axioms

MDPI

Article

On Totally Geodesic Submanifolds

Antonella Nannicini * and Donato Pertici

Dipartimento di Matematica e Informatica, University of Florence, Viale Morgagni 67/a, 50134 Firenze, Italy; donato.pertici@unifi.it
* Correspondence: antonella.nannicini@unifi.it

Abstract

We give a proof of Cartan's Theorem on totally geodesic submanifolds for real analytic manifolds endowed with a real analytic, torsion-free, affine connection. We apply the theorem to real analytic Hadamard manifolds and, more generally, to real analytic manifolds with a torsion-free, analytic, affine connection, such that at a manifold point $p \in M$, the exponential map is a real analytic diffeomorphism from the tangent space $T_p(M)$ to M. Examples of manifolds with this property are statistical manifolds with a cubic form divisible by the metric, as was recently proven. We also give examples of totally geodesic submanifolds obtained as fixed points of affine transformations of M and, moreover, as certain submanifolds of connected Lie groups with the 0-connection of Cartan–Schouten. Finally, we also determine all connected complete totally geodesic surfaces of the Riemannian manifold $(\mathcal{P}_2, g)$ of symmetric positive definite 2×2 real matrices, endowed with the trace metric g.

Keywords: affine connections; Cartan's Theorem; totally geodesic submanifolds; Cartan–Hadamard Theorem

MSC: 53A15; 53A35; 53B12; 53B30

1. Introduction

The main purpose of this paper is to give a direct proof of Cartan's Theorem on totally geodesics submanifolds of manifolds endowed with an affine connection and to describe some applications. In the case of real analytic Riemannian manifolds with the Levi-Civita connection, the theorem is originally attributed to Elie Cartan [1]. Cartan's Theorem states necessary and sufficient conditions, in terms of the curvature tensor and its derivatives, for the existence of totally geodesic submanifolds through a point with a given tangent space. We provide a direct, rigorous proof of the theorem for real analytic manifolds endowed with a real analytic, torsion-free, affine connection. To the best of our knowledge, it is not easy to find a rigorous proof, for this general case, in the mathematical literature. We give examples of totally geodesic submanifolds obtained as fixed points of affine transformations of M and, moreover, as certain submanifolds of connected Lie groups with the 0-connection of Cartan–Schouten. Then we apply the theorem to real analytic Hadamard manifolds with the Levi-Civita connection and to real analytic manifolds M, with a torsion-free, analytic, affine connection ∇, such that at a point $p \in M$, the exponential map is a real analytic diffeomorphism from the tangent space at p, denoted as $T_p(M)$, to the manifold M. Examples of manifolds with this property are statistical manifolds with a cubic form divisible by the metric, as recently proven in [2]. Totally geodesic submanifolds are very

Academic Editors: Zhigang Wang, Yanlin Li and Juan De Dios Pérez

Received: 29 April 2026
Revised: 4 June 2026
Accepted: 11 June 2026
Published: 13 June 2026

important in physics, and in this sense, this research provides a theoretical foundation for future integrable systems and special submanifolds in general relativity [3].

This paper is organized as follows. In Section 2, we briefly recall fundamental definitions and state the notations used in this paper, and then we prove the preliminary results on totally geodesic submanifolds. Section 3 is devoted to the proof of Cartan's Theorem in the general setting. Also, an application to real analytic, affine, locally symmetric spaces is given. In Section 4, we apply the main theorem, first to fixed points of certain affine transformations and then to connected Lie groups with the 0-connection of Cartan–Schouten. Moreover we describe applications to real analytic Hadamard manifolds and to certain real analytic manifolds endowed with a real analytic, torsion-free, affine connection. Finally, we determine all connected complete totally geodesic surfaces of the 3-manifold $(\mathcal{P}_2, g)$ of symmetric positive definite 2×2 real matrices, endowed with the trace metric g. Note that there exist Riemannian or pseudo-Riemannian manifolds, or manifolds with an affine connection, which have no proper geodesic submanifolds of a dimension greater than 1 (see, for instance, [4,5]). It is then crucial to have tools for constructing such submanifolds, even locally. Interesting papers on the subject are [6–12].

2. Preliminaries

In this section, we briefly recall some definitions. For further information and proofs of what will be stated, see, for example, [13,14].

Let (M, ∇) be a smooth n-dimensional manifold endowed with an affine connection.

(a) The *torsion* tensor T of ∇ is defined as follows: $T(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y]$ for every pair of tangent vector fields X, Y of M. The connection ∇ is said to be *torsion-free* when $T = 0$.

The *curvature* tensor R of type $(1, 3)$ of ∇ is defined as follows: $R(X, Y, Z) := \nabla_{[X,Y]} Z - \nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z$, for every triplet of tangent vector fields X, Y, Z of M. It is well known that if ∇ is torsion-free, then the curvature tensor R satisfies the following first Bianchi's identity: $R(X, Y, Z) + R(Y, Z, X) + R(Z, X, Y) = 0$, for every triplet of tangent vector fields X, Y, Z of M.

An *affine transformation* of (M, ∇) is a diffeomorphism $f : M \to M$ such that $f_*(\nabla_X Y) = \nabla_{f_*(X)} f_*(Y)$ for every pair of tangent vector fields X, Y of M. Here f_* denotes the differential map of f, while $f_*(Z)$ denotes the f-related tangent vector field of any tangent vector field Z of M. We will denote by $Fix(f)$ the set consisting of all fixed points of diffeomorphism f, if it is an affine transformation.

For any $m \geq 0$, we denote by $\nabla^m R$ the m-th *covariant derivative* of R defined inductively as follows: $\nabla^0 R := R$ and $\nabla^m R := \nabla(\nabla^{m-1} R)$ when $m \geq 1$. Consequently, for any $m \geq 1$ we obtain $(\nabla^m R)(X_1, X_2, \cdots, X_{m+2}, X_{m+3}) := \nabla_{X_{m+3}} \big((\nabla^{m-1} R)(X_1, \cdots, X_{m+2})\big)$ $- \sum_{j=1}^{m+2} (\nabla^{m-1} R)(X_1, \cdots, X_{j-1}, \nabla_{X_{m+3}} X_j, X_{j+1}, \cdots, X_{m+2})$, whatever the tangent vector fields $X_1, X_2, \cdots, X_{m+2}, X_{m+3}$ of M. As is known, if diffeomorphism f is any affine transformation of (M, ∇), then we have $f_*\big((\nabla^m R)(X_1, \cdots, X_{m+3})\big) = (\nabla^m R)(f_*(X_1), \cdots, f_*(X_{m+3}))$, for every $m \geq 0$ and for all tangent vector fields $X_1, \cdot, X_{m+3}$ of M.

For each $p \in M$, we denote by $T_p(M)$ the tangent space to M at the point p and by $\gamma_{(p,v)}(t)$ the maximal geodesic of (M, ∇) such that $\gamma_{(p,v)}(0) = p$ and $\frac{d\gamma_{(p,v)}}{dt}(0) = v$, where v is any vector of $T_p(M)$. Furthermore, $\exp_p$ denotes the exponential map of (M, ∇) at the point p, while Ω_p denotes the maximal subset of $T_p(M)$ on which the map $\exp_p$ is defined. It is known that Ω_p is open in $T_p(M)$ and star-shaped with respect to the zero vector 0_p of $T_p(M)$. Recall that $\exp_p : \Omega_p \to M$ is the smooth map defined by $\exp_p(v) := \gamma_{(p,v)}(1)$ for every $v \in \Omega_p$ and that we can write $\gamma_{(p,v)}(t) = \exp_p(tv)$ for every t belonging to the domain of $\gamma_{(p,v)}$ and for every $v \in T_p(M)$. It is well known that if f is an affine transformation of (M, ∇), then, for every point $p \in M$, we have $f_*(\Omega_p) = \Omega_{f(p)}$ and $f \circ \exp_p = \exp_{f(p)} \circ f_*$ on the set Ω_p.

(b) It is well known that there exists a star-shaped (with respect to 0_p) open neighborhood $\widetilde{N}_p \subseteq \Omega_p$ of 0_p, such that the restriction of $\exp_p$ to $\widetilde{N}_p$ is a diffeomorphism from $\widetilde{N}_p$ onto the open neighborhood $N_p := \exp_p(\widetilde{N}_p)$ of p in M; then N_p is called a *normal neighborhood* of p in (M, ∇). Therefore, if $\{e_1, \cdots, e_n\}$ is any basis of $T_p(M)$, then the mapping $\exp_p(\sum_{i=1}^{n} x^i e_i) \mapsto (x^1, \cdots, x^n)$ defines a coordinate system on N_p, and the real functions $x^1, \cdots, x^n$ are called *normal coordinates* on N_p. If V is any s-dimensional vector subspace of $T_p(M)$ with $\{e_1, \cdots, e_s\}$ as a basis, then any basis $\{e_1, \cdots, e_s, e_{s+1}, \cdots, e_n\}$ of $T_p(M)$ that completes the basis $\{e_1, \cdots, e_s\}$ determines, on each normal neighborhood N_p, normal coordinates that we will call *V-adapted normal coordinates*. In this case, $\exp_p(V \cap \widetilde{N}_p)$ is the s-dimensional submanifold of M consisting of the points of N_p whose last $(n-s)$ V-adapted normal coordinates $x^{s+1}, \cdots, x^n$ are all zero.

(c) We say that a subset W of M is *star-shaped in* (M, ∇) *with respect to* $p \in W$ (or more simply that W is *p-star-shaped in* (M, ∇)) if, for every $q \in W$, there exists a geodesic segment of (M, ∇) all contained in W whose endpoints are p and q. Clearly, for any $p \in M$, the set $\exp_p(\Omega_p)$ and all normal neighborhoods of p are p-star-shaped in (M, ∇).

