LEVELS OF GENERALIZATION IN THE OBJECTIFICATION OF THE RECURSION STEP

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In this study, we focus on the students' objectification of the recursion step, intended as a process of generalization in the sense of Radford (2001). Building on Radford's model, we elaborate levels of generalization of the recursion step and use them to analyze the processes activated by secondary school students during collaborative activities with geometric recursive sequences. The analysis allows us to identify different levels reached by the students in grasping the recursion step and their transitions between these levels.

INTRODUCTION

This study is part of wider research on mathematical induction and its teaching/learning through significant and exploratory tasks (Antonini & Nannini, 2021; Telloni & Malara, 2021). It has been recognized that recursion and induction are strongly related and a deep understanding of recursion could support a meaningful learning of mathematical induction (Leron & Zazkis, 1986; Harel, 2001). Indeed, recursion, as a process which contains itself as a subprocess, has been intended as a more accessible and "executable version of induction", hence a good "stepping-stone" to teach and learn mathematical induction (Leron & Zazkis, 1986, p. 28). In tune with this idea, we focus on the students' objectification of the recursion step (RS), i.e. their understanding of the possibility to describe a sequence from a given basis in terms of a generic step which connects two consecutive elements of the sequence. In this study, we interpret the objectification of the RS as a form of generalization in the sense of Radford (2001): according to a process moving from particular to general, the RS in a geometric recursive sequence could be recognized as a placeholder for each specific step from a figure to the subsequent one.

The research goal we pursue is to investigate how the objectification of RS emerges in collaborative activities involving secondary school students.

THEORETICAL FRAMEWORK

Assuming a socio-cultural semiotic perspective, Radford (2001, 2003) distinguished different levels of generalization which novice students accomplish when they are involved in generalization of geometric patterns. These levels of generalization, called *factual*, *contextual*, and *symbolic*, are revealed from semiotic means of objectification, i.e. linguistic and non-linguistic signs conveying relations between particular and general, which students use in mathematical generalizing processes. The *factual level* consists in a generalization of iterated actions on concrete objects, possibly linked to

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numeric operations. The *contextual level* no longer operates on specific objects, but on *concrete non-material objects*, such as “the figure”. At this level students perceive the emergence of a general structure, as a mathematical object, genetically arising from the actions performed. The *symbolic level* arises when students produce a non context-bounded explanation and the mathematical meanings are elaborated in general terms. The levels of generalization introduced by Radford refer to the generalization of geometric patterns, aimed at describing the n^{th} figure of the sequence. In this study, we build on Radford’s model to elaborate levels of generalization concerning the objectification of the RS, focused on the link between figure n and figure n+1 in a sequence. According to our perspective, the objectification of the RS in a geometric recursive sequence can be interpreted as a generalization process: from viewing many different steps connecting a figure with the next one to grasping one generic step as a placeholder which represents all the steps. In tune with Radford (2010, p. 55), we expect that the generalization of the RS consists of: 1) “noticing commonality” in some given transition from a figure to the subsequent one; 2) “forming a general concept” of the transition from a figure to the subsequent one by generalizing the noticed commonality to all the transitions within the sequence.

In the following, we rewrite the levels of generalization of the RS and identify the corresponding semiotic means of objectification. At the factual level of generalization, the students, although focused on each individual step to pass from a figure to the next one, begin to identify common features of the different steps. These features emerge from the actions to be done on a specific figure to obtain the next one. This is highlighted by expressions such as “always” and “so on”, referring to something continuing in space and time, which can be iteratively described. Other semiotic means characterizing the factual level are ostensive signs and verbs indicating actions or perceptions. Moreover, the rhythm of the utterances and the ostensive movements can create a cadence revealing the factual level of generalization in a non-linguistic way. Within the contextual level, the different RSs are grasped in terms of a generic representative step, dynamically viewed as a transition process between any two consecutive figures. The semiotic means revealing the contextual generalization are locative and generic terms such as “the next/previous figure”, referring to non-material objects, although spatially situated. The language is hybrid, including abstract and situated elements. The grotesque pointing to concrete objects, typical of the factual level, becomes a refined pointing to non-specific objects, which testifies a new perception field. The symbolic level is revealed by a depersonalized description of the RS, independent from individual actions and perceptions, without reference to space and time. At this level the transition from a figure to the subsequent one is algebraically represented as a static connection between figures n and figure n+1.

