Commun Nonlinear Sci Numer Simulat 154 (2026) 109588

Contents lists available at ScienceDirect

Communications in Nonlinear Science and Numerical Simulation

journal homepage: www.elsevier.com/locate/cnsns

Research paper

Stability of solutions in impulsive integro-differential equations with applications to fading memory systems

Tiziana Cardinali [a], Serena Matucci [b], Paola Rubbioni [a,*]

[a] Department of Mathematics and Computer Science, University of Perugia,
[b] Department of Mathematics and Computer Science U. Dini", University of Florence,

A R T I C L E I N F O

2020 MSC:
Primary: 34D20
34G20
34A37
Secondary: 35B35

Keywords:
Uniform asymptotic stability
Exponential stability
Integro-differential equations with impulses
Gronwall inequality
Distributed delay
Evolution systems

A B S T R A C T

In this paper, we analyze the stability of a class of nonlinear integro-differential equations in Banach spaces, incorporating both integral terms with fading memory effects and impulsive actions occurring at fixed moments in time. These impulses lead to piecewise continuous solutions, thus broadening the range of phenomena that can be modeled. Here we present results on uniform stability on the whole space and on the uniform either asymptotic or exponential stability of solutions over bounded sets of initial data. Further, also the global exponential stability of the null solution is analyzed. The presence of a Volterra integral term allows to consider physical systems which are influenced by their past states. To illustrate the applicability of our theoretical findings, we discuss models arising in population dynamics and robotics, where the memory effect plays some role in shaping the system's evolution.

1. Introduction

The problem of asymptotic stability of solutions to differential equations is a topic of enduring relevance, as in a vast range of situations studied in applied sciences, it is essential to ensure that, at least in the long run, the studied phenomenon does not lead to multiple steady states, let alone chaos. It is well known that the literature offers various definitions of stability, such as exponential stability, or Hyers-Ulam stability, or Lyapunov stability, or uniform asymptotic stability. For a - not exhaustive - overview on recent results on this matter we refer, e.g., to [2,5,11,15,18,20]. Here, we follow the approach of [3] and [9] respectively, and provide results concerning the uniform stability over a set of initial data not necessarily bounded, and the uniform asymptotic or exponential stability of solutions over a bounded set of initial data, together with a result on global exponential stability of the null solution, if it exists.

The system considered in this paper is described by the following integro-differential equation in a Banach space E, subject to impulses distributed over time at fixed instants

$$\begin{aligned} &y'(t) = A(t)y(t) + f\left(t, y(t), \int_{t_0}^{t} k(t,s)y(s)ds\right),\ t \in [t_0, +\infty[\,,\ t \neq t_m,\ m \in \mathbb{N}^+, \\ &y(t_m^+) = y(t_m) + I_m(y(t_m)),\ m \in \mathbb{N}^+, \end{aligned} \tag{IE}$$

* Corresponding author.
E-mail addresses: tiziana.cardinali@unipg.it (T. Cardinali), serena.matucci@unifi.it (S. Matucci), paola.rubbioni@unipg.it (P. Rubbioni).

https://doi.org/10.1016/j.cnsns.2025.109588
Received 15 September 2025; Received in revised form 3 December 2025; Accepted 7 December 2025
Available online 12 December 2025
1007-5704/

where $\{A(t)\}_{t\geq 0}$ is a family of linear operators on E, f is a nonlinear function, $t_0 \geq 0$, k is a continuous kernel, and I_m, $m \in \mathbb{N}^+$, are the impulse functions. The presence of impulses leads to piecewise continuous solutions, which evidently broadens the class of phenomena that can be modeled by the given equation compared to the non-impulsive case. Within the framework of differential equations, the existence problems for solutions of impulsive differential equations have been studied using various methods. For example, the approach based on upper and lower solutions (see, e.g., [4,21,23]), or the "extension with memory" process (see, e.g., [8,19]), the latter being applicable without imposing any conditions on the impulse functions and underlying our existence, uniqueness, and continuous dependence result.

The differential equation under consideration includes a Volterra integral term, whose kernel weights the influence of past states on the present and can have various interpretations in applications describing systems with memory, delay, or nonlocal dependence. Commonly used kernels include exponential kernels, which appear frequently in viscoelasticity, population dynamics, and neural modeling; power-law kernels, often employed in epidemiology to model long-tailed infectious periods, fluid flow in porous media, or viscoelastic materials; Gamma kernels, used in biology, neural dynamics, and chemical kinetics; and distributed-delay kernels, which are widely applied in population dynamics and cell-cycle modeling. In this article, we examine in particular two models – one in population dynamics and another in robotics – where, due to an exponential probability kernel, the Volterra integral represents a form of distributed delay. The basic idea is that the past evolution of the solution trajectory influences the present and future states of the system. The exponential kernel represents short-term memory with exponential decay, ensuring that events further in the past have a lesser impact on the system's evolution compared to the more recent ones, leading to the so-called "fading memory" phenomenon, see, for instance, [10,17].

As far as we know, the stability analysis of a nonlinear model that simultaneously includes a Volterra-type integral term and impulsive effects has not yet been addressed. Asymptotic stability in the presence of impulses has been studied, for instance, in [14,18] (see also the references therein). However, in both works the delay is of functional type, and moreover [14] focuses on periodic solutions. Regarding exponential stability in the presence of a Volterra integral term, this issue has been examined in the linear case without impulses, e.g., in [10,17]. In the nonlinear case without impulses, asymptotic and exponential stability have been investigated only for the special case of the model described in Section 5.2, as discussed in [6].

The article is structured as follows. In Section 2, we introduce the problem framework, providing the necessary notation and definitions for reading the article, along with the problem formulation. Next, in Section 3, we present a preliminary result concerning the existence, uniqueness, and continuous dependence on initial data of mild solutions to the problem. This theorem relies on a Gronwall-type inequality tailored to the impulsive integro-differential case. To the best of our knowledge, this inequality was not previously known in the literature in our setting, and therefore we provide its proof. Section 4 is entirely devoted to the stability analysis - first addressing different types of uniform stability of solutions over given sets of initial data, and subsequently global exponential stability of the null solution, provided it exists. We also state sufficient conditions for the Volterra integral kernel to satisfy the properties required to achieve the stability and demonstrate that some of the most commonly used kernels in the literature fulfill these conditions. Finally, in Section 5 we show how some physical problems can be reformulated in abstract form leading to an impulsive integro-differential equation of type (IE) and we apply our results to achieve the stability of solutions. More in details, we analyze two distinct models: one from population dynamics and another involving flexible robotics, both subject to impulses and exhibiting fading memory effects.

2. Position of the problem

Let E be a real Banach space endowed with the norm $\|\cdot\|$ and J a compact interval in $\mathbb{R}$.
We denote by:

$C(J,E)$ the space of E-valued continuous functions defined on J endowed with the usual sup-norm;

$L^p(J,E)$ the space of all measurable functions $v : J \to E$ such that $\|v\|^p$ is summable, endowed with the norm $\|v\|_{L^p(J,E)} = \left(\int_J \|v(z)\|^p\,dz\right)^{\frac{1}{p}}$ (if $E=\mathbb{R}$, we denote $L^p(J)$ and $\|v\|_{L^p}$ respectively), $p \geq 1$;

$L^1_{loc}([a,+\infty[,E)$ ($a \in \mathbb{R}$) the space of all functions $v : [a,+\infty[\to E$ such that $v \in L^1(I,E)$ for every compact interval $I \subset [a,+\infty[$ (shortly, $L^1_{loc}([a,+\infty[)$ if $E=\mathbb{R}$).

Also, as usual we put

$$H^k(0,1) := \{y : [0,1] \to \mathbb{R} : y, y^{(1)}, \dots, y^{(k)} \in L^2([0,1])\}.$$

Further, for any function $y : [a,+\infty[\to E$ and value $t \in [a,+\infty[$, we will use the symbol

$$y(t^+) := \lim_{h\to 0^+} y(t+h),$$

whenever the limit exists in E.

We recall that a family $\{U(t,s)\}_{t\geq s\geq 0}$ of bounded linear operators on E is said to be a *(strongly continuous) evolution system* on the half-line (see, e.g. [16]) if $U(s,s)=I$ and $U(t,r)U(r,s)=U(t,s)$, $t \geq r \geq s \geq 0$, and the map $\xi_x : (t,s) \mapsto U(t,s)x$ is continuous for every $x \in E$.

