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TORIC EXTREMAL KÄHLER-RICCI SOLITONS ARE KÄHLER-EINSTEIN

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Abstract. In this short note, we prove that a Calabi extremal Kähler-Ricci soliton on a compact toric Kähler manifold is Einstein. This solves for the class of toric manifolds a general problem stated by the authors that they solved only under some curvature assumptions.

Introduction

Let M^{2n} be a compact Kähler manifold and let $\Omega \in H^{1,1}(M)$ be a Kähler class. In the attempt to identify "special" representatives of Ω, several notions of "canonical" Kähler metrics have been introduced. A natural choice are of course Kähler-Einstein metrics, generalized by *extremal* metrics and *Kähler-Ricci solitons (KRS)*. Extremal metrics are defined to be critical points of the *Calabi functional*

$$\omega \mapsto \int_M s_\omega^2 \omega^n$$

that maps the Kähler metric ω to the L^2-norm of its scalar curvature. The Euler-Lagrange equation of the Calabi functional is

$$\operatorname{grad}_\omega(s_\omega) \text{ is holomorphic.} \tag{1}$$

Kähler-Ricci solitons are Kähler metrics that satisfy the relation

$$\rho + c\omega = \mathcal{L}_X \omega \tag{2}$$

with their Ricci form ρ, for some vector field X that is holomorphic and, in the compact case, is the gradient of a smooth function $f\colon M \to \mathbb{R}$. The KRS equation forces ω to lie in the class $2\pi c_1(M)$.

In [2] we addressed the problem whether the same $\omega \in 2\pi c_1(M)$ can be extremal and a KRS without being Einstein and we proved the following.

Theorem 1 ([2]). *A compact extremal KRS with positive holomorphic sectional curvature is Kähler-Einstein.*

Toric manifolds are compact Kähler $2n$-manifolds admitting an effective Hamiltonian action of an n-torus $\mathbb{T}$ by Kähler automorphism. Although in an algebraic geometric context, Fulton calls them a "remarkably fertile testing ground for general theories" and, also from the Kähler geometric point of view, their richness of symmetries makes them a large park of examples.

As compact symplectic manifolds, they are characterized by the image of their moment map, that is a *Delzant polytope*, i.e. a convex polytope $\Delta \subset \mathbb{R}^n$ with

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certain combinatoric properties. Given a compact symplectic toric manifold with moment image Δ, all possible compatible complex structure are described by a single function, as we explain below.

The $\mathbb{T}$-invariant Kähler geometry on a dense subset is well described in the coordinates given by the moment map itself. In these coordinates, the extremal condition (1) has a particularly simple description, see e.g. [1].

Separately, it is known that every toric Fano manifolds admits a KRS, see e.g. [4] and references therein where, in addition, Donaldson explains also the relation between the soliton field X and the Delzant polytope. The existence of extremal metrics in the toric setting is discussed in [1].

The purpose of this note is to prove the following result.

Theorem 2. *A compact toric Calabi-extremal Kähler-Ricci soliton is Kähler-Einstein.*

This solves the problem stated in [2] for the class of toric Kähler metrics, that can have holomorphic sectional curvature of any sign and so are not included in Theorem 1.

The proof of Theorem 2 is based on the combinatoric properties of Delzant polytopes and the boundary behavior of the Abreu potential. The problem in its full generality remains open.

Problem. *Prove that every extremal Kähler-Ricci soliton is Einstein or find a counterexample.*

Another class of manifold related to toric Kähler manifolds is given by toric bundles, where the existence of KRS has been studied in [5]. It would be interesting to apply the techniques of toric geometry from [1, 4] to study the existence of extremal or constant scalar curvature Kähler metrics in this class of manifolds and establish an analogue of Theorem 2.

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1. Proof of Theorem 2

Let (M, g, ω) be a toric Kähler manifold, with moment map $\mu\colon M \to \Delta = \mu(M) \subset \mathbb{R}^n$. The moment image can be written as

$$\Delta = \{x \in \mathbb{R}^n : \ell_k(x) \geq b_k, 1 \leq k \leq d\} \tag{3}$$

as intersection of the d half-spaces $\{x \in \mathbb{R}^n : \ell_k(x) - b_k \geq 0\}$.

The linear functions ℓ_k are defined by $\ell_k(x) = \langle u_k, x\rangle$, where v_k is the normal to the *facet* $\{\ell_k(x) = 0\} \cap \Delta$. The combinatoric property of being Delzant implies the following.

