

THE DERIVED MODULI STACK OF SHIFTED SYMPLECTIC STRUCTURES

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ABSTRACT. We introduce and study the derived moduli stack $\mathbf{Syp}(X, n)$ of n -shifted symplectic structures on a given derived stack X , as introduced in [PTVV]. In particular, under reasonable assumptions on X , we prove that $\mathbf{Syp}(X, n)$ carries a canonical quadratic form, in the sense of [Ve]. This generalizes a classical result of [FH], which was established in the classical C^∞ -setting, to the broader context of derived algebraic geometry, thus proving a conjecture stated in [Ve].

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INTRODUCTION

Let M be a closed smooth manifold. One natural question in symplectic geometry is to classify all possible symplectic structures on M : a reasonable approach to this is to study the moduli space $\mathbf{Syp}(M)$ of symplectic structures on M . The space $\mathbf{Syp}(M)$ can be studied from the point of view of symplectic topology (see for example [MT], [Sm], [Vi] to name a few), but in this paper we will rather be interested in its geometry. One of the main result in this direction is given by Fricke and Habermann, who in [FH] construct a (pseudo-)Riemannian structure on $\mathbf{Syp}(M)$.

The purpose of this paper is to extend the results of [FH] to the setting of derived algebraic geometry. Derived algebraic geometry can be informally understood as studying of generalized spaces (i.e. derived stacks), whose local models are derived commutative algebras, that is to say simplicial commutative algebras. If we suppose to be working over a base field k of characteristic zero, the local models can also be taken to be non-positively graded commutative dg algebras. We refer for example to [To] (and references therein) for a more precise survey.

In the seminal paper [PTVV], the authors introduced the notion of n -shifted symplectic structure on a given derived stack X , where n is any integer. On the other hand, there is a parallel theory of shifted quadratic forms on derived stacks, developed in [Ve] and [Ba].

Building on these works, we construct a derived moduli stack $\mathbf{Symp}(X, n)$ of n -shifted symplectic structures on X , which has to be thought as a derived enhancement (in the algebraic setting) of the moduli space of symplectic structures studied in [FH].

The main result of the present work can be stated as follows.

Theorem 0.1. *Let X be a nice enough derived stack, and let n be an integer. Then the derived moduli stack $\mathbf{Symp}(X, n)$ of n -shifted symplectic structures on X carries a natural quadratic form, in the sense of [Ve] and [Ba], extending the one of [FH].*

In particular, the above Theorem thus proves a conjecture which was stated in [Ve, Remark 3.15].

The paper is organized as follows. In Section 1, we recall some preliminary notions that will be used later on. In Section 2, we give a construction of the derived moduli stack of Lie coalgebroids $\mathbf{CoAlgd}(X)$ on a derived stack X . Moreover, passing through the important notion of symplectic Lie coalgebroid, we arrive to the definition of the derived moduli stack of n -shifted pre-symplectic structures $\mathbf{PrSymp}(X, n)$. Section 3 is devoted to the computation of the cotangent complex of $\mathbf{Symp}(X, n)$, which is the sub-stack of $\mathbf{PrSymp}(X, n)$ whose points correspond to n -shifted symplectic structures on X . The techniques used here are similar to the ones in [Ba]. Finally, in the last section we construct a shifted quadratic form on $\mathbf{Symp}(X, n)$, proving our main result.

Let us mention that in [FHH], the authors gave a criterion for the non-degeneracy of the pseudo-Riemannian structure of [FH]. It would be interesting to see if one can generalize their arguments to derived algebraic geometry. In other words, one should be able to characterize the non-degeneracy of the shifted quadratic structure we construct in the present paper. We plan to come back to this question in a future work.

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Notations and conventions.

- k denotes a base field of characteristic 0. We denote by dg_k the ∞ -category of cochain complexes of k -vector spaces. The standard tensor product of complexes makes it a symmetric monoidal ∞ -category. More generally, if A is a commutative dg algebra over k , then dg_A will denote the symmetric monoidal ∞ -category of dg A -modules.
- The ∞ -category of commutative dg algebras concentrated in degree zero will be denoted $\mathrm{cdga}^{\leq 0}$. The category dAff of derived affines is simply the opposite category $(\mathrm{cdga}^{\leq 0})^{op}$.
- The ∞ -category of simplicial sets is denoted by sSet .
- The ∞ -category of *derived stacks* over k is denoted dSt .
- Given $A \in \mathrm{dAff}^{op}$ we denote by $\mathrm{QCoh}(A)$ the ∞ -category of dg A -modules, while $\mathrm{Perf}(A)$ will denote the full subcategory of *perfect* dg A -modules.
- If X is a derived stack, we define the categories

$$\mathrm{QCoh}(X) := \lim_{\mathrm{Spec} A \rightarrow X} \mathrm{QCoh}(A) \quad \text{and} \quad \mathrm{Perf}(X) := \lim_{\mathrm{Spec} A \rightarrow X} \mathrm{Perf}(A).$$

- If \mathcal{C} is an ∞ -category, we denote by \mathcal{C}^\sim the maximal groupoid contained in \mathcal{C} . We use the notation $\mathrm{Arr}(\mathcal{C})$ for the ∞ -category of morphisms in \mathcal{C} .