Furthermore, a family of linear operators $\{A(t)\}_{t\geq 0}$, $A(t) : D(A) \subset E \to E$, $D(A)$ dense subset of E not depending on t, *generates* an evolution system on the half-line $\{U(t,s)\}_{t\geq s\geq 0}$ if (see, e.g. [12])

$$\frac{\partial U(t,s)}{\partial t} = A(t)U(t,s) \quad \text{and} \quad \frac{\partial U(t,s)}{\partial s} = -U(t,s)A(s), \quad t \geq s \geq 0.$$

Moreover, put $\mathcal{L}(E)$ the Banach space of all bounded linear operators from E to E furnished with the usual operator norm, say $\|\cdot\|_{\mathcal{L}(E)}$, an evolution system $\{U(t,s)\}_{t\geq s\geq 0}$ is said to be *exponentially stable* if (cf. [22, Definition 2.1])

$$\exists D \geq 1,\ \omega > 0\ :\ \|U(t,s)\|_{\mathcal{L}(E)} \leq De^{-\omega(t-s)},\ t \geq s \geq 0. \tag{1}$$

It is well known (cf., e.g. [7] and [16]) that, in the case when A does not depend on t and generates a C_0-semigroup $\{S(t)\}_{t\geq 0}$, then the constant family $\{A(t)\}_{t\geq 0} = \{A\}$, generates the evolution system $\{U(t,s)\}_{t\geq s\geq 0}$, where $U(t,s) := S(t-s)$.

Let $\mathcal{T} := \{t_m\}_{m\in\mathbb{N}}$ be an increasing sequence of real nonnegative numbers diverging to $+\infty$. By the symbol $\mathcal{PCT}([t_0,+\infty[,E)$ we denote the set of functions

$$\mathcal{PCT}([t_0,+\infty[,E) := \left\{ \begin{array}{lll} y : [t_0,+\infty[\to E & : & y_{|\,[t_0,t_1]} \text{ is continuous,} \\ & & y_{|\,]t_m,t_{m+1}]} \text{ is continuous and} \\ & & \exists\, y\left(t_m^+\right) \in E, \text{ for all} m \in \mathbb{N}^+ \end{array} \right\}. \tag{2}$$

Let $v_0 \in E$ and consider the corresponding initial value problem driven by a semilinear integro-differential equation subject to impulses

$$(P)_{v_0} \begin{cases} y'(t) = A(t)y(t) + f\left(t, y(t), \int_{t_0}^t k(t,s)y(s)ds\right),\ t \geq t_0\,,\ t \neq t_m\,,\ m \in \mathbb{N}^+, \\ y(t_0) = v_0, \\ y(t_m^+) = y(t_m) + I_m(y(t_m)),\ m \in \mathbb{N}^+. \end{cases}$$

Here $\{A(t)\}_{t\geq 0}$ is a family of linear operators in E defined on a dense subset of E, not depending on t, generating an evolution system on the half-line $\{U(t,s)\}_{t\geq s\geq 0}$; $f : [t_0,+\infty[\times E \times E \to E$ and $k : \Delta_\infty := \{(t,s) \in \mathbb{R}^2 : t \geq s \geq t_0\} \to \mathbb{R}^+$ are given functions; $I_m : E \to E$, $m \in \mathbb{N}^+$, are impulse functions.

A function $y \in \mathcal{PCT}([t_0,+\infty[,E)$ is an *impulsive mild solution* to $(P)_{v_0}$ if

$$y(t) = U(t,t_0)v_0 + \sum_{t_0<t_m<t} U(t,t_m)I_m(y(t_m)) + \int_{t_0}^t U(t,s)f\left(s, y(s), \int_{t_0}^s k(s,r)y(r)dr\right)ds,\ t \geq t_0,$$

where it is agreed that $\sum_{t_0<t_m<t} U(t,t_m)I_m(y(t_m)) = 0$ if $t \in [t_0,t_1]$.

We assume the next hypotheses:

(H0) $\{U(t,s)\}_{t\geq s\geq 0}$ is an exponentially stable evolution system (cf. (1));

(H1) for every $v,w \in E$ the map $f(\cdot,v,w)$ is strongly measurable;

(H2) there exists a continuous function $C : [t_0,+\infty[\to \mathbb{R}^+$ such that, for a.e. $t \geq t_0$ the map $f(t,\cdot,\cdot)$ is $C(t)$-lipschitzian, i.e.,

$$\|f(t,v_1,w_1) - f(t,v_2,w_2)\| \leq C(t)(\|v_1 - v_2\| + \|w_1 - w_2\|),$$

for all $v_1, v_2, w_1, w_2 \in E$;

(H3) $\|f(\cdot,0,0)\| \in L^1_{loc}([t_0,+\infty[)$;

(H4) k is continuous;

(H5) there exists a sequence $(a_m)_{m\in\mathbb{N}^+}$ of positive numbers such that

(i) $\|I_m(v) - I_m(w)\| \leq a_m\|v - w\|$, for all $v,w \in E$, and all $m \in \mathbb{N}^+$,

(ii) $\sum_{m=1}^\infty a_m$ converges.

3. Gronwall-type inequality and uniqueness of solutions

In this section we present a Gronwall type result in the impulsive setting, which will be used both in the existence and in the stability theorems. This result is a modification of [13, Theorem 1.5.4], and extends [13, Theorem 1.5.1]. In particular, no piecewise differentiability is required to the function to be estimated and T can be finite.

Theorem 3.1. *Let $(t_m)_{m\in\mathbb{N}}$ be a nonnegative increasing sequence converging to T, $t_0 < T \leq +\infty$, and $\gamma : [t_0, T) \to \mathbb{R}_0^+ := [0, +\infty)$ is a piecewise continuous function with jump discontinuities at $t = t_m$, $m \in \mathbb{N}^+$, γ continuous from the left.*
Suppose that $H : \Delta_T := \{(t, s) : t_0 \leq s \leq t < T\} \to \mathbb{R}_0^+$ is such that $t \mapsto H(t, s)$ is continuously differentiable, $s \mapsto H(t, s)$ is continuous, and $\frac{\partial H}{\partial t}(t, s) \geq 0$ in Δ_T.
Further, assume that for every $t \in [t_0, T)$ the following inequality holds

$$\gamma(t) \leq c_0 + \sum_{t_0<t_m<t} \beta_m \gamma(t_m) + \int_{t_0}^{t} H(t, s)\gamma(s)\, ds,$$

where $\beta_m \geq 0$ for every $m \in \mathbb{N}^+$ and $c_0 \geq 0$.
Then,

$$\gamma(t) \leq c_0 \prod_{t_0<t_m<t} (1 + \beta_m) \exp\left(\int_{t_0}^{t} \left(H(s, s) + \int_{t_0}^{s} \frac{\partial H}{\partial s}(s, r)\, dr \right) ds \right),\ t \in [t_0, T). \tag{3}$$

Proof. Let $v : [t_0, T) \to \mathbb{R}^+$ be the auxiliary function

$$v(t) := c_0 + \sum_{t_0<t_m<t} \beta_m \gamma(t_m) + \int_{t_0}^{t} H(t, s)\gamma(s)\, ds,\ t \in [t_0, T).$$

Then v is differentiable for every $t \neq t_m$, $m \in \mathbb{N}^+$, and

$$\begin{cases} v'(t) = H(t,t)\gamma(t) + \int_{t_0}^{t} \frac{\partial H}{\partial t}(t, s)\gamma(s)\, ds, & t \neq t_m, \\ v(t_0) = c_0, \\ v(t_m^+) = v(t_m) + \beta_m \gamma(t_m). \end{cases}$$

The sign assumptions on γ, H and $\partial H/\partial t$ imply that v is nondecreasing. Since $\gamma(t) \leq v(t)$, we obtain for $t \neq t_m$

$$v'(t) \leq H(t,t)v(t) + \int_{t_0}^{t} \frac{\partial H}{\partial t}(t, s)v(s)\, ds \leq \left(H(t,t) + \int_{t_0}^{t} \frac{\partial H}{\partial t}(t, s)\, ds \right) v(t),$$

and

$$v(t_m^+) \leq (1 + \beta_m)v(t_m).$$

By applying [13, Theorem 1.4.1], we have that v satisfies

$$v(t) \leq c_0 \prod_{t_0<t_m<t} (1 + \beta_m) \exp\left(\int_{t_0}^{t} \left(H(s, s) + \int_{t_0}^{s} \frac{\partial H}{\partial s}(s, r)\, dr \right) ds \right),\ t \in [t_0, T).$$

The statement (3) now follows recalling that $\gamma(t) \leq v(t)$. □

Remark 3.1. In the case when the function H depends only on the second variable, i.e., $H(t, s) = p(s)$, then Theorem 3.1 reduces to [1, Lemma 1], and (3) reads as

$$\gamma(t) \leq c_0 \prod_{t_0<t_m<t} (1 + \beta_m) e^{\int_{t_0}^{t} p(s)\, ds},\ t \in [t_0, T).$$

As a first consequence of the above Gronwall-type inequality, we provide the following result on the existence, uniqueness, and continuous dependence of the impulsive mild solutions of $(P)_{v_0}$.

Theorem 3.2. *Assume that properties (H0)-(H5) are satisfied.*
Then, for every $v_0 \in E$ the problem $(P)_{v_0}$ has a unique impulsive mild solution, which continuously depend on the initial data.

