



Full length article

Shear- or traction-induced motions in incompressible transversely isotropic dispersive materials

Rebecca Tozzi ^a,* , Júlia de Castro Motta ^b, Lorenzo Fusi ^a, Luigi Vergori ^c

^a Dipartimento di Matematica e Informatica “Ulisse Dini”, Università degli Studi di Firenze, Italy

^b Dipartimento di Ingegneria Civile, Università degli Studi di Salerno, Italy

^c Dipartimento di Ingegneria, Università degli Studi di Perugia, Italy

ARTICLE INFO

Keywords:

Transversely isotropic bodies

Dispersive materials

Shear- or traction- induced motions

Projection method

ABSTRACT

In this paper, we analyze a linearized model for incompressible transversely isotropic dispersive fiber-reinforced materials in which both the elastic and dispersive properties depend on the direction of the fibers. We investigate the motions in bodies made of these materials that are subjected to prescribed shear or normal action on part of the rigid boundaries. In the case where the motion is induced by shear stresses acting at the boundary, we provide the analytical solution for the displacement field from which we derive the components of the stress tensor. In the case where the motion is due to the action of normal stresses at the boundary, the initial and boundary value problem is solved using a projection algorithm based on spectral methods. In both cases, we quantify how anisotropic dispersion and material stiffness affect the displacement field.

1. Introduction

Wave-based techniques have become standard tools in non-invasive medical diagnostics, with ultrasound imaging and elastography being two of the most widely used methods. Despite their extensive use in practice, the underlying wave behavior in biological soft tissues is still not completely characterized from a theoretical point of view. In particular, predicting the time-dependent mechanical response of soft tissues under loading remains a non-trivial task and is the subject of ongoing research.

Biological soft tissues do not behave as purely elastic solids, and their mechanical response often depends on the frequency of external actions. This behavior may arise from viscous dissipation as well as from elastic dispersion associated with microstructural scattering. For example, shear-wave dispersion ultrasound vibrometry (SDUV) has been used to infer shear viscosity in addition to stiffness (W. Urban et al., 2012). At the same time, certain diagnostic configurations allow the dispersive contribution linked to microstructure to be examined largely independently of dissipative effects. Recent experiments (Annio et al., 2024) indicate that vascular architecture can cause a substantial non-dissipative component of shear-wave attenuation. In view of these experimental evidences, it is natural to design an analytical study by focusing primarily on elastic dispersive mechanisms, and to include viscoelastic effects in future theoretical treatments.

Within the theory of elastic continua, dispersive effects can be incorporated by enriching the Cauchy stress tensor with suitable higher-order contributions. A series of works by Saccomandi and co-authors (Amendola et al., 2024; Destrade & Saccomandi, 2006, 2008; Saccomandi & Vergori, 2023, 2024) developed constitutive models for hyperelastic solids, both isotropic and anisotropic, in which the total stress is split into the sum of a purely elastic part and an additional dispersive one. This dispersive contribution

* Corresponding author.

E-mail address: rebecca.tozzi@unifi.it (R. Tozzi).

<https://doi.org/10.1016/j.ijengsci.2026.104677>

Received 24 July 2026; Received in revised form 4 September 2026; Accepted 6 September 2026

Available online 16 September 2026

0020-7225/© 2026 The Authors. Published by Elsevier Ltd. This is an open access article under the CC BY-NC-ND license (<http://creativecommons.org/licenses/by-nc-nd/4.0/>).

can, in turn, be separated into the sum of two terms: one that can be derived from a thermodynamic potential, and the other that is associated with powerless internal forces (Amendola et al., 2025; Saccomandi & Vergori, 2024). The resulting models lead to nonlinear dispersive partial differential equations (Klein & Saut, 2021; Linares & Ponce, 2014) and show that the corresponding materials can sustain coherent wave patterns, such as solitary, kink, or compact-like waves.

Biological soft tissues are often idealized as incompressible, and many of them are conveniently modeled as transversely isotropic, as is the case for skin, tendons, ligaments and muscles. Mechanical anisotropy in these materials can manifest itself through dispersion properties that depend on the direction of wave propagation. In this perspective, W. Urban et al. (2012) analyzed wave dispersion in muscle using a viscoelastic model of differential type with distinct stiffness and viscosity parameters along and across the fibers thanks to which they obtained different theoretical dispersion curves for different orientations. Although the constitutive law in that work is formally isotropic, the directional dependence of the material parameters effectively represents transverse isotropy. From an applied perspective, boundary driven transient motions provide a useful canonical setting for isolating the mechanical consequences of fiber-induced anisotropic dispersion.

The recent work of Amendola et al. (2024) is particularly relevant in this topic. There, the authors derived a constitutive model for incompressible, transversely isotropic dispersive materials, neglecting dissipation and accounting for dispersion through a characteristic length scale. This framework offers a convenient setting for describing wave propagation in tissues with pronounced fiber-induced directionality. In that context, the analysis focused on specific classes of motions that admit explicit wave-type solutions.

In the present work, we focus on specific one- and two-dimensional prototype problems for this class of materials. More precisely, we consider the linearized version of the constitutive model proposed in Amendola et al. (2024) and study the two-dimensional incompressible motions of a transversely isotropic dispersive solid subjected to prescribed shear or normal action on part of the rigid boundaries. Although the geometry of the configuration occupied by the solid is very simple, it allows us to recognize how elastic stiffness, dispersive length-scale effects, and fiber orientation combine to shape the time-dependent response. The governing equations are supplemented with appropriate boundary and initial conditions and cast in dimensionless form.

The one dimensional initial and boundary value problem is treated analytically. Specifically, we determine the shearing motion caused by prescribed tangential force on one of the planar boundaries. To the best of our knowledge, this is the first time that the constitutive model introduced in Amendola et al. (2024) is analyzed in terms of an initial and boundary value problem, rather than through special traveling-wave solutions. We document how the components of the displacement field depend on the angle between direction of the fibers and that of the forces prescribed at the boundary, as well as on the dispersive and stiffness parameters.