1. PRELIMINARIES

1.1. **Graded mixed complexes.** Let $A \in \text{cdga}^{\leq 0}$ be a commutative dg algebra. The category $\text{dg}_A^{gr,\epsilon}$ is the ∞ -category of graded mixed A -modules. We refer to [CPTVV, Section 1] for a detailed construction of this ∞ -category. Its objects are graded complex $\{M(i)\}_{i \in \mathbb{Z}}$, together with a mixed structure ϵ , that is to say a series of maps of complexes

$$\epsilon : M(i) \rightarrow M(i+1)$$

of degree 1, such that $\epsilon^2 = 0$. Unless otherwise specified, we will only be interested in *perfect* graded mixed complexes. In other words, we will always suppose that all the $M(i)$ are perfect complexes of A -modules.

Consider the functor

$$\text{triv} : \text{dg}_A \rightarrow \text{dg}_A^{gr,\epsilon}$$

sending an A -module M to the same object with trivial graded mixed structure. In other terms, $\text{triv}(M)$ is concentrated in weight 0, and ϵ is identically zero. Following [CPTVV, Section 1.1] and [MS, Section 1.2], we give the following definition.

Definition 1.1. • The right adjoint to the functor triv is called the *realization functor*

$$|-| : \text{dg}_A^{gr,\epsilon} \rightarrow \text{dg}_A.$$

• The left adjoint to the functor triv is called the *left realization functor*

$$|-|^l : \text{dg}_A^{gr,\epsilon} \rightarrow \text{dg}_A.$$

Let $M \in \text{dg}_A^{gr,\epsilon}$ be a graded mixed complex. Then one has the following explicit model for the realization functor:

$$|M| \simeq \prod_{p \geq 0} M(p),$$

where the differential is twisted by the mixed structure of M . Similarly, one also have an analogous model for the left realization functor

$$|M|^l \simeq \bigoplus_{p \leq 0} M(p),$$

where again the differential is twisted by the the mixed structure of M .

The category $\text{dg}_A^{gr,\epsilon}$ admits a natural symmetric monoidal structure, defined weight-wise by

$$(M \otimes_A M')(p) := \bigoplus_{i+j=p} M(i) \otimes_A M(j),$$

where the mixed structure on $M \otimes_A M'$ is the natural one. Moreover, given two objects $M, M' \in \text{dg}_A^{gr,\epsilon}$, we can consider an internal object of morphisms $\underline{\text{Hom}}_{\text{dg}_A^{gr,\epsilon}}(M, M')$, whose weight components are defined by

$$\underline{\text{Hom}}_{\text{dg}_A^{gr,\epsilon}}(M, M')(p) := \prod_{q \in \mathbb{Z}} \underline{\text{Hom}}_A(M(q), M'(p+q)),$$

where $\underline{\text{Hom}}_A(-, -)$ denotes the internal Hom object in A -modules. In the special case where $M' \simeq \text{triv}(A)$ is the monoidal unit of $\text{dg}_A^{gr,\epsilon}$, we use the shorter notation $M^\vee = \underline{\text{Hom}}_{\text{dg}_A^{gr,\epsilon}}(M, A)$ for the dual of M .

Recall that we are implicitly assuming that the weight components of our graded mixed complexes are perfect. In particular, the tensor product interacts nicely with duals, and we get natural identifications

$$(1) \quad (M^\vee)^\vee \simeq M, \quad (M \otimes_A N)^\vee \simeq M^\vee \otimes_A N^\vee,$$

for every $M, N \in \mathrm{dg}_A^{gr, \epsilon}$. Moreover, an easy computation also shows that

$$(2) \quad \underline{\mathrm{Hom}}_{\mathrm{dg}_A^{gr, \epsilon}}(M, N) \simeq \underline{\mathrm{Hom}}_{\mathrm{dg}_A^{gr, \epsilon}}((N)^\vee, M^\vee),$$

again for all $M, N \in \mathrm{dg}_A^{gr, \epsilon}$.