Proof. First of all, by arguing similarly as in the proof of Theorem 3.1 in [6], we can say that all the assumptions of the single-valued version of Theorem 3.1 in [19] are satisfied. We leave the simple details to the reader. Thus, for every $w_0, v_0 \in E$ arbitrarily fixed, there exist z, y impulsive mild solutions of $(P)_{w_0}$, $(P)_{v_0}$ respectively.
For every $t \geq t_0$, by (1), (H5-i), (H2), we obtain

$$\begin{aligned} \|z(t) - y(t)\| &\leq \left\|U(t, t_0)(w_0 - v_0)\right\| + \sum_{t_0<t_m<t} \left\|U(t, t_m)[I_m(z(t_m)) - I_m(y(t_m))]\right\| \\ &+ \int_{t_0}^{t} \left\| U(t, s) \left[f\left(s, z(s), \int_{t_0}^{s} k(s, r)z(r)dr \right) - f\left(s, y(s), \int_{t_0}^{s} k(s, r)y(r)dr \right) \right] \right\| ds \\ &\leq De^{-\omega(t-t_0)}\|w_0 - v_0\| + \sum_{t_0<t_m<t} De^{-\omega(t-t_m)}\|I_m(z(t_m)) - I_m(y(t_m))\| \\ &+ \int_{t_0}^{t} De^{-\omega(t-s)} \left\| f\left(s, z(s), \int_{t_0}^{s} k(s, r)z(r)dr \right) - f\left(s, y(s), \int_{t_0}^{s} k(s, r)y(r)dr \right) \right\| ds \end{aligned}$$

$$\leq De^{-\omega(t-t_0)}\|w_0 - v_0\| + \sum_{t_0<t_m<t} De^{-\omega(t-t_m)}a_m\|z(t_m) - y(t_m)\|$$
$$+\int_{t_0}^{t} De^{-\omega(t-s)}C(s)\left[\|z(s)-y(s)\| + \left\|\int_{t_0}^{s} k(s,r)z(r)dr - \int_{t_0}^{s} k(s,r)y(r)dr\right\|\right] ds$$
$$\leq De^{-\omega(t-t_0)}\|w_0 - v_0\| + \sum_{t_0<t_m<t} De^{-\omega(t-t_m)}a_m\|z(t_m) - y(t_m)\|$$
$$+\int_{t_0}^{t} De^{-\omega(t-s)}C(s)\left[\|z(s)-y(s)\| + \int_{t_0}^{s} k(s,r)\|z(r)-y(r)\|dr\right] ds.$$

Therefore,

$$\|z(t)-y(t)\|e^{\omega(t-t_0)} \leq D\|w_0 - v_0\| + \sum_{t_0<t_m<t} Da_m\|z(t_m)-y(t_m)\|e^{\omega(t_m-t_0)}$$
$$+D\int_{t_0}^{t} C(s)\|z(s)-y(s)\|e^{\omega(s-t_0)}\,ds$$
$$+D\int_{t_0}^{t} C(s)\left(\int_{t_0}^{s} k(s,r)\|z(r)-y(r)\|e^{\omega(s-t_0)}dr\right)ds.$$

Put

$$\gamma(t) := \|z(t)-y(t)\|e^{\omega(t-t_0)} \text{ and } c_0 := D\|w_0 - v_0\|, \tag{4}$$

the previous inequality reads as

$$\gamma(t) \leq c_0 + \sum_{t_0<t_m<t} Da_m\gamma(t_m)$$
$$+D\int_{t_0}^{t} C(s)\gamma(s)\,ds + D\int_{t_0}^{t} C(s)\left(\int_{t_0}^{s} k(s,r)\gamma(r)e^{\omega(s-r)}dr\right)ds. \tag{5}$$

Since

$$\int_{t_0}^{t} C(s)\left(\int_{t_0}^{s} k(s,r)\gamma(r)e^{\omega(s-r)}dr\right)ds = \int_{t_0}^{t}\left(\int_{s}^{t} C(\sigma)k(\sigma,s)e^{\omega(\sigma-s)}d\sigma\right)\gamma(s)\,ds,$$

we get

$$\gamma(t) \leq c_0 + \sum_{t_0<t_m<t} Da_m\gamma(t_m)$$
$$+\int_{t_0}^{t} D\left(C(s) + \int_{s}^{t} C(\sigma)k(\sigma,s)e^{\omega(\sigma-s)}d\sigma\right)\gamma(s)\,ds. \tag{6}$$

Now, put

$$H(t,s) := D\left(C(s) + \int_{s}^{t} C(\sigma)k(\sigma,s)e^{\omega(\sigma-s)}d\sigma\right),\ (t,s)\in\Delta_\infty.$$

Then H is a positive function, and from (H2), (H4) it is continuously differentiable with respect to t and continuous in s, for all $(t,s)\in\Delta_\infty$. Further,

$$H(t,t) = DC(t), \qquad \frac{\partial H(t,s)}{\partial t} = DC(t)k(t,s)e^{\omega(t-s)} \geq 0,\ (t,s)\in\Delta_\infty.$$

Then we can apply our Gronwall type result, with $T = +\infty$ to the function γ in (6), leading to

$$\gamma(t) \leq c_0 \prod_{t_0<t_m<t}(1+Da_m)\exp\left(D\int_{t_0}^{t} C(s)\left(1+\int_{t_0}^{s} k(s,r)e^{\omega(s-r)}dr\right)ds\right),\ t\geq t_0. \tag{7}$$

Let us note that, for fixed $t > t_0$

$$\prod_{t_0<t_m<t}(1+Da_m) = \exp\left(\sum_{t_0<t_m<t}\ln(1+Da_m)\right) \leq \exp\left(\sum_{k=1}^{\infty}\ln(1+Da_k)\right) =: M. \tag{8}$$

Indeed, the series in (8) converges, having the same behavior of the converging series $\sum_{k=1}^{\infty} a_k$ (see hypothesis (H5-ii)). Then, from (7) and (8) we get

$$\gamma(t) \leq c_0 M\exp\left\{D\int_{t_0}^{t} C(s)\left(1+\int_{t_0}^{s} k(s,r)e^{\omega(s-r)}dr\right)ds\right\}, \tag{9}$$

and from (4)

$$\|z(t)-y(t)\| \le DM\|w_0-v_0\| \exp\left\{\int_{t_0}^{t}\left[DC(s)\left(1+\int_{t_0}^{s}k(s,r)e^{\omega(s-r)}dr\right)-\omega\right]ds\right\}, \tag{10}$$

for every $t > t_0$.
Now, we can say that, for every fixed $t_1 > t_0$, there exists a constant $M_1 > 0$ such that

$$\|z(t)-y(t)\| \le M_1\|w_0-v_0\|, \quad t_0 \le t \le t_1.$$

This proves the continuous dependence on the initial data and, obviously, the uniqueness of the impulsive mild solutions of $(P)_{v_0}$ when $w_0 = v_0$. □

4. Stability results

In this section we first provide the definitions of uniform stability, uniform asymptotic stability, and uniform exponential stability that we here adopt. These are generalizations to the present setting of classical ones, see for instance [3,6,9,18].

Definition 4.1. Let Ω be a nonempty subset of E. The impulsive mild solutions of the integro-differential equation

$$y'(t) = A(t)y(t) + f\left(t, y(t), \int_{t_0}^{t}k(t,s)y(s)ds\right),\ t \ge t_0,\ t \ne t_m,\ m \in \mathbb{N}^+, \tag{11}$$

subject to the impulses I_m at the corresponding times t_m, $m \in \mathbb{N}^+$, are said to be:

- *uniformly stable on* Ω if for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that for every $w_0, v_0 \in \Omega$ with $\|w_0 - v_0\| < \delta(\varepsilon)$ then $\|z(t) - y(t)\| \le \varepsilon,$ for every$t \ge t_0$ and $z, y \in \mathcal{PCT}([t_0, +\infty[, E)$ impulsive mild solutions of $(P)_{w_0}, (P)_{v_0}$ respectively;

- *uniformly asymptotically stable on* Ω if

  for every $\varepsilon > 0$ there exists $t(\varepsilon) > t_0$ such that $\|z(t) - y(t)\| \le \varepsilon,$ for every$t \ge t(\varepsilon)$

  and $z, y \in \mathcal{PCT}([t_0, +\infty[, E)$ impulsive mild solutions of $(P)_{w_0}, (P)_{v_0}$ respectively, for every $w_0, v_0 \in \Omega$;

- *uniformly exponentially stable on* Ω if there exist $\Lambda, \Gamma > 0$ such that $\|z(t) - y(t)\| \le \Lambda e^{-\Gamma(t-t_0)}$, for every$t \ge t_0$ and $z, y \in \mathcal{PCT}([t_0, +\infty[, E)$ impulsive mild solutions of $(P)_{w_0}, (P)_{v_0}$ respectively, for every $w_0, v_0 \in \Omega$.

From now on, we denote by $\mathcal{S}(\Omega)$ the set of all impulsive mild solutions of problems $(P)_{v_0}$ with v_0 which varies in Ω.

