial_2 u| = |\operatorname{adj}_2 \nabla u|$, where $\operatorname{adj}_2 \nabla u$ is the vector in $\mathbb{R}^3$ with entries given by the determinant of the 2×2 minors of ∇u. Also, Stokes theorem implies the null-boundary condition

$$\langle \partial G_u, \eta \rangle = 0\,, \quad \forall\, \eta \in \mathscr{D}^1(\Omega \times \mathbb{R}^3). \tag{2.5}$$

More generally, if $u \in W^{1,p}(\Omega, \mathbb{R}^3)$ for some $p \ge 2$, the graph current is defined by $G_u := (\mathrm{Id} \bowtie u)_\# [\![\Omega]\!]$, where $(\mathrm{Id} \bowtie u)(x) := (x, u(x))$ is the graph map. More precisely, the pullback being defined $\mathscr{L}^2$-a.e. in Ω in terms of the approximate gradient of u, we set

$$\langle G_u, \omega \rangle := \int_\Omega (\mathrm{Id} \bowtie u)^\# \omega, \quad \omega \in \mathscr{D}^2(\Omega \times \mathbb{R}^3).$$

Since $\partial_1 u \times \partial_2 u \in L^{p/2}(\Omega, \mathbb{R}^3)$, where $p/2 \ge 1$, it turns out that G_u belongs to $\mathscr{R}_2(\Omega \times \mathbb{R}^3)$, and the mass formula (2.4) still holds true. Moreover, a density argument implies that the null-boundary condition (2.5) is still valid.

If $v \in BV(\Omega, \mathbb{S}^2)$ is such that $\partial_1 v \times \partial_2 v \in L^1(\Omega, \mathbb{R}^3)$, the graph current G_v is defined as in the Sobolev case, in terms of the approximate gradient ∇v, and actually,

by Federer's flatness theorem, $G_\nu \in \mathscr{R}_2(\Omega \times \mathbb{S}^2)$. Its finite mass is given again by formula (2.4), with $u = \nu$. However, even in the Sobolev case, in general the null-boundary condition (2.5) is violated.

Thus, we have $\mathbf{M}(\partial G_\nu) = \infty$ if $|D^C \nu|(\Omega) > 0$. A necessary and sufficient condition ensuring that a BV map $\nu : \Omega \to \mathbb{S}^2$ belongs to $SBV(\Omega, \mathbb{S}^2)$, with a jump set of finite size, namely $\mathscr{H}^1(S(\nu)) < \infty$, is given by the requirement that the currents G_{ν^j} in $\mathscr{R}_1(\Omega \times \mathbb{R})$ carried by the graph of the component BV functions ν^j have finite boundary mass (Giaquinta et al. 1998a), that is,

$$\mathbf{M}(\partial G_{\nu^j}) < \infty\,, \quad \forall\, j \in \{1, 2, 3\}\,. \tag{2.6}$$

Furthermore, if (2.6) holds, a result by Ambrosio et al. (1998) implies that the trace functions $\nu^\pm : S(\nu) \to \mathbb{S}^2$ are approximately differentiable $\mathscr{H}^1$-almost everywhere on the jump set $S(\nu)$.

2.4 Size Bounded Vector-Valued Currents

We will exploit $\mathbb{R}^3$-valued size bounded rectifiable currents defined on a bounded domain $U \subset \bar{\mathbb{R}}^3 \simeq \mathbb{R}^3$, where $\mathbf{x} = (x_1, x_2, x_3)$ (compare (Mariano and Mucci 2023a)).

First recall that k-forms $\overline{\omega}$ in $[\mathscr{D}^k(U)]^3$, with $0 \le k \le 3$ integer, are triplets $\overline{\omega} = (\omega_1, \omega_2, \omega_3)$, where $\omega_j \in \mathscr{D}^k(U)$, for $j = 1, 2, 3$. More precisely, vector fields ϕ in $C_c^\infty(U, \mathbb{R}^3)$ agree with 0-forms $\overline{\omega}$ in $[\mathscr{D}^0(U)]^3$, say $\overline{\omega} = \omega^{(0)}_\phi$. A 1-form $\overline{\omega} \in [\mathscr{D}^1(U)]^3$ is identified by a fully covariant tensor-valued field $\psi \in C_c^\infty(U, \mathbb{R}^{3\times 3})$ with components ψ_{jA} by letting $\omega_j = \sum_{A=1}^3 \psi_{jA} \mathrm{d}x^A$ for $j = 1, 2, 3$. In this case, we write $\overline{\omega} = \omega^{(1)}_\psi$. In a similar way, a 2-form $\overline{\omega} \in [\mathscr{D}^2(U)]^3$ is identified by a tensor-valued field $\zeta \in C_c^\infty(U, \mathbb{R}^{3\times 3})$ with components ζ_{jA} by letting $\omega_j = \sum_{A=1}^3 (-1)^{A-1} \zeta_{jA} \widehat{\mathrm{d}x^A}$, where $\widehat{\mathrm{d}x^A}$ is the 2-vector that complements $\mathrm{d}x^A$ to complete $\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$, so that $\mathrm{d}x^A \wedge \widehat{\mathrm{d}x^A} = (-1)^{A-1} \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$. In this case, we write $\overline{\omega} = \omega^{(2)}_\zeta$. Finally, a 3-form $\overline{\omega} \in [\mathscr{D}^3(U)]^3$ is identified by a covector field $\eta \in C_c^\infty(U, \mathbb{R}^3)$ with $\eta = (\eta_1, \eta_2, \eta_3)$, by letting $\omega_j = \eta_j \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$, and we write $\overline{\omega} = \omega^{(3)}_\eta$. We thus have

$$\mathrm{d}\omega^{(0)}_\phi = \omega^{(1)}_{\nabla\phi}\,, \qquad \mathrm{d}\omega^{(1)}_\psi = \omega^{(2)}_{\mathrm{curl}\,\psi}\,, \qquad \mathrm{d}\omega^{(2)}_\zeta = \omega^{(3)}_{\mathrm{div}\zeta}$$

and hence, the identities $\mathrm{curl}\,\nabla\phi = 0$ and $\mathrm{div}(\mathrm{curl}\,\psi) = 0$ are equivalent to the closure relations $\mathrm{d} \circ \mathrm{d} = \mathrm{d}^2 = 0$ for 0-forms and 1-forms, respectively.

For $k = 1, 2$, a *size bounded* current $\bar{T} \in [\mathscr{S}_k(U)]^3$ is identified by a triplet $(\mathscr{N}, \mathbf{b}, \xi)$, where $\mathscr{N} \subset U$ is a k-rectifiable set of U satisfying $\mathscr{H}^k(\mathscr{N}) < \infty$, the $\mathbb{R}^3$-valued multiplicity function $\mathbf{b} : \mathscr{N} \to \mathbb{R}^3$ is $\mathscr{H}^k \llcorner \mathscr{N}$-summable, and $\xi : \mathscr{N} \to \bigwedge^k(\mathbb{R}^3)$ is an $\mathscr{H}^k \llcorner \mathscr{M}$-measurable function such that $\xi(w)$ is a simple unit k-vector yielding an orientation to the approximate tangent k-space to $\mathscr{N}$ at w, for $\mathscr{H}^k$-a.e. $w \in \mathscr{N}$. Without loss of generality, we assume $\mathbf{b}(w) \neq 0_{\mathbb{R}^3}$ for $\mathscr{H}^k$-a.e. $w \in \mathscr{N}$, so

that the *size* of the current $\bar{T}$ is equal to the k-measure of $\mathscr{N}$,

$$\mathbf{S}(\bar{T}) = \mathscr{H}^k(\mathscr{N}) < \infty.$$

The action of $\bar{T}$ is defined componentwise by the formula

$$\langle \bar{T}, \overline{\omega} \rangle := \sum_{j=1}^{3} \langle T^j, \omega_j \rangle, \quad \overline{\omega} \in [\mathscr{D}^k(U)]^3,$$

where $T^j = [\![\mathscr{N}, |b^j|, \sigma^j \xi]\!]$ if $\mathbf{b} = (b^1, b^2, b^3)$, and $\sigma^j := 1$ if $b^j \geq 0$, whereas $\sigma^j := -1$ if $b^j < 0$. Therefore, we have

$$\langle T^j, \omega_j \rangle = \int_{\mathscr{N}} |b^j| \langle \omega_j, \sigma^j \xi \rangle \, \mathrm{d}\mathscr{H}^k, \quad \forall\, \omega_j \in \mathscr{D}^k(U).$$

We thus assume that every component T^j has finite *mass*, namely

$$\mathbf{M}(T^j) = \int_{\mathscr{N}} |b^j| \, \mathrm{d}\mathscr{H}^k < \infty, \quad j = 1, 2, 3,$$

and the mass of $\bar{T}$ is given by $\mathbf{M}(\bar{T}) := \sum_{j=1}^{3} \mathbf{M}(T^j)$. With the previous notations, we thus write $\bar{T} = (T^1, T^2, T^3) = [\![\mathscr{N}, \mathbf{b}, \xi]\!]$.

Finally, setting $\mathrm{d}\overline{\omega} := (\mathrm{d}\omega_1, \mathrm{d}\omega_2, \mathrm{d}\omega_3)$, the boundary current is defined by

$$\langle \partial \bar{T}, \overline{\omega} \rangle := \langle \bar{T}, \mathrm{d}\overline{\omega} \rangle, \quad \forall\, \overline{\omega} \in [\mathscr{D}^{k-1}(U)]^3 .$$

2.5 Gauss Graphs of Smooth Surfaces

Given a smooth (say C^2), bounded, and oriented surface $\mathscr{M} \subset \mathbb{R}^3$, with smooth boundary $\partial\mathscr{M}$, the *Gauss map* $\nu : \mathscr{M} \to \mathbb{S}^2$ associates with each point $y \in \mathscr{M}$ the unit normal $\nu(y) \in \mathbb{S}^2$. The graph of such a map, or *Gauss graph*, is the 2-dimensional surface in $\mathbb{R}^6$ given by

$$\mathscr{G}\mathscr{M} := \{(y, z) \mid y \in \mathscr{M}\,,\ z = \nu(y)\} \subset \mathscr{M} \times \mathbb{S}^2 \subset \mathbb{R}^3 \times \hat{\mathbb{R}}^3 \simeq \mathbb{R}^6\,,$$

where $\hat{\mathbb{R}}^3$ is the isomorphic copy of $\mathbb{R}^3$ in which we consider embedded the sphere $\mathbb{S}^2$. We are interested in the case of a parametric surface

$$\mathscr{M} = \mathscr{M}_u := u(\Omega) \quad u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^3\,.$$

Therefore, we shall denote by $(\varepsilon_1, \varepsilon_2)$ and (e_1, e_2, e_3) the canonical bases of $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively, and by $(\varepsilon^1, \varepsilon^2)$ and (e^1, e^2, e^3) the natural dual bases defined by $\varepsilon_i \cdot \varepsilon^j = \delta_i^{\,j}$ and $e_h \cdot e^k = \delta_h^{\,k}$, where the interposed dot means dual pairing. In particular, with $x = (x_1, x_2)$ and $y = (y_1, y_2, y_3)$, we will identify the covector bases

in the cotangent spaces to Ω and $\mathbb{R}^3$ at every $x \in \Omega$ and $y \in \mathbb{R}^3$ with $(\mathrm{d}x^1, \mathrm{d}x^2)$ and $(\mathrm{d}y^1, \mathrm{d}y^2, \mathrm{d}y^3)$; they are naturally dual to the bases $(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2})$ and $(\frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2}, \frac{\partial}{\partial y^3})$ in the pertinent tangent spaces. We also correspondingly denote by $(\epsilon_1, \epsilon_2, \epsilon_3)$ the canonical basis of $\hat{\mathbb{R}}^3$, and by $(\mathrm{d}z^1, \mathrm{d}z^2, \mathrm{d}z^3)$ the associated covector basis, where $z = (z_1, z_2, z_3)$.

Referring to Anzellotti et al. (1990) or Anzellotti (1988) for further details, we only recall that if $\Phi : \mathscr{M} \to \mathbb{R}^3 \times \hat{\mathbb{R}}^3$ is the graph map $\Phi(y) := (y, \nu(y))$, the *tangential Jacobian* $J_\Phi^{\mathscr{M}}$ is given by

$$J_\Phi^{\mathscr{M}}(y) = \Big(1 + (\mathbf{k}_1{}^2 + \mathbf{k}_2{}^2) + (\mathbf{k}_1\mathbf{k}_2)^2\Big)^{1/2}, \qquad y \in \mathscr{M}$$

where $\mathbf{k}_{1,2} = \mathbf{k}_{1,2}(y)$ are the *principal curvatures* at $y \in \mathscr{M}$. *Mean curvature* and *Gauss curvature* are given by

$$\mathbf{H} := \frac{1}{2}(\mathbf{k}_1 + \mathbf{k}_2), \qquad \mathbf{K} := \mathbf{k}_1\mathbf{k}_2,$$

so that $\mathbf{k}_{1,2} = \mathbf{H} \pm \sqrt{\mathbf{H}^2 - \mathbf{K}}$. Therefore, we equivalently have

$$(J_\Phi^{\mathscr{M}})^2 = 1 + (2\mathbf{H})^2 - 2\mathbf{K} + \mathbf{K}^2 = 4\mathbf{H}^2 + (1 - \mathbf{K})^2$$

and hence, by the area formula, the area of the Gauss graph is

$$\begin{aligned} \mathscr{H}^2(\mathscr{G}\mathscr{M}) &= \int_{\mathscr{M}} \Big(1 + (\mathbf{k}_1{}^2 + \mathbf{k}_2{}^2) + (\mathbf{k}_1\mathbf{k}_2)^2\Big)^{1/2} \mathrm{d}\mathscr{H}^2 \\ &= \int_{\mathscr{M}} \sqrt{1 + (4\mathbf{H}^2 - 2\mathbf{K}) + \mathbf{K}^2}\, \mathrm{d}\mathscr{H}^2 < \infty\,. \end{aligned} \tag{2.7}$$

It agrees with the mass of the current $[\![\, \mathscr{G}\mathscr{M} \,]\!] = \Phi_\#[\![\, \mathscr{M} \,]\!]$ in $\mathscr{R}_2(\mathbb{R}^3 \times \mathbb{S}^2)$ carried by the Gauss graph.

We finally recall that the *curvature functional* $\|\mathscr{M}\|$ of a smooth surface $\mathscr{M} \subset \mathbb{R}^3$ is defined in reference (Anzellotti et al. 1990) to be

$$\|\mathscr{M}\| := \mathscr{H}^2(\mathscr{M}) + \int_{\mathscr{M}} \sqrt{\mathbf{k}_1^2 + \mathbf{k}_2^2}\, \mathrm{d}\mathscr{H}^2 + \int_{\mathscr{M}} |\mathbf{k}_1\mathbf{k}_2|\, \mathrm{d}\mathscr{H}^2\,. \tag{2.8}$$

It can be equivalently written as

$$\|\mathscr{M}\| := \int_{\mathscr{M}} \big(1 + \sqrt{4\mathbf{H}^2 - 2\mathbf{K}} + |\mathbf{K}|\big)\, \mathrm{d}\mathscr{H}^2\,,$$

so that by formula (2.7) we obtain bounds such as

$$\frac{1}{2}\|\mathscr{M}\| \le \mathscr{H}^2(\mathscr{G}\mathscr{M}) \le \|\mathscr{M}\|, \qquad \mathscr{H}^2(\mathscr{G}\mathscr{M}) = \mathbf{M}([\![\, \mathscr{G}\mathscr{M} \,]\!]).$$

3 Smooth Shell Deformations

Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain, $\mathfrak{h} > 0$ a fixed small thickness parameter, and

$$U = U_{\mathfrak{h}} := \Omega \times I_{\mathfrak{h}} , \quad I_{\mathfrak{h}} = (-\mathfrak{h}/2, \mathfrak{h}/2) .$$

We adopt the notation $x = (x_1, x_2) \in \mathbb{R}^2$ and $\mathbf{x} = (x_1, x_3, x_3) \in \bar{\mathbb{R}}^3 = \mathbb{R}^2 \times \mathbb{R}$, with $\bar{\mathbb{R}}^3$ an isomorphic copy of $\mathbb{R}^3$, the isomorphism being simply the identification. This choice is justified by the will of maintaining distinguished, although identical, the space in which we select the reference configuration with the one in which we detect the deformed configurations. We also identify $\mathrm{d}x$ with $\mathrm{d}\mathscr{L}^2$ or $\mathrm{d}\mathbf{x}$ with $\mathrm{d}\mathscr{L}^3$ inside the integrals defined over Ω or U, respectively.