Notice however that since M, N are in general unbounded in the weight direction, we cannot expect to be able to identify the internal Hom $\underline{\mathrm{Hom}}_{\mathrm{dg}_A^{gr, \epsilon}}(M, N)$ with the tensor product $M^\vee \otimes_A N$. On the other hand, a straightforward check tells us that if we suppose M to be bounded in the weight direction, then we do have

$$(3) \quad \underline{\mathrm{Hom}}_{\mathrm{dg}_A^{gr, \epsilon}}(M, N) \simeq M^\vee \otimes_A N.$$

1.2. \mathcal{O} -compact derived stacks. Let $p : F \rightarrow G$ be a map of derived stacks. Then there is an induced pushforward functor

$$p_* : \mathrm{QCoh}(F) \rightarrow \mathrm{QCoh}(G),$$

which is lax-monoidal, as it is right-adjoint to a monoidal functor. Notice that f_* does not preserve perfect complexes, in the sense that in general it doesn't restrict to a functor $\mathrm{Perf}(F) \rightarrow \mathrm{Perf}(G)$.

Given any derived stack F , there is a structural canonical morphism $p : X \rightarrow \mathrm{Spec} k$. Then the pushforward

$$p_* : \mathrm{QCoh}(X) \rightarrow \mathrm{QCoh}(k)$$

of quasi-coherent sheaves along p_X corresponds to taking global sections.

We recall here the notion of \mathcal{O} -compactness for derived stacks, following [PTVV, Section 2.1].

Definition 1.2 (see [PTVV], Definition 2.1). Let X be a derived stack. We say that X is \mathcal{O} -compact if for every derived affine stack $Y = \mathrm{Spec} A$, the following two conditions are satisfied:

- (1) $\mathcal{O}_{X \times Y}$ is compact in $\mathrm{QCoh}(X \times Y)$.
- (2) If $p_A : X \times Y \rightarrow Y$ is the natural projection, the pushforward p_* sends perfect modules to perfect modules.

Notice that if X is supposed to be \mathcal{O} -compact, then in this case the global section functor p_* sends perfect \mathcal{O}_X -modules to perfect k -modules.

1.3. Quadratic and symplectic structures on derived stacks. In this section we recall the notion of quadratic and symplectic structures on derived Artin stacks, following [Ve] and [PTVV].

The following is essentially Definition 3.14 in [Ve].

Definition 1.3. Let X be a derived Artin stack, and let $\mathbb{L}_X \in \mathrm{QCoh}(X)$ be its cotangent complex. The space of n -shifted quadratic structures on X is

$$\mathbf{QF}(X, n) := \mathrm{Map}_{\mathrm{QCoh}(X)}(\mathcal{O}_X[-n], \mathrm{Sym}_{\mathcal{O}_X}^2(\mathbb{L}_X)).$$

Remark 1.4. The above Definition is a very mild generalization of [Ve, Definition 3.14]. The only difference here is that we allow the cotangent complex \mathbb{L}_X of X to possibly not be perfect. In the special case of X being locally finitely presented, then the two definitions are clearly equivalent. The situation here is totally analogous to the case of shifted Poisson structures (see [Me] and [CPTVV, Remark 1.4.10]).

Let X be a derived Artin stack. Consider the graded dg-module

$$\mathrm{DR}(X) := \Gamma(X, \mathrm{Sym}_{\mathcal{O}_X}(\mathbb{L}_X[-1]))$$

where the additional weight grading is given by the Sym . Since as already mentioned the functor $\Gamma(X, -)$ is lax monoidal, $\mathrm{DR}(X)$ is naturally a commutative graded dg algebra. Then the de Rham differential turns $\mathrm{DR}(X)$ into a graded mixed algebra, that is to say a commutative algebra in the category $\mathrm{dg}^{gr, \epsilon}$.

The following is essentially Definition 1.12 in [PTVV] (see also [CPTVV])

Definition 1.5. The space of *closed p -forms of degree n on X* is the mapping space

$$\mathcal{A}^{p, cl}(X, n) := \mathrm{Map}_{\mathrm{dg}^{gr, \epsilon}}(k(p)[-n-p], \mathrm{DR}(X)),$$

where $k(p)[-n-p]$ is the trivial graded mixed module k sitting in weight p and cohomological degree $n+p$.

