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ERRATA TO: THE ALGEBRA OF SLICE FUNCTIONS

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ABSTRACT. We correct the statement and proof of [4, Proposition 4.10] and straighten out [4, Example 4.13] accordingly. We take this chance to correct a sentence within [4, Examples 1.13].

This note corrects a few errors in the article [4].

The first correction concerns a single sentence within [4, Examples 1.13, page 4736]: we must add to the sentence "The set $\mathscr{Q}^6$ is the intersection of $\mathscr{S}^7$ with the hyperplane $q_0 = 0$" the words "minus $\mathbb{R} + \epsilon\,\mathrm{Im}(\mathbb{H})^*$." For a proof of this fact, see [3, page 5513].

The second correction originates from [2, Remark 6.3]. We propose a new statement and proof of [4, Proposition 4.10]. In $\mathbb{R}_3$, let us adopt the notation $\omega_\pm := \frac{1}{2}(1 \pm e_{123})$. We recall that $\omega_+\omega_- = 0 = \omega_-\omega_+$. We notice that $\omega_\pm^2 = \omega_\pm$, that $\omega_+ + \omega_- = 1$, and that $\omega_+ - \omega_- = e_{123}$. As a consequence, $\mathbb{R}_3 = \omega_+\mathbb{R}_2 + \omega_-\mathbb{R}_2$.

Proposition 4.10. *If $A = \mathbb{R}_3$, if $f \in \mathcal{S}(\Omega)$ and if $x \in \Omega \setminus \mathbb{R}$, then one of the following happens:*

1. $V(f) \cap \mathbb{S}_x = \emptyset$;
2. $V(f) \cap \mathbb{S}_x = \{y\}$, $f'_s(x) \in C_A^*$ *and* $y = \mathrm{Re}(x) - f_s^\circ(x) f'_s(x)^{-1}$;
3. $V(f) \cap \mathbb{S}_x$ *is not empty and* $f'_s(x) \in \omega_\pm\mathbb{R}_2^*$; *for all* $y \in V(f) \cap \mathbb{S}_x$, *it holds* $V(f) \cap \mathbb{S}_x = \{\omega_\pm y + \omega_\mp z : z \in \mathbb{S}_x \cap \mathbb{R}_2\}$;
4. $V(f) \supseteq \mathbb{S}_x$ *and* $f'_s(x) = 0$.

In each of the aforementioned cases, respectively:

1. $\mathbb{S}_x$ *does not intersect* $V(f^c)$ *nor* $V(N(f))$;
2. $\mathbb{S}_x \subseteq V(N(f))$ *and* $V(f^c) \cap \mathbb{S}_x = \{f'_s(x)^{-1} y^c f'_s(x)\}$;
3. $\mathbb{S}_x \subseteq V(N(f))$ *and* $V(f^c) \cap \mathbb{S}_x = \{h^{-1} y^c h : y \in V(f) \cap \mathbb{S}_x\}$, *where* $h \in \mathbb{R}_2^*$ *is such that* $f'_s(x) = \omega_\pm h$;
4. $\mathbb{S}_x$ *is included both in* $V(f^c)$ *and in* $V(N(f))$.

Proof. Let us assume $\mathbb{S}_x = \alpha + \beta\mathbb{S}_{\mathbb{R}_3}$, whence $f(\alpha + \beta I) = a_1 + I a_2$ for each $I \in \mathbb{S}_{\mathbb{R}_3}$, where $a_1 = f_s^\circ(x), a_2 = \beta f'_s(x)$. We can apply [4, Theorem 4.1], taking into account the following facts: $\mathbb{R}_3$ is nonsingular; $\mathbb{R}_3$ is compatible; $C_{\mathbb{R}_3}$ is $\mathbb{R}_3$ minus the set $\omega_+\mathbb{R}_2^* \cup \omega_-\mathbb{R}_2^*$ of its zero divisors. We derive the following properties.