Consider a deformation $u : \bar{\Omega} \to \mathbb{R}^3$ of class $C^2(\bar{\Omega}, \mathbb{R}^3)$ such that

$$\partial_1 u(x) \times \partial_2 u(x) \neq 0_{\mathbb{R}^3} \tag{3.1}$$

for every $x \in \Omega$. The unit normal to the oriented deformed surface $\mathscr{M}_u := u(\Omega)$ at $y = u(x)$ is given by

$$z = \nu_u(x) := \frac{\partial_1 u(x) \times \partial_2 u(x)}{|\partial_1 u(x) \times \partial_2 u(x)|} \in \mathbb{S}^2 \subset \hat{\mathbb{R}}^3. \tag{3.2}$$

3.1 Avoiding Self Penetration

When we consider the deformation of a $3D$ body into $3D$ space, two equivalent conditions express a constraint that avoids self-penetration of matter. One is due to Ciarlet and Nečas (1987), the other to Giaquinta et al. (1989) (see also (Giaquinta et al. 1998b, Sec. 2.3.2)).

For a sufficiently smooth map $v : U \to \mathbb{R}^3$ with $\det \nabla v > 0$ a.e. in U, the *global invertibility condition* above mentioned reads

$$\int_U f(\mathbf{x}, v(\mathbf{x})) \det \nabla v(\mathbf{x}) \, \mathrm{d}\mathbf{x} \leq \int_{\mathbb{R}^3} \sup_{\mathbf{x} \in U} f(\mathbf{x}, y) \, \mathrm{d}y \tag{3.3}$$

for every compactly supported smooth function $f : U \times \mathbb{R}^3 \to [0, +\infty)$. It is preserved by weak convergence in terms of currents and implies $\mathscr{L}^3$-a.e. injectivity (Giaquinta et al. 1989).

For a given parameter $0 < s < 1$, we thus denote by $v_{u,s} : U \to \mathbb{R}^3$ the deformation map

$$v_{u,s}(\mathbf{x}) := u(x_1, x_2) + s \, x_3 \, \nu_u(x_1, x_2) . \tag{3.4}$$

Writing $u = (u^1, u^2, u^3)$ and $v_u = (v_u^1, v_u^2, v_u^3)$, we have

$$\nabla v_{u,s} = \begin{pmatrix} \partial_1 u^1 + s\, x_3\, \partial_1 v_u^1 & \partial_2 u^1 + s\, x_3\, \partial_2 v_u^1 & s\, v_u^1 \\ \partial_1 u^2 + s\, x_3\, \partial_1 v_u^2 & \partial_2 u^2 + s\, x_3\, \partial_2 v_u^1 & s\, v_u^2 \\ \partial_1 u^3 + s\, x_3\, \partial_1 v_u^3 & \partial_2 u^3 + s\, x_3\, \partial_2 v_3 & s\, v_u^3 \end{pmatrix}, \tag{3.5}$$

so that

$$\lim_{s\to 0^+} \frac{1}{s} \int_U \nabla v_{u,s}\, \mathrm{d}\mathbf{x} = \mathfrak{h} \int_\Omega \mathbf{G}\, \mathrm{d}x\,, \tag{3.6}$$

where $\mathbf{G} \in C^\infty(\bar{\Omega}, \mathbb{R}^{3\times 3})$ has components

$$\mathbf{G}_A^j(x) := \begin{cases} \partial_A u^j(x) & \text{if} \quad A = 1, 2 \\ 0 & \text{if} \quad A = 3\,, \end{cases} \qquad j = 1, 2, 3\,. \tag{3.7}$$

We thus compute for each s

$$\det \nabla v_{u,s}(\mathbf{x}) = s\, |(\partial_1 u \times \partial_2 u)(x)| + s^2\, x_3\, \lambda_u^{(1)}(x) + s^3\, x_3^2\, \lambda_u^{(2)}(x)\,, \tag{3.8}$$

where

$$\begin{aligned} \lambda_u^{(1)}(x) :=\ & v_u^1 \left(\partial_1 u^2\, \partial_2 v_u^3 - \partial_2 u^2\, \partial_1 v_u^3 + \partial_1 v_u^2\, \partial_2 u^3 - \partial_2 v_u^2\, \partial_1 u^3\right) \\ & + v_u^2 \left(\partial_1 u^3\, \partial_2 v_u^1 - \partial_2 u^3\, \partial_1 v_u^1 + \partial_1 v_u^3\, \partial_2 u^1 - \partial_2 v_u^3\, \partial_1 u^1\right) \\ & + v_u^3 \left(\partial_1 u^1\, \partial_2 v_u^2 - \partial_2 u^1\, \partial_1 v_u^2 + \partial_1 v_u^1\, \partial_2 u^2 - \partial_2 v_u^1\, \partial_1 u^2\right), \\ \lambda_u^{(2)}(x) :=\ & v_u^1 \left(\partial_1 v_u^2\, \partial_2 v_u^3 - \partial_2 v_u^2\, \partial_1 v_u^3\right) \\ & + v_u^2 \left(\partial_1 v_u^3\, \partial_2 v_u^1 - \partial_2 v_u^3\, \partial_1 v_u^1\right) \\ & + v_u^3 \left(\partial_1 v_u^1\, \partial_2 v_u^2 - \partial_2 v_u^1\, \partial_1 v_u^2\right), \end{aligned} \tag{3.9}$$

so that we have

$$\lim_{s\to 0^+} \frac{1}{s} \int_U \det \nabla v_{u,s}\, \mathrm{d}\mathbf{x} = \mathfrak{h} \int_\Omega |\partial_1 u \times \partial_2 u|\, \mathrm{d}x\,. \tag{3.10}$$

Furthermore, since the functions $\lambda_u^{(k)}(x)$ are bounded, we get the uniform convergence $s^{-1} \det \nabla v_{u,s} \to |\partial_1 u \times \partial_2 u|$ as $s \to 0^+$ on U, and hence, we find

$$\det \nabla v_{u,s}(\mathbf{x}) > 0, \qquad \forall\, \mathbf{x} \in U,$$

if $s > 0$ is sufficiently small. Therefore, injectivity is recovered by requiring property (3.3), with $v = v_{u,s}$, for $s > 0$ sufficiently small (Mariano and Mucci 2023c)

3.2 Gauss Graphs of Smooth Parametric Surfaces

In the present setting, the Gauss graph of the surface $\mathscr{M}_u$ is described by

$$\mathscr{GM}_u = \{\Phi_u(x) \mid x \in \Omega\}\,, \tag{3.11}$$

where $\Phi_u : \Omega \to \mathbb{R}^3 \times \hat{\mathbb{R}}^3$ is the (smooth) Gauss graph map

$$\Phi_u(x) := (u(x), \nu_u(x)) \,. \tag{3.12}$$

The mean curvature of $\mathscr{M}_u$ at $u(x)$ becomes

$$\mathbf{H}_u = \frac{1}{2g_u} \left(|\partial_1 u|^2 \partial^2_{2,2} u + |\partial_2 u|^2 \partial^2_{1,1} u - 2(\partial_1 u \bullet \partial_2 u) \partial^2_{1,2} u \right) \bullet \nu_u \,, \tag{3.13}$$

where

$$g_u := |\partial_1 u \times \partial_2 u|^2 = |\operatorname{adj}_2 \nabla u|^2 \tag{3.14}$$

and the interposed bullet means scalar product. The Gauss curvature $\mathbf{K}_u$ reads

$$\mathbf{K}_u = \frac{1}{{g_u}^{3/2}} \left((\partial^2_{1,1} u \bullet \nu_u)(\partial^2_{2,2} u \bullet \nu_u) - (\partial^2_{1,2} u \bullet \nu_u)^2 \right). \tag{3.15}$$

The tangent space at a point $(y, z) = (u(x), \nu_u(x))$ in the Gauss graph $\mathscr{GM}_u$ is oriented for every $x \in \Omega$ by the wedge product

$$\xi_u(x) := \partial_1 \Phi_u(x) \wedge \partial_2 \Phi_u(x) \in \bigwedge\nolimits^2 (\mathbb{R}^3 \times \hat{\mathbb{R}}^3) \,. \tag{3.16}$$

According to the basis chosen, we can write the stratification

$$\xi_u = \xi_u^{(0)} + \xi_u^{(1)} + \xi_u^{(2)},$$

where on account of (3.12) we have:

$$\begin{aligned}
\xi_u^{(0)} &= \sum_{1 \le i < j \le 3} \det \begin{pmatrix} \partial_1 u^i & \partial_2 u^i \\ \partial_1 u^j & \partial_2 u^j \end{pmatrix} e_i \wedge e_j \,, \\
\xi_u^{(1)} &= \sum_{i,j=1}^{3} \det \begin{pmatrix} \partial_1 u^i & \partial_2 u^i \\ \partial_1 {\nu_u}^j & \partial_2 {\nu_u}^j \end{pmatrix} e_i \wedge \epsilon_j \,, \\
\xi_u^{(2)} &= \sum_{1 \le i < j \le 3} \det \begin{pmatrix} \partial_1 {\nu_u}^i & \partial_2 {\nu_u}^i \\ \partial_1 {\nu_u}^j & \partial_2 {\nu_u}^j \end{pmatrix} \epsilon_i \wedge \epsilon_j \,.
\end{aligned} \tag{3.17}$$

Moreover, by the expression (2.7), the area formula implies that the Gauss graph area $\mathscr{H}^2(\mathscr{GM}_u)$ is equal to

$$\begin{aligned}
&\int_\Omega \sqrt{g_u} \sqrt{1 + (4\mathbf{H}_u^2 - 2\mathbf{K}_u) + \mathbf{K}_u^2} \, \mathrm{d}x = \\
&\qquad \int_{\mathscr{M}_u} \sqrt{1 + (4\mathbf{H}_u^2 - 2\mathbf{K}_u) + \mathbf{K}_u^2} \, \mathrm{d}\mathscr{H}^2 = \int_\Omega |\xi_u| \, \mathrm{d}x \,.
\end{aligned} \tag{3.18}$$

Furthermore, on account of (2.8), the curvature functional $\|\mathscr{G}\mathscr{M}_u\|$ coincides with

$$\mathscr{E}(u) := \int_\Omega \sqrt{g_u}\left(1+\sqrt{4\mathbf{H}_u^2-2\mathbf{K}_u}+|\mathbf{K}_u|\right)\mathrm{d}x\,.$$

We thus get

$$\mathscr{E}(u) = \int_\Omega \left(|\xi_u^{(0)}|+|\xi_u^{(1)}|+|\xi_u^{(2)}|\right)\mathrm{d}x\,, \tag{3.19}$$

where by (3.17) we obtain

$$\begin{aligned}
|\xi_u^{(0)}|^2 &= g_u = |\partial_1 u\times\partial_2 u|^2 = |\operatorname{adj}_2\nabla u|^2\,,\\
|\xi_u^{(1)}|^2 &= g_u\,(4\mathbf{H}_u^2-2\mathbf{K}_u) = \sum_{i,j=1}^{3}\left(\partial_1 u^i\,\partial_2 {\nu_u}^j-\partial_2 u^i\,\partial_1 {\nu_u}^j\right)^2\,,\\
|\xi_u^{(2)}|^2 &= g_u\,\mathbf{K}_u^2 = \sum_{1\le i<j\le 3}\left(\partial_1 {\nu_u}^i\,\partial_2 {\nu_u}^j-\partial_2 {\nu_u}^i\,\partial_1 {\nu_u}^j\right)^2.
\end{aligned} \tag{3.20}$$

Let us now consider the 2-forms Θ_1 and Θ_2 in $\mathbb{R}^3\times\hat{\mathbb{R}}^3$ given by

$$\begin{aligned}
\Theta_1 &:= \frac{1}{2}\Big(z^1(\mathrm{d}y^2\wedge\mathrm{d}z^3-\mathrm{d}y^3\wedge\mathrm{d}z^2)+z^2(\mathrm{d}y^3\wedge\mathrm{d}z^1-\mathrm{d}y^1\wedge\mathrm{d}z^3)\\
&\qquad +z^3(\mathrm{d}y^1\wedge\mathrm{d}z^2-\mathrm{d}y^2\wedge\mathrm{d}z^1)\Big)\,,\\
\Theta_2 &:= z^1\mathrm{d}z^2\wedge\mathrm{d}z^3+z^2\mathrm{d}z^3\wedge\mathrm{d}z^1+z^3\mathrm{d}z^1\wedge\mathrm{d}z^2\,,
\end{aligned} \tag{3.21}$$

the latter a volume form in $\mathbb{S}^2$. The pullback through the graph map Φ_u is given by

$$-{\Phi_u}^{\#}\Theta_1 = \sqrt{g_u}\,\mathbf{H}_u\,\mathrm{d}x^1\wedge\mathrm{d}x^2\,,\qquad {\Phi_u}^{\#}\Theta_2 = \sqrt{g_u}\,\mathbf{K}_u\,\mathrm{d}x^1\wedge\mathrm{d}x^2\,, \tag{3.22}$$

where g_u is in formula (3.14).

Finally, on account of properties (3.5), (3.8), (3.9), definition (3.12), and relations (3.20), we readily obtain the estimate

$$|\nabla v_{u,s}|+|\operatorname{adj}_2\nabla v_{u,s}|+|\det\nabla v_{u,s}|\le c\left(|D\Phi_u|+|\xi_u^{(0)}|+|\xi_u^{(1)}|+|\xi_u^{(2)}|\right) \tag{3.23}$$

where $c>0$ is some absolute real constant not depending on u.

3.3 Absence of Cancellations

Assume now the shell thickness to be vanishing, namely infinitesimal. Consider also the circumstance in which two distant pieces of the deformed shell $\mathscr{M}_u = u(\Omega)$ are in contact, with the same or opposite orientation. In that case, we have two pairwise disjoint and connected open sets A, $B\subset\Omega$ such that both the restrictions $u_{|A}$ and $u_{|B}$ are injective, but $u(A)=u(B)$.

Therefore, one of the two following alternatives holds: either $u_\#[\![A]\!] = u_\#[\![B]\!]$, so that the image current $u_\#[\![A \cup B]\!]$ has multiplicity two, or $u_\#[\![A]\!] = -u_\#[\![B]\!]$, whence $u_\#[\![A \cup B]\!] = 0$. In the second case, a cancellation occurs and actually

$$\mathbf{M}(u_\#[\![\Omega]\!]) < \int_\Omega |\xi_u^{(0)}| \, \mathrm{d}x, \quad |\xi_u^{(0)}| = |\partial_1 u \times \partial_2 u|.$$

However, since the unit normal ν_u at corresponding points in $u(A)$ and $u(B)$ has opposite sign, by the expressions (3.17) it turns out that no cancellation occurs in the Gauss graph current $[\![\mathscr{G}\mathscr{M}_u]\!]$ (Mariano and Mucci 2023c). In conclusion, even in the presence of cancellations for the current $u_\#[\![\Omega]\!]$, we always have

$$\mathbf{M}([\![\mathscr{G}\mathscr{M}_u]\!]) = \mathscr{H}^2(\mathscr{G}\mathscr{M}_u) = \int_\Omega |\xi_u| \, \mathrm{d}x < \infty \tag{3.24}$$

and hence, on account of the expression (3.19) and (3.24),

$$\frac{1}{2} \mathscr{E}(u) \leq \mathbf{M}([\![\mathscr{G}\mathscr{M}_u]\!]) \leq \mathscr{E}(u).$$

4 New Summability Properties

Relevant to what we discuss here is the class of convex surfaces. For them, a summability result of the Gauss graph map holds (Theorem 4.1) together with some properties of conical points recently analyzed in Mucci and Saracco (2024). These results motivate the assumptions we shall adopt.