We can now prove a sufficient condition for the various types of uniform stability introduced above. To this aim, assumption (H2) needs to be strengthened by assuming in addition that the function $C(\cdot)$ in upper bounded, i.e.,

$$C := \sup_{t \ge t_0} C(t) < +\infty. \tag{12}$$

Theorem 4.1. *Assume that properties (H0)-(H5) and (12) are satisfied.*
Let Ω be a bounded subset of E and $K : [t_0, +\infty[\to \mathbb{R}_0^+$ be the function defined by

$$K(t) := \int_{t_0}^{t}k(t,s)e^{\omega(t-s)}\,ds,\ t \ge t_0. \tag{13}$$

Put

$$F(t) := \int_{t_0}^{t}[CD(1+K(s))-\omega]\,ds,\ t \ge t_0, \tag{14}$$

the following holds.

1. *If F is upper bounded, then the impulsive mild solutions of Eq. (11) are uniformly stable on E.*
2. *If*

$$\lim_{t\to+\infty} F(t) = -\infty, \tag{15}$$

then the impulsive mild solutions of Eq. (11) are uniformly asymptotically stable on Ω.

3. *If*

$$\text{there exists}\,\Gamma > 0 : \quad \sup_{t \ge t_0}\left[\Gamma(t-t_0)+F(t)\right] < +\infty, \tag{16}$$

then the impulsive mild solutions of Eq. (11) are uniformly exponentially stable on Ω.

Proof. As in the previous Theorem 3.2, for every z, y impulsive mild solutions of $(P)_{w_0}$ and $(P)_{v_0}$ respectively, $w_0, v_0 \in \Omega$, we achieve the estimate (10). By using (12) and (14), we get

$$\|z(t) - y(t)\| \le DM\,\|w_0 - v_0\|\, e^{F(t)}, \text{ for every} t \ge t_0. \tag{17}$$

Case 1. From (17), the property of uniform stability on Ω is immediately satisfied if, for every $\varepsilon > 0$ we take $\delta(\varepsilon) = \frac{\varepsilon}{M_2}$, where $M_2 := DMe^{\sup_{t\ge t_0} F(t)}$.

Case 2. Taking into account the boundedness of Ω and the inequality (17), the next estimate holds

$$\|z(t) - y(t)\| \le DMQ\, e^{F(t)}, \text{ for every} t \ge t_0, \tag{18}$$

where $Q := \sup_{w_0, v_0 \in \Omega} \|w_0 - v_0\|$. So, by (15), we get

$$\lim_{t\to+\infty} \|z(t) - y(t)\| = 0 \text{ uniformly in} \mathcal{S}(\Omega),$$

i.e., the uniform asymptotic stability on Ω.

Case 3. Put $\Gamma_1 := \sup_{t\ge t_0} \left[\Gamma(t - t_0) + F(t)\right]$, we have

$$F(t) \le \Gamma_1 - \Gamma(t - t_0), \text{ for every} t \ge t_0.$$

Moreover, as in the previous case, we achieve the estimate (18). As a consequence, we can write

$$\|z(t) - y(t)\| \le DMQ\, e^{\Gamma_1 - \Gamma(t-t_0)}, \text{ for every} t \ge t_0.$$

Therefore,

$$\lim_{t\to+\infty} \|z(t) - y(t)\| = 0 \quad \text{exponentially in} \mathcal{S}(\Omega),$$

i.e., the uniform exponential stability on Ω. □

The next two propositions give sufficient conditions for the properties (15) and (16) in Theorem 4.1 to be satisfied, respectively.

Proposition 4.1. *If there exists $\alpha \in]0, 1]$ such that the function K defined in (13) satisfies*

$$\Lambda := \liminf_{t\to+\infty} t^\alpha \left(\frac{\omega - CD}{CD} - K(t)\right) > 0, \tag{19}$$

for some positive constants C, D, ω, with $CD < \omega$, then (15) holds.

Proof. From (19), there exists $t^* \ge t_0$ such that

$$t^\alpha \left(\frac{\omega - CD}{CD} - K(t)\right) \ge \frac{\Lambda}{2}, \quad t \ge t^*.$$

Then, for $t \ge t^*$, the function F defined in (14) satisfies

$$F(t) \le \int_{t_0}^{t^*} (CD - \omega + CDK(s))\, ds - \frac{\Lambda CD}{2} \int_{t^*}^{t} s^{-\alpha}\, ds,$$

and (15) follows from $\alpha \le 1$. □

Proposition 4.2. *Let K be the function defined in (13). If*

$$\int_{t_0}^{\infty} K(t)\, dt < +\infty \quad \text{and} \quad CD < \omega, \tag{20}$$

or

$$A := \sup_{t\in(t_0,+\infty)} \frac{1}{t - t_0} \int_{t_0}^{t} K(s)\, ds < \infty \quad \text{and} \quad CD(1 + A) < \omega, \tag{21}$$

for some positive constants C, D, ω, then (16) holds.

Proof. Suppose that (20) holds. Then there exists $\Gamma > 0$ such that $CD - \omega + \Gamma \le 0$. From this inequality, for any $t \ge t_0$ we obtain

$$\Gamma(t - t_0) + \int_{t_0}^{t} [CD(1 + K(s)) - \omega]\, ds = \int_{t_0}^{t} [CD - \omega + \Gamma + CDK(s)]\, ds \le \int_{t_0}^{+\infty} CDK(s)\, ds,$$

which implies condition (16).

Analogously, if (21) is satisfied, then $\Gamma > 0$ exists such that $CD - \omega + CDA + \Gamma \le 0$. Hence, for any $t > t_0$ we have

$$\begin{aligned}\int_{t_0}^{t} [CD - \omega + \Gamma + CDK(s)]\, ds &= (t - t_0)\left[CD - \omega + \Gamma + \frac{CD}{t - t_0} \int_{t_0}^{t} K(s)\, ds\right] \\ &\le (t - t_0)[CD - \omega + \Gamma + CDA] \le 0,\end{aligned}$$

and again condition (16) is satisfied. □

By strengthening the conditions on the kernel k, we can use the previous Proposition 4.2 and provide the following corollary to Theorem 4.1.

Corollary 4.1. *Assume that properties (H0)-(H5) and (12) are satisfied and one of the following properties holds:*

(k1) there exist $g_1 \in L^1_+([0,+\infty))$, $g_2 \in L^1_+([t_0,+\infty))$ *such that*

$$k(t,s) \le e^{-\omega(t-s)} g_1(t-s) g_2(s),\ (t,s) \in \Delta_\infty, \tag{22}$$

and

$$CD < \omega; \tag{23}$$

(k2) there exist $g_1 \in L^1_+([0,+\infty))$, $g_2 \in L^1_+([t_0,+\infty))$ *and* $A_1 > 0$ *such that*

$$k(t,s) \le A_1 e^{-\omega(t-s)} g_1(t-s),\ (t,s) \in \Delta_\infty, \tag{24}$$

or

$$k(t,s) \le A_1 e^{-\omega(t-s)} g_2(t),\ (t,s) \in \Delta_\infty, \tag{25}$$

or

$$k(t,s) \le A_1 e^{-\omega(t-s)} g_2(s),\ (t,s) \in \Delta_\infty, \tag{26}$$

with

$$CD(1+\tilde{A}) < \omega, \tag{27}$$

where $\tilde{A} = A_1\|g_1\|_{L_1}$ *if (24) holds, while* $\tilde{A} = A_1\|g_2\|_{L_1}$ *if (25) or (26) is satisfied, and* C, D, ω *are from (12) and (1).*

Then, for every $v_0 \in E$ *the problem* $(P)_{v_0}$ *has a unique impulsive mild solution, which continuously depend on the initial data. Moreover, for every bounded* $\Omega \subset E$ *the impulsive mild solutions of Eq. (11) are uniformly exponentially stable on* Ω.

Proof. To achieve the thesis, it is sufficient to show that if (k1) or (k2) holds, then at least one of the conditions (20) or (21) in Proposition 4.2 is satisfied for the constants C, D, ω given in (12) and (1) (see assumptions (H2) and (H0)). More precisely, we prove that (k1) implies (20), while (k2) implies (21).

If (k1) is fulfilled, then by (22) and taking into account (13), for all $t \ge t_0$ we have

$$\int_{t_0}^{t} K(s)\,ds \le \int_{t_0}^{t}\int_{t_0}^{s} g_1(s-r)g_2(r)\,dr\,ds = \int_{t_0}^{t} g_2(r)\left(\int_{r}^{t} g_1(s-r)\,ds\right)dr$$
$$= \int_{t_0}^{t} g_2(r)\left(\int_{0}^{t-r} g_1(\tau)\,d\tau\right)dr \le \|g_1\|_{L_1}\|g_2\|_{L_1},$$

therefore, by (23) we have that (20) holds.