Suppose moreover that X is locally of finite presentation, so that its cotangent complex \mathbb{L}_X is perfect. In particular, consider its dual $\mathbb{T}_X = \mathbb{L}_X^\vee$ in $\mathrm{QCoh}(X)$. We say that a closed 2-form of degree n is an n -symplectic structure if the induced map $\mathbb{T}_X \rightarrow \mathbb{L}_X[-n]$ is an equivalence.

2. THE MODULI STACK OF LIE COALGEBROIDS

Let \mathbf{Perf} be the classifying stack of perfect complexes, as studied in [TVa]. As a functor, \mathbf{Perf} has the following explicit description:

$$\begin{aligned} \mathbf{Perf}: \quad \mathrm{dAff}^{op} &\longrightarrow \mathrm{sSet} \\ \mathrm{Spec} A &\longmapsto \mathrm{Perf}(A)^\sim, \end{aligned}$$

where $\mathrm{Perf}(A)^\sim$ is the maximal groupoid contained in the ∞ -category $\mathrm{Perf}(A)$ of perfect complexes on $\mathrm{Spec} A$.

Definition 2.1. Let X be a derived Artin stack. The moduli stack $\mathbf{Perf}(X)$ of perfect complexes on X is the internal mapping stack

$$\mathbf{Perf}(X) := \mathrm{Map}_{\mathrm{dSt}}(X, \mathbf{Perf})$$

in the closed ∞ -category dSt of derived stacks.

Similarly, we can define a classifying stack of graded algebras

$$\begin{aligned} \mathbf{Alg}_{\mathbf{Perf}}^{gr}: \quad \mathrm{dAff}^{op} &\longrightarrow \mathrm{sSet} \\ \mathrm{Spec} A &\longmapsto (\mathrm{Alg}_{\mathbf{Perf}}^{gr}(A))^\sim, \end{aligned}$$

where $\mathrm{Alg}_{\mathbf{Perf}}^{gr}(A)$ is the ∞ -category of graded A -algebras, whose weight components are all perfect A -modules.

Definition 2.2. If X is a derived stack, then the *classifying stack* $\mathbf{Alg}_{\mathbf{Perf}}^{gr}(X)$ of *graded perfect algebras on X* is the mapping stack

$$\mathbf{Alg}_{\mathbf{Perf}}^{gr}(X) := \mathbf{Map}_{\mathbf{dSt}}(X, \mathbf{Alg}_{\mathbf{Perf}}^{gr}).$$

There is a natural map of derived stacks

$$\mathbf{Perf}(X) \rightarrow \mathbf{Alg}_{\mathbf{Perf}}^{gr}(X),$$

which sends a perfect complex \mathcal{F} to the graded \mathcal{O}_X -algebra $\mathrm{Sym}_{\mathcal{O}_X}(\mathcal{F}[-1])$.

In particular, if we suppose that X is moreover \mathcal{O} -compact in the sense of Definition 1.2, we get a morphism of stacks

$$\mathbf{Perf}(X) \rightarrow \mathbf{Alg}_{\mathbf{Perf}}^{gr},$$

which corresponds to sending a perfect complex \mathcal{F} to the perfect graded algebra $\Gamma(X, \mathrm{Sym}_{\mathcal{O}_X}(\mathcal{F}[-1]))$.

Moreover, let $\mathbf{dg}^{gr, \epsilon}$ and $\mathbf{Alg}^{\epsilon, gr}$ be the classifying stack of graded mixed complexes and of graded mixed algebras respectively, constructed in the same way as \mathbf{Perf} and $\mathbf{Alg}_{\mathbf{Perf}}^{gr}$. Then we have a natural forgetful map

$$\mathbf{Alg}^{\epsilon, gr} \rightarrow \mathbf{Alg}^{gr}$$

which simply forgets the mixed structure.

Definition 2.3. Let X be a derived stack. The *stack* $\mathbf{dg}^{gr, \epsilon}$ of *graded mixed complexes on X* is the mapping stack

$$\mathbf{dg}^{gr, \epsilon}(X) := \mathbf{Map}_{\mathbf{dSt}}(X, \mathbf{dg}^{gr, \epsilon}).$$

Similarly, the *stack* $\mathbf{Alg}^{\epsilon, gr}$ of *graded mixed algebras on X* is the mapping stack

$$\mathbf{Alg}_{\mathbf{Perf}}^{\epsilon, gr}(X) := \mathbf{Map}_{\mathbf{dSt}}(X, \mathbf{Alg}_{\mathbf{Perf}}^{\epsilon, gr}).$$

Using these stacks, we can now give the following definition.