We notice that the evolution from the contextual to the symbolic level of generalization of the RS is in line with the transition described by Harel (2001) from the *inference step view*, typical of “quasi-induction”, to the *inference form view*, typical of mathematical induction. The link between the above described levels of generalization also recalls the passage from the *local implications* "P(1)→P(2)", "P(2)→P(3)",..., to the *generic implication* "P(k)→P(k+1)" and finally to the *general implication* "P(k)→P(k+1) for all natural numbers k" discussed in Telloni & Malara (2021) to foster an aware and meaningful learning of the Principle of Mathematical Induction.

METHODOLOGY AND TASK DESIGN

This is a qualitative study, whose data were collected during an educational path carried out in distance learning. The path involved 24 voluntary students (grade 11^{th}-13^{th}, from 7 secondary schools in the centre of Italy) in a series of activities, preliminary to the introduction of mathematical induction. Participants, who agreed to take part in an experimental study and to be video-recorded, were divided in 6 groups of 3-5 students. Students of each group interacted through Meet platform under the supervision of a researcher; they were provided with a shared board and a shared document, where they were required to provide their solutions to some tasks. The researcher did not intervene during the activity, except to give the tasks and to answer specific questions by the students. Collected data consist of the recordings of the whole development of students' activities and not only the final products: the video-calls, the shared board and shared text document. The two researchers separately analyzed the video-recordings, with specific attention on speeches, inscriptions and gestures as semiotic means of objectification of the RS at specific levels of generalization. Then they discuss the outcomes of the individual analyses up to reach an agreement.

In this paper we focus on the first activity of the path, concerning geometric patterns. Specifically, the first five figures of a recursive sequence were given to the students (first task, Figure 1a), followed by two separated requests: (1) "Draw the figures 6 and 7 which continue the sequence", (2) "Describe the sequence of figures in a way that, a reader, following your words, could draw as many figures as she wants starting from fig.1". Later, the first five figures of another recursive sequence were presented to students (second task, Figure 1b), followed by the same requests (1) and (2).

A few comments on the task design are necessary. In the request (1) we asked students to draw the *two consecutive* figures which continue the sequence. We hypothesized that students could draw figure 6 and *then* figure 7 by using the previous drawing, perhaps with some 'copy-and-paste' strategies. In other terms, we thought that this request could support students in focusing on the relationships between two consecutive figures of the sequence, fostering an important *shift of attention* (Mason, 1989): from how to draw one figure of the sequence to how to modify one figure to obtain the following one. In this way an initial objectification of the RS could be attained. Request (2) introduces two new elements to the task: the need to describe the sequence up to a non-specific figure ("as many figures as she wants"), and the need to address the explanation to someone who cannot see the sequence itself. We hypothesized that these two elements, in tune with what happened in Radford's study

(2003), could create an effective joint labour (Radford, 2016) in the group, leading students to transitions from factual to contextual or symbolic generalizations.

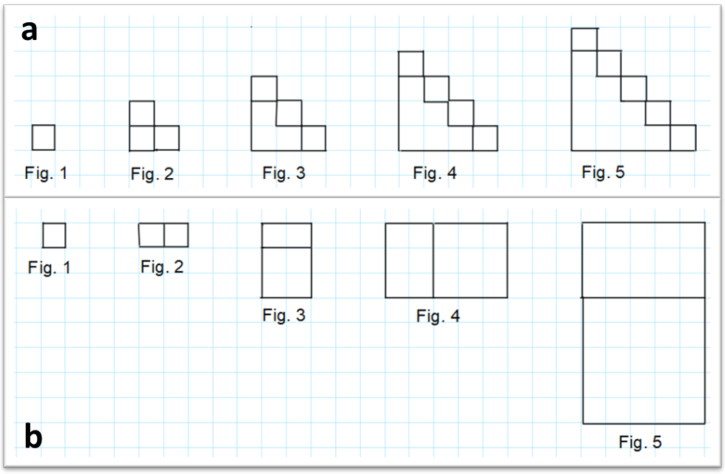

Figure 1: The two sequences of figures, presented as given to the students.

The two sequences were intentionally created with a difference. In the first one the connection between the shape of each figure and its number in the sequence is quite evident: the horizontal and vertical sides of the n-th figure are composed by n squares each. Thus, any figure could be potentially described only in terms of its numerical position in the sequence. In the second sequence, instead, this is not easily possible (it would require a closed form formula for the Fibonacci sequence). We thought that, for this reason, at least in this second case, students would have felt the necessity to describe any figure of the sequence after the first one in terms of the previous one.