(d) Let S be a smooth submanifold of M. Suppose that there exists an affine connection ∇^S on S such that $\nabla^S_X Y = \nabla_X Y$ for every pair of tangent vector fields X, Y of S. We will then say that the connection ∇^S is the *connection induced by* ∇ on the submanifold S. Suppose then that ∇ induces the connection ∇^S on S. We denote by R^S the curvature $(1,3)$-tensor of (S, ∇^S), while $\nabla^m R^S$ is the m-th covariant derivative of R^S with respect to ∇^S (for $m \geq 0$). For any $p \in S$ and $v \in T_p(S) \subseteq T_p(M)$, denote by $\gamma^S_{(p,v)}(t)$ the maximal geodesic of (S, ∇^S) satisfying the initial conditions $\gamma^S_{(p,v)}(0) = p$, $\frac{d\gamma^S_{(p,v)}}{dt}(0) = v$, while $\exp^S_p$ denotes the exponential map of (S, ∇^S) at p, and Ω^S_p is the maximal subset of $T_p(S)$ on which the map $\exp^S_p$ is defined. So we have $\exp^S_p(v) = \gamma^S_{(p,v)}(1)$ for every $v \in \Omega^S_p \subset T_p(S)$.

(e) Let M be a real analytic manifold, ∇ be a real analytic affine connection on M, and S be a real analytic submanifold of M. Suppose that ∇ induces the connection ∇^S on S, and let $p \in S$. Clearly, ∇^S is a real analytic connection on S. Moreover, both exponential maps $\exp_p : \Omega_p \to M$ and $\exp^S_p : \Omega^S_p \to S$ are real analytic, so normal coordinates on any normal neighborhood of p are real analytic.

(f) An affine connection ∇ on M is said to be *complete* if each of its geodesics can be extended to a geodesic $\gamma = \gamma(t)$ defined for every $t \in \mathbb{R}$. Clearly, the affine connection ∇ is complete if and only if we have $\Omega_p = T_p(M)$ for every $p \in M$.

(g) Let S be a (smooth) submanifold of (M, ∇). S is said to be *totally geodesic at* $p \in S$ if the maximal geodesic $\gamma_{(p,v)}(t)$ of M is contained in S for small values of $|t|$, for every $v \in T_p(S)$. If S is totally geodesic at every point $p \in S$, then S is called a *totally geodesic submanifold* of (M, ∇). A smooth submanifold S of (M, ∇) is called *auto-parallel* if, for every $p \in S$, $X \in T_p(S)$ and for every curve γ in S starting from p, the parallel displacement of X along γ (with respect to ∇) is tangent to S. In particular, if S is auto-parallel, then ∇ induces, in a natural way, an affine connection ∇^S on S.

The following facts are known from before.

(g1) Every auto-parallel submanifold of (M, ∇) is totally geodesic.

(g2) If ∇ is torsion-free and S is any totally geodesic submanifold of (M, ∇), then S is auto-parallel, and so ∇ induces a natural affine connection ∇^S on S.

(h) We say that (M, ∇) is an *affine locally symmetric space* if for every $p \in M$, there exists a normal neighborhood U_p of p and an affine transformation $\sigma_p : U_p \to U_p$ which transforms each point $\exp_p(v) \in U_p$ into $\exp_p(-v)$. The map σ_p is called *symmetry* at

p. Furthermore, if for every $p \in M$, the symmetry σ_p can be extended to a global affine transformation of (M, ∇), the space (M, ∇) is said to be *affine symmetric*. We also say that a space (M, ∇) (affine locally symmetric or affine symmetric) is *real analytic* when both the manifold M and the connection ∇ are real analytic. It is known that:

(h1) (M, ∇) is an affine locally symmetric space if and only if the affine connection ∇ is torsion-free and $\nabla R = 0$.

(h2) If (M, ∇) is an affine symmetric space, then the connection ∇ is complete.

(h3) If (M, ∇) is an affine locally symmetric space with M simply connected and ∇ complete, then necessarily, (M, ∇) is an affine symmetric space.

From now on, we will always use the notations established before.

In the following proposition, we describe the properties of totally geodesic submanifolds of a smooth manifold endowed with a torsion-free affine connection, also in terms of the curvature operator.

Proposition 1. *Let M be a smooth manifold endowed with a torsion-free affine connection ∇, and let S be a totally geodesic submanifold of (M, ∇) passing through p. Let ∇^S denote the connection induced by ∇ on S.*

(a) We have $\Omega_p^S \subset \Omega_p$, and $\exp_p^S$ agrees with the restriction of the map $\exp_p$ to Ω_p^S.

(b) For every $m \geq 0$ and for every $X_1, X_2, \cdots, X_{m+2}, X_{m+3} \in T_p(S)$, we have $(\nabla^m R)(X_1, X_2, \cdots, X_{m+2}, X_{m+3}) \in T_p(S)$.

(c) Let U be any neighborhood of 0_p in $T_p(S)$ such that $U \subset \Omega_p^S$. Then $N := \exp_p^S(U)$ is a neighborhood of p in S, and we have $U \subset \Omega_p, \quad N = \exp_p(U)$.

(d) Suppose that ∇ is the Levi-Civita connection of any Riemannian metric on M. If the totally geodesic submanifold S is connected and complete, then we have $T_p(S) \subset \Omega_p$ and $S = \exp_p(T_p(S))$.

Proof. (a) Let $v \in \Omega_p^S$, and let (α, β) be the domain of the maximal geodesic $\gamma_{(p,v)}^S$ so that $1 \in (\alpha, \beta)$. Since S is totally geodesic in (M, ∇), the path $\gamma_{(p,v)}^S$ is also a geodesic of (M, ∇), so the domain of the maximal geodesic $\gamma_{(p,v)}$ contains the interval (α, β), and $\gamma_{(p,v)}$ agrees with $\gamma_{(p,v)}^S$ on (α, β). Hence $v \in \Omega_p$ and $\exp_p^S(v) = \gamma_{(p,v)}^S(1) = \gamma_{(p,v)}(1) = exp_p(v)$.

This proves part (a).

(b) Since S is totally geodesic in (M, ∇), we have $\nabla_X^S Y = \nabla_X Y$ for every pair of tangent vector fields X, Y of S. From this, it is easy to get $(\nabla^m R^S)(X_1, X_2, \cdots, X_{m+2}, X_{m+3}) = (\nabla^m R)(X_1, X_2, \cdots, X_{m+2}, X_{m+3})$ for every $m \geq 0$ and every $X_1, \cdots, X_{m+3} \in T_p(S)$. This implies part (b).

(c) Clearly, N contains at least one normal neighborhood of p in (S, ∇^S), so it is a neighborhood of p in S. From (a), we get $U \subset \Omega_p^S \subset \Omega_p$ and $N = \exp_p^S(U) = \exp_p(U)$.

This proves part (c).

(d) Since S is connected and complete, we get $\Omega_p^S = T_p(S)$ and $\exp_p^S(T_p(S)) = S$ (see [13] Thm. 4.2, p. 172). Hence, part (c) implies that $T_p(S) \subset \Omega_p$ and $S = \exp_p(T_p(S))$. Therefore, the proof is complete. □

The following corollary describes the uniqueness of totally geodesic submanifolds under some assumptions.

Corollary 1. *Let p be a common point of two totally geodesic submanifolds S, R of a manifold M endowed with a torsion-free affine connection ∇, and suppose $T_p(S) = T_p(R)$.*

(a) The set $\exp_p(\Omega_p^S \cap \Omega_p^R)$ is a p-star-shaped neighborhood of p in both S and R.

(b) If ∇ is the Levi-Civita connection of any Riemannian metric on M, and both S and R are connected and complete, then we necessarily have $S = R$.

Proof. The set $\Omega_p^S \cap \Omega_p^R$ is open and star-shaped (with respect to 0_p) in $T_p(S) = T_p(R)$, so from Proposition 1 (c), the set $\exp_p(\Omega_p^S \cap \Omega_p^R)$ is a p-star-shaped neighborhood of p in both S and R.

From Proposition 1 (d), we get $T_p(S) = T_p(R) \subset \Omega_p$, and also $S = \exp_p(T_p(S)) = \exp_p(T_p(R)) = R$, from which part (b) follows. $\square$

3. The Cartan Theorem

This section is devoted to the proof of the main theorem.

We state some notations.

Let (M, ∇) be a smooth manifold with a torsion-free affine connection, ∇. Let N_p be any normal neighborhood of the point $p \in M$. We denote by $x^1, \cdots, x^n$ the normal coordinates on N_p induced by a basis $e_1, \cdots, e_n$ of $T_p(M)$ and by $\partial_1 := \dfrac{\partial}{\partial x^1}, \cdots, \partial_n := \dfrac{\partial}{\partial x^n}$ the local frame on the tangent bundle of M induced by these coordinates. It is clear that $x^1(p) = x^2(p) = \cdots = x^n(p) = 0$, and $\partial_{1|_p} = e_1, \cdots, \partial_{n|_p} = e_n$. We denote by Γ_{ij}^h, R_{ijk}^h, $R_{ijk;l_1\cdots l_m}^h$ the components, with respect to the coordinates $x^1, \cdots, x^n$, of the affine connection ∇, of the curvature tensor R, and of the m-th covariant derivative of R ($m \geq 1$), respectively. More explicitly, for every $m \geq 1$ and every $i, j, k, l_1, \cdots, l_m \in \{1, \cdots, n\}$, we set : $\nabla_{\partial_i}\partial_j := \sum_{h=1}^n \Gamma_{ij}^h \partial_h$; $R(\partial_i, \partial_j, \partial_k) := \sum_{h=1}^n R_{ijk}^h \partial_h$; $(\nabla^m R)(\partial_i, \partial_j, \partial_k, \partial_{l_1}, \cdots, \partial_{l_m}) := \sum_{h=1}^n R_{ijk;l_1\cdots l_m}^h \partial_h$.

As is known, for all $t \geq 1$ and $i, j, k, h, l_1, \cdots, l_t \in \{1, \cdots, n\}$, we have the following:

$$R_{ijk}^h = \frac{\partial \Gamma_{ik}^h}{\partial x^j} - \frac{\partial \Gamma_{jk}^h}{\partial x^i} + \sum_{r=1}^n (\Gamma_{ik}^r \Gamma_{jr}^h - \Gamma_{jk}^r \Gamma_{ir}^h); \tag{1}$$

$$\begin{aligned} R_{ijk;l_1\cdots l_t}^h = {} & \frac{\partial R_{ijk;l_1\cdots l_{t-1}}^h}{\partial x^{l_t}} + \sum_{r=1}^n \Gamma_{r\,l_t}^h R_{ijk;l_1\cdots l_{t-1}}^r \\ & - \sum_{r=1}^n (\Gamma_{i\,l_t}^r R_{rjk;l_1\cdots l_{t-1}}^h + \Gamma_{j\,l_t}^r R_{irk;l_1\cdots l_{t-1}}^h + \Gamma_{k\,l_t}^r R_{ijr;l_1\cdots l_{t-1}}^h) \\ & - \sum_{v=1}^{t-1} \sum_{r=1}^n \Gamma_{l_v\,l_t}^r R_{ijk;l_1\cdots \underset{\substack{\uparrow \\ v\text{–}th \\ place}}{r} \cdots l_{t-1}}^h . \end{aligned} \tag{2}$$

In (2), we agree that when $t = 1$, the last sum $\sum_{q=1}^{t-1} \sum_{r=1}^n (\cdots)$ does not appear, and the term $R_{ijk;l_1\cdots l_{t-1}}^h$ is equal to R_{ijk}^h, for all indices i, j, k, h.