Now, we consider the hypotheses in (k2). If (24) is satisfied, then for all $s \ge t_0$ we have

$$K(s) \le A_1 \int_{t_0}^{s} g_1(s-r)\,dr \le A_1\|g_1\|_{L_1}.$$

Similarly, if (26) is satisfied, then it implies $K(s) \le A_1\|g_2\|_{L_1}$ for all $s \ge t_0$.
Finally, if (25) holds, then

$$K(s) \le A_1 \int_{t_0}^{s} g_2(s)\,dr \le A_1 g_2(s)(s-t_0),$$

for all $s \ge t_0$. Hence, if any of conditions (24), (25), (26) is satisfied, then there exist a constant $c_1 \ge 0$ and a function $\tilde{k} \in L^+_1(t_0,+\infty)$ such that

$$K(s) \le c_1 + \tilde{k}(s)(s-t_0), \quad s \ge t_0.$$

From the above inequality and the positivity of $\tilde{k}$, we have

$$\int_{t_0}^{t} K(s)\,ds \le c_1(t-t_0) + \int_{t_0}^{t} \tilde{k}(s)(s-t_0)\,ds \le c_1(t-t_0) + (t-t_0)\int_{t_0}^{t}\tilde{k}(s)\,ds$$
$$\le (c_1 + \|\tilde{k}\|_{L_1})(t-t_0), \quad t \ge t_0,$$

and

$$\sup_{t\in(t_0,+\infty)} \frac{1}{t-t_0}\int_{t_0}^{t} K(s)\,ds < \infty.$$

Therefore, taking also into account (27) and that $\tilde{A} \ge A$, condition (21) is fulfilled.

Now, the statements follow from Theorem 3.2 and Theorem 4.1 (case 3). □

In the literature, different kinds of kernels appear in order to describe via integro-differential equations some models of natural phenomena. The next examples consider some of them to illustrate properties (k1), (k2) presented in Corollary 4.1, and (19) in Proposition 4.1.

Example 4.1. Let $T > 0$ be a given constant. The distributed probability kernel

$$k(t,s) = \frac{e^{-(t-s)/T}}{T}, \quad t \geq s \geq 0,$$

satisfies property (k2) if

$$T < \frac{\omega - 2CD}{\omega(\omega - CD)}, \quad \omega > 2CD. \tag{28}$$

Indeed, put

$$A_1 = \frac{1}{T}, \quad g_1(u) = e^{-(\frac{1}{T}-\omega)u}, \ u \geq 0,$$

we have $k(t,s) = A_1 e^{-\omega(t-s)} g_1(t-s)$, i.e., (24) holds. Note that $g_1 \in L^1([0,+\infty)$ only if $\frac{1}{T} - \omega > 0$, and in this case $\|g_1\|_{L^1} = (\frac{1}{T} - \omega)^{-1}$. So $A = \frac{1}{1-T\omega}$ and (27) reads as

$$CD + \frac{CD}{1 - T\omega} < \omega,$$

which is equivalent to condition (28).

Example 4.2. Let $T > 0$ be a given constant. The kernel

$$k(t,s) = \frac{1}{T}\frac{e^{-(t-s)/T}}{s^2}, \quad t \geq s \geq 1, \ T > 0,$$

satisfies (22) in (k1) if

$$T < \frac{1}{\omega}. \tag{29}$$

Indeed, it is sufficient to take $g_1(u) = e^{-(\frac{1}{T}-\omega)u}$, $u \geq 0$, and $g_2(u) = (Tu^2)^{-1}, u \geq 1$. Both these functions belong to $L^1([1,+\infty))$ if (29) holds, and $k(t,s) = e^{-\omega(t-s)} g_1(t-s) g_2(s)$.

Example 4.3. Let $A_1 > 0$. The kernels

$$k_1(t,s) = A_1 \frac{e^{-\omega(t-s)}}{t^2}, \quad k_2(t,s) = A_1 \frac{e^{-\omega(t-s)}}{s^2}, \quad t \geq s \geq 1,$$

both satisfy property (k2) if

$$A_1 < \frac{\omega - CD}{CD}. \tag{30}$$

Moreover, the kernel k_2 satisfies (19) if either (30) or

$$A_1 = \frac{\omega - CD}{CD} \tag{31}$$

holds.

Indeed, if (30) holds, put $g_2(u) = 1/u^2$, $u \geq 1$, then k_1 satisfies (25) and k_2 satisfies (26). Further, $\|g_2\|_{L^1} = 1$ and (27) is satisfied.

Moreover, being $K(t) = A_1(1 - t^{-1})$, $t \geq 1$, then by choosing $\alpha = 1$ in (19), we have that (30) implies $\Lambda = +\infty$ while (31) implies $\Lambda = A_1$, and (19) is satisfied in both cases.

Remark 4.1. More in general, if $A > 0$ and $g \in L^1_+([t_0,\infty)) \cap C([t_0,\infty))$ exist, such that

$$k(t,s) \leq Ae^{-\omega(t-s)} g(s), \quad t \geq s \geq t_0,$$

then (19) is satisfied if $CD(1 + A\|g\|_{L_1}) < \omega$, as in this case (13) holds (see Corollary 4.1 and Proposition 4.2) or

$$CD(1 + A\|g\|_{L_1}) = \omega \quad \text{and} \quad \liminf_{t\to\infty} t^{\alpha+1} g(t) > 0,$$

for some $0 < \alpha \leq 1$. Indeed, as for the function K defined in (13) it results

$$K(t) \leq A\left(\|g\|_{L_1} - \int_t^\infty g(s)\,ds\right),$$

then

$$\Lambda \geq t^\alpha \left(\frac{\omega - CD}{CD} - A\|g\|_{L_1} + A\int_t^\infty g(s)\,ds\right),$$

and the assertion follows by noticing that, from the L'Hôpital's rule, $\liminf_{t\to+\infty} t^\alpha \int_t^\infty g(s)\,ds \geq \liminf_{t\to\infty} \alpha^{-1} t^{\alpha+1} g(t)$.

We conclude the section with a result on the existence of attractors and one on the global exponential stability of the null solution, if it exists. The definition of attractor we use is an adaptation to the present setting of the one in [6].

Definition 4.2. A function $y(t) = \mathbf{c}$, where $\mathbf{c} \in E$, is said to be an *attractor* for an impulsive mild solution z of the integro-differential equation (11) if

$$\lim_{t\to+\infty} \|z(t) - \mathbf{c}\| = 0.$$

Theorem 4.2. *Let (H0) be satisfied. Assume that the kernel k satisfies (H4) and the function f has properties (H1), (H2), (12), and*

(H3) there exists $\tilde{\mu} \in L^1_+([t_0, +\infty[)$ such that*

$$\|f(t, 0, 0)\| \le \tilde{\mu}(t)e^{-\omega t}, \ \text{a.e. } t \ge t_0.$$

Moreover, assume that the impulse functions I_m, $m \in \mathbb{N}^+$, satisfy (H5) and

(H6) $I_m(0) = 0$, for every $m \in \mathbb{N}^+$.

Finally, assume that condition (15) holds.

Then, the function $y(t) = 0$ is an attractor for the impulsive mild solution of $(P)_{v_0}$, for all $v_0 \in E$. In addition, for every bounded set $\Omega \subset E$, the solution set $S(\Omega)$ is bounded.

Proof. For any $v_0 \in E$, let y_{v_0} be the unique mild solution of $(P)_{v_0}$, since all the assumptions of Theorem 3.2 are satisfied.

Notice that from (H2), (H3*) we have for any $v, w \in E$ and $s \ge t_0$,

$$\|f(s, v, w)\| \le \|f(s, v, w) - f(s, 0, 0)\| + \|f(s, 0, 0)\| \le C(\|v\| + \|w\|) + \tilde{\mu}(s)e^{-\omega s},$$

and from (H5-i) and (H6), for every $m \in \mathbb{N}^+$ we get

$$\|I_m(v)\| \le \|I_m(v) - I_m(0)\| + \|I_m(0)\| \le a_m\|v\|, \ v \in E\,.$$

So, with similar arguments as in the proof of Theorem 3.2, we have

$$\begin{aligned}\|y_{v_0}(t)\|e^{\omega(t-t_0)} \le\ & D\|v_0\| + \sum_{t_0<t_m<t} Da_m\left\|y_{v_0}(t_m)\right\|e^{\omega(t_m-t_0)} \\ &+CD\int_{t_0}^{t} \|y_{v_0}(s)\|e^{\omega(s-t_0)}\,ds \\ &+CD\int_{t_0}^{t}\left(\int_{t_0}^{s} k(s,r)\left\|y_{v_0}(r)\right\|e^{\omega(s-t_0)}dr\right)ds + \\ &+D\int_{t_0}^{+\infty} \tilde{\mu}(s)ds.\end{aligned}$$

This inequality can be written as (5) by defining

$$\gamma(t) := \|y_{v_0}(t)\|e^{\omega(t-t_0)} \text{ and } c_0 := D(\|v_0\| + \|\tilde{\mu}\|_{L^1}),\ t \ge t_0,$$

and from (9) we obtain

$$\|y_{v_0}(t)\| \le DM(\|v_0\| + \|\tilde{\mu}\|_{L^1})e^{F(t)}, \quad t \ge t_0, \tag{32}$$

where M, F are defined in (8) and (14), respectively.