Since we use local arguments, we restrict to the nonparametric case, where the deformation is given by the graph map $x \mapsto u(x) := (x, g(x))$ of some concave, positive, and bounded function $g : B^2 \to \mathbb{R}$ defined on the unit disk B^2. The image $u(B^2)$, that is, the graph $\mathscr{G}_g = \{(x, g(x)) \mid x \in B^2\}$ of g, may be seen as a piece of the boundary of a convex set

$$K(g) := \{(x, t) \in \mathbb{R}^3 \mid x \in B^2, \ 0 \leq t \leq g(x)\}.$$

The function g is continuous, it belongs to the Sobolev class $W^{1,1}(B^2)$, and its gradient $\nabla g := (\partial_1 g, \partial_2 g)$ is a vector-valued function of bounded variation, that is, $\nabla g \in BV(B^2, \mathbb{R}^2)$. By the chain rule formula, the outward unit normal in (3.2) is given for $\mathscr{L}^2$-a.e. $x \in B^2$ by

$$\nu_u(x) = \frac{(-\partial_1 g(x), -\partial_2 g(x), 1)}{\sqrt{1 + |\nabla g(x)|^2}}, \tag{4.1}$$

and actually $\nu_u \in BV(B^2, \mathbb{S}^2)$. However, we cannot conclude that ν_u is a special function of bounded variation, in general.

Example 4.1 If $V : (0, 1) \to \mathbb{R}$ is the classical Cantor–Vitali function, and $v(\rho) := \int_0^\rho V(t) \, dt$, the positive function $g(x) = 1 - v(|x|)$ is concave on B^2, and $\nabla g(x) =$

$-V(|x|)\,x/|x|$ for $x \neq 0_{\mathbb{R}^2}$. We thus have

$$\nu_u(x) = \frac{\big(V(\rho)x_1,\, V(\rho)x_2,\, \rho\big)}{\rho\sqrt{1+V(\rho)^2}}, \quad \rho = |x|, \quad x \neq 0_{\mathbb{R}^2},$$

so that by the chain rule formula $D^C \nu_u \neq 0$, and hence, $\nu_u \notin SBV(B^2, \mathbb{S}^2)$.

Using known results (Anzellotti et al. 1990; Anzellotti and Ossanna 1995; Mucci 2021), we obtain new summability properties concerning the approximate derivatives of the unit normal ν_u.

Theorem 4.1 *Given $g : B^2 \to \mathbb{R}$ a bounded, positive, and concave function, let Φ_u denote the Gauss graph map in (3.12), where $u(x) = (x, g(x))$ and ν_u is given by (4.1). Moreover, let $\xi_u(x)$ be the 2-vector given for $\mathscr{L}^2$-a.e. $x \in B^2$ by (3.16). Then, $\xi_u \in L^1(B^2, \bigwedge^2(\mathbb{R}^3 \times \hat{\mathbb{R}}^3))$. In particular, we have*

i) $\partial_1 g\, \partial_2 \nu_u^j - \partial_2 g\, \partial_1 \nu_u^j \in L^1(B^2)$, *for* $j = 1, 2, 3$;

ii) $\partial_1 \nu_u^{j_1}\, \partial_2 \nu_u^{j_2} - \partial_2 \nu_u^{j_1}\, \partial_1 \nu_u^{j_2} \in L^1(B^2)$, *for* $1 \leq j_1 < j_2 \leq 3$.

Proof For $h \in \mathbb{N}^+$, denote by $K(g)_h$ the set of points in $\mathbb{R}^3$ whose distance from the convex set $K(g)$ is lower than $1/h$. Its boundary $\partial K(g)_h$ is a surface of class $C^{1,1}$. By setting $B^2_r := \{x \in \mathbb{R}^2 : |x| < r\}$, we have

$$K(g)_h = \{(x,t) \in \mathbb{R}^3 \mid x \in B^2_{1+1/h}, \quad -\alpha_h(x) \leq t \leq \beta_h(x)\},$$

where $\alpha_h(x) = 1/h$ if $|x| \leq 1$, and $\alpha_h(x) = \sqrt{h^{-2} - 1 + 2|x| - |x|^2}$ if $1 \leq |x| \leq 1 + 1/h$, whereas $\beta_h : B^2_{1+1/h} \to \mathbb{R}$ is a smooth, positive, concave function.

Therefore, the currents Σ_h carried by the Gauss graph of the (naturally oriented) smooth surfaces $\partial K(g)_h$ are i.m. rectifiable currents with equibounded masses and zero boundary. By Federer and Fleming's closure theorem, a subsequence of $\{\Sigma_h\}$ weakly converges to a current Σ (compare (Anzellotti et al. 1990)). On the other hand, setting $g_h := \beta_{h|B^2}$, the sequence $\{g_h\}$ of Lipschitz-continuous functions converges to g uniformly in B^2.

As a consequence, by the relationship between Σ_h and g_h, it turns out that the concave function g has finite relaxed energy in the sense of Mucci (2021), and hence, the assertion readily follows. Expressions (i) and (ii) in the theorem follow from (3.17). Further details are omitted. □

Something more can be said concerning conical points of the convex surface given by $u(B^2) = \mathscr{G}_g$. To this end, we recall that g is differentiable at $\mathscr{L}^2$-a.e. point $x \in B^2$, and that the unit normal at regular points is given by the Lebesgue value $\nu_u(x) \in \mathbb{S}^2$, see (4.1).

If g is not differentiable at x, a weak notion of unit normal at the point $P = u(x) = (x, g(x))$ is given in Mucci and Saracco (2024). Denote by $\mathbf{c}_P$ the geodesically convex spherical curve obtained by intersecting the tangent cone to $u(B^2) = \mathscr{G}_g$ at P with the unit sphere centered at P. By exploiting (through approximating inscribed geodesically convex polygonals) a polarity argument concerning the curve $\mathbf{c}_P$, it is possible to define a function $\nu_P : [0, 2\pi[\to \mathbb{S}^2$ satisfying the following properties:

- ν_P is a function of bounded variation, $\nu_P \in BV((0, 2\pi), \mathbb{S}^2)$;
- the geodesically convex hull of the rectifiable image of ν_P in $\mathbb{S}^2$ gives a good notion of normal cone $\mathcal{N}_P$ to $u(B^2) = \mathcal{G}_g$ at P;
- with $\mathscr{AD}(P)$ the area of normal cone $\mathcal{N}_P$, also called an *angle defect*, if P is a ridge or a regular point, $\mathscr{AD}(P) = 0$ and $\mathscr{AD}(P) > 0$ otherwise, that is, at any conical point P;
- for every $P \in u(B^2)$, we can find a sequence $\{\Sigma_h\}$ of smooth surfaces converging in Hausdorff distance to the tangent cone to $u(B^2)$ at P and such that, denoting by $\mathbf{K}_h$ the Gauss curvature of Σ_h, we have

$$\lim_{h\to\infty} \int_{\Sigma_h} |\mathbf{K}_h| \, d\mathscr{H}^2 = \mathscr{AD}(P).$$

Referring to Mucci and Saracco (2024) for further details, it turns out that at every point $P = u(x)$, a Dirac measure with weight equal to the angle defect $\mathscr{AD}(P)$ is well defined, and $\mathscr{AD}(P) = 0$ if and only if P is either a regular or a ridge point.

5 Possibly Irrecoverable Sharp Folds

We consider the occurrence of possibly irrecoverable sharp folds. For them, we propose a description in terms of vector-valued measures concentrated on 1-dimensional sets where jumps of the unit normal to the deformed shell may appear. Such measures are introduced in terms of size bounded currents $\bar{T}$ in $[\mathscr{S}_2(U)]^3$, where $U \subset \bar{\mathbb{R}}^3$ is a bounded open set.

To this purpose, we first point out that a tensor-valued measure $\bar{F} = \bar{F}(\bar{T})$ in $\mathscr{M}_b(U, \mathbb{R}^{3\times 3})$ is naturally associated with a current $\bar{T} \in [\mathscr{S}_2(U)]^3$, by letting

$$\langle \bar{F}, \zeta \rangle := \langle \bar{T}, \omega^{(2)}_\zeta \rangle, \quad \forall\, \zeta \in C^\infty_c(U, \mathbb{R}^{3\times 3}) \, .$$

With the previous notation, we denote $\bar{T} = [\![\mathcal{N}, \mathbf{b}, \xi]\!]$, where $\xi = \xi_1 \varepsilon_2 \wedge \varepsilon_3 + \xi_2 \varepsilon_3 \wedge \varepsilon_1 + \xi_3 \varepsilon_1 \wedge \varepsilon_2$, with $(\varepsilon_A)_{A=1}^3$ the canonical basis in $\bar{\mathbb{R}}^3$. Then, writing $\bar{\omega} = \omega^{(2)}_\zeta$ we have

$$\langle T^j, \omega_j \rangle = \int_{\mathcal{N}} b^j \sum_{A=1}^{3} (-1)^{A-1} \zeta_{jA} \xi_A \, \mathrm{d}\mathscr{H}^2 \, , \quad j = 1, 2, 3 \, .$$

Therefore, recalling that $*\xi = \sum_{A=1}^3 \xi_A \varepsilon_A$, where $*$ is the Hodge operator in $\bar{\mathbb{R}}^3$, we get

$$\bar{F} = \mathbf{b} \otimes \mathbf{m} \, \mathscr{H}^2 \llcorner \mathcal{N} \quad \mathbf{m} := *\xi \, .$$

Finally, for every $\psi \in C^\infty_c(U, \mathbb{R}^{3\times 3})$ we have:

$$\langle \operatorname{curl} \bar{F}, \psi \rangle := \langle \bar{F}, \operatorname{curl} \psi \rangle = \langle \bar{T}, \omega^{(2)}_{\operatorname{curl} \psi} \rangle = \langle \partial \bar{T}, \omega^{(1)}_\psi \rangle \, . \tag{5.1}$$

5.1 A Relevant Example

Definition 5.1 Let $\mathscr{PF}$ be the class of triplets $(\mathscr{I}, \mathbf{n}, \Theta)$ endowed with the following properties:

(1) $\mathscr{I}$ is a 1-rectifiable subset of Ω;
(2) $\mathbf{n} : \mathscr{I} \to \mathbb{R}^3$ is an $\mathscr{H}^1 \llcorner \mathscr{I}$-measurable unit vector field that is normal to the approximate tangent 1-space to $\mathscr{I}$ at $\mathscr{H}^1$-a.e. $(x_1, x_2) \in \mathscr{I}$;
(3) $\Theta : \mathscr{I} \to \mathbb{R}^3$ is $\mathscr{H}^1 \llcorner \mathscr{I}$-summable, with components $\Theta = (\theta^1, \theta^2, \theta^3)$, such that $|\Theta| \leq 2$.

For a given triplet $(\mathscr{I}, \mathbf{n}, \Theta)$ in $\mathscr{PF}$, we denote by $\mathbf{t} = \mathbf{n}^\perp$ the unit vector

$$\mathbf{t} := (t_1, t_2) = (-n_2, n_1) \quad \text{if} \quad \mathbf{n} = (n_1, n_2)\,, \tag{5.2}$$

that is tangent to $\mathscr{I}$ at $\mathscr{H}^1 \llcorner \mathscr{I}$-a.e. point.

An $\mathbb{R}^{3\times 2}$-valued rectifiable measure $\mu_{\mathscr{I},\mathbf{n}^\perp,\Theta}$ is naturally associated to the triplet $(\mathscr{I}, \mathbf{n}, \Theta)$, with $\mu_{\mathscr{I},\mathbf{n}^\perp,\Theta}$ defined by

$$\mu_{\mathscr{I},\mathbf{n}^\perp,\Theta} := \Theta \otimes \mathbf{n}^\perp \, \mathscr{H}^1 \llcorner \mathscr{I}\,.$$

More precisely, a test function $\bar{f} \in C_b^0(\Omega, \mathbb{R}^{3\times 2})$ is given by $\bar{f} = (\bar{f}_{jA})$, where $j = 1, 2, 3$ and $A = 1, 2$, and each $\bar{f}_{jA}$ is a bounded and continuous function on Ω. With $\mathbf{t} = \mathbf{n}^\perp$, we have

$$\langle \mu_{\mathscr{I},\mathbf{n}^\perp,\Theta}, \bar{f} \rangle = \sum_{j=1}^{3} \sum_{A=1}^{2} \int_{\mathscr{I}} \theta^j \, \bar{f}_{jA} t_A \, \mathrm{d}\mathscr{H}^1\,. \tag{5.3}$$

Moreover, for any open set $V \subset \Omega$, the total variation of $\mu_{\mathscr{I},\mathbf{n}^\perp,\Theta}$ is bounded, namely

$$|\mu_{\mathscr{I},\mathbf{n}^\perp,\Theta}|(V) = \int_{V\cap\mathscr{I}} |\Theta| \, \mathrm{d}\mathscr{H}^1 < \infty\,.$$

For $0 < s < 1$, we let $\bar{T}_s = (T_s^1, T_s^2, T_s^3) := [\![\, \mathscr{N}, \mathbf{b}_s, \xi \,]\!] \in [\mathscr{S}_2(U)]^3$, where $\mathscr{N} := \mathscr{I} \times I_\mathfrak{h}$, $\mathbf{b}_s(x_1, x_2, x_3) := s\,\Theta(x_1, x_2)$, and $\xi := *\mathbf{m}$, where $\mathbf{m} := (\mathbf{n}, 0)$, so that $\xi = n_1\,\varepsilon_2 \wedge \varepsilon_3 + n_2\,\varepsilon_3 \wedge \varepsilon_1$, and $*\xi = \mathbf{m}$. In particular, the tensor-valued measure in $\mathscr{M}_b(U, \mathbb{R}^{3\times 3})$ naturally associated with $\bar{T}_s$ is given by

$$\bar{F}_s = s\, x_3\, \Theta \otimes \mathbf{m}\, \mathscr{H}^2 \llcorner (\mathscr{I} \times I_\mathfrak{h})\,. \tag{5.4}$$

Finally, we choose a smooth test function $\varphi \in C_c^\infty(I_\mathfrak{h})$ that satisfies the constraint $\int_{I_\mathfrak{h}} \varphi(x_3)\,\mathrm{d}x_3 = 1$, so that

$$\int_{I_\mathfrak{h}} x_3\, \varphi'(x_3)\,\mathrm{d}x_3 = -\int_{I_\mathfrak{h}} \varphi(x_3)\,\mathrm{d}x_3 = -1\,, \tag{5.5}$$

for a given $\bar f \in C^0_b(\Omega, \mathbb{R}^{3\times 2})$ we introduce the notation $\psi = \bar f\varphi$ if $\psi_{jA}(\mathbf{x}) := \bar f_{jA}(x_1, x_2)\,\varphi(x_3)$, when $A = 1, 2$, and $\psi_{j3}(\mathbf{x}) \equiv 0$, for $j = 1, 2, 3$. Notice that $f\varphi \in C^\infty_c(U, \mathbb{R}^{3\times 3})$ if $f \in C^\infty_c(\Omega, \mathbb{R}^{3\times 2})$.