Subsequently, it will be convenient to set $\Gamma_{ij,l_1\cdots l_m}^h := \dfrac{\partial^m \Gamma_{ij}^h}{\partial x^{l_1} \cdots \partial x^{l_m}}$ (where $m \geq 1$), whatever the indices $i, j, h, l_1, \cdots, l_m \in \{1, \cdots, n\}$ are.

As above in (2), we agree that when $m = 0$, any term of the form $\Gamma_{ij,l_1\cdots l_m}^h$ reduces to Γ_{ij}^h, for all indices i, j, h. From now on, we will always use this agreement on the coefficients $R_{ijk;l_1\cdots l_t}^h$ and $\Gamma_{ij,l_1\cdots l_m}^h$.

The following lemma is a useful tool for simplifying computations in the proof of the main theorem.

Lemma 1. *If $x^1, \cdots, x^n$ are normal coordinates on a normal neighborhood of p in (M, ∇) and the affine connection ∇ is torsion-free, then we have*

(a) $\Gamma_{ij}^h(0, \cdots, 0) = 0$, *for every* $i, j, h \in \{1, \cdots, n\}$;

(b) $\sum_{i,j,l_1,\cdots,l_m=1}^{n} \Gamma^h_{ij,l_1\cdots l_m}(0,\cdots,0)\, v^i v^j v^{l_1}\cdots v^{l_m} = 0$, *for every* $(v^1,\cdots,v^n)\in\mathbb{R}^n$, *where* $h\in\{1,\cdots,n\}$ *and* $m\geq 1$.

Proof. If we fix $(v^1,\cdots,v^n)\in\mathbb{R}^n$, the curve in (M,∇) which, in the coordinates $x^1,\cdots,x^n$, is expressed as $x^1=tv^1,\cdots,x^n=tv^n$ is a geodesic for $|t|$ small enough, i.e., the equations $\frac{d^2x^h}{dt^2}+\sum_{i,j=1}^{n}\Gamma^h_{ij}(tv^1,\cdots,tv^n)\frac{dx^i}{dt}\frac{dx^j}{dt}=\sum_{i,j=1}^{n}\Gamma^h_{ij}(tv^1,\cdots,tv^n)\,v^iv^j=0$ are satisfied, for every $h=1,\cdots,n$ and for every $|t|$ small enough. Since ∇ is torsion-free, we have $\Gamma^h_{ij}=\Gamma^h_{ji}$. Hence, for $t=0$, we get $2\sum_{1\leq i<j\leq n}\Gamma^h_{ij}(0,\cdots,0)\,v^iv^j+\sum_{i=1}^{n}\Gamma^h_{ii}(0,\cdots,0)(v^i)^2=0$, for $h=1,\cdots,n$. Since these equations hold for every $(v^1,\cdots,v^n)\in\mathbb{R}^n$, from the identity theorem for polynomials, we obtain (a).

If we differentiate m times with respect to t the identity $\sum_{i,j=1}^{n}\Gamma^h_{ij}(tv^1,\cdots,tv^n)\,v^iv^j=0$ and evaluate the result for t=0, we obtain (b). □

The following theorem is essentially attributed to Elie Cartan, in the case where ∇ is the Levi-Civita connection of a Riemannian metric on M [1].

Theorem 1. *Let* (M,∇) *be a real analytic manifold with a torsion-free real analytic affine connection. Let* $p\in M$ *and* V *be a vector subspace of* $T_p(M)$ *such that, for any* $m\geq 0$ *and any* $X_1,X_2,\cdots,,X_{m+3}\in V$, *we have* $(\nabla^m R)(X_1,X_2,\cdots,X_{m+3})\in V$. *Consider any normal neighborhood* $N_p=\exp_p(\widetilde{N}_p)$ *of* p *in* (M,∇); *then* $\exp_p(\widetilde{N}_p\cap V)$ *is a totally geodesic real analytic submanifold of* (M,∇) *that passes through* p *and has* V *as its tangent space at* p.

Proof. Let $x^1,\cdots,x^n$ be V-adapted real analytic normal coordinates on N_p. So, we have $\exp_p(\widetilde{N}_p\cap V)=\{q\in N_p : x^{s+1}(q)=\cdots=x^n(q)=0\}$, where $1\leq s=\dim V<n=\dim M$. We now denote by A the subset of $\mathbb{R}^n$ defined as follows: $A:=\{\underline{x}:=(x^1,\cdots,x^s,\underbrace{0,\cdots,0}_{n-s}) : x^1,\cdots,x^s,\underbrace{0,\cdots,0}_{n-s}$ are coordinates of some $q\in N_p\}$.

We also set $\underline{0}:=(\underbrace{0,\cdots,0}_{n})$. As usual, let Γ^h_{ij}, R^h_{ijk}, $R^h_{ijk;l_1\cdots l_m}$ denote, respectively, the components of ∇, R and $\nabla^m R$ $(m\geq 1)$ with respect to the coordinates $x^1,\cdots,x^n$. Using the V-adapted normal coordinates $x^1,\cdots,x^n$, the assumptions about R and its covariant derivatives at the point p can be expressed in the following form:

(α_1) $R^h_{ijk}(\underline{0})=0$ and $R^h_{ijk;l_1\cdots l_m}(\underline{0})=0$, for every $m\geq 1$, every $h\in\{s+1,\cdots,n\}$, and every $i,j,k,l_1,\cdots,l_m\in\{1,\cdots,s\}$.

Recalling Lemma 1 (a) and equalities (1) and (2), the conditions (α_1) are equivalent to

(α_2) $\Gamma^h_{ik,j}(\underline{0})=\Gamma^h_{jk,i}(\underline{0})$, $\frac{\partial R^h_{ijk}}{\partial x^{l_1}}(\underline{0})=0$, and $\frac{\partial R^h_{ijk;l_1\cdots l_{m-1}}}{\partial x^{l_m}}(\underline{0})=0$, for every $m\geq 2$, every $i,j,k,l_1,\cdots,l_m\in\{1,\cdots,s\}$, and every $h\in\{s+1,\cdots,n\}$.

To prove the theorem, it is sufficient to prove that $\Gamma^h_{ij}(\underline{x})=0$, for every $\underline{x}\in A$, every $i,j\in\{1,\cdots,s\}$, and every $h\in\{s+1,\cdots,n\}$. By Lemma 1 (a), we have $\Gamma^h_{ij}(\underline{0})=0$ for every $i,j,h\in\{1,\cdots,n\}$. Since M and ∇ are real analytic, all functions Γ^h_{ij} are real analytic (with respect to $x^1,\cdots,x^s$) on the connected set A. Hence, it suffices to prove the following:

(α_3) $\Gamma^h_{ij,l_1\cdots l_m}(\underline{0})=0$, for every $m\geq 1$, every $i,j,l_1,\cdots,l_m\in\{1,\cdots,s\}$, and every $h\in\{s+1,\cdots,n\}$.

We will prove the conditions (α_3) in several steps.

(*Step i*) Fix an integer $m\geq 2$. Suppose the following two conditions hold:

$$\Gamma^c_{ab,\alpha_1}(\underline{0})=\Gamma^c_{ab,\alpha_1\alpha_2}(\underline{0})=\cdots=\Gamma^c_{ab,\alpha_1\cdots\alpha_m}(\underline{0})=0 \tag{3}$$

for every $a, b, \alpha_1, \cdots, \alpha_m, \in \{1, \cdots, s\}$ and every $c \in \{s+1, \cdots, n\}$;

$$\frac{\partial^d R^{\,r}_{abc\,;\,u_1 \cdots u_z}}{\partial x^{w_1} \cdots \partial x^{w_d}}\,(\underline{0}) = 0, \tag{4}$$

for every $1 \leq d \leq m-1$, every $z \geq 0$, every $r \in \{s+1, \cdots, n\}$, and every $a, b, c, u_1, \cdots, , u_z, w_1, \cdots, , w_d \in \{1, \cdots, s\}$.

Fix any integer $t \geq 1$, any $i, j, k, l_1, \cdots, l_{t-1}, l_t, \zeta_1, \cdots \zeta_m \in \{1, \cdots, s\}$, and any $h \in \{s+1, \cdots, n\}$. Then we assert that we have the following:

$$\frac{\partial^{m+1} R^{\,h}_{ijk\,;\,l_1 \cdots l_{t-1}}}{\partial x^{l_t} \partial x^{\zeta_1} \cdots \partial x^{\zeta_m}}\,(\underline{0}) = \frac{\partial^m R^{\,h}_{ijk\,;\,l_1 \cdots l_t}}{\partial x^{\zeta_1} \cdots \partial x^{\zeta_m}}\,(\underline{0}) = 0. \tag{5}$$

Now, we will prove equality (5). Let q be an integer such that $1 \leq q \leq m$.