The two dimensional initial and boundary value problem is instead treated numerically and it is induced by the application of a normal stress at the upper boundary. For spatial discretization, we adopt spectral methods based on Chebyshev polynomials (see, e.g. Fusi & Giovinetto, 2024; Fusi et al., 2025, 2023), while time integration is carried out using a leapfrog scheme. These ingredients are combined within a projection-type algorithm, which is widely used for incompressible motions in fluid dynamics (Bell et al., 1989, 1991; Guermond et al., 2006; Timmermans et al., 1996) and elastodynamics (Feng et al., 2007; Qiang et al., 2017; Talbot & Crampton, 2006).

While the present work is primarily theoretical, it is motivated by the need to further develop the analysis of the constitutive model proposed in Amendola et al. (2024). Our contribution complements and extends this previous work by broadening its mathematical framework, thereby providing a deeper understanding of the range of phenomena that the model is capable of describing, highlighting its potential for future applications.

The paper is organized as follows: the constitutive model is presented and discussed in Section 2; in Section 3 we derive the governing equations, and introduce appropriate initial and boundary conditions. Next, we non-dimensionalize the governing equations and the initial and boundary conditions and obtain a non-dimensional system of time-dependent partial differential equations, which is solved numerically in Section 5. The shearing motion is analyzed in Section 4, where analytical solutions for displacement field are determined and some observations on the stress components are presented. In Section 5.2 we discuss the results, showing plots of the behavior of the main variables, and finally draw some conclusions in Section 6.

2. Mathematical model

Let $\mathcal{B}$ be an incompressible transversely isotropic dispersive body whose mechanical response is modeled within the theory of simple material of differential type as in Amendola et al. (2024), and let

$$\Omega^* = \{(x^*, y^*, t^*) \mid (x^*, y^*) \in [-L^*, L^*] \times [-H^*, H^*], t^* \in [0, T^*]\}, \quad (1)$$

where the superscript $*$ is appended to denote dimensional quantities, be the region in the space–time in which we shall study the motions of $\mathcal{B}$.

To make the analytical treatment as simple as possible, we limit our analysis to the linear regime, that is, to motions in which the displacement gradient is very small. More precisely, denoted by $\mathbf{u}^*$ the displacement vector, we consider motions during which

$$\varpi \equiv \max_{\Omega^*} |\nabla^* \mathbf{u}^*(x^*, y^*, t^*)| \ll 1. \quad (2)$$

Bear in mind that here x^* and y^* denote *material* coordinates and not spatial ones. Likewise, the differential operator div^* , which will be used throughout the paper, denotes the divergence computed with respect to the material coordinates, consistently with the definition of the gradient operator ∇^* introduced above. Indeed, the spatial coordinates differ from the material ones by the displacement. However, in the framework of the linearized theory of elasticity, this difference affects the Lagrangian and Eulerian

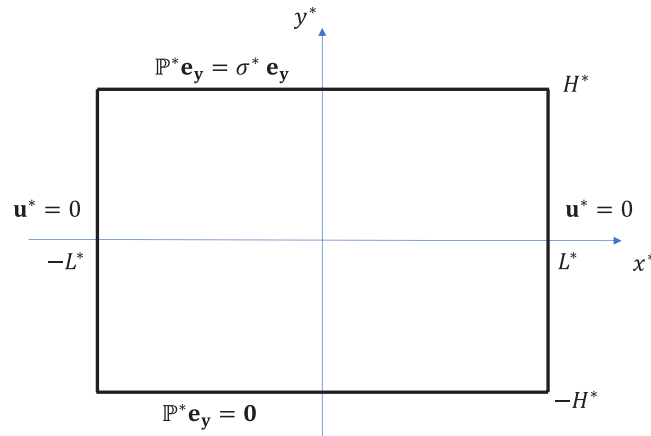


Fig. 1. Sketch of the geometry of the two-dimensional problem.

formulations only through higher-order terms, which are neglected here. We have therefore made the *unconventional* choice to denote the material coordinates by lower-case letters rather than upper-case ones.

The transversely isotropic dispersive body $\mathcal{B}$ is supposed to be made of a composite material consisting of a hyperelastic ground matrix reinforced by a single family of fibers whose orientation is described by a unit vector $\mathbf{M}$ which is usually called the *preferred direction*. As in Amendola et al. (2024), both the strain energy function and the dispersion potential are assumed to depend explicitly on the preferred direction as well as on the appropriate invariants, whence also the material properties are somehow affected by the presence of the fibers. Specifically, the stiffnesses of the ground matrix in directions parallel and normal to $\mathbf{M}$ are given by the elastic moduli μ_L^* and μ_T^* , respectively, and the amount of the stiffness of the fibers is characterized by the constant β^* . As the dispersion potential is concerned, the constants ν_L^* and ν_T^* denote dispersion material parameters of the ground matrix in directions parallel and normal to $\mathbf{M}$, respectively, and γ^* provides a measure of the material dispersion in the fibers. In order that the elastic stored energy attains a minimum in the reference configuration the stiffnesses must satisfy the inequalities

$$\mu_L^* > 0, \quad \mu_T^* > 0, \quad \frac{\beta^*}{2} + 2\mu_L^* > \mu_T^*, \quad (3a)$$

while to guarantee that the dispersion potential is minimum at equilibrium, the dispersion parameters must be so that Amendola et al. (2024)

$$\nu_L^* > 0, \quad \nu_T^* > 0, \quad \frac{\gamma^*}{2} + 2\nu_L^* > \nu_T^*. \quad (3b)$$

Within the infinitesimal theory, the stress tensor $\mathbb{T}^*$ in these dispersive anisotropic bodies approximates as

$$\mathbb{T}^* \approx \mathbb{P}^* = -p^* \mathbb{I} + \mathbb{S}^*, \quad (4a)$$

where $-p^* \mathbb{I}$ is the indeterminate reactive part of the stress tensor, with p^* being the Lagrange multiplier associated with the constraint of incompressibility, and the extra stress

$$\begin{aligned} \mathbb{S}^* = & \beta^* (\mathbb{E}^* \mathbf{M} \cdot \mathbf{M}) \mathbf{M} \otimes \mathbf{M} + 2\mu_T^* \mathbb{E}^* + 2(\mu_L^* - \mu_T^*) (\mathbb{E}^* \mathbf{M} \otimes \mathbf{M} + \mathbf{M} \otimes \mathbb{E}^* \mathbf{M}) \\ & + \rho^* \gamma^* (\mathbb{E}^* \mathbf{M} \cdot \mathbf{M}) \mathbf{M} \otimes \mathbf{M} + 2\rho^* \nu_T^* \mathbb{E}^* + 2\rho^* (\nu_L^* - \nu_T^*) (\mathbb{E}^* \mathbf{M} \otimes \mathbf{M} + \mathbf{M} \otimes \mathbb{E}^* \mathbf{M}) \end{aligned} \quad (4b)$$

represents the active part of $\mathbb{P}^*$.