Theorem 5.1 *With the previous notation, for any* $(\mathscr{I}, \mathbf{n}, \Theta) \in \mathscr{PF}$ *we have*

$$\frac{1}{s}\,\langle \operatorname{curl} \bar F_s,\, \bar f\varphi\rangle = \langle \mu_{\mathscr{I},\mathbf{n}^\perp,\Theta},\, \bar f\rangle \quad \forall\, \bar f \in C^0_b(\Omega, \mathbb{R}^{3\times 2})\,.$$

Proof Using that $T^j_s = [\![\, \mathscr{N}, s\,|\theta^j|, \sigma^j \xi \,]\!]$, if $\overline{\omega} = \omega^{(2)}_\zeta$, we get

$$\langle T^j_s, \omega_j\rangle = s\int_{\mathscr{N}} x_3\,\theta^j\left(n_1\,\zeta_{j1} + n_2\zeta_{j2}\right)\mathrm{d}\mathscr{H}^2$$

for $j = 1, 2, 3$, and hence, taking $\overline{\omega} = \omega^{(1)}_\psi \in [\mathscr{D}^1(U)]^3$, we compute

$$\langle \partial T^j_s, \omega_j\rangle := \langle T^j_s, \mathrm{d}\omega_j\rangle = s\int_{\mathscr{N}} x_3\,\theta^j\left(n_1\,(\operatorname{curl}\psi_j)^1 + n_2\,(\operatorname{curl}\psi_j)^2\right)\mathrm{d}\mathscr{H}^2$$

where $(\operatorname{curl}\psi_j)^1 = \partial_{x_2}\psi_{j3} - \partial_{x_3}\psi_{j2}$ and $(\operatorname{curl}\psi_j)^2 = \partial_{x_3}\psi_{j1} - \partial_{x_1}\psi_{j3}$. Therefore, with $\psi = f\varphi$, since $f_{j3} = 0$, we get

$$\langle \partial T^j_s, \omega_j\rangle = s\int_{\mathscr{N}} x_3\,\theta^j\left(-n_1\,f_{j2}\,\varphi' + n_2\,f_{j1}\,\varphi'\right)\mathrm{d}\mathscr{H}^2.$$

Recalling that $\mathscr{N} = \mathscr{I}\times I_{\mathfrak{h}}$, by Fubini's theorem and (5.5) we obtain

$$\langle \partial T^j_s, \omega_j\rangle = s\sum_{A=1}^{2}\int_{\mathscr{I}} \theta^j\,f_{jA}\,t_A\,\mathrm{d}\mathscr{H}^1 \quad \forall\, j = 1, 2, 3\,.$$

On account of equations (5.1) and (5.3), we have just shown the thesis when $f \in C^\infty_c(U, \mathbb{R}^{3\times 2})$. Since the involved measures have finite total variation, the assertion follows through a density argument. □

5.2 Possibly Irrecoverable Sharp Folds in the Deformed Surface

Let now $u : \bar\Omega \to \mathbb{R}^3$ be a continuous function such that

(1) $u \in W^{1,1}(\Omega, \mathbb{R}^3)$;
(2) property (3.1) holds true for $\mathscr{L}^2$-a.e. $x \in \Omega$;
(3) the unit normal ν_u given by (3.2) belongs to $SBV(\Omega, \mathbb{S}^2)$;
(4) the jump set of ν_u is a 1-rectifiable subset of Ω, that is $\mathscr{H}^1(S(\nu_u)) < \infty$;
(5) with Φ_u the Gauss graph map in (3.12) for $\mathscr{L}^2$-a.e. $x \in \Omega$, and ξ_u the corresponding 2-vector in (3.16), then $\xi_u \in L^1(\Omega, \bigwedge^2(\mathbb{R}^3 \times \hat{\mathbb{R}}^3))$.

As already shown, properties (1)–(5) are automatically verified if $\Omega = B^2$ and $u(x) = (x, g(x))$ for some bounded and concave function $g : B^2 \to \mathbb{R}$, provided that condition (2.6) holds, with $\nu = \nu_u$. Moreover, if (1)–(5) hold, the distributional derivative of $\nu = \nu_u$ satisfies the decomposition formula (2.1), where $n = 2$ and $D^C \nu_u = 0$, the approximate gradient $\nabla \nu_u$ belongs to $L^1(\Omega, \mathbb{R}^{3\times 2})$, and the jump component $D^J \nu_u$ satisfies (2.2), where $n = 2$ and $\mathbf{n} = \mathbf{n}_u \in \mathbb{R}^2$ is a given unit normal to $S(\nu_u)$.

For $0 < s < 1$, we denote by $v_{u,s} : U \to \mathbb{R}^3$ the deformation map defined for $\mathscr{L}^3$-a.e. $\mathbf{x} \in U$ through the expression (3.4). Since $\nabla \Phi_u \in L^1(\Omega, \mathbb{R}^{6\times 2})$, it turns out that $v_{u,s} \in SBV(U, \mathbb{R}^3)$, so that we can write

$$Dv_{u,s} = \nabla v_{u,s}\, \mathscr{L}^3 \llcorner U + D^J v_{u,s}\,.$$

The approximate gradient $\nabla v_{u,s} \in L^1(U, \mathbb{R}^{3\times 3})$ is given by the expression (3.5), for $\mathscr{L}^3$-a.e. $\mathbf{x} \in U$, so that the limit (3.6) holds, with $\mathbf{G} \in L^1(\Omega, \mathbb{R}^{3\times 3})$ given $\mathscr{L}^3$-a.e. in Ω by formula (3.7). Moreover, the jump component satisfies

$$D^J v_{u,s} = s\, x_3\, [\nu_u] \otimes \mathbf{m}_u\, \mathscr{H}^2 \llcorner (S(\nu_u) \times I_\mathfrak{h})\,, \quad \mathbf{m}_u := (\mathbf{n}_u, 0)\,.$$

The triplet $(S(\nu_u), \mathbf{n}_u, [\nu_u])$ belongs to the class $\mathscr{PF}$ (see Definition 5.1) and we thus introduce the measure

$$\mu^J_{\nu_u} := \mu_{S(\nu_u), \mathbf{n}_u^\perp, [\nu_u]} = [\nu_u] \otimes \mathbf{n}_u^\perp\, \mathscr{H}^1 \llcorner S(\nu_u)\,. \tag{5.6}$$

According to (5.2), with $\mathbf{n} = \mathbf{n}_u$, by (5.3) we have

$$\langle \mu^J_{\nu_u}, \bar f\rangle = \sum_{j=1}^{3}\sum_{A=1}^{2} \int_{S(\nu_u)} [\nu_u]^j\, \bar f_{jA} t_A \,\mathrm{d}\mathscr{H}^1\,, \quad \forall\, \bar f \in C^0_b(\Omega, \mathbb{R}^{3\times 2}).$$

With the latter choice, on account of (5.4), we also have $\bar F_s = D^J v_{u,s}$. Therefore, Theorem 5.1 gives for each $s > 0$ the equation

$$\frac{1}{s}\,\langle \operatorname{curl} \bar F_s, \bar f \varphi\rangle = \langle [\nu_u] \otimes \mathbf{n}_u^\perp\, \mathscr{H}^1 \llcorner S(\nu_u), \bar f\rangle\,, \quad \bar F_s := D^J v_{u,s}\,, \tag{5.7}$$

for every $\bar f \in C^0_b(\Omega, \mathbb{R}^{3\times 2})$, where, we recall, φ is any function in $C^\infty_c(I_\mathfrak{h})$ that satisfies $\int_{I_\mathfrak{h}} \varphi(x_3)\,\mathrm{d}x_3 = 1$.

Theorem 5.1 allows us to describe the occurrence of possibly irrecoverable sharp folds in the deformed surface.

To this aim, we fix a triplet $(\mathscr{J}, \mathbf{n}, \Theta)$ in $\mathscr{PF}$; based on definitions (5.3) and (5.6), we require the validity of the total variation bound

$$|\mu^J_{\nu_u}|(V) \le |\mu_{\mathscr{J}, \mathbf{n}^\perp, \Theta}|(V) \quad \forall\, V \subset \Omega\,, \quad V \text{ open}\,. \tag{5.8}$$

By the Borel regularity of the involved measures, property (5.8) implies that the jump set $S(\nu_u)$ is $\mathscr{H}^1$-essentially contained in the 1-rectifiable set $\mathscr{I}$, that is,

$$\mathscr{H}^1(S(\nu_u) \setminus \mathscr{I}) = 0\,. \tag{5.9}$$

Therefore, without loss of generality we can require that $\mathbf{n} = \mathbf{n}_u$ at $\mathscr{H}^1$-a.e. $x \in S(\nu_u)$. It suffices to replace Θ with $-\Theta$ at points in $S(\nu_u)$ where $\mathbf{n} = -\mathbf{n}_u$.

Since $[\nu_u] = \nu_u^+ - \nu_u^-$, with $\nu_u^\pm \in \mathbb{S}^2$, the angle α between the one-sided normals,

$$\alpha(x) := 2\,\arcsin\frac{|[\nu_u](x)|}{2},$$

describes the emergence of a wedge on the deformed surface $u(\Omega)$ at $u(x)$, for $\mathscr{H}^1$-a.e. $x \in S(\nu_u)$. Since property (5.8) implies that $|[\nu_u]| \le |\Theta|$ at $\mathscr{H}^1$-a.e. $x \in S(\nu_u)$, we obtain the angle bound:

$$\alpha \le 2\,\arcsin\frac{|\Theta|}{2} \le \frac{\pi}{2}|\Theta|, \quad \mathscr{H}^1\text{-a.e. on } S(\nu_u).$$

The inequality $\mathscr{H}^1(\mathscr{I} \setminus S(\nu_u)) > 0$ may describe the occurrence of irrecoverable sharp folds at previous steps of a process, where at the present stage the deformation is unfolded. No sharp folds appear in the case $\mathscr{H}^1(S(\nu_u)) = 0$, so that $\nu_u \in W^{1,1}(\Omega, \mathbb{S}^2)$, a sufficient condition to its validity being given by equation $\mathscr{H}^1(\mathscr{I}) = 0$.

5.3 A Multiplicative Decomposition Formula

For the sake of completeness, even if we shall not pursue the pertinent path, we finally point out a multiplicative decomposition formula (see (5.10)).

Assume that $u \in C(\overline{\Omega}, \mathbb{R}^3)$ satisfies properties (1)–(5). Then, $\det \nabla v_{u,s}(\mathbf{x})$ is given for $\mathscr{L}^3$-a.e. $\mathbf{x} \in U$ by formula (3.8), with $\lambda_u^{(k)}(x)$ given for $\mathscr{L}^2$-a.e. $x \in \Omega$ by (3.9), for $k = 1, 2$. Therefore, $\det \nabla v_{u,s} \in L^1(U)$ provided that $\lambda_u^{(k)} \in L^1(\Omega)$ for $k = 1, 2$. More generally, it turns out that the pointwise inequality (3.23) holds true $\mathscr{L}^3$-a.e. on U, where $c > 0$ is some absolute real constant. Therefore, besides the validity of the limit (3.10), property (5) gives the summability of all minors of the approximate gradient $\nabla v_{u,s}$.

Assume now that the gradient matrix $\nabla v_{u,s}$ is invertible $\mathscr{L}^3$-a.e. on U, for $s > 0$ small. We have already seen that this property holds true if u is smooth. In this case (compare (Mariano and Mucci 2023a)), we can write the distributional derivative measure $F_s = Dv_{u,s}$ through a multiplicative decomposition:

$$F_s = F_s^e F_s^p\,, \quad F_s^e := G_s \mathscr{L}^3 \llcorner U\,. \tag{5.10}$$

The vector field $G_s \in L^1(U, \mathbb{R}^{3\times 3})$ appearing in the factor F_s^e satisfies $G_s(\mathbf{x}) = \nabla v_{u,s}(\mathbf{x})$ for $\mathscr{L}^3$-a.e. $\mathbf{x} \in U \setminus \mathscr{N}$, with $\mathscr{N} := S(\nu_u) \times I_\mathfrak{h}$, and $G_s(\mathbf{x}) = \mathbf{I}_3$ if $\mathbf{x} \in \mathscr{N}$.

The plastic factor is given by the sum

$$F_s^p = (\nabla v_{u,s})^{-1}\, \mathscr{L}^3 \llcorner U + \bar{F}_s\,, \tag{5.11}$$

with $\bar{F}_s$ the measure in $\mathscr{M}_b(U, \mathbb{R}^{3\times 3})$ given by (5.4), where $\Theta := [\nu_u]$ and $\mathbf{m} := \mathbf{m}_u$. Therefore, in general curl $F_s = 0$, but equation (5.7) implies that curl $F_s^e \neq 0$ and curl $F_s^p \neq 0$, if $\mathscr{H}^1(S(\nu_u)) > 0$, that is, if $\nu_u \notin W^{1,1}(\Omega, \mathbb{S}^2)$. In this sense, from a purely kinematic perspective, in the presence of (possibly) irrecoverable sharp folds, we are reasonably entitled to call F_s^e and F_s^p the *elastic* and *plastic* components of deformation gradient.

6 Cusps

The occurrence of cusps in the deformed shell $\mathscr{M}_u = u(\Omega)$ can be described in terms of atomic measures. Our starting point is the assumption that any physically plausible configuration is obtained as the limit of a sequence of smooth deformations. Through a minimum energy principle, we then describe the situation when the deformed shell has *folds without wrinkles*. We also require that the folding in the deformed shell *does not degenerate*. Under such ansatz, the theory of currents carried by Gauss graphs implies that there is an essentially unique way to "fill in the jumps" of the unit normal ν_u.

As an example, we analyze concentration phenomena of the Gauss curvature, in the case of convex deformations. The basic idea is that around a cusp the deformation is locally convex.

6.1 Deformed Shells as Weak Limits of Gauss Graphs

Let $\psi :]0, +\infty[\to \mathbb{R}^+$ be a positive and convex function with linear growth at $+\infty$ and such that $\psi(\rho) \to +\infty$ when $\rho \to 0^+$, as, for example, $\psi(\rho) = \rho^{-1} \vee \rho$. For any smooth map $u : \Omega \to \mathbb{R}^3$, we correspondingly define

$$\mathscr{F}_p(u) := \int_\Omega \big(|\nabla u|^p + \psi(|\partial_1 u \times \partial_2 u|)\big)\, \mathrm{d}x\,, \quad p > 2\,. \tag{6.1}$$

Moreover, Dirichlet-type conditions are established through the choice of a smooth and injective function $\widetilde{u} \in C^2(\widetilde{\Omega}, \mathbb{R}^3)$ defined in an open set $\widetilde{\Omega} \subset \mathbb{R}^2$ such that the closure of the domain Ω is well contained in $\widetilde{\Omega}$. As usual, $\|u\|_\infty$ denotes the L^∞ norm of u in Ω.

Definition 6.1 Let $p > 2$. A function $u : \Omega \to \mathbb{R}^3$ is said to belong to the class $\mathscr{A}_p(\Omega)$, if u is the pointwise $\mathscr{L}^2$-a.e. limit of a sequence $\{u_h\} \subset C^2(\Omega, \mathbb{R}^3)$ of smooth functions that satisfy the following properties:

- every function u_h can be smoothly extended to $\widetilde{\Omega}$ so that $u_h = \widetilde{u}$ on $\widetilde{\Omega} \setminus \Omega$;
- every function u_h satisfies (3.1) for all $x \in \Omega$;

- on account of (3.19) and (6.1),

$$\sup_h \Big(\|u_h\|_\infty + \mathscr{F}_p(u_h) + \mathscr{E}(u_h) + |Dv_{u_h}|(\Omega) \Big) < \infty;$$

- for each h, there exists $s_h \in (0,1)$ such that the deformation map v_{u_h,s_h} given by (3.4) satisfies the global invertibility condition (3.3), for every $f \in C_c^\infty(U \times \mathbb{R}^3, [0,+\infty))$.

Arguments taken from Mariano and Mucci (2023c) imply what follows:

Proposition 6.1 *Take $u \in \mathscr{A}_p(\Omega)$, with $p > 2$. Then,*

- *$u \in W^{1,p}(\Omega, \mathbb{R}^3)$, and the property (3.1) holds true for $\mathscr{L}^2$-a.e. $x \in \Omega$;*
- *the unit normal ν_u given by (3.2) belongs to $BV(\Omega, \mathbb{S}^2)$;*
- *with Φ_u the Gauss graph map in (3.12) for $\mathscr{L}^2$-a.e. $x \in \Omega$, and ξ_u the corresponding 2-vector in (3.16), we have $\xi_u \in L^1(\Omega, \bigwedge^2(\mathbb{R}^3 \times \hat{\mathbb{R}}^3))$.*

By Proposition 6.1, for every $u \in \mathscr{A}_p(\Omega)$ we can define the energy $\mathscr{F}_p(u)$ by (6.1), but in terms of the approximate gradient. Moreover, if $\{u_h\}$ is any smooth sequence from Definition 6.1, arguing as in Mariano and Mucci (2023c) we also obtain a lower semicontinuity inequality:

$$\begin{aligned} &\|u\|_\infty + \mathscr{F}_p(u) + \mathscr{E}(u) + |D\nu_u|(\Omega) \le \\ &\qquad \liminf_{h\to\infty} \Big(\|u_h\|_\infty + \mathscr{F}_p(u_h) + \mathscr{E}(u_h) + |Dv_{u_h}|(\Omega) \Big) < \infty \,. \end{aligned} \tag{6.2}$$

If $u \in \mathscr{A}_p(\Omega)$ and $\{u_h\}$ is taken as above, on account of the definition (3.4) of the deformation map $v_{u,s}$, we could assume in addition that

$$\sup_h \int_U |\det \nabla v_{u_h,s_h}|^{-r} \, \mathrm{d}\mathbf{x}, \quad r > 0,$$

for some sequence $\{s_h\} \subset (0,1)$ such that $s_h \ge \bar{s} > 0$ for every h. We thus obtain $\det \nabla v_{u,\bar{s}}(\mathbf{x}) > 0$ for $\mathscr{L}^3$-a.e. $\mathbf{x} \in U$. However, since in general ν_u fails to be a Sobolev map, unlike the case analyzed in Mariano and Mucci (2023c), we cannot conclude that $v_{u,\bar{s}}$ satisfies the global invertibility condition (3.3).