Differentiating equality (2), with the same agreements used in (2), we obtain the following:

$$\begin{aligned}
\frac{\partial^q R^{\,h}_{ijk\,;\,l_1 \cdots l_t}}{\partial x^{\zeta_1} \cdots \partial x^{\zeta_q}} &= \frac{\partial^{q+1} R^{\,h}_{ijk\,;\,l_1 \cdots l_{t-1}}}{\partial x^{l_t} \partial x^{\zeta_1} \cdots \partial x^{\zeta_q}} + \sum_{r=1}^{n} \frac{\partial^q (\Gamma^{\,h}_{r\,l_t} R^{\,r}_{ijk\,;\,l_1 \cdots l_{t-1}})}{\partial x^{\zeta_1} \cdots \partial x^{\zeta_q}} \\
-\sum_{r=1}^{n} \Bigg[\frac{\partial^q (\Gamma^{\,r}_{i\,l_t} R^{\,h}_{rjk\,;\,l_1 \cdots l_{t-1}})}{\partial x^{\zeta_1} \cdots \partial x^{\zeta_q}} &+ \frac{\partial^q (\Gamma^{\,r}_{j\,l_t} R^{\,h}_{irk\,;\,l_1 \cdots l_{t-1}})}{\partial x^{\zeta_1} \cdots \partial x^{\zeta_q}} + \frac{\partial^q (\Gamma^{\,r}_{k\,l_t} R^{\,h}_{ijr\,;\,l_1 \cdots l_{t-1}})}{\partial x^{\zeta_1} \cdots \partial x^{\zeta_q}} \Bigg] \\
&- \sum_{v=1}^{t-1} \sum_{r=1}^{n} \frac{\partial^q (\Gamma^{\,r}_{l_v\,l_t} R^{\,h}_{ijk\,;\,l_1 \cdots\, r\, \cdots l_{t-1}})}{\partial x^{\zeta_1} \cdots \partial x^{\zeta_q}}.
\end{aligned} \tag{6}$$

Any partial derivative $\dfrac{\partial^q (\Gamma^{\,h}_{r\,l_t} R^{\,r}_{ijk\,;\,l_1 \cdots l_{t-1}})}{\partial x^{\zeta_1} \cdots \partial x^{\zeta_q}}\,(\underline{0})$ is a linear combination of terms of the form $D_\beta(\Gamma^{\,h}_{r\,l_t})\,(\underline{0}) \cdot D_{q-\beta}(R^{\,r}_{ijk\,;\,l_1 \cdots l_{t-1}})\,(\underline{0})$ with $0 \leq \beta \leq q$, where D_β and $D_{q-\beta}$ are suitable partial derivatives of order β and $q-\beta$, respectively (with the agreement that $D_0(f) = f$, for any function f).

Assume $1 \leq r \leq s$. From Lemma 1 (a) and condition (1), we obtain $D_\beta(\Gamma^{\,h}_{r\,l_t})\,(\underline{0}) = 0$, for every $0 \leq \beta \leq m$, Thus, since $q \leq m$, we conclude that

$$\frac{\partial^q (\Gamma^{\,h}_{r\,l_t} R^{\,r}_{ijk\,;\,l_1 \cdots l_{t-1}})}{\partial x^{\zeta_1} \cdots \partial x^{\zeta_q}}\,(\underline{0}) = 0, \tag{7}$$

for every $r \in \{1, \cdots, s\}$.

Similarly, the following equalities are also obtained:

$$\begin{aligned}
\frac{\partial^q (\Gamma^{\,r}_{i\,l_t} R^{\,h}_{rjk\,;\,l_1 \cdots l_{t-1}})}{\partial x^{\zeta_1} \cdots \partial x^{\zeta_q}}\,(\underline{0}) &= \frac{\partial^q (\Gamma^{\,r}_{j\,l_t} R^{\,h}_{rjk\,;\,l_1 \cdots l_{t-1}})}{\partial x^{\zeta_1} \cdots \partial x^{\zeta_q}}\,(\underline{0}) \\
= \frac{\partial^q (\Gamma^{\,r}_{k\,l_t} R^{\,h}_{ijr\,;\,l_1 \cdots l_{t-1}})}{\partial x^{\zeta_1} \cdots \partial x^{\zeta_q}}\,(\underline{0}) &= \frac{\partial^q (\Gamma^{\,r}_{l_v\,l_t} R^{\,h}_{ijk\,;\,l_1 \cdots\, r\, \cdots l_{t-1}})}{\partial x^{\zeta_1} \cdots \partial x^{\zeta_q}}\,(\underline{0}) = 0,
\end{aligned} \tag{8}$$

for every $r \in \{s+1, \cdots, n\}$.

Now assume $s+1 \leq r \leq n$.

Since $q - \beta \leq m-1$, from (α_1) and condition (4), we get $D_{q-\beta}(R^{\,r}_{ijk\,;\,l_1 \cdots l_{t-1}})\,(\underline{0}) = 0$, for every $1 \leq \beta \leq q$, while when $\beta = 0$, we get $D_\beta(\Gamma^{\,h}_{r\,l_t})\,(\underline{0}) = 0$, from Lemma 1 (a).

In any case, we have $D_\beta(\Gamma^{\,h}_{r\,l_t})\,(\underline{0}) \cdot D_{q-\beta}(R^{\,r}_{ijk\,;\,l_1 \cdots l_{t-1}})\,(\underline{0}) = 0$, for every $0 \leq \beta \leq q$.

So we conclude that

$$\frac{\partial^q (\Gamma^{\,h}_{r\,l_t} R^{\,r}_{ijk\,;\,l_1 \cdots l_{t-1}})}{\partial x^{\zeta_1} \cdots \partial x^{\zeta_q}}\,(\underline{0}) = 0, \tag{9}$$

for every $r \in \{1, \cdots, s\}$. Similarly, we obtain the following:

$$\frac{\partial^q (\Gamma^r_{i\, l_t} R^h_{rjk;l_1\cdots l_{t-1}})}{\partial x^{\zeta_1} \cdots \partial x^{\zeta_q}}(\underline{0}) = \frac{\partial^q (\Gamma^r_{j\, l_t} R^h_{rjk;l_1\cdots l_{t-1}})}{\partial x^{\zeta_1} \cdots \partial x^{\zeta_q}}(\underline{0}) \tag{10}$$
$$= \frac{\partial^q (\Gamma^r_{k\, l_t} R^h_{ijr;l_1\cdots l_{t-1}})}{\partial x^{\zeta_1} \cdots \partial x^{\zeta_q}}(\underline{0}) = \frac{\partial^q (\Gamma^r_{l_v\, l_t} R^h_{ijk;l_1\cdots\, r\, \cdots l_{t-1}})}{\partial x^{\zeta_1} \cdots \partial x^{\zeta_q}}(\underline{0}) = 0,$$

for every $r \in \{s+1, \cdots, n\}$. Therefore we conclude

$$\frac{\partial^{q+1} R^h_{ijk;l_1\cdots l_{t-1}}}{\partial x^{l_t} \partial x^{\zeta_1} \cdots \partial x^{\zeta_q}}(\underline{0}) = \frac{\partial^q R^h_{ijk;l_1\cdots l_t}}{\partial x^{\zeta_1} \cdots \partial x^{\zeta_q}}(\underline{0}), \tag{11}$$

for any integer q such that $1 \le q \le m$.

In particular, we have the following:

$$\frac{\partial^{m+1} R^h_{ijk;l_1\cdots l_{t-1}}}{\partial x^{l_t} \partial x^{\zeta_1} \cdots \partial x^{\zeta_m}}(\underline{0}) = \frac{\partial^m R^h_{ijk;l_1\cdots l_t}}{\partial x^{\zeta_1} \cdots \partial x^{\zeta_m}}(\underline{0}). \tag{12}$$

If $m = 1$, from (α_2) we obtain (5).

Note that (11) holds for any integer $t \ge 1$, any $i, j, k, l_1, \cdots, l_{t-1}, l_t, \zeta_1, \cdots \zeta_m \in \{1, \cdots, s\}$, and any $h \in \{s+1, \cdots, n\}$.

Therefore, if $m \ge 2$, from (11) (with $q = m-1 \ge 1$), and from condition (4), we deduce the following:

$$\frac{\partial^m R^h_{ijk;l_1\cdots l_t}}{\partial x^{\zeta_1} \cdots \partial x^{\zeta_m}}(\underline{0}) = \frac{\partial^{m-1} R^h_{ijk;l_1\cdots l_t \zeta_1}}{\partial x^{\zeta_2} \cdots \partial x^{\zeta_m}}(\underline{0}) = 0; \tag{13}$$

Thus, (12) and (13) imply that condition (5) also holds for $m \ge 2$.

(*Step ii*) Fix an integer $m \ge 0$. When $m \ge 2$, we assume that the following two conditions hold:

$$\frac{\partial^m R^h_{ijk}}{\partial x^{\zeta_1} \cdots \partial x^{\zeta_m}}(\underline{0}) = 0, \tag{14}$$

for every $i, j, k, \zeta_1, \cdots \zeta_m \in \{1, \cdots, s\}$, and every $h \in \{s+1, \cdots, n\}$;

$$\Gamma^c_{ab,\alpha_1}(\underline{0}) = \Gamma^c_{ab,\alpha_1\alpha_2}(\underline{0}) = \cdots = \Gamma^c_{ab,\alpha_1\cdots\alpha_{m-1}}(\underline{0}) = 0, \tag{15}$$

for every $a, b, \alpha_1, \cdots, \alpha_{m-1} \in \{1, \cdots, s\}$ and every $c \in \{s+1, \cdots, n\}$.

Then we assert that, for any $m \ge 0$, we have $\Gamma^h_{ik,j\beta_1\cdots\beta_m}(\underline{0}) = \Gamma^h_{jk,i\beta_1\cdots\beta_m}(\underline{0})$, for every $i, j, k, \beta_1, \cdots \beta_m \in \{1, \cdots, s\}$, and every $h \in \{s+1, \cdots, n\}$.

We now prove this statement. When $m = 0$, it follows from (α_1) and Lemma 1 (a), keeping in mind equality (1).