Now, assumption (15) assures that

$$\lim_{t\to+\infty} \|y_{v_0}(t)\| = 0,$$

for all $v_0 \in E$, i.e., $y(t) \equiv 0$ is an attractor. Further, since F is continuous on $[t_0, +\infty[$, from (15) we have that F is upper bounded. Then, if Ω is a bounded set in E and $v_0 \in \Omega$, from (32) we have

$$\|y_{v_0}(t)\| \le DM\left(\sup_{u\in\Omega}\|u\| + \|\tilde{\mu}\|_{L^1}\right)e^{\max_{t\ge t_0} F(t)},\ v_0 \in \Omega,$$

and therefore $S(\Omega)$ is bounded. □

From the proof of Theorem 4.2, we infer that, if condition (15) is replacd by (16), then any impulsive mild solution y_{v_0}, with $v_0 \in E$, tends to 0 exponentially as t goes to infinity. So, by strengthening assumption (H3*) in order to have that $y \equiv 0$ is a solution, the next result holds.

Corollary 4.2. *Let (H0) be satisfied. Assume that the kernel k satisfies (H4), the impulse functions I_m, $m \in \mathbb{N}^+$, satisfy (H5) and (H6), and the function f has properties (H1), (H2), (12), and*

*(H3**) $f(\cdot, 0, 0) = 0$.*

Finally, assume that condition (16) holds.

Then, the solution $y(t) = 0$ is globally exponentially stable.

Taking into account that either condition (k1) or (k2) implies property (16) (see the proof of Corollary 4.1), we can reformulate the Corollary 4.2 in the following stronger form.

Corollary 4.3. *Let (H0) be satisfied. Assume that the kernel k satisfies (H4), the impulse functions I_m, $m \in \mathbb{N}^+$, satisfy (H5) and (H6), and the function f has properties (H1), (H2), (12), and (H3**). Finally, suppose that either condition (k1) or (k2) holds.*

Then, the solution $y(t) = 0$ is globally exponentially stable.

5. Applications

The results obtained in the previous section allow for a unified treatment of the uniform asymptotic stability and of the exponential stability of various classes of differential models drawn from applied sciences.

In this section, we illustrate the application of our main results to a population dynamics model and to a model of a flexible robotic arm.

We point out that the models presented here are considered solely from a mathematical perspective and do not aim to delve into the specific aspects of the applied disciplines there involved.

5.1. A population dynamics model

Let us consider a population with density $u = u(t, x)$, varying over time and space, subject to the action of instantaneous external forces occurring at predetermined time instants. For example, think to a pest population being targeted with pesticides. It is assumed that the population's evolution is influenced not only by the current state but also by past events, as in cases where the maturation time of individuals is significant, and their entry into the adult population occurs some time after birth. The impact of these past events diminishes as they lie further back in the biological system's history. Thus, the model can be described by the following impulsive problem:

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = -b(t,x)u(t,x) + g\left(t, u(t,x), \displaystyle\int_{t_0}^{t} \frac{e^{-(t-s)/T}}{T}\, u(s,x)\, ds\right), \\ \qquad\qquad t \geq t_0\,,\ t \neq t_m\,,\ m \in \mathbb{N}^+\,,\ a.e.\, x \in [0,1], \\ u(t_0, x) = u_0(x)\,,\ a.e.\, x \in [0,1], \\ u\big(t_m^+, x\big) = u\big(t_m, x\big) + \mathcal{I}_m\big(u\big(t_m, x\big)\big)\,,\quad m \in \mathbb{N}^+\,,\ a.e.\, x \in [0,1]. \end{cases} \tag{33}$$

Here, $(t_m)_{m\in\mathbb{N}}$ is an increasing sequence of non negative real numbers diverging to $+\infty$, $b(t,x)$ is the removal coefficient (including migration rate and death rate), and g is the nonlinear development law of the population. Further, the Volterra integral $\int_{t_0}^{t} \frac{e^{-(t-s)/T}}{T}\, u(s,x)\, ds$ is the distributed delay term, being associated to the exponential distribution of probability kernel $\mathfrak{k}(\tau) = \frac{e^{-\tau/T}}{T}$ leading to a fading memory setup, where the number T represents the time span of the delay and the interval $[0,1]$ is a normalized spatial range. The impulsive external action on the system is described by the functions $\mathcal{I}_m$, $m \in \mathbb{N}^+$.

This model can be formulated as an abstract problem like $(P)_{v_0}$ in the function space $E = L^2([0,1])$, making the positions: $v_0 := u_0 \in L^2([0,1])$, $A(t) : L^2([0,1]) \to L^2([0,1])$, $t \geq 0$, with

$$[A(t)y](x) = -b(t,x)y(x),\ a.e.\, x \in [0,1], \tag{34}$$

$f : [t_0, +\infty[\times L^2([0,1]) \times L^2([0,1]) \to L^2([0,1])$, with

$$f(t, v, w)(x) = g(t, v(x), w(x)),\ a.e.\, x \in [0,1], \tag{35}$$

and $I_m : L^2([0,1]) \to L^2([0,1])$, $m \in \mathbb{N}$, with

$$I_m(v)(x) = \mathcal{I}_m(v(x)),\ a.e.\, x \in [0,1].$$

Suppose on the function $b : [t_0, +\infty[\times [0,1] \to \mathbb{R}^+$ the assumptions

(b.1) b is measurable;

(b.2) there exist $\beta > 0$ and $s \in L^1_{loc}([t_0, +\infty[)$ such that

$$\beta \leq b(t,x) \leq s(t)\,,\ \text{for every } t \geq t_0,\ \text{a.e. } x \in [0,1];$$

(b.3) for every $x \in [0,1]$, the function $b(\cdot, x) : [t_0, +\infty[\to \mathbb{R}^+$ is continuous.

Then, assume that $g : [t_0, +\infty[\times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies the following properties.

(g.0) $g(t, v(\cdot), w(\cdot)) \in L^2([0,1])$, for every $t \geq t_0$, $v, w \in L^2([0,1])$;

(g.1) for every $v, w \in L^2([0,1])$, the function $t \mapsto g(t, v(\cdot), w(\cdot))$ is strongly measurable;

(g.2) there exists $C > 0$ such that

$$\left| g(t, p_1, \hat{p}_1) - g(t, p_2, \hat{p}_2) \right| \le C\big(|p_1 - p_2| + |\hat{p}_1 - \hat{p}_2|\big),$$

for a.e. $t \ge t_0$ and every $p_1, p_2, \hat{p}_1, \hat{p}_2 \in \mathbb{R}$.

(g.3) $g(\cdot, 0, 0) \in L^1_{loc}([t_0, +\infty[)$.

Finally, on the maps $\mathcal{I}_m$ we assume the property

(I.1) for every $m \in \mathbb{N}^+$ there exists $a_m > 0$ such that

$$\left| \mathcal{I}_m(p_1) - \mathcal{I}_m(p_2) \right| \le a_m |p_1 - p_2|, \quad \text{for every } p_1, p_2 \in \mathbb{R},$$

with $\sum_{m=1}^{+\infty} a_m < +\infty$.

In this framework it is known that the next result holds (cf. [18, Proposition 5.1]).

Proposition 5.1. *Under assumptions (b.1)-(b.3) the family $\{A(t)\}_{t\ge 0}$ defined by (34) generates the exponentially stable evolution system*

$$[U(t,s)y](x) = e^{\int_s^t -b(\sigma,x)d\sigma} y(x), \ \ y \in L^2([0,1]), \ x \in [0,1], \ t \ge s \ge 0.$$

By applying Corollary 4.1 and Corollary 4.3, we can state the following theorem on the uniform asymptotic stability and on the exponential stability for the impulsive mild solutions of the population dynamics model (33).

Theorem 5.1. *Assume (b.1)-(b.3), (g.0)-(g.3), (I.1) and condition (28).*
Then, for every $u_0 \in L^2([0,1])$ the problem (33) has a unique impulsive mild solution $u : [t_0, +\infty[\times[0,1] \to \mathbb{R}$, $u(t,\cdot) \in L^2([0,1])$ for every $t \ge t_0$ and $u(\cdot, x) \in \mathcal{PCT}([t_0,+\infty[, \mathbb{R})$ (see (2)) for all $x \in [0,1]$, such that

$$u(t,x) = e^{\int_{t_0}^t -b(\sigma,x)d\sigma} u_0(x) + \sum_{t_0<t_m<t} e^{\int_{t_m}^t -b(\sigma,x)d\sigma} \mathcal{I}_m(u(t_m,x)) +$$
$$+ \int_{t_0}^t e^{\int_s^t -b(\sigma,x)d\sigma} g\Big(s, u(s,x), \int_{t_0}^s \frac{e^{-(s-\tau)/T}}{T} u(\tau,x)\Big)\, d\tau\, ds, \quad t \ge t_0, \ x \in [0,1],$$

which continuously depends on the initial data.

Moreover, for every bounded $\Omega \subset L^2([0,1])$ the impulsive mild solutions of the partial differential equation driving (33) are uniformly exponentially stable on Ω.

Further, if in addition we assume properties

(g.4) $g(t,0,0) = 0, \ t \ge t_0$,

(I.2) $\mathcal{I}_m(0) = 0, \ m \in \mathbb{N}^+$,

then the solution $u \equiv 0$ is globally exponentially stable.