The approximation process described by Definition 6.1 admits the occurrence of wrinkled folds: a concentration excess of total variation of the unit normal along its jump set is allowed, indeed. Therefore, folds without wrinkles are obtained by requiring energy minimality.

To describe this aspect, for a given map $\nu \in BV(\Omega, \mathbb{S}^2)$ that admits the decomposition formulae (2.1) and (2.2), with $n = 2$, we consider the variation of ν with respect to the geodesic distance $\mathrm{dist}_{\mathbb{S}^2}$ in $\mathbb{S}^2$, that is,

$$\mathrm{Var}_{\mathbb{S}^2}(\nu) := \int_\Omega |\nabla \nu| \, dx + |D^C \nu|(\Omega) + \int_{S(\nu)} \mathrm{dist}_{\mathbb{S}^2}(\nu^-(x), \nu^+(x)) \, \mathrm{d}\mathscr{H}^1(x) \,.$$

By (2.3), we have

$$\frac{2}{\pi}\,\mathrm{Var}_{\mathbb{S}^2}(\nu) \le |D\nu|(\Omega) \le \mathrm{Var}_{\mathbb{S}^2}(\nu) \quad \forall\, \nu \in BV(\Omega, \mathbb{S}^2)$$

and the strict inequality $|D\nu|(\Omega) < \mathrm{Var}_{\mathbb{S}^2}(\nu)$ holds as soon as the jump component $D^J\nu$ is non-trivial.

Definition 6.2 Let $p > 2$. We denote by $\mathscr{A}_p^\infty(\Omega)$ the class of maps u in $\mathscr{A}_p(\Omega)$ for which we can find a sequence $\{u_h\} \subset C^2(\Omega, \mathbb{R}^3)$ as in Definition 6.1 that satisfies in addition the identity

$$\lim_{h\to\infty} \int_\Omega |\nabla \nu_{u_h}|\,\mathrm{d}x = \mathrm{Var}_{\mathbb{S}^2}(\nu_u)\,. \tag{6.3}$$

6.2 Weak Limit Currents

Take $u \in \mathscr{A}_p^\infty(\Omega)$ and a smooth sequence $\{u_h\}$ as in Definition 6.2. By Federer and Fleming's closure theorem, we find a (not relabeled) subsequence such that $G_{\nu_{u_h}} \rightharpoonup T$ weakly in $\mathscr{D}_2(\Omega \times \mathbb{S}^2)$ to some *Cartesian current* T in $\mathrm{cart}(\Omega \times \mathbb{S}^2)$, with underlying function equal to ν_u (see (Giaquinta et al. 1998a)). Denote by $\mathscr{T}_{\nu_u}$ the class of currents in $\mathrm{cart}(\Omega \times \mathbb{S}^2)$ obtained in this way. Additional notations are instrumental for analyzing the structure of currents T in $\mathscr{T}_{\nu_u}$. They are here in order.

Any 2-form $\omega \in \mathscr{D}^2(\Omega \times \mathbb{S}^2)$ is decomposed into the sum $\omega = \omega^{(0)} + \omega^{(1)} + \omega^{(2)}$ according to the number of differentials $\mathrm{d}z^j$. Then, a generic current $T \in \mathscr{D}_2(\Omega \times \mathbb{S}^2)$ is equal to the sum $T = T_{(0)} + T_{(1)} + T_{(2)}$, where we define

$$\langle T_{(k)}, \omega\rangle := \langle T, \omega^{(k)}\rangle, \quad \forall\, \omega \in \mathscr{D}^2(\Omega \times \mathbb{S}^2),$$

for $k = 0, 1, 2$. We also recall that $\widehat{\mathrm{d}x^i}$ is the covector in $\mathbb{R}^2$ such that $\widehat{\mathrm{d}x^i} \wedge \mathrm{d}x^i = (-1)^i \mathrm{d}x$, for $i = 1, 2$, where $\mathrm{d}x := \mathrm{d}x^1 \wedge \mathrm{d}x^2$.

If ν_u is the unit normal to a given map $u \in \mathscr{A}_p^\infty(\Omega)$, we have denoted by G_{ν_u} the current in $\mathscr{R}_2(\Omega \times \mathbb{S}^2)$ carried by the rectifiable graph of ν_u (see Sect. 2.3). Moreover, a current $T^C_{\nu_u}$ in $\mathscr{D}_2(\Omega \times \mathbb{S}^2)$ is naturally associated with the Cantor component $D^C\nu_u$ of the derivative of ν_u. It satisfies $T^C_{\nu_u(0)} = 0$, $T^C_{\nu_u(2)} = 0$ and

$$\langle T^C_{\nu_u}, \psi(x,z)\,\widehat{\mathrm{d}x^i} \wedge \mathrm{d}z^j\rangle := (-1)^i \int_\Omega \psi(x, \nu_u(x))\,\mathrm{d}(D^C\nu_u)^j_i\,, \tag{6.4}$$

for any test function $\psi \in C^\infty_c(\Omega \times \mathbb{R}^3)$, $i = 1, 2$, $j = 1, 2, 3$, where $\nu_u(x)$ is a precise representative. Indeed, we have

$$\mathbf{M}(T^C_{\nu_u}) = |D^C\nu_u|(\Omega) < \infty\,. \tag{6.5}$$

Finally, concerning the jump component $D^J\nu_u$, we denote by T^J a generic current in $\mathscr{D}_2(\Omega \times \mathbb{S}^2)$ such that $T^J_{(0)} = 0$, $T^J_{(2)} = 0$, and for any choice of ψ, i and j as above

we write

$$\langle T^J, \psi(x,z)\,\widehat{\mathrm{d}x^i}\wedge \mathrm{d}z^j\rangle = -\int_{S(v_u)}\Big(\int_{\Gamma_x}\psi(x,z)\,\mathrm{d}z^j\Big)\,\widehat{t_i}(x)\,\mathrm{d}\mathscr{H}^1(x)\,, \tag{6.6}$$

where Γ_x is a rectifiable and oriented arc in $\mathbb{S}^2$ from $v_u^-(x)$ to $v_u^+(x)$, for $\mathscr{H}^1$-a.e. $x\in S(v_u)$. In formula (6.6), referring to (2.1) and (5.2), we considered $\widehat{t_1}=t_2=n_1$ and $\widehat{t_2}=t_1=-n_2$. We thus obtain

$$\mathbf{M}(T^J)=\int_{S(v_u)}\mathscr{H}^1(\Gamma_x)\,\mathrm{d}\mathscr{H}^1(x)\,. \tag{6.7}$$

The following result motivates the previous notation.

Lemma 6.1 *Let $\bar T$ be a current of the form $\bar T = G_{v_u}+T^C_{v_u}+T^J$. Then, for every $\eta\in\mathscr{D}^1(\Omega\times\mathbb{S}^2)$ such that $\eta=\eta^{(0)}$ we have $\langle\partial\bar T,\eta\rangle=0$.*

Proof We show that $\bar T(\mathrm{d}\eta)=0$ when $\eta=(-1)^{i-1}\varphi(x,y)\,\widehat{\mathrm{d}x^i}$, where $i=1,2$ and $\varphi\in C_c^\infty(\Omega\times\mathbb{S}^2)$. The assertion follows by linearity. Since

$$\mathrm{d}\eta=\partial_{x_i}\varphi(x,y)\,\mathrm{d}x+(-1)^i\sum_{j=1}^3\partial_{y_j}\varphi(x,y)\,\widehat{\mathrm{d}x^i}\wedge\mathrm{d}y^j,$$

using that for $\mathscr{L}^2$-a.e. $x\in\Omega$

$$\partial_{x_i}\left[\varphi(x,v_u(x))\right]=\partial_{x_i}\varphi(x,v_u(x))+\sum_{j=1}^3\partial_{y_i}\varphi(x,v_u(x))\,\partial_i v_u^j(x)$$

and by the chain rule formula

$$(D^C[\varphi(x,v_u(x))])_i=\sum_{j=1}^3\partial_{y_i}\varphi(x,v_u(x))\,(D^Cv_u)_i^j\,,$$

whereas for $\mathscr{H}^1$-a.e. $x\in S(v_u)$

$$\int_{\Gamma_x}\sum_{j=1}^3\partial_{y_j}\varphi(x,y)\,\mathrm{d}y^j=\varphi(x,v_u^+(x))-\varphi(x,v_u^-(x)),$$

in terms of the BV function $v(x):=\varphi(x,v_u(x))$ we obtain

$$\langle\bar T,\mathrm{d}\eta\rangle=\int_\Omega\partial_{x_i}v(x)\,\mathrm{d}x+\int_\Omega\mathrm{d}\,(D^Cv)_i+\int_{S(v)}\big(v^+(x)-v^-(x)\big)\,n_i\,\mathrm{d}\mathscr{H}^1(x)$$

and the assertion follows from the chain rule formula of v. □

The following structure result holds:

Theorem 6.1 *Assume $T \in \mathscr{T}_{v_u}$ for some $u \in \mathscr{A}_p^\infty(\Omega)$ (see (6.12)). Then, T is an i.m. rectifiable current in $\mathscr{R}_2(\Omega \times \mathbb{S}^2)$ that satisfies the null-boundary condition*

$$\langle \partial T, \eta \rangle = 0 \quad \forall\, \eta \in \mathscr{D}^1(\Omega \times \mathbb{S}^2)\,. \tag{6.8}$$

Moreover, we can decompose T as $T = G_{v_u} + T^C_{v_u} + T^J + S_T$ so that

$$\mathbf{M}(T) = \mathbf{M}(G_{v_u}) + \mathbf{M}(T^C_{v_u}) + \mathbf{M}(T^J) + \mathbf{M}(S_T) < \infty\,, \quad \textit{where} \tag{6.9}$$

- *G_{v_u} is the current carried by the graph of v_u;*
- *$T^C_{v_u}$ is the Cantor component defined as above, see (6.4);*
- *T^J is the jump component given as above, where in (6.6) we have*

$$\Gamma_x \textit{ is a geodesic arc from } v^-(x) \textit{ to } v^+(x) \textit{ for } \mathscr{H}^1\textit{-a.e. } x \in S(v)\,; \tag{6.10}$$

- *$S_T \in \mathscr{D}_2(\Omega \times \mathbb{R}^2)$ is completely vertical, that is, $S_{T\,(0)} + S_{T\,(1)} = 0$.*

Proof For a given current $R \in \mathscr{R}_2(\Omega \times \mathbb{S}^2)$, we let

$$\mathscr{F}_1(R) := \sup_{\omega \in \mathscr{F}^1} \langle R, \omega \rangle$$

where $\mathscr{F}^1$ denotes the class of 2-forms $\omega \in \mathscr{D}^2(\Omega \times \mathbb{S}^2)$ such that $\omega = \omega^{(1)}$ and $\|\omega\| \le 1$. We thus equivalently have $\mathscr{F}_1(R) = \mathbf{M}(R_{(1)})$.

By the properties of the class $\mathrm{cart}(\Omega \times \mathbb{S}^2)$ (Giaquinta et al. 1998b), we infer that $T \in \mathscr{R}_2(\Omega \times \mathbb{S}^2)$, the null-boundary condition (6.8) holds and there exists $\Sigma_T \in \mathscr{R}_2(\Omega \times \mathbb{S}^2)$ such that $\Sigma_{T\,(0)} = 0$ and

$$T = G_{v_u} + \Sigma_T, \quad \text{where} \quad \mathbf{M}(T) = \mathbf{M}(G_{v_u}) + \mathbf{M}(\Sigma_T) < \infty\,.$$

Let $\{u_h\} \subset C^2(\Omega, \mathbb{R}^3)$ be the approximating sequence from Definition 6.2. Possibly passing to a not relabeled subsequence, we can assume that $G_{v_{u_h}} \rightharpoonup T$ and that the strict convergence (6.3) holds. Since v_u is a BV function, by the structure properties of the class $\mathrm{cart}(\Omega \times \mathbb{S}^2)$ we can decompose Σ_T into the sum of three terms, namely

$$\Sigma_T = T^C_{v_u} + T^J + S_T\,,$$

where $T^C_{v_u}$ and T^J are given as above. With the previous notation, we thus have

$$\mathscr{F}_1(G_{v_u}) = \int_\Omega |\nabla v_u|\, \mathrm{d}x\,, \quad \mathscr{F}_1(T^C_{v_u}) = \mathbf{M}(T^C_{v_u})\,, \quad \mathscr{F}_1(T^J) = \mathbf{M}(T^J)\,,$$

where all terms are finite. Moreover, the null-boundary condition in Lemma 6.1 implies that

$$\mathscr{F}_1(T^C_{v_u} + T^J) + \mathscr{F}_1(S_T) \leq \mathscr{F}_1(\Sigma_T).$$

In addition, according to (6.5) and (6.7), by the mutual singularity of the measures $D^C v_u$ and $D^J v_u$ we obtain the decomposition

$$\mathscr{F}_1(T^C_{v_u} + T^J) = \mathscr{F}_1(T^C_{v_u}) + \mathscr{F}_1(T^J) < \infty,$$

where by the definition of the component T^j we clearly have

$$\int_{S(v_u)} \operatorname{dist}_{\mathbb{S}^2}(v_u^-, v_u^+)\, \mathrm{d}\mathscr{H}^1 \leq \mathscr{F}_1(S_T)\,. \tag{6.11}$$

Since $G_{v_{u_h}} \rightharpoonup T$, by lower semicontinuity we have

$$\mathscr{F}_1(G_{v_u}) + \mathscr{F}_1(\Sigma_T) = \mathscr{F}_1(T) \leq \liminf_{h\to\infty} \mathscr{F}_1(G_{v_{u_h}}),$$

where the first equality follows from the mutual singularity of the measures associated with the components G_{v_u} and Σ_T of T. Moreover, since

$$\mathscr{F}_1(G_{v_{u_h}}) = \int_\Omega |\nabla v_{u_h}|\, \mathrm{d}x \quad \forall\, h,$$

the strict convergence in (6.3) is equivalent to the limit

$$\lim_{h\to\infty} \mathscr{F}_1(G_{v_{u_h}}) = \mathscr{F}_1(G_{v_u}) + \mathscr{F}_1(T^C_{v_u}) + \int_{S(v_u)} \operatorname{dist}_{\mathbb{S}^2}(v_u^-, v_u^+)\, \mathrm{d}\mathscr{H}^1.$$

Since $\mathscr{F}_1(G_{v_u}) < \infty$, by the previous properties we obtain the inequalities

$$\mathscr{F}_1(T^C_{v_u}) + \mathscr{F}_1(T^J) + \mathscr{F}_1(S_T) \leq \mathscr{F}_1(\Sigma_T) \leq \mathscr{F}_1(T^C_{v_u}) + \mathscr{F}_1(T^J) < \infty\,,$$

and hence $\mathscr{F}_1(S_T) = 0$. Recalling that $S_{T(0)} = 0$, the identity $\mathscr{F}_1(S_T) = 0$ gives $S_{T(1)} = 0$ and hence $S_T = S_{T(2)}$. As a consequence, the property (6.9) holds true. Finally, since $\mathscr{F}_1(T^C_{v_u}) < \infty$, the equality sign holds true in (6.11). Therefore, we have $\mathscr{H}^1(\Gamma_x) = \operatorname{dist}_{\mathbb{S}^2}(v_u^-(x), v_u^+(x))$ for $\mathscr{H}^2$-a.e. $x \in S(v_u)$. Recalling that Γ_x is an oriented arc in $\mathbb{S}^2$ from $v_u^-(x)$ to $v_u^+(x)$, the property (6.10) holds and the proof is complete. □

6.3 Mass Minimizing Currents

Take $u \in \mathscr{A}_p^\infty(\Omega)$. By Federer and Fleming's closure theorem and Theorem 6.1, there exists $T_{\min} \in \mathscr{T}_{\nu_u}$ such that

$$\mathbf{M}(T_{\min}) = \min\left\{\mathbf{M}(T) \mid T \in \mathscr{T}_{\nu_u}\right\}. \tag{6.12}$$

Localized possibly irrecoverable strain in a small neighborhood of discrete points and cusps will be described in terms of a mass minimizing current in $\mathscr{T}_{\nu_u}$. However, we now see that it fails to be unique, in general, even in the case when the unit normal ν_u is a Sobolev map.