Now assume $m \ge 1$. Differentiating equality (1), we get the following:

$$\frac{\partial^m R^h_{ijk}}{\partial x^{\beta_1} \cdots \partial x^{\beta_m}}(\underline{0}) = \Gamma^h_{ik,j\beta_1\cdots\beta_m}(\underline{0}) - \Gamma^h_{jk,i\beta_1\cdots\beta_m}(\underline{0}) + \tag{16}$$
$$\sum_{r=1}^n \left[\frac{\partial^m (\Gamma^r_{ik}\Gamma^h_{jr})}{\partial x^{\beta_1} \cdots \partial x^{\beta_m}}(\underline{0}) - \frac{\partial^m (\Gamma^r_{jk}\Gamma^h_{ir})}{\partial x^{\beta_1} \cdots \partial x^{\beta_m}}(\underline{0}) \right].$$

When $m = 1$, equality (16) becomes the following:

$$\frac{\partial R^{\,h}_{ijk}}{\partial x^{\beta_1}}\,(\underline{0}) \;=\; \Gamma^{\,h}_{ik\,,\,j\,\beta_1}\,(\underline{0}) \;-\; \Gamma^{\,h}_{jk\,,\,i\,\beta_1}\,(\underline{0}) \,+ \tag{17}$$

$$\sum_{r=1}^{n}\big(\,\Gamma^{\,r}_{ik\,,\,\beta_1}\,(\underline{0})\,\Gamma^{\,h}_{jr}\,(\underline{0}) + \Gamma^{\,r}_{ik}\,(\underline{0})\,\Gamma^{\,h}_{jr\,,\,\beta_1}(\underline{0}) - \Gamma^{\,r}_{jk\,,\,\beta_1}\,(\underline{0})\,\Gamma^{\,h}_{ir}\,(\underline{0}) - \Gamma^{\,r}_{jk}\,(\underline{0})\,\Gamma^{\,h}_{ir\,,\,\beta_1}(\underline{0})\,\big).$$

Therefore the statement follows from (α_2) and Lemma 1 (a).

Finally, let $m \geq 2$. Consider equality (16) again.

Any partial derivative $\dfrac{\partial^{\,m}(\Gamma^{\,r}_{ik}\Gamma^{\,h}_{jr})}{\partial x^{\beta_1}\cdots\partial x^{\beta_m}}\,(\underline{0})$ is a linear combination of terms of the form $D_\beta(\Gamma^{\,r}_{ik})\,(\underline{0})\cdot D_{m-\beta}(\Gamma^{\,h}_{jr})\,(\underline{0})$, with $0 \leq \beta \leq m$. As in (*Step i*), and with the same agreements, D_β and $D_{m-\beta}$ are suitable partial derivatives of order β and $m-\beta$, respectively. We have $D_\beta(\Gamma^{\,r}_{ik})\,(\underline{0}) = 0$ for $\beta = 0$, and $D_{m-\beta}(\Gamma^{\,h}_{jr})\,(\underline{0}) = 0$ for $\beta = m$, both by Lemma 1 (a). Meanwhile, for $1 \leq \beta \leq m-1$, by condition (3), we have $D_\beta(\Gamma^{\,r}_{ik})\,(\underline{0}) = 0$ with $r \in \{s+1,\cdots,n\}$, and $D_{m-\beta}(\Gamma^{\,h}_{jr}) = 0$ with $r \in \{1,\cdots,s\}$. Therefore, for every integer $\beta \in [\,0,\,m\,]$, we have $D_\beta(\Gamma^{\,r}_{ik})\,(\underline{0})\cdot D_{m-\beta}(\Gamma^{\,h}_{jr})\,(\underline{0}) = 0$, and we obtain the following:

$$\frac{\partial^{\,m}(\Gamma^{\,r}_{ik}\Gamma^{\,h}_{jr})}{\partial x^{\beta_1}\cdots\partial x^{\beta_m}}\,(\underline{0}) = 0, \tag{18}$$

for every $r \in \{1,\cdots,n\}$, and similarly,

$$\frac{\partial^{\,m}(\Gamma^{\,r}_{jk}\Gamma^{\,h}_{ir})}{\partial x^{\beta_1}\cdots\partial x^{\beta_m}}\,(\underline{0}) = 0, \tag{19}$$

for every $r \in \{1,\cdots,n\}$. Therefore, from condition (14), we obtain the statement for $m \geq 2$.

(*Step iii*) Let $m \geq 0$. Assume that the following conditions hold:

$$\Gamma^{\,h}_{ik\,,\,j\,\beta_1\cdots\,\beta_m}(\underline{0}) = \Gamma^{\,h}_{jk\,,\,i\,\beta_1\cdots\,\beta_m}(\underline{0}), \tag{20}$$

for every $i,j,k,\beta_1,\cdots\beta_m \in \{1,\cdots,s\}$, and every $h \in \{s+1,\cdots,n\}$.

Then necessarily, $\Gamma^{\,h}_{ij\,,\,k\,\beta_1\cdots\,\beta_m}(\underline{0}) = 0$, for every $i,j,k,\beta_1,\cdots\beta_m \in \{1,\cdots,s\}$, and every $h \in \{s+1,\cdots,n\}$.

First of all, we note that, after exchanging m with $m+1$ and after setting $v^{s+1} = v^{s+2} = \cdots = v^n = 0$, the conditions of part (b) of Lemma 1 become the following:

$$\sum_{i,j,l_1,\cdots,l_{m+1}=1}^{s}\Gamma^{\,h}_{ij\,,\,l_1\cdots\,l_{m+1}}(\underline{0})\;v^iv^jv^{l_1}\cdots v^{l_{m+1}} = 0, \tag{21}$$

for every $(v^1,\cdots,v^s) \in \mathbb{R}^s$, for every $h \in \{1,\cdots,n\}$.

Clearly, the coefficients $\Gamma^{\,h}_{ij\,,\,l_1\cdots\,l_{m+1}}(\underline{0})$ are symmetric with respect to the indices i,j and with respect to the indices $l_1,\cdots,l_{m+1}$ (for every fixed $h \in \{1,\cdots,n\}$). Since condition (20) holds, we deduce that, for every fixed $h \in \{s+1,\cdots,n\}$, the coefficients $\Gamma^{\,h}_{ij\,,\,l_1\cdots\,l_{m+1}}(\underline{0})$ are symmetric with respect to all indices $i,j,l_1,\cdots,l_m$. Therefore, from (21) and from the identity theorem for polynomials, we obtain $\Gamma^{\,h}_{ij\,,\,l_1\cdots\,l_{m+1}}(\underline{0}) = 0$, for every $i,j,l_1,\cdots l_{m+1} \in \{1,\cdots,s\}$, and every $h \in \{s+1,\cdots,n\}$. That is what we wanted to prove.

(*Step iv*) We assert that, for every integer $m \geq 1$, the following two conditions hold:

$$\Gamma^{\,c}_{ab\,,\,\alpha_1}(\underline{0}) = \Gamma^{\,c}_{ab\,,\,\alpha_1\alpha_2}(\underline{0}) = \cdots = \Gamma^{\,c}_{ab\,,\,\alpha_1\cdots\,\alpha_m}(\underline{0}) = 0, \tag{22}$$

for every $a, b, \alpha_1, \cdots, \alpha_m, \in \{1, \cdots, s\}$, and every $c \in \{s+1, \cdots, n\}$; when $m \geq 2$, we have

$$\frac{\partial^d R^r_{abc;u_1\cdots u_z}}{\partial x^{w_1}\cdots\partial x^{w_d}}(\underline{0}) = 0, \tag{23}$$

for every $1 \leq d \leq m-1$, every $z \geq 0$, every $r \in \{s+1, \cdots, n\}$, and every $a, b, c, u_1, \cdots, , u_z, w_1, \cdots, , w_d \in \{1, \cdots, s\}$.

If $m = 1$, the above conditions follow from (*Step ii*) and (*Step iii*) (both for $m = 0$).

If $m = 2$, we obtain the same condition again from (*Step ii*) and (*Step iii*) (both for $m = 1$) and from (α_2).

We will now prove the statement by induction on the integer $m \geq 2$.

Therefore, assuming that conditions (1) and (2) are true for $m \geq 2$, we will prove that they are also true for the next integer $m+1$.

In fact, from (*Step i*), conditions (22) and (23) imply

$$\frac{\partial^{m+1} R^h_{ijk;l_1\cdots l_{t-1}}}{\partial x^{l_t}\partial x^{\zeta_1}\cdots\partial x^{\zeta_m}}(\underline{0}) = \frac{\partial^m R^h_{ijk;l_1\cdots l_t}}{\partial x^{\zeta_1}\cdots\partial x^{\zeta_m}}(\underline{0}) = 0, \tag{24}$$

for any integer $t \geq 1$, any $i, j, k, l_1, \cdots, l_{t-1}, l_t, \zeta_1, \cdots \zeta_m \in \{1, \cdots, s\}$, and any $h \in \{s+1, \cdots, n\}$. In particular, condition (23) also holds for the integer $m+1$. Furthermore, from (*Step ii*) and (*Step iii*), it can be deduced that, for the integer $m+1$, condition (22) also holds. Therefore, the statement in (*Step iv*) is fully proven.

Clearly, (*Step iv*) implies condition (20), and as already noted, this is sufficient to prove Theorem 1. □

The following proposition describes totally geodesic submanifolds of a real analytic affine locally symmetric space, with a given tangent space at a point, under assumptions on the curvature operator.

Proposition 2. *Let (M, ∇) be a real analytic affine locally symmetric space. Let p be a point of M, let $N_p = \exp_p(\widetilde{N}_p)$ be a normal neighborhood of p in (M, ∇), and let V be a vector subspace of $T_p(M)$ such that we have $R(X, Y, X) \in V$ for every $X, Y \in V$. Then $\exp_p(\widetilde{N}_p \cap V)$ is a totally geodesic real analytic submanifold of (M, ∇) that passes through p and has V as its tangent space at p.*

Proof. From (h1) in the Preliminaries, the affine connection ∇ is torsion-free, and $\nabla R = 0$. Therefore, by Theorem 1, it is sufficient to prove that we have $R(X, Y, Z) \in V$ for every $X, Y, Z \in V$. Let X, Y, Z be arbitrary vectors of V. From the hypotheses, we obtain $R(X + Z, Y, X + Z) = R(Z, Y, Z) + R(X, Y, Z) + R(Z, Y, X) + R(X, Y, X) \in V$ or equivalently the following (always keeping the hypotheses in mind):

$$R(X, Y, Z) + R(Z, Y, X) \in V. \tag{25}$$

From Bianchi's first identity, we have $R(Z, Y, X) = -R(X, Z, Y) - R(Y, X, Z)$, so we get

$$R(Z, Y, X) = -R(X, Z, Y) + R(X, Y, Z). \tag{26}$$

Consequently, from (25) and (26), we obtain the following:

$$2\, R(X, Y, Z) - R(X, Z, Y) \in V, \tag{27}$$

or equivalently,

$$4\, R(X, Y, Z) - 2\, R(X, Z, Y) \in V. \tag{28}$$

By exchanging the vectors Y and Z in (27), we get the following:

$$2\,R(X,Z,Y) - R(X,Y,Z) \in V. \tag{29}$$

Finally, adding the terms of (28) and (29), we get $3\,R(X,Y,Z) \in V$ or equivalently $R(X,Y,Z) \in V$. This concludes the proof. □

4. Some Consequences

In this section, we describe some applications of previous results.