Lemma 1.1. *Let Δ be a Delzant polytope in $\mathbb{R}^n$. Then the vertices of Δ cannot lie on the any affine hyperplane.*

Proof. Let P be a vertex of Δ. By definition of Delzant polytope, the exactly n edges meeting at P are of the form tv_i for $t \in [0, a_i]$ and the v_i can be taken to be a basis of $\mathbb{Z}^n$. Further n vertices are of the form $P_i = a_i v_i$ and they cannot lie on the same affine hyperplane of $\mathbb{R}^n$ as the v_i are linearly independent over $\mathbb{R}$. □

Given a compact toric symplectic manifold (M, ω) with Delzant polytope Δ, consider the dense subset

$$M^0 = \{p \in M : \text{ the } \mathbb{T}\text{-action is free at } p\} \simeq \Delta^0 \times \mathbb{T},$$

where Δ^0 is the interior of Δ and $(x, y) \in \Delta^0 \times \mathbb{T}$ are the *symplectic coordinates*. In these coordinates, the $\mathbb{T}$-action is just the group multiplication on the second component. In particular, $\mathbb{T}$-invariant tensor fields on M^0 depend only on $x \in \Delta^0$.

All $\mathbb{T}$-invariant complex structures compatible with ω are determined by the *Abreu potential*, a function $g\colon \Delta^0 \to \mathbb{R}$ given by

$$2g(x) = \sum \ell_k(x) \log \ell_k(x) + h(x), \tag{4}$$

on the interior of Δ, where the ℓ_k are from (3) and h is a smooth function on Δ.

In the (x, y)-coordinates, the symplectic form is the canonical $\omega = dx_i \wedge dy_i$ and the Kähler metric corresponding g as in (4) it is $g_{,ij}(x) dx_i \cdot dx_j$, where $G = (g_{,ij})$ is the (Euclidean) Hessian of g. The matrix G has to be singular on the boundary of Δ in order for the metric to extend smoothly on the whole M. However, it is possible to describe the behavior of G on the vertices of Δ.

Lemma 1.2. *The inverse of the Hessian matrix G vanishes at the vertices of Δ.*

Proof. Without loss of generality, up to translations and to a transformation of $\mathrm{SL}(n, \mathbb{Z})$, we can assume that 0 is a vertex and that the edges meeting there are the coordinate axes $x_1, \dots, x_n$.

The transformed polytope is then given by

$$\Delta = \bigcap_{i=1}^{n} \{x \in \mathbb{R}^n : x_i \geq 0\} \cap \bigcap_{i=n+1}^{d} \{x \in \mathbb{R}^n : \ell_k(x) \geq 0\}$$

and the linear functions ℓ_k do not vanish at zero.

The Abreu potential g is given by

$$2g(x) = \sum_{i=1}^{n} x_i \log x_i + \underbrace{\sum_{i=n+1}^{d} \ell_i(x) \log \ell_i(x) + h(x)}_{=:\tilde{h}(x)}$$

and its Hessian matrix is

$$G_{ij} = \frac{\delta_{ij}}{x_j} + \tilde{h}_{,ij}(x) \tag{5}$$

where the function $\tilde{h}_{,ij}$ is given by

$$\tilde{h}_{,ij} = \sum_{k=n+1}^{d} \frac{\ell_{k,i}(x)\ell_{k,j}(x)}{\ell_k(x)} + h_{,ij}. \tag{6}$$

From [1, Thm. 2.8], the determinant of G is given by

$$\frac{1}{\det G} = \delta(x) x_1 \cdots x_n \cdot \ell_{n+1}(x) \cdots \ell_d(x)$$

for some function δ *strictly positive and smooth on the whole* Δ.

The entry g^{ij} of G^{-1} is given by

$$g^{ij}(x) = \frac{1}{\det G} \operatorname{cof}(G)_{ij}$$

where $\operatorname{cof}(G)$ is the cofactor matrix of G.

The conclusion follows from the claim that

$$\operatorname{cof}(G)_{ij} = o\left(\frac{1}{x_1 \cdots x_n}\right).$$

From (5) one can see that, after eliminating the i-th row and the j-th column, the variables x_i and x_j can appear at the denominator only in the derivatives of $\tilde{h}$, but from (6) we see that their limit for $x \to 0$ is finite, so the claim is true. □

Abreu's characterization [1] of toric extremal metrics relies on the fact that a $\mathbb{T}$-invariant function has a holomorphic gradient if, and only if, it is an affine function in the symplectic coordinates. We use this on the scalar curvature and on the potential f of X.

Proof of Theorem 2. From the preserved quantity (in our notation)

$$s + |\nabla f|^2 + 2f = \text{const}$$

that holds for every Ricci soliton, see e.g. [3], plus the extremal assumption, it follows that both f and $|\nabla f|^2$ are affine functions in the interior of Δ.

If $f = a \cdot x$, then one has that

$$|\nabla f|^2 = a^T G^{-1}(x) a$$

is an affine function as well. If we consider its extension to the whole $\mathbb{R}^n$, it is zero in all the vertices of Δ by the Lemma 1.2. On the other hand, the zeros of a nonzero affine function is a proper affine hyperplane, so by Lemma 1.1 we can conclude that the length of X must be the zero function. So $X = 0$ and the metric is Einstein. □

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