In (4) the infinitesimal strain tensor $\mathbb{E}^*$ coincides with the symmetric part of the displacement gradient, the superposed dots denote differentiation with respect to time, and, assuming that the body is homogeneous, ρ^* is the constant mass density.

3. Governing equations

In what follows, we investigate the one- and two-dimensional infinitesimal motions of an incompressible transversely isotropic dispersive material whose mechanical response is described by the constitutive model (4) in a rectangular domain of length $2L^*$ and width $2H^*$ (see Fig. 1).

Both the displacement vector $\mathbf{u}^*$ and the preferred direction $\mathbf{M}$ lie in the x^*y^* -plane and have components

$$\mathbf{u}^* = u^*(x^*, y^*, t^*) \mathbf{e}_x + v^*(x^*, y^*, t^*) \mathbf{e}_y, \quad \mathbf{M} = \cos \theta \mathbf{e}_x + \sin \theta \mathbf{e}_y, \quad (5)$$

where, due to the head-tail symmetry of fibers, the angle θ that parametrizes the preferred direction belongs to the interval $[0, \pi[$.

In the absence of body forces, the equations that govern the motions of $\mathcal{B}$ are the equation of the constraint of incompressibility and the balance of linear momentum:

$$\begin{cases} \rho^* \ddot{\mathbf{u}}^* = \text{div}^* \mathbb{P}^*, \\ \text{div}^* \mathbf{u}^* = 0. \end{cases} \quad (6)$$

Clearly, this system of PDEs has to be solved under prescribed initial and boundary conditions. As the initial conditions are concerned, we assume that at the initial time $t^* = 0$ the body $\mathcal{B}$ is at equilibrium in the reference configuration, viz

$$\mathbf{u}^*(x^*, y^*, 0) = \mathbf{0} \quad \text{and} \quad \dot{\mathbf{u}}^*(x^*, y^*, 0) = \mathbf{0} \quad \text{for all } (x^*, y^*) \in [-L^*, L^*] \times [-H^*, H^*]. \quad (7)$$

With regard to restrictions on motion at $\partial\Omega^*$, we consider two distinct sets of boundary conditions.

- Under the assumption that the aspect ratio $\delta = H^*/L^*$ is very small, the rectangular domain $[-L^*, L^*] \times [-H^*, H^*]$ can be approximated to a strip that is unbounded in the x^* -direction, so it makes sense to study the one-dimensional motions of the form $\mathbf{u}^* = u^*(y^*, t^*) \mathbf{e}_x$ that are caused by a constant shear stress τ^* applied at $y^* = H^*$, with the lower boundary $y^* = -H^*$ fixed (see Fig. 2). In formulae, we require that

$$\begin{cases} \mathbb{P}^* \mathbf{e}_y = \tau^* \mathbf{e}_x & \text{on } y^* = H^*, \\ \mathbf{u}^* = \mathbf{0} & \text{on } y^* = -H^*. \end{cases} \quad (8)$$

- For two-dimensional motions we take, for simplicity, the aspect ratio δ equal to unity and suppose that the inlet and outlet cross-sections $x^* = \pm H^*$ are fixed. The upper and lower boundaries $y^* = \pm H^*$ are assumed to be subjected to a prescribed constant normal stress σ^* and stress-free condition, respectively. More precisely, we impose the following boundary conditions

$$\begin{cases} \mathbf{u}^* = 0 & \text{on } x^* = \pm H^*, \\ \mathbb{P}^* \mathbf{e}_y = \sigma^* \mathbf{e}_y & \text{on } y^* = H^*, \\ \mathbb{P}^* \mathbf{e}_y = \mathbf{0} & \text{on } y^* = -H^*. \end{cases} \quad (9)$$

3.1. Non-dimensionalization

To ease the analytical and numerical treatments of the initial and boundary-value problem (6), (7) and (8) or (9), we find it convenient to non-dimensionalize the governing equations and the initial and boundary conditions by choosing $U_c^* = \varpi H^*$ and $t_c^* = \sqrt{\rho^* \nu_T^* / \mu_L^*}$ as characteristic displacement and time, respectively, and introducing the following dimensionless quantities

$$\begin{aligned} x &= \frac{x^*}{H^*}, & y &= \frac{y^*}{H^*}, & u &= \frac{u^*}{U_c^*}, & v &= \frac{v^*}{U_c^*}, & \nabla &= H^* \nabla^* \\ \text{div} &= H^* \text{div}^*, & \mathbb{E} &= \frac{H^*}{U_c^*} \mathbb{E}^*, & \mathbb{S} &= \frac{H^*}{\mu_L^* U_c^*} \mathbb{S}^*, & p &= \frac{H^*}{\mu_L^* U_c^*} p^*, & t &= \frac{t^*}{t_c^*}, \\ \beta &= \frac{\beta^*}{\mu_L^*}, & \gamma &= \frac{\gamma^*}{\nu_T^*}, & \mu_T &= \frac{\mu_T^*}{\mu_L^*}, & \nu_L &= \frac{\nu_L^*}{\nu_T^*}. \end{aligned} \quad (10)$$