Definition 2.4. Let X be a derived \mathcal{O} -compact stack. The *moduli stack of perfect Lie coalgebroids on X* is the fiber product

$$\begin{array}{ccc} \mathbf{CoAlgd}(X) & \longrightarrow & \mathbf{Perf}(X) \\ \downarrow & & \downarrow \\ \mathbf{Alg}_{\mathbf{Perf}}^{\epsilon, gr} & \longrightarrow & \mathbf{Alg}_{\mathbf{Perf}}^{gr} \end{array}$$

By definition, a k -point of $\mathbf{CoAlgd}(X)$ corresponds to a perfect complex \mathcal{F} on X with a mixed structure on $\Gamma(X, \mathrm{Sym}_{\mathcal{O}_X}(\mathcal{F}[-1]))$. These are precisely Lie co-algebroids on X , that is to say perfect complexes on X whose duals are Lie algebroids. The mixed structure here is the data corresponding to the Chevalley-Eilenberg differential on the CE algebra for the Lie algebroid.

In general, A -points of $\mathbf{CoAlgd}(X)$ are given by a perfect complex \mathcal{F} on $X \times \mathrm{Spec} A$, together with a A -linear mixed structure on the graded algebra

$$(p_A)_*(\mathrm{Sym}_{\mathcal{O}_{X \times \mathrm{Spec} A}}(\mathcal{F}[-1])),$$

where $p_A : X \times \mathrm{Spec} A \rightarrow \mathrm{Spec} A$ is the natural projection.

Remark 2.5. Suppose that X is a derived Artin stack. It follows that there is a distinguished point of $\mathbf{CoAlgd}(X)$, i.e. a canonical map

$$\mathrm{Spec} k \rightarrow \mathbf{CoAlgd}(X)$$

representing the Lie co-algebroid \mathbb{L}_X (that is to say, the dual of the tangent Lie algebroid \mathbb{T}_X).

Now consider the induced map of stacks

$$\mathbf{CoAlgd}(X) \rightarrow \mathbf{Alg}_{\mathrm{Perf}}^{\varepsilon, gr}(k) \rightarrow \mathbf{dg}^{gr, \varepsilon}(k)$$

where the first map is the one coming from the definition of $\mathbf{CoAlgd}(X)$, and the second simply forgets the algebra structure, and just retains the underlying graded mixed module. Let us call ϕ this composition.

We denote by $(\mathbf{dg}^{gr, \varepsilon})^{\Delta^1}$ the stack of morphisms of $\mathbf{dg}^{gr, \varepsilon}$. Concretely, it can be defined as

$$(\mathbf{dg}^{gr, \varepsilon})^{\Delta^1}: \begin{array}{ccc} \mathrm{dAff}^{op} & \longrightarrow & \mathrm{sSet} \\ \mathrm{Spec} A & \longmapsto & (\mathrm{Arr}(\mathrm{dg}^{gr, \varepsilon}(A)))^{\sim}, \end{array}$$

where $\mathrm{Arr}(\mathrm{dg}_A^{gr, \varepsilon})$ is the ∞ -category of morphisms in $\mathrm{dg}_A^{gr, \varepsilon}$. By definition, the stack $(\mathbf{dg}^{gr, \varepsilon})^{\Delta^1}$ comes equipped with two natural maps s and t (for “source” and “target”)

$$\mathbf{dg}^{gr, \varepsilon} \xleftarrow{s} (\mathbf{dg}^{gr, \varepsilon})^{\Delta^1} \xrightarrow{t} \mathbf{dg}^{gr, \varepsilon}$$

which remembers only the source or the target of the points of $(\mathbf{dg}^{gr, \varepsilon})^{\Delta^1}$.

Definition 2.6. The moduli stack \mathcal{Y}_n of n -pre-symplectic Lie co-algebroids on X is the fiber product

$$\begin{array}{ccc} \mathcal{Y}_n & \longrightarrow & \mathbf{CoAlgd}(X) \\ \downarrow & & \downarrow (k(2)[-n-2], \phi) \\ (\mathbf{dg}^{gr, \varepsilon})^{\Delta^1} & \xrightarrow{(s, t)} & \mathbf{dg}^{gr, \varepsilon} \times \mathbf{dg}^{gr, \varepsilon} \end{array}$$

Again by definition, a k -point of \mathcal{Y}_n is given by the following data

- a perfect module \mathcal{F} on X
- a mixed structure on the graded algebra $\Gamma(X, \mathrm{Sym}_{\mathcal{O}_X}(\mathcal{F}[-1]))$
- a map of graded mixed modules

$$k[-n-2](2) \longrightarrow \Gamma(X, \mathrm{Sym}_{\mathcal{O}_X}(\mathcal{F}[-1])).$$