Example 6.1 Let $\Omega = B^2$ and u be such that the deformed shell $u(B^2)$ forms a cusp, i.e., $u(x) = (x_1, x_2, f(|x|))$, where

$$f(\rho) := 1 - \sqrt{2\rho - \rho^2}\,, \quad \rho \in [0,1]\,, \tag{6.13}$$

in such a way that with $\rho := |x| > 0$ we have

$$\nu_u(x) = \frac{1}{\sqrt{1 + f'(\rho)^2}}\left(-f'(\rho)\, x_1/\rho,\ -f'(\rho)\, x_2/\rho,\ 1\right).$$

The unit normal ν_u belongs to the class $W^{1,q}(B^2, \mathbb{S}^2)$ for every $q < 2$, and around the origin $0_{\mathbb{R}^2}$ it describes the circle $\mathbb{S}^1 := \{z \in \mathbb{S}^2 \mid z_3 = 0\}$, which can be oriented in such a way that

$$\partial G_{\nu_u} \llcorner B^2 \times \mathbb{S}^2 = -\delta_{0_{\mathbb{R}^2}} \times [\![\, \mathbb{S}^1 \,]\!].$$

Denote by $\mathbb{S}^2_\pm := \{z \in \mathbb{S}^2 \mid \pm z_3 > 0\}$ the naturally oriented upper and lower half spheres, so that

$$\partial [\![\, \mathbb{S}^2_+ \,]\!] = [\![\, \mathbb{S}^1 \,]\!], \quad \partial [\![\, \mathbb{S}^2_- \,]\!] = -[\![\, \mathbb{S}^1 \,]\!].$$

Then, both the currents

$$T_+ := G_{\nu_u} + \delta_{0_{\mathbb{R}^2}} \times [\![\, \mathbb{S}^2_+ \,]\!], \quad T_- := G_{\nu_u} - \delta_{0_{\mathbb{R}^2}} \times [\![\, \mathbb{S}^2_- \,]\!]$$

are mass minimizers in the class $\mathscr{T}_{\nu_u}$.

We have (see Theorem 6.2)

$$T_+ - T_- = \delta_{0_{\mathbb{R}^2}} \times [\![\, \mathbb{S}^2 \,]\!].$$

Moreover, if Θ_2 the 2-form in $\mathbb{S}^2$ given by (3.21), we obtain

$$\langle T_\pm, \varphi(x)\, \Theta_2(z)\rangle = \pm\langle [\![\, \mathbb{S}^2_\pm \,]\!], \Theta_2\rangle \cdot \langle \delta_{0_{\mathbb{R}^2}}, \varphi\rangle = \pm\pi\, \varphi(0_{\mathbb{R}^2}) \tag{6.14}$$

for every $\varphi \in C_c^\infty(B^2)$—see definition (6.18).

Example 6.2 Let $\Omega = B^2$ and u be such that the deformed shell $u(B^2)$ forms a sharp fold, i.e., $u(x) = (x_1, x_2, f(|x_1|))$, with f given by (6.13), so that

$$\nu_u(x) = \frac{1}{\sqrt{1 + f'(|x_1|)^2}} \left(-f'(|x_1|)\, x_1/|x_1|,\ 0,\ 1\right), \quad x_1 \neq 0\,.$$

The unit normal ν_u belongs to the class $SBV(B^2, \mathbb{S}^2)$, with jump set $S(\nu_u) = B^2 \cap \{x_1 = 0\}$ and one-sided limits $\nu_u^{\pm} \equiv (\pm 1, 0, 0)$, if we choose $\mathbf{n} \equiv (1, 0)$. Any mass minimizer in $\mathscr{T}_{\nu_u}$ is equal to $T = G_{\nu_u} + T^J$, where in the jump component we can take Γ_x equal to any fixed geodesic arc of $\mathbb{S}^2$ from ν_u^- to ν_u^+.

However, in Theorem 6.2 we shall see that the arc Γ_x must be orthogonal to the orienting unit vector to the fold $u(S(\nu_u))$ of the deformed shell at $u(x)$, and definitely it is parameterized by $\Phi(t) = (-\cos t, 0, \sin t)$, $t \in [0, \pi]$, for every $x \in S(\nu_u)$.

The latter property applies when the sharp fold does not degenerate. An opposite circumstance is described in the following example.

Example 6.3 Let $\Omega = B^2$ and $u \in W^{1,\infty}(\Omega, \mathbb{R}^3)$ be given by

$$u(x) = \left(\frac{x_1|x_2|}{|x|},\ \frac{x_2|x_2|}{|x|},\ 0\right), \quad x = (x_1, x_2), \quad x_2 \neq 0\,,$$

so that the continuous representative satisfies $u(x_1, 0) \equiv 0_{\mathbb{R}^3}$. Thus, we have $\partial_1 u \times \partial_2 u = (0, 0, x_2|x_2|/|x|^2)$ for $x_2 \neq 0$, whence $\nu_u(x) = (0, 0, 1)$ if $x_2 > 0$, and $\nu_u(x) = (0, 0, -1)$ if $x_2 < 0$, so that $S(\nu_u) = B^2 \cap \{x_2 = 0\}$. Also, $u(S(\nu_u)) = \{0_{\mathbb{R}^3}\}$ and u maps the half disks $B^2 \cap \{\pm x_2 > 0\}$ onto the disk

$$D = \{y \in \mathbb{R}^3 \mid {y_1}^2 + (y_2 - 1/2)^2 < 1/4,\ \ y_3 = 0\},$$

but with opposite orientation. In particular, with $\mathbf{t} = (1, 0)$ the unit tangent vector to $S(\nu_u)$, we have $\partial_{\mathbf{t}} u(x) = 0$ for every $x \in S(\nu_u)$.

We aim at excluding situations like the previous one. So, for $u \in \mathscr{A}_p^{\infty}(\Omega)$, we denote by G_{Φ_u} the current carried by the graph of the map Φ_u given by (3.12), and we notice that if it is endowed with finite mass, $G_{\Phi_u} \in \mathscr{R}_2(U)$, where $U := \Omega \times \mathbb{R}^3 \times \mathbb{S}^2$. We recall that the unit vector $\mathbf{t} = \mathbf{t}(x) = \mathbf{n}_u(x)^{\perp}$ fixes the orientation to the approximate tangent 1-space to the jump set $S(\nu_u)$ at $\mathscr{H}^1$-a.e. $x \in S(\nu_u)$.

Theorem 6.2 *Let $u \in \mathscr{A}_p^{\infty}(\Omega)$ be such that*

$$\mathbf{M}(G_{\Phi_u}) + \mathbf{M}(\partial G_{\Phi_u}) < \infty\,. \tag{6.15}$$

Then, u is approximately differentiable $\mathscr{H}^1$-a.e. on $S(\nu_u)$. Assume in addition that the tangential gradient $\partial_{\mathbf{t}} u \neq 0_{\mathbb{R}^3}$ for $\mathscr{H}^1$-a.e. $x \in S(\nu_u)$. Then, for any couple $T_1,\ T_2$ of mass minimizing currents in $\mathscr{T}_{\nu_u}$, we can find a current $L \in \mathscr{R}_0(\Omega)$ such that

$$T_1 - T_2 = L \times [\![\, \mathbb{S}^2 \,]\!]\,. \tag{6.16}$$

Proof Setting $S := T_1 - T_2$, it suffices to show that S is a completely vertical integral 2-cycle; that is, $S \in \mathscr{R}_2(\Omega \times \mathbb{S}^2)$ satisfies $\partial S = 0$ and $S = S_{(2)}$. Indeed, following (Giaquinta et al. 1998b), any completely vertical 2-form $\omega = \omega^{(2)} \in \mathscr{D}^2(\Omega \times \mathbb{S}^2)$ can be written as

$$\omega^{(2)}(x, z) = \phi(x)\Theta_2(z) + \mathrm{d}_z\eta(x, z),$$

where $\phi \in C_c^\infty(\Omega)$, Θ_2 is the volume 2-form in (3.21) and $\eta \in \mathscr{D}^1(\Omega \times \mathbb{S}^2)$ is of the type $\eta = \eta^{(1)}$. Writing $\mathrm{d}\eta = \mathrm{d}_x\eta + \mathrm{d}_z\eta$, we have $(\mathrm{d}_x\eta)^{(2)} = 0$ and hence $\langle S, \mathrm{d}_x\eta\rangle = 0$. Therefore, we obtain:

$$\langle S, \mathrm{d}_z\eta\rangle = \langle \partial S, \eta\rangle - \langle S, \mathrm{d}_x\eta\rangle = 0,$$

whence the action of S is possibly nonzero only on the term $\phi\Theta_2$, and definitely (6.16) holds.

By Theorem 6.1, for $\ell = 1, 2$ we can write $T_\ell = G_{\nu_u} + T^C_{\nu_u} + T^J_\ell + S_{T_\ell}$, so that $S = S_{T_1} - S_{T_2} + T_1^J - T_2^J$. Therefore, if we show that the jump components are equal, that is,

$$T_1^J = T_2^J\,, \tag{6.17}$$

we obtain that $S \in \mathscr{R}_2(\Omega \times \mathbb{S}^2)$ satisfies $\partial S = 0$ and $S = S_{(2)}$.

The property (6.17) is trivial if $\nu_u \in W^{1,1}(\Omega, \mathbb{S}^2)$. More generally, if (6.15) holds, a result by Ambrosio et al. (1998) implies that Φ_u is a function in SBV whose trace functions are approximately differentiable $\mathscr{H}^1$-almost everywhere on the jump set of Φ_u. Since $u \in W^{1,p}(\Omega, \mathbb{R}^2)$, with $p > 2$, the first assertion follows.

Assume that $\partial_{\mathbf{t}} u \neq 0_{\mathbb{R}^3}$ for $\mathscr{H}^1$-a.e. $x \in S(\nu_u)$. Then, the unit vector $\mathbf{T}(x) := \partial_{\mathbf{t}} u(x)/|\partial_{\mathbf{t}} u(x)|$ provides an orientation to the fold $u(S(\nu_u))$ of the deformed shell $u(\Omega)$ at $u(x)$, for $\mathscr{H}^1$-a.e. $x \in S(\nu_u)$.

By the properties of weak limits of the Gauss graphs (see (Anzellotti et al. 1990; Mariano and Mucci 2023a)), it turns out that, for $\mathscr{H}^1$-a.e. $x \in S(\nu_u)$, the arc Γ_x appearing in the definition (6.6), for $T = T_\ell$, must belong to the great circle obtained by the intersection of the Gauss sphere $\mathbb{S}^2$ with the 2-space $\Pi_x^\perp$ orthogonal to the vector $\mathbf{T}(x)$, and the orientation of Γ_x is consistent with the one induced by $\mathbf{T}(x)$ on $\Pi_x^\perp$. Indeed, when $\nu_u^-(x) \neq -\nu_u^+(x)$ we have $\mathbf{T}(x) = \lambda(\nu_u^-(x) \times \nu_u^+(x))$ for some $\lambda > 0$. Instead, when $\nu_u^-(x) = -\nu_u^+(x)$, up to a rotation in $\mathbb{R}^3$ we may assume that $\nu_u^-(x) = (-1, 0, 0)$, $\nu_u^+(x) = (1, 0, 0)$, $\mathbf{T}(x) = (0, 1, 0)$ and the oriented arc Γ_x is parameterized by $\Phi(t) = (-\cos t, 0, \sin t)$, $t \in [0, \pi]$. As a consequence, the geodesic arcs Γ_x only depend on the deformation map u and on its normal ν_u. We thus definitely obtain equation (6.17), and the proof is complete. □

6.4 Need for Atomic Measures

Let T be a mass minimizer in the class $\mathscr{T}_{\nu_u}$ for some $u \in \mathscr{A}_p^\infty(\Omega)$. Recalling that Θ_2 is the volume 2-form in (3.21), we denote by $\boldsymbol{\mu}_T$ the Borel regular signed measure on Ω generated by the finite mass distribution

$$\boldsymbol{\mu}_T(\phi) := \langle T\,, \phi(x)\,\Theta_2(z)\rangle\,, \quad \phi \in C_c^\infty(\Omega)\,. \tag{6.18}$$

If the minimizer T in $\mathscr{T}_{\nu_u}$ is unique, we shall write $\boldsymbol{\mu}_u := \boldsymbol{\mu}_T$. In general, we decompose $\boldsymbol{\mu}_T$ into mutually singular components, namely, $\boldsymbol{\mu}_T = \boldsymbol{\mu}_T^a + \boldsymbol{\mu}_T^s$, where $\boldsymbol{\mu}_T^a$ is absolutely continuous with respect to the Lebesgue measure $\mathscr{L}^2$. Hence, we have

$$\boldsymbol{\mu}_T^a(\phi) = \langle G_{\nu_u}, \phi(x)\,\Theta_2(z)\rangle \quad \forall\,\phi \in C_c^\infty(\Omega)\,.$$

Since it only depends on the unit normal ν_u to u, we shall write $\boldsymbol{\mu}_u^a = \boldsymbol{\mu}_T^a$. On account of (3.22) and the area formula, we have

$$\boldsymbol{\mu}_u^a(\phi) = \int_\Omega \phi(x)\,\sqrt{g_u(x)}\,\mathbf{K}_u(x)\,\mathrm{d}x$$

with $\mathbf{K}_u(x)$ given by (3.15) for $\mathscr{L}^2$-a.e. $x \in \Omega$, and we may say that the measure $\boldsymbol{\mu}_T$ describes concentration of the Gauss curvature energy of the shell surface. Indeed, in the smooth case we have $\boldsymbol{\mu}_T = \boldsymbol{\mu}_u^a$, so that since $[\![\, \mathscr{G}\mathscr{M}_u \,]\!] = \Phi_{u\#}[\![\, \Omega \,]\!]$, we obtain

$$\int_{\mathscr{M}_u} \mathbf{K}_u\,\mathrm{d}\mathscr{H}^2 = \int_\Omega \sqrt{g_u}\,\mathbf{K}_u \mathrm{d}x = \int_\Omega {\Phi_u}^\# \Theta_2 = \langle [\![\, \mathscr{G}\mathscr{M}_u \,]\!],\, \Theta_2\rangle.$$

On account of Theorem 6.1, the singular part satisfies

$$\boldsymbol{\mu}_T^s(\phi) = \langle \Sigma_T, \phi(x)\,\Theta_2(z)\rangle \quad \forall\,\phi \in C_c^\infty(\Omega)\,.$$

We finally denote by $\boldsymbol{\mu}_T^{at}$ the *atomic component* of $\boldsymbol{\mu}_T$. Since the measure $\boldsymbol{\mu}_T$ has finite total variation, the Schwartz theorem implies that $\boldsymbol{\mu}_T^{at}$ is concentrated on a countable set of points.

To avoid deformed surfaces with spread wrinkles, we require that the singular component $\boldsymbol{\mu}_T^s$ be concentrated on a 1-rectifiable set of Ω. However, in general this is not the case even for convex shells. Indeed, as in case of the map u from Example 4.1, with $T = G_{\nu_u} + T^C_{\nu_u}$, it turns out that $\boldsymbol{\mu}_T^s$ is concentrated on the set $\Sigma = \{x \in B^2 : |x| \in C\}$, where C is the Cantor middle-thirds set, whence Σ has Hausdorff dimension $1 + \log_3 2$. Therefore, we shall work with deformation maps whose normal ν_u is a special function of bounded variation.