Proof. The function f introduced in (35) is well-defined and $f(\cdot, v, w)$, for every $v, w \in E$, is strongly measurable (see respectively (g.0) and (g.1)).
Further, the map $f(t,\cdot,\cdot)$, for a.e. $t \ge t_0$, is C-lipschitzian with C the constant in (g.2). In fact, by (g.2) and the Hölder's inequality for any $v_1, v_2, w_1, w_2 \in L^2([0,1])$ and a.e. $t \ge t_0$ we get

$$\|f(t,v_1,w_1) - f(t,v_2,w_2)\|^2_{L^2([0,1])} = \int_0^1 [g(t,v_1(x),w_1(x)) - g(t,v_2(x),w_2(x))]^2 dx \le$$
$$\le C^2 \int_0^1 \big\{[v_1(x)-v_2(x)]^2 + [w_1(x)-w_2(x)]^2 + 2|v_1(x)-v_2(x)||w_1(x)-w_2(x)|\big\} dx \le$$
$$\le C^2 \Big\{ \|v_1 - v_2\|^2_{L^2([0,1])} + \|w_1 - w_2\|^2_{L^2([0,1])} + 2\|v_1 - v_2\|_{L^2([0,1])}\|w_1 - w_2\|_{L^2([0,1])} \Big\} =$$
$$= C^2 \big\{ \|v_1 - v_2\|_{L^2([0,1])} + \|w_1 - w_2\|_{L^2([0,1])} \big\}^2.$$

Moreover, by (g.3) we have $\|f(\cdot,0,0)\|_{L^2([0,1])} \in L^1_{loc}([t_0,+\infty[)$.
About the kernel $k(t,s) = \frac{e^{-(t-s)/T}}{T}$, it is obviuosly continuous and, by Example 4.1 and assumption (28), it also satisfies property (k2)

in Corollary 4.1.
Finally, by (I.1) for every $v_1, v_2 \in L^2([0,1])$, we get

$$\|I_m(v) - I_m(w)\|_{L^2([0,1])} = \left\{ \int_0^1 \left|\mathcal{I}_m\big(v_1(x)\big) - \mathcal{I}_m\big(v_2(x)\big)\right|^2 dx \right\}^{1/2} \leq$$

$$\leq \left\{ \int_0^1 a_m^2 |v_1(x) - v_2(x)|^2 dx \right\}^{1/2} = a_m \|v_1 - v_2\|_{L^2([0,1])}.$$

Hence, all the assumptions of Corollary 4.1 are satisfied.

If in addition we assume (g.4) and (I.2), then properties (H6) and (H3**) are immediately satisfied. Hence, all the assumptions of Corollary 4.3 are fulfilled as well, and the statements follow. □

5.2. A flexible robotic arm model

In a recent paper [6], we have studied a model which can represent the movement of a robotic arm having fading memory of the past deflections or of a flexible beam clamped at one end and controlled at the free one. In both cases it is meaningful to add the presence of instantaneous impulses, modeling disturbances such as mechanical shocks or electrical impulses coming from the surrounding environment, obtaining the following problem.

$$\begin{cases} u_{tt}(t,x) = -u_{xxxx}(t,x) + G\left(t, u(t,x), \int_0^t \frac{e^{-(t-s)/T}}{T} u(s,x)\,ds\right), \ t \geq 0, \\ \qquad\qquad t \neq t_m, \ m \in \mathbb{N}^+, \ a.e.\ x \in [0,1], \\ u(t,0) = u_x(t,0) = u_{xx}(t,1) = 0, \\ M u_{tt}(t,1) - u_{xxx}(t,1) = -\alpha u_t(t,1) + \beta u_{xxxt}(t,1), \\ u(0,x) = \bar{p}(x), \ a.e.\ x \in [0,1], \\ u_t(0,x) = \bar{q}(x), \ a.e.\ x \in [0,1], \\ -u_{xxx}(0,1) + \frac{M}{\beta} u_t(0,1) = -\bar{p}_{xxx}(1) + \frac{M}{\beta}\bar{q}(1), \\ u\left(t_m^+, x\right) = u\left(t_m, x\right) + \mathcal{I}_m\left(u\left(t_m, x\right)\right), \ m \in \mathbb{N}^+, \ a.e.\ x \in [0,1]. \end{cases} \tag{36}$$

Here $u(t,x)$ is the vibration amplitude, G is a nonlinear forcing term with fading memory given by the Volterra integral inside, $M > 0$ stands for a tip mass, α, β are positive constants, $\bar{p} \in H^4(0,1), \bar{q} \in H^1(0,1)$ are given functions, $(t_m)_{m\in\mathbb{N}^+}$ is an increasing sequence of positive real numbers diverging to $+\infty$, and $\mathcal{I}_m$, $m \in \mathbb{N}^+$ are the impulses.

Now, considered the auxiliary function (see [6,9])

$$\eta(t) := -u_{xxx}(t,1) + \frac{M}{\beta} u_t(t,1), \ t \geq 0,$$

the model can be rewritten as the first order system

$$\begin{cases} u_t(t,x) = v(t,x), \\ v_t(t,x) = -u_{xxxx}(t,x) + G\left(t, u(t,x), \int_0^t \frac{e^{-(t-s)/T}}{T} u(s,x)\,ds\right), \\ \eta'(t) = -\frac{1}{\beta}\eta(t) - \frac{1}{\beta}\left(\alpha - \frac{M}{\beta}\right) v(t,1), \end{cases}$$

with boundary conditions

$$u(t,0) = u_x(t,0) = 0, \quad u_{xx}(t,1) = 0,$$

in the Hilbert function space

$$\mathcal{H} := \{y := (p,\ q,\ \eta)^{\mathrm{T}} : p \in \mathcal{V},\ q \in L^2([0,1]),\ \eta \in \mathbb{R}\},$$

where the superscript T stands for the transpose and

$$\mathcal{V} := \{p \in H^2(0,1) : p(0) = p_x(0) = 0\}.$$

The problem (36) can be restated in the function space $\mathcal{H}$ as an impulsive problem of type $(P)_{v_0}$, where $v_0 := (\bar{p}, \bar{q}, \bar{\eta})^{\mathrm{T}} \in \mathcal{H}$, $\bar{\eta} := -\bar{p}_{xxx}(1) + \frac{M}{\beta}\bar{q}(1)$, defining $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ with

$$Ay = A\begin{pmatrix} p \\ q \\ \eta \end{pmatrix} := \begin{pmatrix} q \\ -p_{xxxx} \\ -\frac{1}{\beta}\eta - \frac{1}{\beta}\left(\alpha - \frac{M}{\beta}\right) q(1) \end{pmatrix}$$

on

$$D(A) := \left\{ y = (p, q, \eta)^{\mathrm{T}} : p \in H^4(0,1) \cap \mathcal{V},\ q \in \mathcal{V},\ p_{xx}(1) = 0,\ \eta = -p_{xxx}(1) + \frac{M}{\beta} q(1) \right\},$$

$f : \mathbb{R}_0^+ \times \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ with

$$f(t, y_1, y_2)(x) := \left(0_{\mathcal{V}}, G(t, p_1(x), p_2(x)), 0\right)^{\mathrm{T}},\ x \in [0,1], \tag{37}$$

and $I_m : \mathcal{H} \to \mathcal{H}$, $m \in \mathbb{N}^+$, as

$$I_m(y)(x) := (0, \mathcal{I}_m(p(x)), 0)^{\mathrm{T}},\ x \in [0,1].$$

It is known (see [9, Theorems 2.1 and 2.2]) that the operator A above defined generates on $\mathcal{H}$ an exponentially stable C_0-semigroup of contractions e^{At}, $t \geq 0$, and therefore an exponentially stable evolution system $U(t,s) = e^{A(t-s)}$, $t \geq s \geq 0$.

We assume on $G : \mathbb{R}_0^+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and on the maps $\mathcal{I}_m : \mathbb{R} \to \mathbb{R}$, $m \in \mathbb{N}^+$, the properties

(G.0) $G\big(t, p_1(\cdot), p_2(\cdot)\big) \in L^2([0,1])$, for every $t \geq 0$, $p_1, p_2 \in \mathcal{V}$;

(G.1) for every $p_1, p_2 \in \mathcal{V}$, the function $t \mapsto G(t, p_1(\cdot), p_2(\cdot))$ is strongly measurable;

(G.2) there exists $C > 0$ such that

$$\left\| G\big(t, p_1(\cdot), p_2(\cdot)\big) - G\big(t, \hat{p}_1(\cdot), \hat{p}_2(\cdot)\big) \right\|_{L^2} \leq C\big(\|p_1 - \hat{p}_1\|_{L^2} + \|p_2 - \hat{p}_2\|_{L^2}\big),$$

for all $t \geq 0$ and $p_1, p_2, \hat{p}_1, \hat{p}_2 \in \mathcal{V}$;

(G.3) $G(\cdot, 0, 0) \in L^1_{loc}([0, +\infty[)$;

($\mathcal{I}$.1) for every $m \in \mathbb{N}^+$ there exists $a_m > 0$ such that

$$\left|\mathcal{I}_m(r_1) - \mathcal{I}_m(r_2)\right| \leq a_m |r_1 - r_2|, \quad \text{for every } r_1, r_2 \in \mathbb{R},$$

with $\sum_{m=1}^{+\infty} a_m < +\infty$.