Inserting the dimensionless quantities (10) into (1), (4) and (6), and using the subscript notation for differentiation, the dimensionless governing equations read as

$$\begin{cases} u_{tt} = -p_x + (S_{11})_x + (S_{12})_y, \\ v_{tt} = -p_y + (S_{12})_x + (S_{22})_y, \\ u_x + v_y = 0, \end{cases} \quad (11)$$

in

$$\Omega = \{(x, y, t) | x \in [-\delta^{-1}, \delta^{-1}], y \in [-1, 1], t \in [0, T]\}, \quad (12)$$

where $T = T^*/t_c^*$ and the non-zero independent components of the dimensionless extra stress tensor $\mathbb{S}$ are as follows:

$$\begin{aligned} S_{11} &= \beta \left[u_x \cos^2 \theta + \frac{u_y + v_x}{2} \sin 2\theta + v_y \sin^2 \theta \right] \cos^2 \theta + 2\mu_T u_x \\ &\quad + (1 - \mu_T) \left[4u_x \cos^2 \theta + (u_y + v_x) \sin 2\theta \right] \\ &\quad + \gamma \left[u_{xtt} \cos^2 \theta + \frac{u_{ytt} + v_{xtt}}{2} \sin 2\theta + v_{ytt} \sin^2 \theta \right] \cos^2 \theta + 2u_{xtt} \\ &\quad + (\nu_L - 1) \left[4u_{xtt} \cos^2 \theta + (u_{ytt} + v_{xtt}) \sin 2\theta \right], \\ S_{12} &= \frac{\beta}{2} \left[u_x \cos^2 \theta + \frac{u_y + v_x}{2} \sin 2\theta + v_y \sin^2 \theta \right] \sin 2\theta + u_y + v_x \\ &\quad + (1 - \mu_T)(u_x + v_y) \sin 2\theta + \frac{\gamma}{2} \left[u_{xtt} \cos^2 \theta + \frac{u_{ytt} + v_{xtt}}{2} \sin 2\theta + v_{ytt} \sin^2 \theta \right] \sin 2\theta \end{aligned} \quad (13a)$$

$$\begin{aligned} S_{12} &= \frac{\beta}{2} \left[u_x \cos^2 \theta + \frac{u_y + v_x}{2} \sin 2\theta + v_y \sin^2 \theta \right] \sin 2\theta + u_y + v_x \\ &\quad + (1 - \mu_T)(u_x + v_y) \sin 2\theta + \frac{\gamma}{2} \left[u_{xtt} \cos^2 \theta + \frac{u_{ytt} + v_{xtt}}{2} \sin 2\theta + v_{ytt} \sin^2 \theta \right] \sin 2\theta \end{aligned} \quad (13b)$$

$$\begin{aligned}
& + v_L(u_{ytt} + v_{xtt}) + (v_L - 1)(u_{xtt} + v_{ytt}) \sin 2\theta, \\
S_{22} = & \beta \left[u_x \cos^2 \theta + \frac{u_y + v_x}{2} \sin 2\theta + v_y \sin^2 \theta \right] \sin^2 \theta + 2\mu_T v_y \\
& + (1 - \mu_T) \left[4v_y \sin^2 \theta + (u_y + v_x) \sin 2\theta \right] \\
& + \gamma \left[u_{xtt} \cos^2 \theta + \frac{u_{ytt} + v_{xtt}}{2} \sin 2\theta + v_{ytt} \sin^2 \theta \right] \sin^2 \theta + 2v_{ytt} \\
& + (v_L - 1) \left[4v_{ytt} \sin^2 \theta + (u_{ytt} + v_{xtt}) \sin 2\theta \right].
\end{aligned} \tag{13c}$$

On the other hand, inserting (10) into (7), (8), and (9) we obtain the dimensionless initial and boundary conditions to append to system (11). In particular, we find that

$$u = v = u_t = v_t = 0 \quad \text{at } t = 0, \tag{14}$$

and for one-dimensional motions the dimensionless version of the boundary conditions (8) reads

$$\begin{cases} S_{12} = \frac{\tau^* H^*}{\mu_L^* U_c^*} \equiv \tau & \text{on } y = 1, \\ -p + S_{22} = 0 & \text{on } y = 1, \\ u = 0 & \text{on } y = -1, \end{cases} \tag{15}$$

and for two-dimensional motions the boundary conditions (9) become

$$\begin{cases} u = v = 0 & \text{on } x = \pm 1, \\ S_{12} = 0 & \text{on } y = \pm 1, \\ -p + S_{22} = 0 & \text{on } y = -1, \\ -p + S_{22} = \frac{\sigma^* H^*}{\mu_L^* U_c^*} \equiv \sigma & \text{on } y = 1. \end{cases} \tag{16}$$

4. One-dimensional motions

We assume that the aspect ratio $\delta \ll 1$, so that the dimensionless domain (12) in which we study the motion can be approximated as $\Omega \approx \mathbb{R} \times [-1, 1] \times [0, T]$.

Moreover, since the domain is unbounded in the x -direction and the boundary conditions (15) prescribed for the one-dimensional motions do not depend on x , the resulting initial-boundary value problem is invariant under translations along x . It is therefore natural to seek a solution that shares this invariance, namely a displacement field of the form $\mathbf{u} = u(y, t) \mathbf{e}_x$; consistently, every other variable, including the Lagrange multiplier p , turns out to depend only on y and t . To obtain the shearing motions that are caused by the application of a shear stress at the upper boundary, with the lower boundary being fixed, we have to solve the following dimensionless initial-boundary value problem (IBVP)

$$\begin{cases} u_{tt} = (S_{12})_y, \\ 0 = -p_y + (S_{22})_y, \\ u(-1, t) = 0, \\ S_{12}(1, t) = \tau, \quad -p(1, t) + S_{22}(1, t) = 0, \\ u(y, 0) = u_t(y, 0) = 0. \end{cases} \tag{17}$$

where the components of the extra stress are

$$S_{11} = \Gamma_\theta^E u_y + \Gamma_\theta^D u_{ytt}, \quad S_{12} = \Lambda_\theta^E u_y + \Lambda_\theta^D u_{ytt}, \quad S_{22} = \Xi_\theta^E u_y + \Xi_\theta^D u_{ytt}, \tag{18a}$$

with

$$\Gamma_\theta^E = \left(\frac{\beta}{2} \cos^2 \theta + 1 - \mu_T \right) \sin 2\theta, \quad \Gamma_\theta^D = \left(\frac{\gamma}{2} \cos^2 \theta + v_L - 1 \right) \sin 2\theta, \tag{18b}$$

$$\Lambda_\theta^E = \frac{\beta}{4} \sin^2 2\theta + 1, \quad \Lambda_\theta^D = \frac{\gamma}{4} \sin^2 2\theta + v_L, \tag{18c}$$

$$\Xi_\theta^E = \left(\frac{\beta}{2} \sin^2 \theta + 1 - \mu_T \right) \sin 2\theta, \quad \Xi_\theta^D = \left(\frac{\gamma}{2} \sin^2 \theta + v_L - 1 \right) \sin 2\theta, \tag{18d}$$