In general, A -points of \mathcal{Y}_n are perfect modules \mathcal{F} on $X \times \mathrm{Spec} A$, such that

$$(p_A)_* \mathrm{Sym}_{\mathcal{O}_{X \times \mathrm{Spec} A}}(\mathcal{F}[-1])$$

is a A -linear graded mixed algebra, together with a map of graded mixed complexes

$$A(2)[-n-2] \longrightarrow (p_A)_* \mathrm{Sym}_{\mathcal{O}_{X \times \mathrm{Spec} A}}(\mathcal{F}[-1]).$$

Definition 2.7. Finally, suppose X is a derived Artin stack. The moduli stack of n -pre-symplectic structures $\mathbf{PrSimp}(X, n)$ is the fiber product

$$\begin{array}{ccc} \mathbf{PrSimp}(X, n) & \longrightarrow & \mathcal{Y}_n \\ \downarrow & & \downarrow \\ \mathrm{Spec} k & \longrightarrow & \mathbf{CoAlg}d(X) \end{array}$$

where the bottom map is the one representing \mathbb{L}_X .

3. THE COTANGENT COMPLEX OF $\mathbf{Simp}(X, n)$

In this section, we study the geometry of the stack $\mathbf{PrSimp}(X, n)$ in more detail: in particular, we compute its cotangent complex, following the explicit definition of [TVe, Section 1.2.1]. As a first remark, we notice that by definition this stack fits in a cartesian square

$$\begin{array}{ccc} \mathbf{PrSimp}(X, n) & \longrightarrow & \mathrm{Spec} k \\ \downarrow & & \downarrow k(2)[-n-2] \times \mathrm{DR}(X) \\ (\mathbf{dg}^{gr, \epsilon})^{\Delta^1} & \xrightarrow{(s, t)} & \mathbf{dg}^{gr, \epsilon} \times \mathbf{dg}^{gr, \epsilon} \end{array}$$

In other words, a k -point of $\mathbf{PrSimp}(X, n)$ is just a map

$$k(2)[-n-2] \rightarrow \mathrm{DR}(X)$$

of graded mixed complexes, i.e. a closed 2-form of degree n on X . More generally, an A -point of $\mathbf{PrSimp}(X, n)$ is a degree n closed 2-form of $X \times \mathrm{Spec} A$ relative to A , or equivalently a map of graded mixed A -modules

$$A(2)[-n-2] \rightarrow p_*(\mathrm{Sym}_{\mathcal{O}_{X \times \mathrm{Spec} A}} \mathbb{L}_{X \times \mathrm{Spec} A / \mathrm{Spec} A}[-1]) \simeq \mathrm{DR}(X) \otimes_k A.$$

Remark 3.1. Even though k -points are the same as k -points of $\mathcal{A}^{2, cl}(X, n)$, the two stacks are not equivalent, as A -points of $\mathcal{A}^{2, cl}(X, n)$ are just degree n closed 2-forms on $X \times \mathrm{Spec} A$ relative to k .

Let us now consider an A -point ω of $\mathbf{PrSimp}(X, n)$, corresponding to a map

$$\omega: A(2)[-n-2] \rightarrow \mathrm{DR}(X) \otimes_k A.$$

With a slight abuse of notation, let us also denote by ω the induced A -point of $(\mathbf{dg}^{gr, \epsilon})^{\Delta^1}$. Using the fact the the above diagram is cartesian, we know that if the bottom map (s, t) has a relative cotangent complex at ω , then also $\mathbb{L}_{\mathbf{PrSimp}(X, n), \omega}$ exists, and moreover we have an equivalence

$$\mathbb{L}_{\mathbf{PrSimp}(X, n), \omega} \simeq \mathbb{L}_{(s, t), \omega}.$$

Proposition 3.2. *Let again (s, t) be the map of derived stacks*

$$(\mathbf{dg}^{gr, \epsilon})^{\Delta^1} \longrightarrow \mathbf{dg}^{gr, \epsilon} \times \mathbf{dg}^{gr, \epsilon}$$

sending a morphism to its source and target. Let $f: \mathrm{Spec} A \rightarrow (\mathbf{dg}^{gr, \epsilon})^{\Delta^1}$ correspond to a map $f: E \rightarrow F$ in $\mathbf{dg}^{gr, \epsilon}(A)$. Then (s, t) admits a cotangent complex at the point f , which is given by

$$\mathbb{L}_{(s, t), f} \simeq |\underline{\mathrm{Hom}}_{\mathbf{dg}^{gr, \epsilon} A}(E, F)^\vee|^l,$$

where $|-|^l$ is the left realization of Section 1.1.