The following lemma describes the local properties of the set of fixed points of affine transformations of a real analytic manifold endowed with a torsion-free real analytic connection.

Lemma 2. *Let (M,∇) be a real analytic manifold with a torsion-free real analytic affine connection. Consider an affine transformation f of (M,∇) and any fixed point p of f. Then there exists a normal neighborhood $N_p = \exp_p(\widetilde{N}_p)$ of p in (M,∇) such that we have $N_p \cap Fix(f) = \exp_p(\widetilde{N}_p \cap E_p(1))$, where $E_p(1)$ is the eigenspace relative to the eigenvalue 1 of the differential map $f_* : T_p(M) \to T_p(M)$.*

Proof. Choose a normal neighborhood $N_p = \exp_p(\widetilde{N}_p)$ of p in (M,∇) such that there exists a neighborhood N_0 of the zero vector of $T_p(M)$ satisfying the following conditions:

(i) $f_*(\widetilde{N}_p)$ and $\widetilde{N}_p$ are both contained in N_0;

(ii) $\exp_p(N_0)$ is a normal neighborhood of p in (M,∇).

If $q \in N_p \cap Fix(f)$, then $f(q) = q$, and $q = \exp_p(v)$ where, $v \in \widetilde{N}_p$. Since f is an affine transformation, we have $\exp_p(f_*(v)) = f(\exp_p(v)) = f(q) = q = \exp_p(v)$. From (i), both vectors v and $f_*(v)$ belong to N_0, while from (ii), the restriction of the map $\exp_p$ to N_0 is injective. Consequently we have $f_*(v) = v$, and so $q \in \exp_p(\widetilde{N}_p \cap E_p(1))$.

Conversely, if $q = \exp_p(v)$ with $v \in \widetilde{N}_p \cap E_p(1)$, then we have $f(q) = f(\exp_p(v)) = \exp_p(f_*(v)) = \exp_p(v) = q$. Finally $q \in \exp(\widetilde{N}_p) \cap Fix(f) = N_p \cap Fix(f)$, and the proof is complete. □

The following proposition describes the global properties of the set of fixed points of an affine transformation of a real analytic manifold endowed with a torsion-free real analytic connection. In particular, it provides examples of totally geodesic submanifolds.

For the analogous result in the particular case of the Levi-Civita connection of a Riemannian manifold, see [6] [Lemma 2.1]. Furthermore, a complete description of the set of fixed points of all isometries of the Riemannian manifold $(\mathcal{P}_n, g)$ of symmetric positive definite real matrices of order n (endowed with the trace metric g) can be found in [15,16].

Proposition 3. *Let (M,∇) be a real analytic manifold with a torsion-free real analytic affine connection, and let f be an affine transformation of (M,∇). Then every connected component S of $Fix(f)$ is a totally geodesic real analytic submanifold of (M,∇). Furthermore, if the affine connection ∇ is complete, then the induced affine connection ∇^S is also complete.*

Proof. Let S be a connected component of $Fix(f)$, and let $p \in S$. From Lemma 2, there is a normal neighborhood $N_p = \exp_p(\widetilde{N}_p)$ of p in (M,∇) such that $N_p \cap Fix(f) = \exp_p(\widetilde{N}_p \cap E_p(1))$, where $E_p(1) = \{u \in T_p(M) : f_*(u) = u\}$. In particular, $N_p \cap Fix(f)$ is connected. Since S is a connected component of $Fix(f)$, we get $N_p \cap S = \exp_p(\widetilde{N}_p \cap E_p(1))$. This implies that $N_p \cap S$ is a real analytic submanifold of M whose dimension is equal to the dimension of the eigenspace $E_p(1)$. The arbitrariness of $p \in S$ and the connectedness of S imply that the dimension of $E_p(1)$ is independent of $p \in S$. We conclude that S is a real analytic submanifold of M whose dimension is equal to the dimension of $E_p(1)$ (where p is an arbitrary point of S). Moreover, we have $T_p(S) = E_p(1)$. Now let $m \geq 0$ and $X_1, X_2, \cdots, X_{m+3} \in T_p(S) = E_p(1)$. Since f is an affine transformation, we have $f_*\big((\nabla^m R)(X_1, X_2, \cdots, X_{m+3})\big) = (\nabla^m R)(f_*(X_1), f_*(X_2), \cdots, f_*(X_{m+3})) = (\nabla^m R)(X_1, X_2, \cdots, X_{m+3})$. Therefore, $(\nabla^m R)(X_1, X_2, \cdots, X_{m+3}) \in E_p(1) = T_p(S)$. From Theorem 1, we conclude that $N_p \cap S = \exp_p(\widetilde{N}_p \cap E_p(1))$ is a totally geodesic real analytic submanifold of (M, ∇). Therefore S is totally geodesic at every point $p \in S$, and so the real analytic submanifold S is totally geodesic in (M, ∇).

Now suppose that ∇ is complete, so $T_p(S) = E_p(1) \subseteq T_p(M) = \Omega_p$, i.e., the map $\exp_p$ is defined at every point of $E_p(1)$. Since $f(p) = p$ and $f \circ \exp_p = \exp_p \circ f_*$, we deduce that $\exp_p\big(E_p(1)\big)$ is a connected subset of $Fix(f)$ containing p. So $\exp_p\big(E_p(1)\big) = \exp_p\big(T_p(S)\big) \subseteq S$. Hence, for every $v \in T_p(S)$ and every $t \in \mathbb{R}$, we have $\exp_p(tv) = \gamma_{(p,v)}(t) \in S$. Consequently, $\gamma_{(p,v)}(t)$ is also a geodesic of (S, ∇^S) which is defined for all $t \in \mathbb{R}$, and so $T_p(S) = \Omega^S_p$, i.e., $\exp^S_p$ is defined on the entire tangent space $T_p(S) = E_p(1)$. Since p is an arbitrary point of S, it follows that the connection ∇^S is complete. □

The following proposition provides local totally geodesic real analytic submanifolds of a real analytic affine locally symmetric space, with a tangent space at a fixed point of an affine transformation f equal to the eigenspace of the -1 eigenvalue of f_*.

Proposition 4. *Let (M, ∇) be a real analytic affine locally symmetric space, f be an affine transformation of (M, ∇), p be a fixed point of f, and $N_p = \exp_p(\widetilde{N}_p)$ be a normal neighborhood of p in (M, ∇). Let $E_p(-1) := \{v \in T_p(M) : f_*(v) = -v\}$. Then $\exp_p\big(\widetilde{N}_p \cap E_p(-1)\big)$ is a totally geodesic real analytic submanifold of (M, ∇) passing through p and with $E_p(-1)$ as its tangent space at p.*

Proof. By Proposition 2, we need to prove that we have $R(u, v, u) \in E_p(-1)$ for every $u, v \in E_p(-1)$. Let $u, v \in E_p(-1)$, so $f_*(u) = -u$ and $f_*(v) = -v$. Since f is an affine transformation, we have $f_*\big(R(u, v, u)\big) = R\big(f_*(u), f_*(v), f_*(u)\big) = R(-u, -v, -u) = -R(u, v, u)$. Therefore $R(u, v, u) \in E_p(-1)$, and so the proposition is proven. □

The results given above can be applied within Lie algebra, and it follows in the next remark.

Remark 1. *Let G be a connected smooth Lie group, with identity e and Lie algebra $\mathfrak{g} \cong T_e(G)$. Denote by ∇^0 the classical $0-$connection of Cartan–Schouten on G which is defined on left-invariant vector fields X, Y by the following formula:*

$\nabla^0_X Y = \frac{1}{2}[X, Y]$. *The following facts are well known.*

(a) (G, ∇^0) is an affine symmetric space; left and right translations and the inverse map $J : g \mapsto g^{-1}$ are affine transformations of (G, ∇^0), while the curvature R^0 of ∇^0 is given at the identity e by the formula $R^0(u, v, w) = \frac{1}{4}\big[[u, v], w\big]$ for every $u, v, w \in T_e(G) \cong \mathfrak{g}$ (see, for example, [17] (pp. 148, 549–550).

(b) If h is any smooth bi-invariant semi-Riemannian metric on G, then the Levi-Civita connection of (G, h) is always the 0-connection of Cartan–Schouten ∇^0 (see again [17] (pp. 148, 549–550)). Therefore, (G, h) is a (complete) symmetric semi-Riemannian manifold.

(c) The smooth Lie group G inherits a natural real analytic structure (compatible with the C^∞-structure of G) that makes it a real analytic Lie group. With respect to this real analytic structure, the $0-$connection of Cartan–Schouten ∇^0 is real analytic.

The following proposition describes totally geodesic submanifolds of a connected Lie group endowed with the 0-connection of Cartan–Schouten, with an assigned tangent space at the identity.

Proposition 5. *Let G be a connected Lie group with identity e, and denote by ∇^0 the $0-$connection of Cartan–Schouten of G. Let $N_e = \exp_e(\widetilde{N}_e)$ be a normal neighborhood of e in (G, ∇^0) and V be a vector subspace of $T_e(G)$ such that we have $\big[[u,v],u\big] \in V$ for every $u, v \in V$. Then $\exp_e(\widetilde{N}_e \cap V)$ is a totally geodesic submanifold of (G, ∇^0) passing through e and with V as its tangent space at e.*

Proof. The statement follows directly from Proposition 2 and Remark 1 (a) and (c). □

Remark 2. *We say that a Riemannian manifold (M, g) is a* Hadamard manifold *if it is simply connected and complete with non-positive sectional curvature everywhere. Recall that the well-known Cartan–Hadamard Theorem states that if (M, g) is any Hadamard manifold, then for every $p \in M$, the exponential map $\exp_p$ is defined on the entire tangent space $T_p(M)$ and is a diffeomorphism from $T_p(M)$ onto M. Consequently, the entire Hadamard manifold M is a normal neighborhood (with respect to the Levi-Civita connection of g) of any of its point p. Furthermore, it is clear that if (M, g) is real analytic, then its Levi-Civita connection is also real analytic.*

In the following proposition, we consider a real analytic Hadamard manifold with the Levi-Civita connection, and we prove that under assumptions on the curvature, there exists a unique and totally geodesic submanifold with assigned tangent space at a point.