4.1. Equilibrium problem

We start by considering the equilibrium problem. In particular, we look for a displacement field parallel to the x -direction that depends only on the spatial variable y , namely $u = u(y)$ and $v \equiv 0$, that solves the equilibrium equations

$$\begin{cases} \frac{dS_{12}}{dy} = 0, \\ \frac{dS_{22}}{dy} = \frac{dp}{dy}, \end{cases} \tag{19a}$$

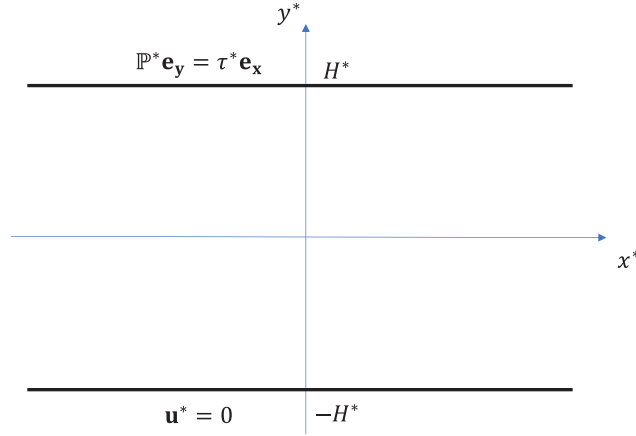


Fig. 2. Geometry of the domain with small aspect ratio δ .

and the boundary conditions

$$\begin{cases} u(-1) = 0, \\ S_{12}(1) = \tau, \quad -p(1) + S_{22}(1) = 0. \end{cases} \quad (19b)$$

For this class of equilibrium configurations, the components of the extra stress tensor (18c) simplify considerably (because the part of the Cauchy stress tensor that accounts for dispersion vanishes as the displacement depends only on the spatial variable y) and read

$$S_{11} = \Gamma_{\theta}^E u_y, \quad S_{12} = \Lambda_{\theta}^E u_y, \quad S_{22} = \Xi_{\theta}^E u_y. \quad (20)$$

Integration of both equations in (19a) with respect to y gives the following expressions for the Lagrange multiplier p and the shear stress S_{12}

$$p = S_{22} + p_0 \quad (21a)$$

and

$$S_{12} = k, \quad (21b)$$

where p_0 and k are integration constants, having taken into account that all the variables involved are independent of x .

From the third boundary condition in (19b), (21a), we deduce that $p_0 = 0$, hence from (20), (21b) and the first and second boundary conditions in (19b) the displacement is found to be

$$u = \frac{\tau}{\Lambda_{\theta}^E} (y + 1). \quad (22)$$

For completeness, we report that the Lagrange multiplier and the components of the extra stress are constant and equal to

$$p = S_{22} = \frac{\tau \Xi_{\theta}^E}{\Lambda_{\theta}^E}, \quad S_{12} = \tau, \quad S_{11} = \frac{\tau \Gamma_{\theta}^E}{\Lambda_{\theta}^E}. \quad (23)$$

4.2. Shearing motions

When the displacement field depends also on time, i.e. $\mathbf{u} = u(y, t) \mathbf{e}_x$, the motion is governed by the following system of linear PDEs

$$\begin{cases} u_{tt} = \Lambda_{\theta}^E u_{yy} + \Lambda_{\theta}^D u_{yyt}, \\ 0 = -p_y + \Xi_{\theta}^E u_{yy} + \Xi_{\theta}^D u_{yyt}, \end{cases} \quad (24)$$

Following similar arguments as in the previous subsection, we deduce $p = S_{22}$, so that the problem reduces to solving the sole linear partial differential equation

$$u_{tt} - \Lambda_{\theta}^E u_{yy} - \Lambda_{\theta}^D u_{yyt} = 0, \quad (25)$$

subject to the following initial and boundary conditions:

$$\begin{cases} u(y, 0) = u_t(y, 0) = 0 & \text{for all } y \in \mathbb{R}, \\ u(-1, t) = 0 & \text{for all } t \geq 0, \\ \Lambda_{\theta}^E u_y(1, t) + \Lambda_{\theta}^D u_{yt}(1, t) = \tau & \text{for all } t \geq 0. \end{cases} \quad (26)$$

In looking for a solution of the IBVP (25)–(26), we find it convenient to take the displacement in the form

$$u(y, t) = \tilde{u}(y) + w(y, t), \quad (27)$$

where $\tilde{u}$ is as in (22) and w is to be determined by solving the following IBVP:

$$\begin{cases} w_{tt} - \Lambda_\theta^E w_{yy} - \Lambda_\theta^D w_{yyt} = 0 & \text{for all } (y, t) \in \mathbb{R} \times \mathbb{R}_0^+, \\ w(y, 0) = -\tilde{u}(y) & \text{for all } y \in \mathbb{R}, \\ w_t(y, 0) = 0 & \text{for all } y \in \mathbb{R}, \\ w(-1, t) = 0 & \text{for all } t \geq 0, \\ \Lambda_\theta^E w_y(1, t) + \Lambda_\theta^D w_{yt}(1, t) = 0 & \text{for all } t \geq 0. \end{cases} \quad (28)$$