- If $f'_s(x) = 0$ then one of the following holds:

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   - $\mathbb{S}_x$ is included in $V(f), V(f^c), V(N(f))$ and $V(N(f^c))$.
   - $\mathbb{S}_x$ does not intersect $V(f), V(f^c)$ nor $V(N(f))$.
- If $f'_s(x)$ is a right zero divisor then one of the following holds:
   - $\mathbb{S}_x$ intersects $V(f)$ at at least one point $y = \alpha+\beta u$ and the intersection is the set of all $y' = \alpha+\beta v \in \mathbb{S}_x$ such that $(v-u)f'_s(x) = 0$. Moreover, $\mathbb{S}_x \subseteq V(N(f))$.
   - $\mathbb{S}_x$ does not intersect $V(f)$.
- If $f'_s(x)$ is neither 0 nor a right zero divisor then one of the following holds:
   - $\mathbb{S}_x$ intersects $V(f)$ exactly at $y = \mathrm{Re}(x) - f^\circ_s(x)f'_s(x)^{-1}$. It holds $\mathbb{S}_x \cap V(f^c) = \{f'_s(x)^{-1}y^c f'_s(x)\}$ and $\mathbb{S}_x \subseteq V(N(f))$.
   - $\mathbb{S}_x$ does not intersect $V(f)$ nor $V(f^c)$. $\mathbb{S}_x$ is not included in $V(N(f))$.

We now observe that, when $\mathbb{S}_x$ is not included in $V(N(f))$, then it does not intersect $V(N(f))$. Suppose, indeed, $N(f)(\alpha+\beta I) = n(a_1) - n(a_2) + It(a_1a_2^c)$ to vanish for some $I \in \mathbb{S}_{\mathbb{R}_3}$. Since the functions $n, t : \mathbb{R}_3 \to \mathbb{R}_3$ take values in the center $\mathbb{R}+e_{123}\mathbb{R}$ of the algebra, there exist $a, b, c, d \in \mathbb{R}$ such that $n(a_1) - n(a_2) = a + e_{123}b$ and $t(a_1a_2^c) = c + e_{123}d$, whence $a + e_{123}b = -I(c + e_{123}d)$. By squaring, we obtain $a^2 + b^2 + 2e_{123}ab = -c^2 - d^2 - 2e_{123}cd$, whence $a^2 + b^2 = -c^2 - d^2$. It follows that $a = b = c = d = 0$ and that $n(a_1) - n(a_2) = t(a_1a_2^c) = 0$. As a consequence, $\mathbb{S}_x$ is included in $V(N(f))$.

We are left with studying in detail the case when $a_2 = \beta f'_s(x)$ is a right zero divisor. We recall that, in $\mathbb{R}_3$, both the set of left zero divisors and the set of right zero divisors coincide with $\omega_+\mathbb{R}_2^* \cup \omega_-\mathbb{R}_2^*$. We suppose henceforth that $a_2 = \omega_\pm h$ with $h \in \mathbb{R}_2^*$.

We first assume $y = \alpha + \beta u \in V(f) \cap \mathbb{S}_x$. We will prove that $V(f) \cap \mathbb{S}_x = \{\omega_\pm y + \omega_\mp z : z \in \mathbb{S}_x \cap \mathbb{R}_2\}$ by solving the equation $(v-u)a_2 = 0$ or, equivalently, $(v-u)\omega_\pm = 0$ for $v \in \mathbb{S}_{\mathbb{R}_3}$. This equation is equivalent to $v - u = \omega_\mp k$ for some $k \in \mathbb{R}_2$. We remark that $\omega_\pm u = \omega_\pm u_\pm$ for appropriate $u_\pm \in \mathbb{R}_2$. Thus,

$$v = u + \omega_\mp k = \omega_\pm u + \omega_\mp u + \omega_\mp k = \omega_\pm u_\pm + \omega_\mp(u_\mp + k)$$

for some $k \in \mathbb{R}_2$. Equivalently, $v = \omega_\pm u_\pm + \omega_\mp u' = \omega_\pm u + \omega_\mp u'$ for some $u' \in \mathbb{R}_2$. Such a v belongs to $\mathbb{S}_{\mathbb{R}_3}$ if, and only if,

$$\begin{aligned} 0 &= t(v) = \omega_\pm t(u) + \omega_\mp t(u') = \omega_\mp t(u'), \\ 1 &= n(v) = \omega_\pm n(u) + \omega_\mp n(u') = \omega_\pm + \omega_\mp n(u'), \end{aligned}$$

where we took into account the fact that $t(u) = 0$ and $n(u) = 1$. Thus, v belongs to $\mathbb{S}_{\mathbb{R}_3}$ if, and only if, $t(u') = 0, n(u') = 1$, i.e., $u' \in \mathbb{S}_{\mathbb{R}_2}$. It follows that the solutions of $(v-u)\omega_\pm = 0$ in $\mathbb{S}_{\mathbb{R}_3}$ are exactly the Clifford numbers $v = \omega_\pm u + \omega_\mp u'$ with $u' \in \mathbb{S}_{\mathbb{R}_2}$. This proves that the elements of $V(f) \cap \mathbb{S}_x$ are the points $\alpha + \beta v = \omega_\pm(\alpha+\beta u) + \omega_\mp(\alpha+\beta u') = \omega_\pm y + \omega_\mp z$ with $z \in \mathbb{S}_x \cap \mathbb{R}_2$.