By Theorem 6.1, we have

$$\boldsymbol{\mu}_T^{at}(\phi) = \langle S_T, \phi(x)\,\Theta_2(z)\rangle \quad \forall\,\phi \in C_c^\infty(\Omega)\,.$$

Therefore, if $\nu_u \in SBV(\Omega, \mathbb{S}^2)$, under the hypotheses of Theorem 6.2, by (6.17) and (6.16), since $\mathbf{M}(T_1) = \mathbf{M}(T_2)$, we obtain what follows:

- the component $\boldsymbol{\mu}_T^s - \boldsymbol{\mu}_T^{at}$ does not depend on the choice of the mass minimizing current T in $\mathscr{T}_{\nu_u}$, and it is concentrated on the jump set $S(\nu_u)$;
- the total variation $|\boldsymbol{\mu}_T^{at}|$ of the atomic component does not depend on the choice of the mass minimizing current T in $\mathscr{T}_{\nu_u}$—see eq. (6.14).

As a consequence, with an abuse of notation we shall write $\boldsymbol{\mu}_u^{at} = \boldsymbol{\mu}_T^{at}$, independently of the choice of $T = T_{\min}$ in (6.12). Indeed, if T_1 and T_2 are mass minimizers in $\mathscr{T}_{v_u}$, we have

$$|\boldsymbol{\mu}_{T_1}|(V) = |\boldsymbol{\mu}_{T_2}|(V) \quad \forall\, V \subset \Omega\,, \quad V \text{ open}.$$

We thus introduce a nonnegative and finite Borel measure $\mathbf{m}_p$ whose atomic part $\mathbf{m}_p^{at}$ is concentrated in a given countable set of points in Ω, in such a way that the atomic part of the measure $\boldsymbol{\mu}_u$ is concentrated in the support of $\mathbf{m}_p^{at}$, namely

$$|\boldsymbol{\mu}_u|(V) \le \mathbf{m}_p(V) \quad \forall\, V \subset \Omega\,, \quad V \text{ open}\,. \tag{6.19}$$

We call $\mathbf{m}_p^{at}$ *the atomic component of a deformation implying possibly irrecoverable strain*. We have

$$|\boldsymbol{\mu}_u^{at}|(\{x\}) := \lim_{r\to 0^+} |\boldsymbol{\mu}_u|(B^2(x,r)) = c > 0 \quad \Longrightarrow \quad \mathbf{m}_p^{at}(\{x\}) \ge c\,. \tag{6.20}$$

At some $x \in \Omega$, we could have $|\boldsymbol{\mu}_u^{at}|(\{x\}) = 0$ but $\mathbf{m}_p^{at}(\{x\}) > 0$. In that case, $u(x)$ may be interpreted as a plastic nucleation point where the deformed surface presents no cusps.

6.5 Examples

In all the examples below, we have $\Omega = B^2$ and $u(x) = (x, g(x))$ for some bounded and concave function $g : B^2 \to \mathbb{R}$ such that $\nu_u \in SBV(B^2, \mathbb{S}^2)$. Moreover, the mass minimizer in $\mathscr{T}_{\nu_u}$ is unique and will be indicated by T_u. Qualitatively different behaviors of the singular measure $\boldsymbol{\mu}_u^s$ appear.

In the first example, the measure $\boldsymbol{\mu}_u^s$ is a Dirac mass at the vertex of a cone, with a weight equal to the angle defect (that is, to the area of the geodesically convex set of $\mathbb{S}^2$ determined by the solid tangent cone at the vertex; see Fig. 2).

Example 6.4 Let g be of the form $g(x) = -m\rho$, where $m > 0$ and $\rho := |x|$, so that for $\rho > 0$ we have $\nu_u(x) = (mx_1/\rho, mx_2/\rho, 1)/\sqrt{1+m^2}$. Therefore, the image of ν_u is a one-dimensional set given by the parallel $\Gamma_m := \{z \in \mathbb{S}^2 \mid z_3 = (1+m^2)^{-1/2}\}$. Moreover, since as currents in $\mathscr{R}_2(B^2 \times \mathbb{S}^2)$

$$\partial G_{\nu_u} = -\delta_{0_{\mathbb{R}^2}} \times [\![\, \Gamma_m \,]\!],$$

with Σ_m the set defined by $\Sigma_m := \{z \in \mathbb{S}^2 \mid z_3 > (1+m^2)^{-1/2}\}$, using the identity $[\![\, \Gamma_m \,]\!] = \partial [\![\, \Sigma_m \,]\!]$ we get $T_u = G_{\nu_u} + S_u$, where

$$S_u = \delta_{0_{\mathbb{R}^2}} \times [\![\, \Sigma_m \,]\!].$$

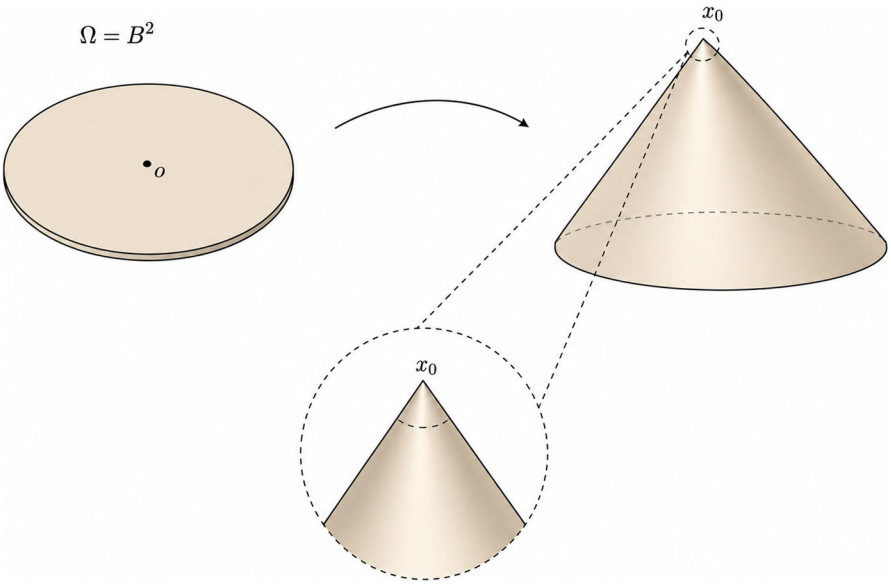

Fig. 2 Conical singularity and atomic Gauss curvature concentration

Hence, we have

$$\langle \boldsymbol{\mu}_u^s, \varphi \rangle = \varphi(0_{\mathbb{R}^2}) \cdot \alpha_m \,, \quad \alpha_m = 2\pi \left(1 - \frac{1}{\sqrt{1+m^2}}\right).$$

We thus have $\boldsymbol{\mu}_u^s = \boldsymbol{\mu}_u^{at} = \alpha_m \, \delta_{0_{\mathbb{R}^2}}$. The deformed shell $u(B^2)$ has no sharp folds, since ν_u is a Sobolev map in $W^{1,p}(B^2, \mathbb{S}^2)$ for every $p < 2$.

The second example describes a developable surface. We have $\boldsymbol{\mu}_u = 0$, even if the jump component of the distributional derivative of ν_u is nonzero. A sharp fold in the deformed shell occurs; it is a straight segment (see Fig. 3).

Example 6.5 Let $g(x) = -m|x_1|$, where $m > 0$. Then, we have $\nu_u \equiv a_m^+$ if $x_1 > 0$ and $\nu_u \equiv a_m^-$ if $x_1 < 0$, where $a_m^\pm := (\pm m, 0, 1)/\sqrt{1+m^2}$. The deformed surface $u(B^2)$ has a sharp fold given by the straight segment $I \times \{0\}$ in $\mathbb{R}^3$, where $I := \{x \in B^2 \mid x_1 = 0\}$. Moreover, since

$$\partial G_{\nu_u} = [\![\, I \,]\!] \times \left(\delta_{a_m^-} - \delta_{a_m^+}\right),$$

we can write $T_u = G_{\nu_u} + S_u$ with

$$S_u = -[\![\, I \,]\!] \times \Gamma,$$

where $\Gamma \in \mathscr{R}_1(\mathbb{S}^2)$ is associated with the oriented geodesic arc in $\mathbb{S}^2$ from a_m^- to a_m^+, whence $\boldsymbol{\mu}_u = 0$.

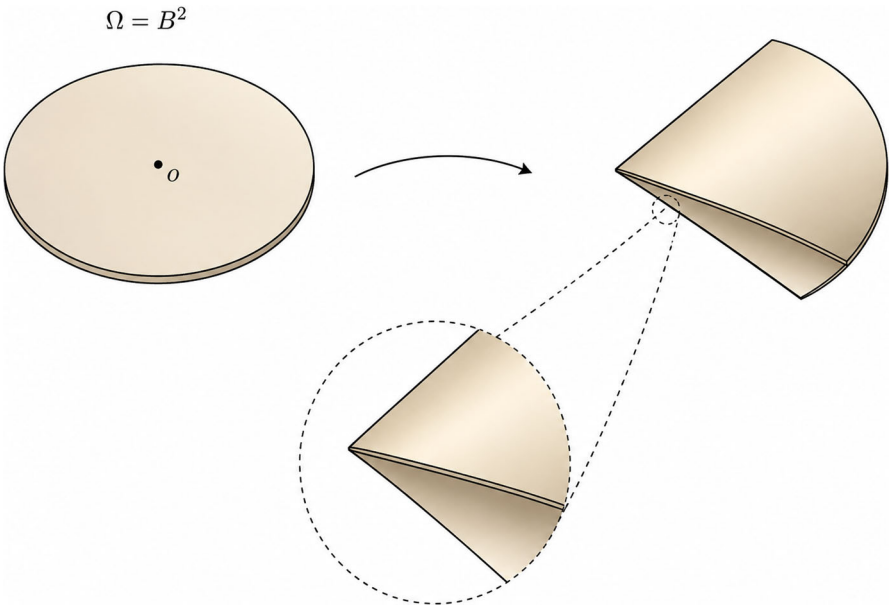

Fig. 3 Developable fold with vanishing Gauss curvature measure

The third example shows that when $D^J v_u \neq 0$, it may happen that the singular measure $\boldsymbol{\mu}_u^s$ is concentrated on a one-dimensional set $\mathscr{J}$. In this case, the sharp fold pertaining to the deformed shell is a line of $\mathbb{R}^3$ with nonzero curvature (see Fig. 4).

Example 6.6 Let g be of the form $g(x) = -|x_1| - x_2^2/2$, so that

$$v_u(x) = \frac{(\mathrm{sgn}(x_1), x_2, 1)}{\sqrt{2 + x_2^2}}, \quad x_1 \neq 0.$$

The deformed surface $u(B^2)$ has a sharp fold given by the support of the curve $\gamma(t) = (0, t, -t^2/2)$, where $t \in [-1, 1]$, whose curvature is nonzero at any point. We have:

$$\partial G_{v_u} = \Psi^-_{\#} [\![-1, 1]\!] - \Psi^+_{\#} [\![-1, 1]\!],$$

where $\Psi^{\pm} : (-1, 1) \to B^2 \times \mathbb{S}^2$ is given by

$$\Psi^{\pm}(t) := \big(0, t, \psi_{\pm}(t)\big), \quad \psi_{\pm}(t) := \frac{(\pm 1, t, 1)}{\sqrt{2 + t^2}}.$$

Therefore, we infer that $T_u = G_{v_u} + S_u$ where

$$S_u = F_{\#} [\![[-1, 1]^2]\!], \quad F(t, \lambda) := \Big(0, t, \frac{(\lambda, t, 1)}{\sqrt{\lambda^2 + t^2 + 1}}\Big).$$

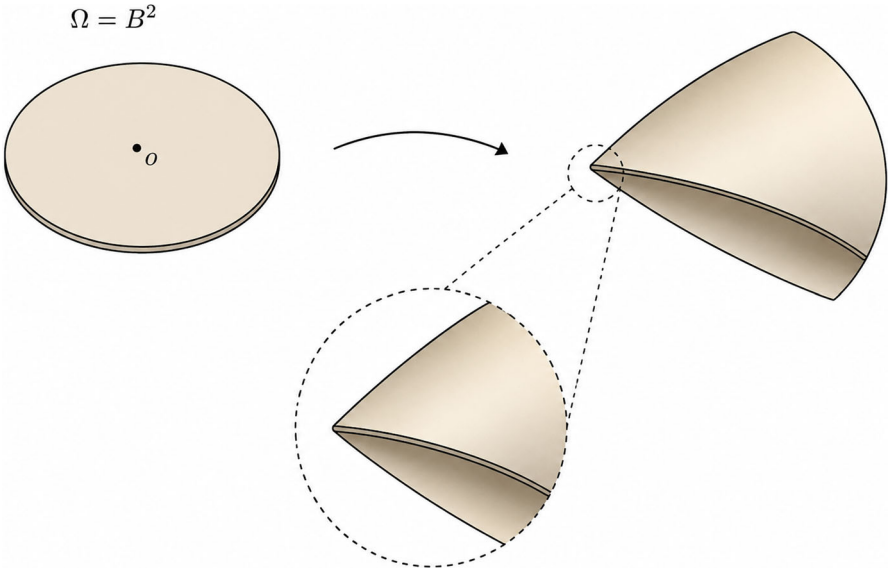

Fig. 4 Curved fold carrying a one-dimensional singular measure

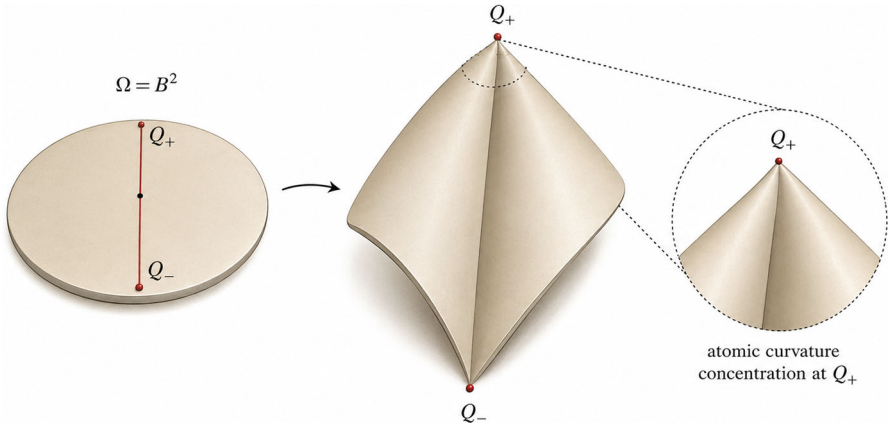

Fig. 5 Combined fold-cusp configuration with atomic concentration

As a consequence, the singular measure $\boldsymbol{\mu}_u^s$ is non-trivial; it is concentrated on the segment $I := \{x \in B^2 \mid x_1 = 0\}$, and we obtain $\boldsymbol{\mu}_u^{at} = 0$ whereas

$$\boldsymbol{\mu}_u^s(B^2) = \mathscr{H}^2(F([-1,1]^2))\,.$$

In the last example, both the measures $\boldsymbol{\mu}_u^{at}$ and $\boldsymbol{\mu}_u^s - \boldsymbol{\mu}_u^{at}$ are non-trivial (see Fig. 5 for a situation falling in the same class of the following example.).