Solving (28) by resorting to the method of the separation of variables gives

$$w(y, t) = \sum_{n=0}^{\infty} a_n \cos(\kappa_n t) \sin[\lambda_n(y+1)], \quad (29a)$$

where

$$\lambda_n^2 = \frac{(2n+1)^2 \pi^2}{16} \quad (n \in \mathbb{N}_0) \quad (29b)$$

are the eigenvalues of the eigenvalue problem

$$\begin{cases} \frac{d^2 Y}{dy^2} + \lambda^2 Y = 0, \\ Y(-1) = Y'(1) = 0, \end{cases} \quad (29c)$$

with the corresponding eigenfunctions $Y_n(y) = \sin[\lambda_n(y+1)]$ constituting a Hilbert basis for the functional space $\mathcal{H} = \{f \in L^2([-1, 1]) : f(-1) = f'(1) = 0\}$,

$$\kappa_n^2 = \frac{\lambda_n^2 \Lambda_\theta^E}{1 + \lambda_n^2 \Lambda_\theta^D} \quad (29d)$$

are the opposite of the constants of separation, and

$$a_n = - \int_{-1}^1 \tilde{u}(y) Y_n(y) dy = \frac{(-1)^{n+1} \tau}{\Lambda_\theta^E \lambda_n^2} \quad (29e)$$

are the Fourier coefficients of the initial datum for $-\tilde{u}$.

We can then conclude that the solution of the IBVP problem (25)–(26) expands as

$$u(y, t) = \frac{\tau}{\Lambda_\theta^E} \left\{ y + 1 + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{\lambda_n^2} \cos(\kappa_n t) \sin[\lambda_n(y+1)] \right\}. \quad (30)$$

Note that since

$$\pi \sqrt{\frac{\Lambda_\theta^E}{16 + \pi^2 \Lambda_\theta^D}} \leq \kappa_n < \sqrt{\frac{\Lambda_\theta^E}{\Lambda_\theta^D}} \quad \text{for all } n \in \mathbb{N}_0 \quad (31)$$

and the order of infinitesimal of the Fourier coefficients a_n as $n \rightarrow +\infty$ is two, the series (29a) and all its time derivatives converge uniformly. By contrast, the first derivative of the series (29a) with respect to y converges pointwise and in the $L^2([-1, 1])$ -norm, but not uniformly. Consequently, (30) must be regarded as a generalized solution of the IBVP (25)–(26) as it is continuous but not continuously differentiable.

Fig. 3 displays the displacement field for different values of the angle θ . By the way, note that since (30) contains only the terms Λ_θ^E and Λ_θ^D that are periodic in θ with period $\pi/2$, we have restricted our analysis to acute angles. In particular, for any $T > 0$, the sup-norm of the solution given in (30),

$$\|u\|_\infty = \max_{(y,t) \in [-1,1] \times [0,T]} |u(y, t)|, \quad (32)$$

depends on the fiber orientation θ only through the coefficients Λ_θ^E and Λ_θ^D , both of which, according to (18c), are functions of $\sin^2(2\theta)$ alone. Hence $\|u\|_\infty$ can be regarded as a function of $\sin^2(2\theta)$, and, by the chain rule, every stationary point of $\theta \mapsto \sin^2(2\theta)$ is necessarily a critical point of $\|u\|_\infty$ as well. These stationary points are the solutions of

$$\frac{d \sin^2(2\theta)}{d\theta} = \sin(4\theta) = 0,$$

which, restricted to the interval of acute angles considered above, are $\theta = 0$ and $\theta = \pi/4$. A direct numerical evaluation of $\|u\|_\infty$ confirms that these are indeed the extremal orientations, the maximum being attained when the preferred direction is aligned with $\mathbf{e}_x$ (i.e. $\theta = 0$) and the minimum at $\theta = \pi/4$.

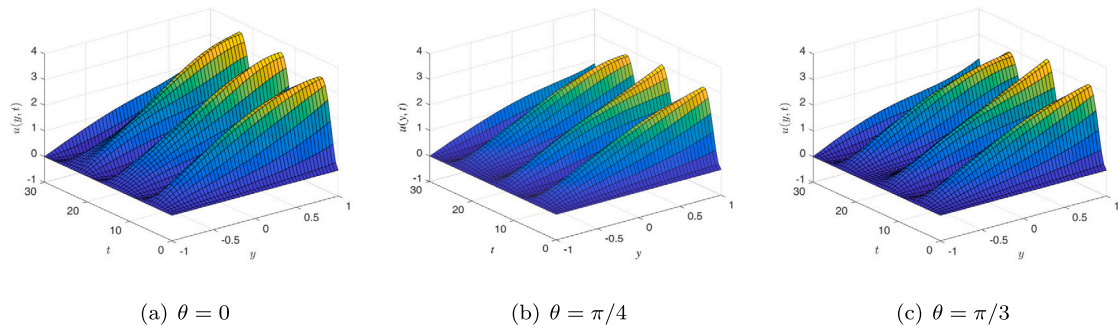


Fig. 3. Displacement field $u(y, t)$ for $(y, t) \in [-1, 1] \times [0, 30]$ for different orientation of the preferred direction. The dimensionless parameters are taken as follows: $\beta = 1$, $\mu_T = 2$, $\gamma = 2$, $\nu_L = 0.5$, and $\tau = 1$.

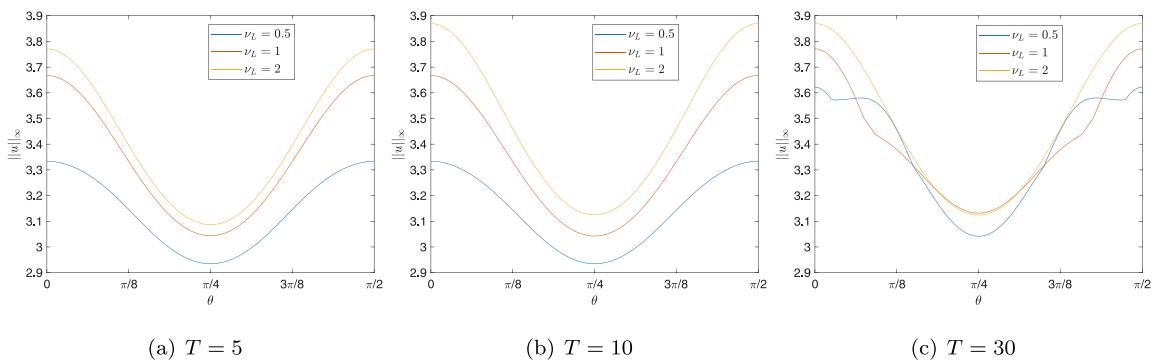


Fig. 4. Plots of the sup-norm (32) of the displacement field $u(y, t)$ as function of θ at different values of ν_L for (a) $T = 5$, (b) $T = 10$, (c) $T = 30$. Parameters are set to $\beta = 1$, $\mu_T = 2$, $\gamma = 2$, $\tau = 1$.