Under the same assumption $y = \alpha + \beta u \in V(f) \cap \mathbb{S}_x$, not only $\mathbb{S}_x \subseteq V(N(f))$ as we already stated; it also holds $y' := h^{-1}y^c h \in V(f^c) \cap \mathbb{S}_x$. To prove this fact, we first observe that y' belongs to $\mathbb{S}_x$ by [4, Remark 1.16]: indeed, $h \in C^*_{\mathbb{R}_3}$. We then observe that $f(y) = a_1 + ua_2 = 0$ implies $a_1 = -ua_2$, whence $f^c(y') = a_1^c - h^{-1}uha_2^c = a_2^c u - h^{-1}uhh^c\omega_\pm = a_2^c u - h^c\omega_\pm u = a_2^c u - a_2^c u = 0$. Taking into account the equalities $f = (f^c)^c$ and $\beta(f^c)'_s(x) = a_2^c$, we conclude that $V(f^c) \cap \mathbb{S}_x = \{h^{-1}y^c h : y \in V(f) \cap \mathbb{S}_x\}$.

Now let us assume, instead, $V(f) \cap \mathbb{S}_x = \emptyset$. We remark that $V(f^c) \cap \mathbb{S}_x = \emptyset$: if f^c had a zero in $\mathbb{S}_x$, then $(f^c)^c = f$ would have a zero in $\mathbb{S}_x$ by what we already proved. We conclude the proof by checking that $V(N(f)) \cap \mathbb{S}_x = \emptyset$. Suppose by contradiction $V(N(f)) \cap \mathbb{S}_x \neq \emptyset$, whence $\mathbb{S}_x \subseteq V(N(f))$. Then $n(a_1) = n(a_2)$ and $t(a_1 a_2^c) = 0$. The fact that $n(a_1) = n(a_2) = \omega_\pm n(h)$ implies that $a_1 = \omega_\pm k$, with $k \in \mathbb{R}_2^*$ having $n(k) = n(h)$. Since $1 = n(k)n(h)^{-1} = n(kh^{-1})$ and $0 = t(a_1 a_2^c) = \omega_\pm t(kh^c) = \omega_\pm t(kh^{-1})n(h)$, if we set $w := -kh^{-1}$, then $n(w) = 1$ and $t(w) = 0$. Thus, $w \in \mathbb{S}_{\mathbb{R}_3}$ and $f(\alpha+\beta w) = \omega_\pm k + w\omega_\pm h = \omega_\pm(k + wh) = 0$, which contradicts the hypothesis $V(f) \cap \mathbb{S}_x = \emptyset$. □

Consequently, we apply a third correction. Namely, we modify [4, Example 4.13] as follows.

Example 4.13. Let $A = \mathbb{R}_3$ and let $f(x) = \left(e_1 - \frac{\mathrm{Im}(x)}{|\mathrm{Im}(x)|}\right)\omega_-$. Then f is slice regular in $Q_A \setminus \mathbb{R}$ and f is constant in $\mathbb{C}_I^+$ for each $I \in \mathbb{S}_{\mathbb{R}_3}$. By direct computation, $f(e_1) = 0$ and $f_s'(e_1) = -\omega_-$. By Proposition 4.10, it holds

$$V(f) = \bigcup_{u \in \mathbb{S}_{\mathbb{R}_2}} \mathbb{C}^+_{\omega_- e_1 + \omega_+ u}.$$

For instance, $\mathbb{C}^+_{e_1}, \mathbb{C}^+_{e_{23}}$ are both included in $V(f)$ because $\omega_- e_1 + \omega_+ e_1 = e_1$ and

$$\omega_- e_1 + \omega_+(-e_1) = (\omega_- - \omega_+)e_1 = -e_{123}e_1 = -e_1^2 e_{23} = e_{23}.$$

An example of $g \in \mathcal{SR}(Q_A \setminus \mathbb{R})$ with the same zero set as f, but which is not constant along the half-slices $\mathbb{C}_I^+$, can be constructed following [1] and letting $g(x) = x \cdot f(x) = xf(x)$.

We take this chance to point out that [5, Example 9.6] must be corrected along the same lines. This is done in [2, Example 6.6], with an approach that is slightly different from ours.

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