Proof. Let M be an A -module. A straightforward computation shows that the space of relative derivations can be expressed as

$$\mathrm{Der}_{(s,t)}(A, M) \simeq \mathrm{Map}_{\mathrm{dg}_A^{gr,\epsilon}}(E, F \otimes_A M),$$

where M is taken with the trivial graded mixed structure. Moreover, in view of the identifications (1) and (2) in the category $\mathrm{dg}_A^{gr,\epsilon}$, we get

$$\mathrm{Map}_{\mathrm{dg}_A^{gr,\epsilon}}(E, F \otimes_k M) \simeq \mathrm{Map}_{\mathrm{dg}_A^{gr,\epsilon}}(\underline{\mathrm{Hom}}_{\mathrm{dg}_A^{gr,\epsilon}}(E, F)^\vee, M).$$

By definition of left realization, we conclude by adjunction that

$$\mathrm{Der}_{(s,t)}(A, M) \simeq \mathrm{Map}_{\mathrm{dg}_A}(|\underline{\mathrm{Hom}}_{\mathrm{dg}_A^{gr,\epsilon}}(E, F)^\vee|^l, M),$$

which proves the proposition. \square

An immediate consequence of the above result is the following corollary.

Proposition 3.3. *The derived stack $\mathbf{PrSymp}(X, n)$ admits a cotangent complex in every point. In particular, given an A -point ω of $\mathbf{PrSymp}(X, n)$, we have*

$$\mathbb{L}_{\mathbf{PrSymp}(X,n),\omega} \simeq \bigoplus_{p \geq 2} \underline{\mathrm{Hom}}_A(A[-n-2], \mathrm{DR}(X)(p) \otimes_k A)^\vee.$$

Proof. This is a simple combination of the above Proposition and the observation that we have an equivalence

$$|\underline{\mathrm{Hom}}_{\mathrm{dg}_A^{gr,\epsilon}}(A(2)[-n-2], \mathrm{DR}(X) \otimes A)^\vee|^l \simeq \bigoplus_{p \geq 2} \underline{\mathrm{Hom}}_A(A[-n-2], \mathrm{DR}(X)(p) \otimes_k A)^\vee,$$

where the right hand side is endowed with the twisted differential coming from the mixed structure on $\mathrm{DR}(X) \otimes A$. \square

Remark 3.4. Suppose X is such that $\mathrm{DR}(X)$ is a bounded graded mixed perfect complex. Then the sum appearing in the statement of the above Proposition is in fact finite, and thus $\mathbf{PrSymp}(X, n)$ has a perfect cotangent complex. In particular, it is now easy to see that one has

$$\mathbb{T}_{\mathbf{PrSymp}(X,n),\omega} \simeq \underline{\mathrm{Hom}}_A(A[-n-2], |\mathrm{DR}(X)^{\geq 2} \otimes_k A|).$$

Notice that this is in line with the content of the conjecture of [Ve, Remark 3.15], as the right hand side is precisely the complex of closed 2-forms of degree n on $X \times \mathrm{Spec} A$, relative to $\mathrm{Spec} A$.

Proposition 3.5. *The derived stack $\mathbf{PrSymp}(X, n)$ has a global cotangent complex.*

Proof. Indeed, suppose we have maps

$$\mathrm{Spec} B \xrightarrow{\phi} \mathrm{Spec} A \xrightarrow{f} \mathbf{PrSymp}(A, n).$$

In view of [TVe, Definition 1.4.1.7], we need to check that the induced map

$$\phi^* \mathbb{L}_{\mathbf{PrSymp}(X,n),f} \rightarrow \mathbb{L}_{\mathbf{PrSymp}(X,n),f \circ \phi}$$

is an equivalence of B -modules. But by the above proposition, this morphism comes from a map of graded mixed complexes

$$\underline{\mathrm{Hom}}_{\mathrm{dg}_A^{\mathrm{gr},\epsilon}}(E, F)^\vee \otimes_A B \rightarrow \underline{\mathrm{Hom}}_{\mathrm{dg}_B^{\mathrm{gr},\epsilon}}(E \otimes_A B, F \otimes_A B)^\vee,$$

where in this particular case $E = A(2)[-n-2]$ and $F = \mathrm{DR}(X) \otimes A$. Since E is bounded, we know from identification (3) that

$$\underline{\mathrm{Hom}}_{\mathrm{dg}_A^{\mathrm{gr},\epsilon}}(E, F) \simeq E^\vee \otimes_A F,$$

and thus the above map is an equivalence, concluding the proof. \square

4. THE QUADRATIC FORM

In this section we state and prove our main result. Namely, we show that one can endow $\mathbf{Symp}(X, n)$ with a canonical quadratic form, extending the one of [FH].