In addition, observe that although the generalized solution (30) is unchanged under the substitutions $\theta \longleftrightarrow \pi - \theta$ and $\theta \longleftrightarrow \theta + \pi/2$, only the shear stress inherits such a symmetry, the extra normal stresses change under these substitutions. This is a direct consequence of (18c) and the fact that only Λ_θ^E and Λ_θ^D are periodic in θ with period $\pi/2$, while Γ_θ^E , Γ_θ^D , Ξ_θ^E and Ξ_θ^D have period π . Note that, when combining (18c) with the derivative with respect to the spatial variable of (30) to determine the components of the stress tensor, in view of the discussion above, the so-obtained series converge only pointwise.

Finally, we focus on how the stiffness and the dispersion parameters of the ground matrix affect the shearing solution (30). First, note that the elastic modulus μ_T does not appear at all in the governing Eq. (25) or in its boundary conditions (26), and therefore does not enter the solution (30) either: as shown by (18c), the coefficients Λ_θ^E and Λ_θ^D , which alone control the shearing motion, depend only on β , γ , ν_L and θ . The stiffness parameter μ_T only enters the coefficients Γ_θ^E and Ξ_θ^E that determine the extra normal stresses S_{11} and S_{22} ; hence, in the present shearing problem, μ_T affects the stress state of the body but has no influence on the displacement field.

As regards the dispersion parameter ν_L of the ground matrix, we observe that neither the static displacement $\tilde{u}$ in (22) nor the Fourier coefficients a_n in (29e) depend on ν_L , both being functions of θ and β alone through Λ_θ^E . The parameter ν_L enters the solution (30) exclusively through Λ_θ^D and, in turn, through the temporal frequencies κ_n defined in (29d), which decrease monotonically as ν_L increases. Fig. 4 shows the sup-norm of the displacement u , defined by (32), as a function of the angle θ at different values of ν_L for three observation times T , i.e. $T = 5$, $T = 10$ and $T = 30$. In particular, for the moderate observation times $T = 5$ and $T = 10$ (Figs. 4(a) and 4(b)), the sup-norm (32) increases monotonically with ν_L for every fiber orientation θ , while preserving the qualitative dependence on θ discussed above, namely periodicity of period $\pi/2$ with a minimum at $\theta = \pi/4$. For the longer observation time $T = 30$ (Fig. 4(c)), the three curves no longer scale with ν_L in the same uniform way: the curve corresponding to $\nu_L = 0.5$ (i.e. to the case in which the dispersion parameter of the ground matrix in the direction parallel to $\mathbf{M}$ is smaller than in the transverse direction) appears to be the one most affected by the increase of the observation time: as T grows from 5, 10 to 30, both its values near $\theta = 0, \pi/2$ and its minimum near $\theta = \pi/4$ increase appreciably and the curve develops additional oscillations, whereas the curves for $\nu_L = 1$ and, especially, $\nu_L = 2$ change comparatively little.

5. Two-dimensional motions

5.1. Numerical approach

To solve the IBVP (11), (14), and (16), we employ the leapfrog method for time integration and spectral methods based on Chebyshev polynomials for spatial discretizations. The problem is solved via the projection algorithm that is useful when dealing with time-dependent incompressible motions. In doing so, at each time step, we solve two decoupled systems of equations: one for an intermediate displacement and one for an auxiliary pressure. Then, we make a correction on the displacement solution obtained from the first system. Going into further detail, at the time step $n + 1$, we define the linear operator corresponding to the leapfrog formula for the second derivative as

$$\mathcal{L}^{(n+1)}(\tilde{u}) := \frac{\tilde{u} - 2u^{(n)} + u^{(n-1)}}{(\Delta t)^2}, \quad \mathcal{L}^{(n+1)}(\tilde{v}) := \frac{\tilde{v} - 2v^{(n)} + v^{(n-1)}}{(\Delta t)^2},$$

where $u^{(j)}$, $v^{(j)}$ are the iterates referred to the j th time step, $\tilde{u}$, $\tilde{v}$ are the solutions of the first system, and Δt is the temporal step. In addition, we use the operators D_x , D_y , D_{xy} , D_{xx} , D_{yy} to represent the discretized first and second derivatives with respect to the spatial variables.

The first step consists of solving a system formed by (11)₁, (11)₂, (16), where we have ignored the pressure terms and whose unknowns are $\tilde{u}$, $\tilde{v}$. Therefore, we end up with the following problem for an intermediate displacement $\tilde{\mathbf{u}} = (\tilde{u}, \tilde{v})^T$ (which is not incompressible)

$$\begin{aligned} \mathcal{L}^{(n+1)}(\tilde{u}) = D_x \left[\beta f_1(\tilde{u}, \tilde{v}) \cos^2 \theta + 2\mu_T f_2(\tilde{u}) + (1 - \mu_T) f_3(\tilde{u}, \tilde{v}) + \gamma g_1(\tilde{u}, \tilde{v}) \cos^2 \theta \right. \\ \left. + 2g_2(\tilde{u}) + (v_L - 1)g_3(\tilde{u}, \tilde{v}) \right] + D_y \left[\frac{\beta}{2} f_1(\tilde{u}, \tilde{v}) \sin 2\theta + \mu_T f_4(\tilde{u}, \tilde{v}) \right. \\ \left. + (1 - \mu_T) f_5(\tilde{u}, \tilde{v}) + \frac{\gamma}{2} g_1(\tilde{u}, \tilde{v}) \sin 2\theta + g_4(\tilde{u}, \tilde{v}) + (v_L - 1)g_5(\tilde{u}, \tilde{v}) \right], \end{aligned} \quad (33a)$$