Example 6.7 Let g be a function given by

$$g(x) = \begin{cases} -\sqrt{x_1^2 + (x_2 - 1/2)^2} & \text{if } \ x_2 \geq 1/2 \\ -|x_1| & \text{if } \ |x_2| \leq 1/2 \\ -\sqrt{x_1^2 + (x_2 + 1/2)^2} & \text{if } \ x_2 \leq -1/2 \end{cases} .$$

For $x_1 \neq 0$, we have

$$v_u(x) = \begin{cases} \dfrac{1}{\sqrt{2}} \Big(\dfrac{x_1}{\sqrt{x_1^2 + (x_2 - 1/2)^2}}, \dfrac{x_2 - 1/2}{\sqrt{x_1^2 + (x_2 - 1/2)^2}}, 1 \Big) & \text{if } \ x_2 \geq 1/2 \\ \dfrac{1}{\sqrt{2}} \big(\mathrm{sgn}(x_1), 0, 1 \big) & \text{if } \ |x_2| \leq 1/2 \\ \dfrac{1}{\sqrt{2}} \Big(\dfrac{x_1}{\sqrt{x_1^2 + (x_2 + 1/2)^2}}, \dfrac{x_2 + 1/2}{\sqrt{x_1^2 + (x_2 + 1/2)^2}}, 1 \Big) & \text{if } \ x_2 \leq -1/2 \end{cases} ,$$

and hence $v_u \in SBV(B^2, \mathbb{S}^2)$. Moreover, setting $a_\pm := (\pm 1, 0, 1)/\sqrt{2}$, $I := \{x \in B^2 \mid x_1 = 0, \ |x_2| \leq 1/2\}$, $Q_\pm := (0, \pm 1/2)$, $\gamma^+(\theta) := (\cos\theta, \sin\theta, 1)/\sqrt{2}$, and $\gamma^-(\theta) := (\cos(\theta + \pi), \sin(\theta + \pi), 1)/\sqrt{2}$, we get

$$\partial G_{v_u} = [\![I]\!] \times \big(\delta_{a_-} - \delta_{a_+} \big) - \delta_{Q_+} \times \gamma^+_{\#} [\![0, \pi]\!] - \delta_{Q_-} \times \gamma^-_{\#} [\![0, \pi]\!]$$

where $\partial [\![I]\!] = \delta_{Q_+} - \delta_{Q_-}$ and $\partial \gamma^+_{\#} [\![0, \pi]\!] = \delta_{a_-} - \delta_{a_+} = -\partial \gamma^-_{\#} [\![0, \pi]\!]$. Therefore, we have $T_u = G_{v_u} + S_u$ with

$$S_u = -[\![I]\!] \times [\![J]\!] + \delta_{Q_+} \times [\![D_+]\!] + \delta_{Q_-} \times [\![D_-]\!]$$

where J is the oriented geodesic arc in $\mathbb{S}^2$ from a_- to a_+, whereas $D_\pm$ is the geodesically convex subset of $\mathbb{S}^2$ enclosed by the curves J and $\gamma^\pm([0, \pi])$, that is, $D_\pm := \{z \in \mathbb{S}^2 \mid z_3 > 1/\sqrt{2}, \ \pm z_2 \geq 0\}$. As a consequence, the singular measure $\boldsymbol{\mu}_u^s$ has a 1-dimensional component $\boldsymbol{\mu}_u^s - \boldsymbol{\mu}_u^{at}$ concentrated on the segment I, and an atomic part given by

$$\boldsymbol{\mu}_u^{at} = \mathscr{H}^2(D_+)\, \delta_{Q_+} + \mathscr{H}^2(D_-) \times \delta_{Q_-} .$$

Further combined fold-cusp configurations can be considered, as those in Fig. 6.

7 An Existence Result

Here, we consider shells equilibrium configurations free of spread wrinkles but with possibly irrecoverable sharp folds and cusps. So, we discuss energy minimization. To this aim we require that the unit normal v_u field over the deformed shell is free of the Cantor component. Thus, on the basis of Theorem 6.2, along energy minimizing

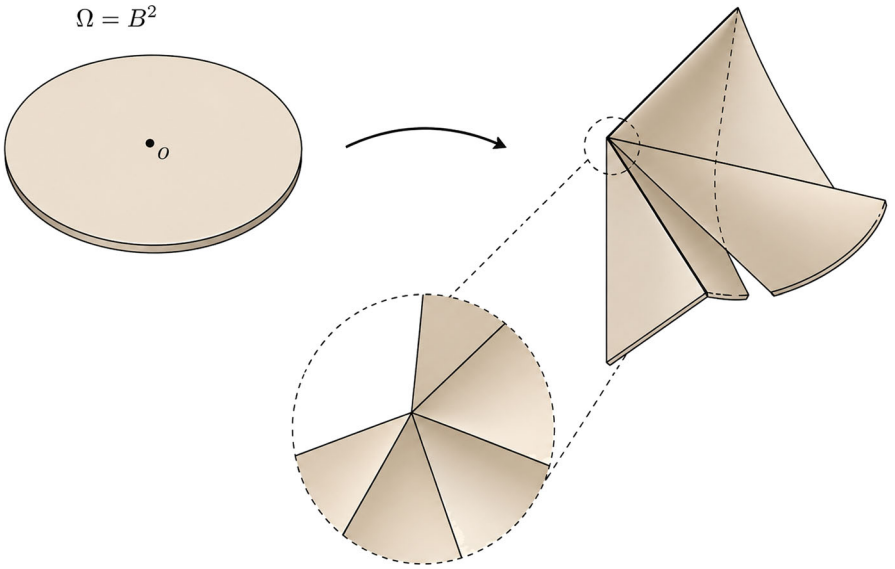

Fig. 6 Combined fold-cusp configuration with atomic concentration

sequences we need to have weak convergence in L^1 of all the minors of approximate gradient $\nabla\Phi_u$ of the Gauss graph map Φ_u in (3.12). Also, we need to apply the closure-compactness theorem in $SBV(\Omega, \mathbb{S}^2)$. We thus introduce a further subclass of competitors in $\mathscr{A}_p^\infty(\Omega)$.

Definition 7.1 For $p > 2$, we denote by $\widehat{\mathscr{A}_p^\infty}(\Omega)$ the subclass of maps $u \in \mathscr{A}_p^\infty(\Omega)$—with $\mathscr{A}_p^\infty(\Omega)$ introduced in Definition 6.2—such that

- the approximate gradient ∇v_u belongs to the Lebesgue space $L^p(\Omega, \mathbb{R}^{3\times 2})$;
- the boundary of the graph current G_{Φ_u} has finite mass, $\mathbf{M}(\partial G_{\Phi_u}) < \infty$;
- for $\mathscr{H}^1$-a.e. $x \in S(v_u)$, the approximate tangential gradient is non-trivial, that is, $\partial_{\mathbf{t}} u \neq 0_{\mathbb{R}^3}$.

By the parallelogram and by Hölder inequalities, a deformation map u in the class $\widehat{\mathscr{A}_p^\infty}(\Omega)$ satisfies $\mathscr{E}(u) < \infty$, where the energy $\mathscr{E}(u)$ is given by (3.19)–(3.20), and also the mass bound $\mathbf{M}(G_{\Phi_u}) < \infty$, so that Theorem 6.2 guarantees the existence of $\partial_{\mathbf{t}} u$ for $\mathscr{H}^1$-a.e. $x \in S(v_u)$. Finally, the bound $\mathbf{M}(\partial G_{\Phi_u}) < \infty$ implies that $\mathbf{M}(\partial G_{v_u}) < \infty$, whence $v_u \in SBV(\Omega, \mathbb{S}^2)$ and $\mathscr{H}^1(S(v_u)) < \infty$.

Four basic ingredients are chosen:

- a class $\widehat{\mathscr{A}_p^\infty}(\Omega)$ as in Definition 7.1, for some exponent $p > 2$;
- a real constant $K > 0$ yielding a uniform bound of the deformation maps;
- a triplet $(\mathscr{I}, \mathbf{n}, \Theta)$ in the class $\mathscr{PF}$, according to Definition 5.1: It accounts for the potential occurrence of possibly irrecoverable sharp folds;
- a nonnegative and finite regular measure $\mathbf{m}_p$ whose atomic component $\mathbf{m}_p^{at}$ is concentrated in a countable subset of Ω: it accounts for the potential occurrence of possibly irrecoverable localized strains in 'small' neighborhoods of isolated points.

Definition 7.2 We denote by $\widetilde{\mathscr{A}}_p(\Omega)$ the class of functions u in $\widehat{\mathscr{A}}_p^\infty(\Omega)$ satisfying the following properties:

- according to the notation in (5.3) and (5.6), the total variation bound (5.8) holds, so that inequality (5.9) is satisfied;
- the measure $\boldsymbol{\mu}_u$ satisfies the local bound (6.19) with respect to the measure $\mathbf{m}_p$, so that the relation (6.20) concerning the atomic measures $\boldsymbol{\mu}_u^{at}$ and $\mathbf{m}_p^{at}$ is satisfied;
- $\|u\|_\infty \le K < +\infty$.

A class of energies that agree with the peculiarities of deformations in $\widetilde{\mathscr{A}}_p(\Omega)$, to within dimensional constants, is given by

$$\mathscr{E}_p(u) := \mathscr{F}_p(u) + \int_\Omega |\nabla v_u|^p \,\mathrm{d}x + \mathbf{M}(\partial G_{\Phi_u}) + \int_{S(v_u)} \psi(|\partial_{\mathbf{t}} u|)\,\mathrm{d}\mathscr{H}^1 + \int_\Omega e(u(x))\,\mathrm{d}x \tag{7.1}$$

where, we recall, the function $\psi :]0, +\infty[\to \mathbb{R}^+$ is convex, nonnegative, with linear growth at $+\infty$, and such that $\psi(\rho) \to +\infty$ when $\rho \to 0^+$. The term $\mathscr{F}_p(u)$ is given by (6.1); it is an adaptation to shells of an energy proposed by Hadamard (1903). The fourth addendum guarantees that the non-degeneracy assumption $\partial_{\mathbf{t}} u \neq 0_{\mathbb{R}^3}$ is preserved in the minimization process. Finally, $e : \mathbb{R}^3 \to \mathbb{R}$ is assumed to be a continuous density function, so that the last addendum represents (in a sense not specified here) the action of external bulk forces. By construction, according to Definitions 7.1 and 7.2, the class $\widetilde{\mathscr{A}}_p(\Omega)$ always contains functions u with finite energy, namely those such that $\mathscr{E}_p(u) < \infty$.

Theorem 7.1 *With the previous notation, the infimum of the problem*

$$\inf\{\mathscr{E}_p(u) \mid u \in \widetilde{\mathscr{A}}_p(\Omega)\}$$

is attained.

Proof We repeatedly pass to not relabeled subsequences. In fact, we recall that the weak convergence of i.m. rectifiable currents with support contained in a compact set is metrizable.

Let $\{u_k\} \subset \widetilde{\mathscr{A}}_p(\Omega)$ be an energy minimizing sequence. By applying Proposition 6.1 to each u_k, and on account of the lower semicontinuity inequality (6.2), we find the existence of a function $u \in \mathscr{A}_p^\infty(\Omega)$ such that $\|u\|_\infty \le K$ and

$$\mathscr{F}_p(u) \le \liminf_{k\to\infty} \mathscr{F}_p(u_k) < \infty\,.$$

Moreover, since for every k

$$\mathscr{H}^1(S(v_{u_k})) \le 2\,\mathbf{M}(\partial G_{v_{u_k}}) \le 2\,\mathbf{M}(\partial G_{\Phi_{u_k}}) \le C < \infty,$$

we have

$$\sup_k \Big(\|v_{u_k}\|_\infty + \int_\Omega |\nabla v_{u_k}|^p \,\mathrm{d}x + \mathscr{H}^1(S(v_{u_k}))\Big) < \infty\,,$$

and the compactness theorem in SBV applies, see (Ambrosio et al. 2000). It implies that

$$\int_\Omega |\nabla v_u|^p \, \mathrm{d}x \le \liminf_{k\to\infty} \int_\Omega |\nabla v_{u_k}|^p \, \mathrm{d}x$$

and that

$$|\mu^J_{v_u}|(V) \le \liminf_{k\to\infty} |\mu^J_{v_{u_k}}|(V) \quad \forall\, V \subset \Omega\,, \quad V \text{ open}$$

so that property (5.8) is preserved in the limit process.

Furthermore, using that $p > 2$, by the parallelogram and Hölder inequalities we obtain the weak convergence in $L^1(\Omega)$ of all the minors of the gradient 6×2 matrix $\nabla \Phi_{u_k}$ to the corresponding minor of $\nabla \Phi_u$. Therefore, using that

$$\sup_k \Big(\mathbf{M}(G_{\phi_{u_k}}) + \mathbf{M}(\partial G_{\phi_{u_k}}) \Big) < \infty,$$

by Federer and Fleming's closure theorem we infer that $G_{\phi_{u_k}} \rightharpoonup G_{\phi_u}$ weakly as currents in $\mathscr{D}_2(U)$, where $U := \Omega \times \mathbb{R}^3 \times \mathbb{S}^2$, and by lower semicontinuity of the mass we have

$$\mathbf{M}(G_{\phi_u}) \le \liminf_{k\to\infty} \mathbf{M}(G_{\phi_{u_k}}) < \infty\,, \quad \mathbf{M}(\partial G_{\phi_u}) \le \liminf_{k\to\infty} \mathbf{M}(\partial G_{\phi_{u_k}}) < \infty\,.$$

As a consequence, by the first assertion in Theorem 6.2 we infer that u is approximately differentiable $\mathscr{H}^1$-a.e. on $S(v_u)$.

Furthermore, denoting by $\partial_{\mathbf{t}_k} u_k$ the approximate tangential gradient of u_k to $S(v_{u_k})$, the weak convergence $\partial G_{\phi_{u_k}} \rightharpoonup \partial G_{\phi_u}$ in $\mathscr{D}_1(U)$ implies the weak convergence in the sense of measures

$$|\partial_{\mathbf{t}_k} u_k| \, \mathscr{H}^1 \llcorner S(v_{u_k}) \rightharpoonup \mu$$

to some nonnegative Borel measure μ greater than $|\partial_{\mathbf{t}} u| \, \mathscr{H}^1 \llcorner S(v_k)$. Therefore, using a lower semicontinuity result due to Reshetnyak (1968), we have

$$\int_{S(v_u)} \psi(|\partial_{\mathbf{t}} u|) \, \mathrm{d}\mathscr{H}^1 \le \liminf_{k\to\infty} \int_{S(v_{u_k})} \psi(|\partial_{\mathbf{t}_k} u_k|) \, \mathrm{d}\mathscr{H}^1 < \infty.$$

Recalling that $\psi(\rho) \to +\infty$ when $\rho \to 0^+$, we thus obtain that $\partial_{\mathbf{t}} u \neq 0_{\mathbb{R}^3}$ for $\mathscr{H}^1$-a.e. $x \in S(v_u)$, and hence, $u \in \widehat{\mathscr{A}}^\infty_p(\Omega)$. In particular, the second part of Theorem 6.2 applies to u.

Let now T_k be a minimizing current in $\mathscr{T}_{v_{u_k}}$, so that $|\boldsymbol{\mu}_{u_k}| = |\boldsymbol{\mu}_{T_k}|$ for every k. Again by Federer and Fleming's closure theorem, a subsequence of $\{T_k\}$ weakly converges to

a Cartesian current $\bar{T}$ in the class $\mathscr{T}_u$. Therefore, denoting by T a minimizing current in $\mathscr{T}_u$, we obtain

$$|\boldsymbol{\mu}_u|(V) = |\boldsymbol{\mu}_T|(V) \leq |\boldsymbol{\mu}_{\bar{T}}|(V) \leq \liminf_{k\to\infty} |\boldsymbol{\mu}_{T_k}|(V) \quad \forall\, V \subset \Omega\,, \quad V \text{ open}.$$

As a consequence, property (6.19) is preserved and definitely $u \in \widetilde{\mathscr{A}_p}(\Omega)$. Finally, by putting the energy terms together, we obtain the lower semicontinuity inequality

$$\mathscr{E}_p(u) \leq \liminf_{k\to\infty} \mathscr{E}_p(u_k) < \infty\,,$$

the term $u \mapsto \int_\Omega e(u(x))\,\mathrm{d}x$ being continuous. □

8 Concluding Remarks

We have provided a functional characterization of shell deformations admitting possibly irrecoverable sharp folds and cusps while maintaining compactness. In this functional setting, we have considered a class of energies accounting for the peculiar characteristics of these singular behavior, proving for it the existence of minimizers. Our results can be generalized to shells with additional through-the-thickness microstructure described by phase fields taking values on a finite-dimensional geodetic complete differentiable manifold not embedded in any linear space. Besides this potential extension, the results collected here underline once again that the selection of a function space for an admissible class of deformations is per se a constitutive choice.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest regarding this manuscript.

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