CASE ANALYSES

Episode I - Factual generalization and the emergence of a contextual generalization

In this episode a group of 4 male students (G, V, O and P) of grade 12^{th} is facing the second task (Figure 1.b), request (1). After a brief observation, G says:

1 G: It takes the previous figure as the little part, let's say…then it puts a square with area (*inaudible*). And it draws a square on the side.

The sentence is not grasped by the others and an exploratory phase follows, during which the students work on their own without coming to an agreement. After about a minute, while students are discussing and trying to describe the sequence (e.g., "It follows the factorial function, 2x1, 2. 3x2, 6…indeed figure 3 is 6"), G suggests and begins to sketch figure 5 on the board, then extends it to create figure 6 (Figure 2). Then V intervenes:

2 V: What are you trying to draw there?

3 G: The figure… the next one [Figure 6].

4 V: Thus, what do you do? You draw again figure 5… but add a side, right?

5 G: You add the square on the bigger side.

6 P: Ok, I understand.

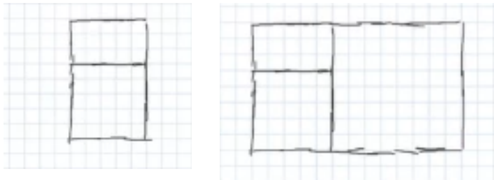

Figure 2: G's drawing of figure 5 and figure 6 on the shared board.

Some other interventions follow, aimed at further clarifying the construction of figure 6, when G is preparing to draw figure 7.

7 G: I think this can be done with the square… wait, I try to put a square over (G writes the label '7', then he inserts a rectangular shape over the drawing of figure 6 (see Figure 3.a)).

8 O: Yes, it's faster. (Meanwhile, G drags the rectangular shape below the label '7' (see Figure 3.b)).

9 P: Thus, what is the criterion?

10 O: You take the bigger side and construct a square over it. That's enough!

After this speech, P and V agree with O and G. Meanwhile, G inserts a square shape with the upper side corresponding to the bigger down side of figure 6, then he drags it under the rectangular shape to create figure 7 (Figure 3.c). Finally the group, answering to request (2), writes on the text document "You take the bigger side and construct a square over it, and so on for every next figure".

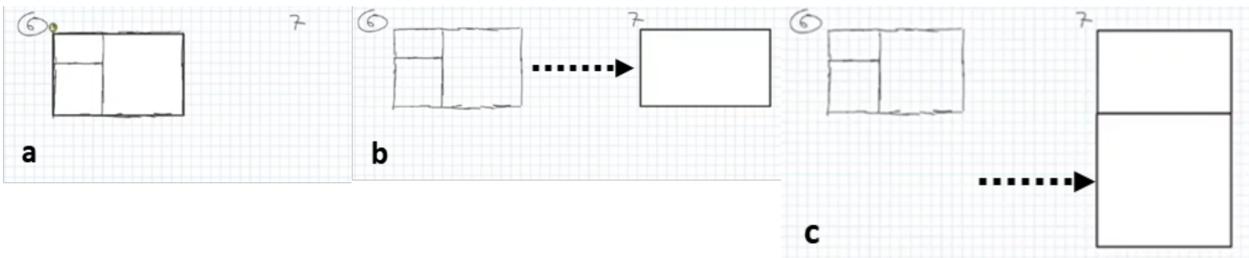

Figure 3: The drawing of figure 7 using figure 6 on the shared board. The arrows indicate the direction of the dragging of the shapes.

The episode shows an initial misalignment between the students in the group: G describes the sequence at a quasi-contextual level of generalization (line 1), highlighted by the expression "previous figure", but the other students do not understand and need to further explore the task. Later, G sketches the figures on the shared board, using figure 5 to draw figure 6 (Figure 2) and then figure 6 to draw figure 7 (Figure 3). Here a factual level of generalization arises, testified by an iterated cycle of actions (trace a figure, re-trace it, extend it to create the next one), where the focus is on the singular link between two consecutive figures (first on the link between figures 5 and 6, then on the link between figures 6 and 7). However, these actions, supported by the joint labour consisting of stimulus questions aimed at the generalization (lines 4 and 9), seem to make a new perceptual field emerge and induce a shared understanding. The transition from a figure to the subsequent one seems to progressively be grasped by students as a placeholder for each transition between consecutive figures in the sequence, towards a contextual level of generalization of the RS. This is testified by line 10 and the final solution written by the group, where the reference to previous and next figures is implicit, but a shift of attention can be highlighted. Indeed, the RS is generically expressed for all the figures in the sequence. Students display satisfaction that the brief description at line 10 is sufficient for a reader to draw the sequence, starting from the first figure ("That's enough!"). Indeed, they finally add "and so on for all the next figures" only because V says that the sentence in line 11 "is too skimpy".