Let X be a derived Artin stack, locally of finite presentation. Let $Y = \mathrm{Spec} A$ be a derived affine scheme, and let $\omega : Y \rightarrow \mathbf{PrSymp}(X, n)$ be represented by an n -shifted pre-symplectic structure on $X \times Y$, relative to Y . Then we say that ω is *non-degenerate* if the induced map of $\mathcal{O}_{X \times Y}$ -modules

$$\omega^\sharp : \mathbb{T}_{X \times Y/Y}[-n] \rightarrow \mathbb{L}_{X \times Y/Y}$$

is an equivalence.

Definition 4.1. Let X be a derived Artin stack, locally of finite presentation. The derived stack $\mathbf{Symp}(X, n)$ of n -shifted symplectic structures on X is the substack of $\mathbf{PrSymp}(X, n)$ composed of non-degenerate pre-symplectic structure.

Notice that $\mathbf{Symp}(X, n)$ is exactly the moduli stack involved in the conjecture in [Ve, Remark 3.15].

Theorem 4.2. *There exists a canonical non-trivial quadratic form on $\mathbf{Symp}(X, n)$, extending the one of [FH].*

Proof. Let again $Y = \mathrm{Spec} A$ be a derived affine scheme. Since $\mathbf{Symp}(X, n)$ is an open substack of $\mathbf{PrSymp}(X, n)$, it follows that if ω is an A -point of $\mathbf{Symp}(X, n)$, then one has an equivalence

$$\mathbb{L}_{\mathbf{Symp}(X, n), \omega} \simeq \mathbb{L}_{\mathbf{PrSymp}(X, n), \omega}$$

of A -modules. In particular, thanks to Proposition 3.3 we have

$$\mathbb{L}_{\mathbf{Symp}(X, n), \omega} \simeq |\underline{\mathrm{Hom}}_{\mathrm{dg}_A^{\mathrm{gr},\epsilon}}(A(2)[-n-2], \mathrm{DR}(X) \otimes A)^\vee|^l.$$

As a consequence, by considering only the weight 0 component of the right hand side we get a morphism

$$\underline{\mathrm{Hom}}_A(A[-n-2], \mathrm{DR}(X)(2) \otimes A)^\vee \longrightarrow \mathbb{L}_{\mathbf{Symp}(X, n), \omega}$$

of A -modules. Remark that the source is in fact by definition the A -linear dual of the complex of 2-forms on $X \times \mathrm{Spec} A$, relative to $\mathrm{Spec} A$. Since X is supposed to be locally of finite presentation, its cotangent complex is in particular perfect. It follows that by adjunction, we get a morphism

$$\underline{\mathrm{Hom}}_A(A[-n-2], \mathrm{DR}(X)(2) \otimes A) \longrightarrow \mathrm{Hom}_{\mathrm{QCoh}(X \times Y)}(\mathbb{T}_{X \times Y/Y}[-n], \mathbb{L}_{X \times Y/Y})$$

of A -modules, where the Hom on the right hand side is the dg_A -enriched Hom of $\mathrm{QCoh}(X \times Y)$. On the other hand, the A -point ω corresponds to a symplectic structure, and therefore it gives an identification $\mathbb{T}_{X \times Y/Y}[-n] \simeq \mathbb{L}_{X \times Y/Y}$ given by ω^\sharp . Hence, we get a map (and in fact an equivalence)

$$\mathrm{Hom}_{\mathrm{QCoh}(X \times Y)}(\mathbb{T}_{X \times Y/Y}[-n], \mathbb{L}_{X \times Y/Y}) \longrightarrow \mathrm{End}(\mathbb{L}_{X \times Y/Y})$$

of A -modules. Dualizing these last morphisms and putting all together, we end up with

$$(\mathrm{End}(\mathbb{L}_{X \times Y/Y}))^\vee \longrightarrow \mathbb{L}_{\mathbf{Sym}(X, n), \omega}$$

of A -modules. Finally, notice that there is a canonical map

$$\mathrm{Sym}^2(\mathrm{End}(\mathbb{L}_{X \times Y/Y})) \rightarrow A$$

simply given by the standard formula

$$(M, N) \mapsto \frac{1}{2} \mathrm{Tr}(MN).$$

Dualizing this map, we eventually get a well defined quadratic structure on $\mathbf{Sym}(X, n)$, which clearly extends the one constructed in [FH]. □

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