Theorem 5.2. *Assume (G.0)-(G.3), ($\mathcal{I}$.1) and condition (28).*
Then, for every $(\bar{p}, \bar{q}, \bar{\eta})^{T} \in \mathcal{H}$ the problem (36) has a unique impulsive mild solution
$u : [0, +\infty[\times [0,1] \to \mathbb{R}$, $u(t, \cdot) \in L^2([0,1])$ *for every* $t \geq 0$ *and* $u(\cdot, x) \in \mathcal{PCT}([0, +\infty[, \mathbb{R})$ *(see (2)) for all* $x \in [0,1]$, *such that*

$$u(t,x) = e^{At}\bar{p}(x) + \sum_{t_0 < t_m < t} e^{A(t-t_m)} \mathcal{I}_m(u(t_m, x)) + \\ + \int_{t_0}^{t} e^{A(t-s)} G\Big(s, u(s,x), \int_{t_0}^{s} \frac{e^{-(s-\tau)/T}}{T} u(\tau, x)\Big)\, d\tau\, ds, \quad t \geq 0,\ x \in [0,1],$$

which continuously depend on the initial data.

Moreover, for every bounded $\Omega \subset \mathcal{H}$ the impulsive mild solutions of the partial differential equation driving (36) are uniformly exponentially stable on Ω.

Further, if in addition we assume properties

(G.4) $G(t, 0, 0) = 0$, $t \geq 0$,

($\mathcal{I}$.2) $\mathcal{I}_m(0) = 0$, $m \in \mathbb{N}^+$,

then the solution $u \equiv 0$ is globally exponentially stable.

Proof. It is easy to see that by (G.0) and (G.1) the function f defined in (5.2) is well defined and satisfies (H.1).
Property (H.2) is satisfied as well, since

$$\|f(t, y_1, y_2) - f(t, \hat{y}_1, \hat{y}_2)\| \leq C(\|p_1 - \hat{p}_1\|_{L^2} + \|p_2 - \hat{p}_2\|_{L^2}) \leq C(\|y_1 - \hat{y}_1\|_{\mathcal{H}} + \|y_2 - \hat{y}_2\|_{\mathcal{H}}),$$

for every $y_1, y_2, \hat{y}_1, \hat{y}_2 \in \mathcal{H}$, where $y_i = (p_i, q_i, \eta_i)$, $\hat{y}_i = (\hat{p}_i, \hat{q}_i, \hat{\eta}_i)$, $i = 1, 2$.
Moreover, since $\|f(\cdot, 0_{\mathcal{H}}, 0_{\mathcal{H}})\|_{\mathcal{H}} = |G(\cdot, 0, 0)|$, then (H.3) is satisfied.
Further, fixed $y_1, y_2 \in \mathcal{H}$ by ($\mathcal{I}$.1), we get

$$\begin{aligned} \|I_m(y_1) - I_m(y_2)\|_{\mathcal{H}} &= \|\mathcal{I}_m(p_1) - \mathcal{I}_m(p_2)\|_{L^2([0,1])} \\ &\leq a_m \|p_1 - p_2\|_{L^2([0,1])} \leq a_m \|y_1 - y_2\|_{\mathcal{H}}, \end{aligned}$$

for every $m \in \mathbb{N}^+$, so that (H.5) is satisfied.
Finally, property (k2) is satisfied thanks to condition (28) (see Example 4.1), so by Corollary 4.1 we obtain the first part of the thesis.
For the second part of the statement we can use Corollary 4.3 since (G.4) and ($\mathcal{I}$.2) imply (H3**) and (H6), respectively. □

Credit Author Statement

All authors contributed equally to the conception, development, and writing of this manuscript. Each author participated in the research, analysis, and preparation of the final text, and all authors approve the final version of the manuscript.

CRediT authorship contribution statement

Tiziana Cardinali: Writing – review & editing, Writing – original draft, Methodology, Funding acquisition, Formal analysis, Conceptualization; **Serena Matucci:** Writing – review & editing, Writing – original draft, Methodology, Funding acquisition, Formal analysis, Conceptualization; **Paola Rubbioni:** Writing – review & editing, Writing – original draft, Methodology, Funding acquisition, Formal analysis, Conceptualization.

Data availability

No data was used for the research described in the article.

Declaration of interests

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

This study was partly funded by : Research project of MUR (italian Ministry of University and Research) PRIN 2022 "Nonlinear differential problems with applications to real phenomena" (Grant Number: 2022ZXZTN2, CUP Master J53D2300390 0006).

The authors are members of the national group GNAMPA of INdAM and, the third one, of the UMI Group T.A.A. "Approximation Theory and Applications".

References

[1] Baĭnov DD, Covachev V, Stamova I. Estimates of the solutions of impulsive quasilinear functional-differential equations. Ann. Fac. Sci. Toulouse Math. (5) 1991;12(2):149–61.
[2] Baleanu D, Kasinathan R, Kasinathan R, Sandrasekaran V. Existence, uniqueness and Hyers-Ulam stability of random impulsive stochastic integro-differential equations with nonlocal conditions. AIMS Math 2023;8(2):2556–75.
[3] Banaś J, Jleli M, Mursaleen M, Samet B, Vetro C (Editors). Advances in Nonlinear Analysis via the Concept of Measures of Noncompactness. Singapore; Springer; 2017.
[4] Cabada A, Liz E. Boundary value problems for higher order ordinary differential equations with impulses. Nonlinear Anal. 1998;32(6):775–86.
[5] Caraballo T, Ezzine F, Hammami MA. Lyapunov functionals and practical stability for stochastic differential delay equations with general decay rate. Electron. J. Qual. Theory Differ. Equ. 2022;60: p. 22.
[6] Cardinali T, Matucci S, Rubbioni P. Uniform asymptotic stability of a PDE's system arising from a flexible robotics model. Math. Methods Appl. Sci. 2025;48(11):11242–51.
[7] Cardinali T, Rubbioni P. On the existence of mild solutions of semilinear evolution differential inclusions. J. Math. Anal. Appl. 2005;308(2):620–35.
[8] Cardinali T, Rubbioni P. The controllability of an impulsive integro-differential process with nonlocal feedback controls. Appl. Math. Comput. 2019;347:29–39.
[9] Conrad F, Morgül Ö. On the stabilization of a flexible beam with a tip mass. SIAM J. Control. Optim. 1998;36(6):1962–86.
[10] Conti M, Dell'Oro F, Pata V. Exponential decay of a first order linear volterra equation. Math. Eng. 2020;2(3):459–71.
[11] D'Apice C, Manzo R, Piccoli B, Vespri V. Lyapunov stability for measure differential equations. Math. Control Relat. Fields 2024;14(4):1391–407.
[12] Kreĭn SG. Linear differential equations in Banach space; Translations of Mathematical Monographs, vol. 29. Providence, R.I.:American Mathematical Society; 1971.
[13] Lakshmikantham V, Baĭnov DD, Simeonov PS. Theory of impulsive differential equations; vol. 6. Teaneck, NJ: World Scientific Publishing Co., Inc; 1989.
[14] Li Q, Wei M. Existence and asymptotic stability of periodic solutions for impulsive delay evolution equations. Adv. Difference Equ. 2019; 51; pp. 19.
[15] Nowak G, Saker SH, Sikorska-Nowak A. Asymptotic stability of nonlinear neutral delay integro-differential equations. Math. Slovaca 2023;73(1):103–18.
[16] Pazy A. Semigroups of linear operators and applications to partial differential equations; Applied Mathematical Sciences, vol. 44. New York: Springer-Verlag; 1983.
[17] Rautian NA. Exponential stability of semigroups generated by Volterra integro-differential equation. Ufa Math. J. 2021;13(4):63–79.
[18] Rubbioni P. Asymptotic stability of solutions for some classes of impulsive differential equations with distributed delay. Nonlinear Anal. Real World Appl. 2021;61:pp.17. Paper No. 103324.
[19] Rubbioni P. Solvability for a class of integro-differential inclusions subject to impulses on the half-line. Mathematics, 2022(2):224.
[20] Salim A, Benchohra M, Karapinar E, Lazreg JE. Existence and Ulam stability for impulsive generalized Hilfer-type fractional differential equations. Adv. Difference Equ. 2020, Paper No. 601; pp. 21.
[21] Tomeček J. Periodic solution of differential equation with ϕ-Laplacian and state-dependent impulses. J. Math. Anal. Appl. 2017;450(2):1029–46.
[22] Van Minh N, Räbiger F, Schnaubelt R. Exponential stability, exponential expansiveness, and exponential dichotomy of evolution equations on the half-line. Integral Equations Operator Theory 1998;32(3):332–53.
[23] Xu M, Sun S, Han Z. The method of upper and lower solutions for integral boundary value problem of semilinear fractional differential equations with non-instantaneous impulses. Math. Slovaca 2020;70(3):625–40.