Episode II – Toward a symbolic generalization.

A group of 4 students, three females (K, V and R) and a male (A), all of grade 13^{th}, are facing the first task (Figure 1a). After V draws figure 6 and then, independently, figure 7 on the shared board through her graphic tablet, answering to request (1), the researcher provides request (2). The group tries to describe the sequence of figures in a non-recursive way, facing some difficulties (K: "it is easy to do, but difficult to say"). Then A tries to support the joint labour of the group, speaking to V:

1 A: Let us think about what you did before [when drawing figures 6 and 7].

2 V: I considered the previous figure and then I added those little square.

After one minute of group discussion on describing the diagonal of squares, V says:

3 V: If you think the next figure is the previous figure plus as many little squares as the number of the figure is.

4 A: I'm not sure.

5 V: For example, figure 3 is an L plus three little squares, figure 4 is figure 3 plus four little squares, figure 5 is figure 4 plus five little squares *(V rhythmically swings her head from left to right, according to the cadence of the phrase, (Figure 4))*… Do you understand?

Figure 4: V's head swinging at line 5.

6 A: Ah, ok…yes, ok.

7 R: Thus, you say that figure n will be figure n-1 plus n little squares.

After this excerpt, the group refines the final sentence and writes "figure n-1 is the contour of figure n with n little squares added" as shared solution in the text document. The episode shows the joint labour of the group, which is the key element through which all the students reach an agreement about the generalization of the RS. Initially students see the sequence in different ways, including non-recursive ones. Then V, stimulated by A's metacognitive intervention aimed at reconstructing the actions she performed (line 1), describes recursively the sequence, making explicit the connection between the drawings of figure 6 and figure 7 (line 2), albeit before she apparently drew independently the two figures. The description is at a contextual level of generalization (line 3), revealed by the use of generic and locative terms ("the figure", "the next figure") and references to space and time ("previous, next"). The RS is expressed for any transition between a figure and the next one. The need of sharing knowledge and the doubts of A (line 4) induce V to use the factual level of generalization (line 5), highlighted by the sentences with the same structure referring to iterative actions and by the body language. Although the group passes to a lower level of generalization, they now regard the sequence with a new perspective, oriented by the goal of "thinking about what [they] did before" (line 1). Then they reach an agreement, going beyond the contextual level. In line 7, as well as in the final solution, the RS is generalized at a symbolic level: the description of the sequence is impersonal, with reference to the generic number of the figures. Moreover, there are no references to space and time nor to individual actions or perceptions of the sequence.

CONCLUDING REMARKS

In this paper we interpreted the objectification of the RS as a process of generalization and, building on the Radford's model (2001), we elaborated three levels of generalization. This theoretical lens has been used to design tasks involving geometric recursive sequences and to analyze the processes activated in secondary school students' collaborative activities with these tasks. The analysis allowed us to identify different levels reached by students in grasping the RS and their transition between these levels. Typically, during the collaborative activities, students passed from lower levels of generalization of the RS to higher ones. In particular, some groups reached a symbolic level of generalization of the RS. A key role in these transitions has been played by some elements of the educational path: the chosen sequences, "easy to do, but difficult to say", together with the interaction at distance, preventing some forms of communication, made the tasks more challenging and induced students to make their thoughts explicit. The students' meta-reflection on the actions performed has been also crucial in supporting the development of an effective joint labour (Radford, 2016) within the groups and favouring a shift of attention (Mason, 1989) in viewing the RS. Our analysis also highlights some transitions from higher levels of generalization of the RS to lower ones. In both the episodes, there is an initial misalignment between the students about the way the sequence is viewed. The students who reach a contextual or quasi-contextual level of generalization feel the need to use a factual generalization to foster their colleagues' understanding. In episode I, the description of the RS at the factual level of generalization is made by means of the drawings, done through "copy-and-paste" techniques; instead, in episode II, the factual level emerges through linguistic means of objectification and the body language. This suggests that the joint labour could support an evolution of the students' objectification of the RS towards higher levels of generalization. However, students seem to rely on lower levels of generalization, namely the factual level, for communicative needs, and to reflect on their own actions.

In the future, we plan to extend the study on large scale to investigate these aspects. Moreover, further research is needed to deepen how the objectification of the RS could be exploited to foster the objectification of the induction step towards a meaningful learning of mathematical induction.

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