Proposition 6. *Let (M, g) be a real analytic Hadamard manifold with ∇ as a Levi-Civita connection. Let p be a point of M and V be a subspace of $T_p(M)$. We denote by $\exp_p : T_p(M) \to M$ the exponential map of (M, g) at the point p. Then:*

(a) There exists a connected complete totally geodesic submanifold S of (M, g) such that $p \in S$ and $T_p(S) = V$ if and only if, for every integer $m \geq 0$ and every $X_1, \cdots, X_{m+3} \in V$, we have $(\nabla^m R)(X_1, \cdots, X_{m+3}) \in V$;

(b) If such a submanifold S exists, it is unique, and we have $S = exp_p(V)$, and S, endowed with the metric induced by g, is a real analytic Hadamard manifold.

Proof. The necessity of the conditions expressed in part (a) is a consequence of Proposition 1 (b). Conversely, if these conditions are satisfied, then Theorem 1 and Remark 2 imply the existence of a totally geodesic real analytic submanifold S of (M, g) such that $p \in S$, $T_p(S) = V$, and $\exp_p(V) = S$. As in the proof of Proposition 3, we can easily deduce that the exponential map $\exp_p^S$ is defined on the entire tangent space $T_p(S) = V$. This is sufficient to conclude that the Riemannian submanifold S is complete (see [13] Cor. 4.3, p. 175). The map ${\exp_p}_{|_V}$ is a diffeomorphism from V onto $S = \exp_p(V)$, since it is the restriction to V of the diffeomorphism $\exp_p$ from $T_p(M)$ onto M. Hence S is simply connected. Since S is totally geodesic in (M, g), all the sectional curvatures of the Riemannian submanifold S are equal to the respective sectional curvatures of (M, g) and are therefore non-positive. Therefore, the Riemannian submanifold S is a Hadamard manifold. Finally, the uniqueness of the submanifold S follows from Corollary 1 (b). This concludes the proof. □

Remark 3. *A more general but much more complicated version of Proposition 6, in the smooth Riemannian case, has been proven by R. Hermann [18].*

We note that for an arbitrary torsion-free affine connection, the classical Cartan–Hadamard Theorem does not hold; however, the proof of Proposition 6 allows us to state the following.

Proposition 7. *Let (M, ∇) be a real analytic manifold with a torsion-free real analytic affine connection. Suppose that there exists $p \in M$ such that the exponential map $\exp_p$ is defined on the entire tangent space $T_p(M)$ and is a diffeomorphism from $T_p(M)$ onto M. Let V be a vector subspace of $T_p(M)$ such that, for every integer $m \geq 0$ and every $X_1, X_2, \cdots, X_{m+3} \in V$, we have $(\nabla^m R)(X_1, X_2, \cdots, X_{m+3}) \in V$. Then there exists a simply connected totally geodesic real analytic submanifold S of (M, ∇) such that $p \in S$ and $T_p(S) = V$. Furthermore, we have $S = exp_p(V)$.*

In order to describe examples to which Proposition 7 can be applied, we remark that, recently, a Cartan–Hadamard-type theorem has been proven for a class of statistical manifolds, precisely for statistical manifolds with cubic forms divisible by the metric [2]. Further generalizations of the Cartan–Hadamard Theorem, which allow us to apply Proposition 7, can be found in [19,20].

The following Example 1 allows us to obtain geodesic triangles that are not contained in any two-dimensional totally geodesic submanifold of the ambient manifold.

Example 1. *We remark that in the case of the Heisenberg group H_3, following [5], we can easily verify that, given a left-invariant metric g on H_3, there exist an orthonormal basis of the Lie algebra of H_3, $\{E_1, E_2, E_3\}$, and a positive constant c, such that the Riemann curvature tensor R of (H_3, g) satisfies $R(E_1, E_3)E_1 = -\frac{1}{4}c^2 E_2$. Therefore, the condition of Theorem 1 is not satisfied (when $m = 0$) for the triplet E_1, E_3, E_1; this implies that there is no two-dimensional totally geodesic submanifold of (H_3, g) containing the geodesic triangle with vertices I_3, $\exp(E_1)$, and $\exp(E_3)$ (where I_3 is the identity of H_3). Actually, much more can be said: it has been proven that in (H_3, g), there are no two-dimensional totally geodesic submanifolds [5].*

Many examples of geodesic triangles that are not contained in any two-dimensional totally geodesic submanifold can be constructed by using the next Proposition 8 that we are about to prove. First we need to recall some known facts about the manifold $(\mathcal{P}_2, g)$ of symmetric positive definite real square matrices of order 2, endowed with the trace metric g. We will denote by $I_2 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in \mathcal{P}_2$ the identity matrix of order two, by Sym_2 the space of symmetric 2×2 real matrices, and by $[\ ,\ ]$ the usual commutator of two square matrices.

The known properties we will use are as follows:

(i) $(\mathcal{P}_2, g)$ is a real analytic symmetric Hadamard manifold.

(ii) At point $I_2 \in \mathcal{P}_2$, we have: $T_{I_2}(\mathcal{P}_2) = Sym_2$ and also $g_{I_2}(X, Y) = \operatorname{trace}(XY)$, $R_{I_2}(X, Y, Z) = \frac{1}{4} \cdot [[X, Y], Z]$, $g_{I_2}(R(X, Y, Z), W) = \frac{1}{4} \cdot \operatorname{trace}([X, Y] \cdot [Z, W])$, for every $X, Y, Z, W \in T_{I_2}(\mathcal{P}_2) = Sym_2$. Furthermore, the Riemannian exponential map of $(\mathcal{P}_2, g)$ at I_2 is the matrix exponential map $\exp : Sym_2 \to \mathcal{P}_2$.

(iii) The map $(A, X) \mapsto \Gamma_A(X) := AXA^T$ (for $A \in GL_2(\mathbb{R})$ and $X \in \mathcal{P}_2$) defines a transitive isometric action of the general linear group $GL_2(\mathbb{R})$ on $(\mathcal{P}_2, g)$.

(iv) For every $c > 0$, the set $\mathcal{P}_2(c) := \{Q \in \mathcal{P}_2 : \det(Q) = c\}$ is a two-dimensional simply connected complete totally geodesic submanifold of $(\mathcal{P}_2, g)$, and for every $A \in GL_2(\mathbb{R})$, we have $\Gamma_A(\mathcal{P}_2(c)) = \mathcal{P}_2(c \cdot \det(A)^2)$.

(v) The maximal geodesics of $(\mathcal{P}_2, g)$ starting from I_2 are exactly the paths of the form $\gamma_{(I_2, X)}(t) := \exp(t\,X)$ $(t \in \mathbb{R})$ where $X \in Sym_2$, so the maximal geodesics of $(\mathcal{P}_2, g)$ are exactly the maps $t \mapsto \gamma_A(\exp(t\,X)) = A\exp(t\,X)A^T$ $(t \in \mathbb{R})$ where $X \in Sym_2$ and $A \in GL_2(\mathbb{R})$. We say that a non-constant maximal geodesic $t \mapsto \gamma_A(\exp(t\,X))$ is *non-vertical* when $X \neq \lambda I_2$ for all $\lambda \in \mathbb{R} \setminus \{0\}$, while it is said to be *vertical* when $X = \lambda I_2$ for some $\lambda \in \mathbb{R} \setminus \{0\}$; clearly, a maximal geodesic is vertical if and only if its image is a half-line of the form $\mathbb{R}^+ \cdot Q := \{t\, Q : t \in \mathbb{R}, t > 0\}$, for some fixed $Q \in \mathcal{P}_2$.

(vi) We finally say that a set $\mathcal{C} \subset \mathcal{P}_2$ is a *vertical geodesic cylinder* if there exists a non-vertical and non-constant maximal geodesic $\alpha(t')$ of $(\mathcal{P}_2, g)$ such that we have $\mathcal{C} = \{t \cdot \alpha(t') : t, t' \in \mathbb{R}, t > 0\}$; in other words, any vertical geodesic cylinder is the union of the vertical maximal geodesics passing through all the points of the image of a non-vertical maximal geodesic of $(\mathcal{P}_2, g)$. Clearly, for each $A \in GL_2(\mathbb{R})$, the isometry Γ_A maps every vertical geodesic cylinder onto a vertical geodesic cylinder.

For details and further information, see, for example, [16,21].

Proposition 8. *The two-dimensional connected complete totally geodesic submanifolds of* $(\mathcal{P}_2, g)$ *are exactly all vertical geodesic cylinders and all submanifolds of the type* $\mathcal{P}_2(c) = \{Q \in \mathcal{P}_2 : \det(Q) = c\}$, *with* $c > 0$. *Furthermore, every vertical geodesic cylinder is isometric to the Euclidean plane, while, for any* $c > 0$, *the Riemannian submanifold* $\mathcal{P}_2(c)$ *is isometric to the hyperbolic plane of curvature* $-\frac{1}{2}$.

Proof. Keeping in mind the previous properties (iv), (v), and (vi), it is clear that it is sufficient to prove the following two statements (a) and (b):

(a) The two-dimensional connected complete totally geodesic submanifolds of $(\mathcal{P}_2, g)$ passing through the identity I_2 are exactly all sets of the form $\mathcal{C}_B := \{s \cdot \exp(t\, B) : s, t \in \mathbb{R}, s > 0\}$ with $B \in Sym_2$ and $B \neq \lambda I_2$ for every $\lambda \in \mathbb{R}$, in addition to the set $\mathcal{P}_2(1)$;

(b) Each set $\mathcal{C}_B$ of condition (a) is a two-dimensional Hadamard manifold with constant curvature equal to zero, while the Riemannian submanifold $\mathcal{P}_2(1)$ has constant curvature equal to $-\frac{1}{2}$.

Let us therefore prove conditions (a) and (b).

From Proposition 6 and properties (i) and (ii), we deduce that the two-dimensional connected complete totally geodesic submanifolds of $(\mathcal{P}_2, g)$ passing through the identity I_2 are exactly all sets of the form $S = \exp(V)$, where V is a two-dimensional subspace of Sym_2 such that

$$[[X, Y], Z] \in V \quad \text{for every} \quad X, Y, Z \in V. \tag{30}$$

Now, fix any two-dimensional subspace V of Sym_2 and any orthonormal basis E, F of V (with respect to g_{I_2}). Note that the 2×2 matrix $[E, F]$ is skew-symmetric; consequently, we have $[E, F] = \mu \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ for some $\mu \in \mathbb{R}$. Remembering property (ii), we get that the sectional curvature $\mathcal{K}(V)$ of the section V is $\mathcal{K}(V) = g_{I_2}\big(R(E, F, E), F\big) = \frac{1}{4} \cdot \operatorname{trace}\left([E, F]^2\right) = \frac{1}{4} \cdot \operatorname{trace}\left(\mu^2 \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^2\right) = -\dfrac{\mu^2}{2}$. Therefore, we have $\mathcal{K}(V) = 0$ if and only if $XY = YX$ for every $X, Y \in V$. Note that the latter condition is equivalent to the condition $I_2 \in V$, since the codimension of V in Sym_2 is equal to 1.

First, assume that $\mathcal{K}(V) = 0$. So, if $B \in V$ is not a multiple of I_2, then $\{I_2, B\}$ is a basis of the vector space V. Furthermore, obviously V satisfies condition (30). Therefore, we conclude that the set $\mathcal{C}_B := \exp(V) = \{\exp(x\, I_2 + y\, B) : x, y \in \mathbb{R}\}$ is a simply connected complete totally geodesic submanifold of $(\mathcal{P}_2, g)$. Furthermore, since I_2 and B commute, we have $\exp(x\, I_2 + y\, B) = \exp(x\, I_2) \cdot \exp(y\, B) = e^x \cdot \exp(y\, B)$, for every $x, y \in \mathbb{R}$. Hence, $\mathcal{C}_B = \{t \cdot \exp(y\, B) : t, y \in \mathbb{R}, t > 0\}$. Since the map $y \mapsto \exp(y\, B)$ is an arbitrary non-vertical maximal geodesic of $(\mathcal{P}_2, g)$ starting from I_2, the submanifold $\mathcal{C}_B$ is an arbitrary vertical geodesic cylinder passing through I_2. Furthermore, since $\mathcal{C}_B$ is a connected complete totally geodesic submanifold of the symmetric Hadamard manifold $(\mathcal{P}_2, g)$, then $\mathcal{C}_B$ is also symmetric, and its curvature at the point I_2 is $\mathcal{K}\big(T_{I_2}(\mathcal{C}_B)\big) = \mathcal{K}(V) = 0$. It follows that the curvature of $\mathcal{C}_B$ is zero at every point.

Now, assume that $\mathcal{K}(V) = -\frac{\mu^2}{2} \neq 0$, where $V := \left\{ \begin{pmatrix} x & y \\ y & z \end{pmatrix} \in Sym_2 : \alpha x + \beta y + \gamma z = 0 \right\}$ (for some $\alpha, \beta, \gamma \in \mathbb{R}$, $\alpha^2 + \beta^2 + \gamma^2 > 0$), while μ is the non-zero constant defined by the equality $[E, F] = \mu \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ (with $\{E, F\}$ arbitrary orthonormal basis of V). Assume that condition (30) holds. The matrix $H_1 := \begin{pmatrix} \gamma & 0 \\ 0 & -\alpha \end{pmatrix}$ belongs to V, and so the matrix $[[E, F], H_1] = \mu(\alpha + \gamma) \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ also belongs to V. The hypothesis $\mathcal{K}(V) \neq 0$ implies that I_2 does not belong to V, and consequently $\alpha + \gamma \neq 0$. This implies that $\beta = 0$. Therefore, the matrices $H_2 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $[[E, F], H_2] = 2\mu \cdot \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ also belong to V, and this implies that $\alpha = \gamma$; hence the space V is necessarily equal to $V_0 := \left\{ \begin{pmatrix} x & y \\ y & z \end{pmatrix} \in Sym_2 : x + z = 0 \right\} = \{ x \in Sym_2 : \text{trace}\,(X) = 0 \}$. Conversely, the vector space $V_0 = \{ x \in Sym_2 : \text{trace}\,(X) = 0 \}$ clearly satisfies condition (30). Remembering the identity $\det\big(\exp(X)\big) = e^{\text{trace}\,(X)}$, we conclude that, in the case where $\mathcal{K}(V) \neq 0$, we obtain only the submanifold $\exp(V_0) = \mathcal{P}_2(1) = \{Q \in \mathcal{P}_2 : \det(Q) = 1\}$. An orthonormal basis of V_0 is given by $\left\{ E_0 := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},\ F_0 := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$, and a simple calculation shows that $[E_0, F_0] = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. So, for the subspace V_0, the constant μ is equal to 1; this implies that $\mathcal{K}(V_0) = -\frac{1}{2}$. Arguing as before, we conclude that the curvature of $\mathcal{P}_2(1)$ is constant and equal to $-\frac{1}{2}$.

Therefore, statements (a) and (b) are fully proven. □

Taking into account that two symmetric real matrices commute if and only if their exponential matrices commute, from Proposition 8 (and its proof), the following can be easily deduced.

Corollary 2. *Let $P, Q \in \mathcal{P}_2$, and suppose that the three points I_2, P, Q do not lie on any geodesic of $(\mathcal{P}_2, g)$. Then the geodesic triangle with vertices I_2, P, Q is contained in a two-dimensional totally geodesic submanifold of $(\mathcal{P}_2, g)$ if and only if $PQ = QP$ or $\det(P) = \det(Q) = 1$.*

We would like to thank the anonymous referees for giving us the opportunity to improve our paper with their suggestions.

Author Contributions: Conceptualization, A.N. and D.P.; methodology, A.N. and D.P.; validation, A.N. and D.P.; investigation, A.N. and D.P.; writing-original draft preparation, A.N. and D.P.; writing-review and editing, A.N. and D.P.; visualization, A.N. and D.P.; supervision, A.N. and D.P. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: No new data were created or analyzed in this study. Data sharing is not applicable to this article.

Conflicts of Interest: The authors declare no conflict of interest

References

1. Cartan, E. *Géometrie des éspaces de Riemann*; Gauthier-Villars: Paris, France, 1946.
2. Ueno, R. Geodesic connectedness on statistical manifolds with cubic forms divisible by the metric. *Info. Geo.* **2026**, *9*, 81–102. https://doi.org/10.1007/s41884-025-00185-0.
3. O'Neill, B. *Semi-Riemannian Geometry and Applications to Relativity*; Academic Press: Cambridge, MA, USA, 1983.
4. Murphy, T.; Wilhelm, F. Random manifolds have no totally geodesic submanifolds. *Mich. Math. J.* **2019**, *68*, 323–335.
5. Inogushi, J.; Kuwabara, K.; Naitoh, H. Grassmann geometry on the 3-dimensional Heisenberg group. *Hokkaido Math. J.* **2005**, *34*, 375–391.
6. Chen, B.Y.; Nagano, T. Totally geodesic submanifolds of symmetric spaces, I. *Duke. Math. J.* **1977**, *44*, 745–755.
7. Chen, B.Y.; Nagano, T. Totally geodesic submanifolds of symmetric spaces, II. *Duke. Math. J.* **1978**, *45*, 405–425.
8. Berndt, J.; Olmos, C. Maximal totally geodesic submanifolds and index of symmetric spaces. *J. Differ. Geom.* **2016**, *104*, 187–217.
9. Pawel, K.; Reckziegel, H. Affine submanifolds and the Theorem of Cartan-Ambrose-Hicks. *Kodai Math. J.* **2002**, *25*, 341–356.
10. Tojo, K. Totally geodesic submanifolds of naturally reductive homogeneous spaces. *Tsukuba J. Math.* **1996**, *20*, 181–190.
11. Tsukada, K. Totally geodesic submanifolds of Riemannian manifolds and curvature-invariant subspaces. *Kodai Math. J.* **1996**, *19*, 395–437.
12. Tumpach, A.B.; Larotonda, G. Totally geodesic submanifolds in the manifold SPD of symmetric positive-definitge real matrices. *Info. Geo.* **2024**, *7*, 913–942. https://doi.org/10.1007/s41884-024-00146-z.
13. Kobayashi, S.; Nomizu, K. *Foundations of Differential Geometry*; Interscience Tracts in Pure and Applied Mathematics; John Wiley & Sons: Hoboken, NJ, USA, 1963; Volume 1.
14. Kobayashi, S.; Nomizu, K. *Foundations of Differential Geometry*; Interscience Tracts in Pure and Applied Mathematics; John Wiley & Sons: Hoboken, NJ, USA, 1969; Volume 2.
15. Dolcetti, A.; Pertici, D. Elliptic isometries of the manifold of positive definite real matrices with the trace metric. *Rend. Circ. Mat. Palermo Ser.* **2021**, *2*, 575–592. https://doi.org/10.1007/s12215-020-00510-9.
16. Pertici, D.; Dolcetti, A. Semi-simple isometries of the manifold of symmetric positive definite real matrices with the trace metric. *Inf. Geom.* **2026**. https://doi.org/10.1007/s41884-026-00201-x.
17. Helgason, S. *Differential Geometry, Lie Groups, and Symmetric Spaces*; Pure and Applied Mathematics; Academic Press: New York, NY, USA; London, UK, 1978; Volume 80.
18. Hermann, R. Existence in the large of totally geodesic submanifolds of Riemannian spaces. *Bull. Amer. Math. Soc.* **1960**, *66*, 59–61.
19. Beem, J.K.; Parker, P.E. Pseudoconvexity and Geodesic Connectedness. *Ann. Mat. Pura Appl.* **1989**, *151*, 137–142.
20. Beem J.K.; Ehrlichley P.E.; Easley K.L. *Global Lorentzian Geometry*; Monograph and textbooks in Pure and Applied Mathematics; Marcel Dekker: New York, NY, USA, 1996; Volume 202.
21. Dolcetti, A.; Pertici, D. Differential properties of spaces of symmetric real matrices. *Rend. Sem. Mat. Univ. Pol. Torino* **2019**, *77*, 25–43.