$$\begin{aligned} \mathcal{L}^{(n+1)}(\tilde{v}) = D_x \left[\frac{\beta}{2} f_1(\tilde{u}, \tilde{v}) \sin 2\theta + \mu_T f_4(\tilde{u}, \tilde{v}) + (1 - \mu_T) f_5(\tilde{u}, \tilde{v}) + \frac{\gamma}{2} g_1(\tilde{u}, \tilde{v}) \sin 2\theta \right. \\ \left. + g_4(\tilde{u}, \tilde{v}) + (v_L - 1)g_5(\tilde{u}, \tilde{v}) \right] + D_y \left[\beta f_1(\tilde{u}, \tilde{v}) \sin^2 \theta + 2\mu_T f_6(\tilde{v}) \right. \\ \left. + (1 - \mu_T) f_7(\tilde{u}, \tilde{v}) + \gamma g_1(\tilde{u}, \tilde{v}) \sin^2 \theta + 2g_6(\tilde{v}) + (v_L - 1)g_7(\tilde{u}, \tilde{v}) \right], \end{aligned} \quad (33b)$$

$$\tilde{u} = \tilde{v} = 0 \quad \text{at } x = \pm 1, \quad (33c)$$

$$\begin{aligned} \frac{\beta}{2} f_1(\tilde{u}, \tilde{v}) \sin 2\theta + \mu_T f_4(\tilde{u}, \tilde{v}) + (1 - \mu_T) f_5(\tilde{u}, \tilde{v}) + \frac{\gamma}{2} g_1(\tilde{u}, \tilde{v}) \sin 2\theta \\ + g_4(\tilde{u}, \tilde{v}) + (v_L - 1)g_5(\tilde{u}, \tilde{v}) = 0 \quad \text{at } y = \pm 1, \end{aligned} \quad (33d)$$

$$\begin{aligned} \beta f_1(\tilde{u}, \tilde{v}) \sin^2 \theta + 2\mu_T f_6(\tilde{v}) + (1 - \mu_T) f_7(\tilde{u}, \tilde{v}) + \gamma g_1(\tilde{u}, \tilde{v}) \sin^2 \theta \\ + 2g_6(\tilde{v}) + (v_L - 1)g_7(\tilde{u}, \tilde{v}) = 0 \quad \text{at } y = -1, \end{aligned} \quad (33e)$$

$$\begin{aligned} \beta f_1(\tilde{u}, \tilde{v}) \sin^2 \theta + 2\mu_T f_6(\tilde{v}) + (1 - \mu_T) f_7(\tilde{u}, \tilde{v}) + \gamma g_1(\tilde{u}, \tilde{v}) \sin^2 \theta \\ + 2g_6(\tilde{v}) + (v_L - 1)g_7(\tilde{u}, \tilde{v}) = \sigma \quad \text{at } y = 1, \end{aligned} \quad (33f)$$

where

$$f_1(\tilde{u}, \tilde{v}) := D_x(\tilde{u}) \cos^2 \theta + \frac{1}{2} \left[D_y(\tilde{u}) + D_x(\tilde{v}) \right] \sin 2\theta + D_y(\tilde{v}) \sin^2 \theta, \quad f_2(\tilde{u}) := D_x(\tilde{u}), \quad (33g)$$

$$f_3(\tilde{u}, \tilde{v}) := 4D_x(\tilde{u}) \cos^2 \theta + \left[D_y(\tilde{u}) + D_x(\tilde{v}) \right] \sin 2\theta, \quad f_4(\tilde{u}, \tilde{v}) := D_y(\tilde{u}) + D_x(\tilde{v}), \quad (33h)$$

$$f_5(\tilde{u}, \tilde{v}) := D_y(\tilde{u}) + D_x(\tilde{v}) + \left[D_x(\tilde{u}) + D_y(\tilde{v}) \right] \sin 2\theta, \quad f_6(\tilde{v}) := D_y(\tilde{v}), \quad (33i)$$

$$f_7(\tilde{u}, \tilde{v}) := 4D_y(\tilde{v}) \sin^2 \theta + \left[D_y(\tilde{u}) + D_x(\tilde{v}) \right] \sin 2\theta, \quad (33j)$$

and, for any $j = 1, \dots, 7$,

$$g_j(\tilde{u}, \tilde{v}) := \mathcal{L}^{(n+1)}(f_j(\tilde{u}, \tilde{v})). \quad (33k)$$

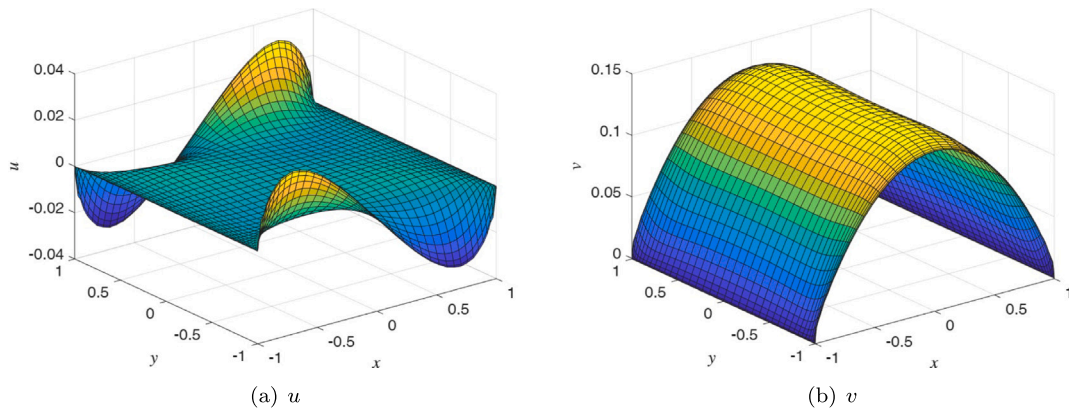


Fig. 5. Components of the displacement field at time $t = 1$ in the case where $\theta = 0$ and the normal stress at the upper boundary $y = 1$ is $\sigma = 1$. Parameters are set to $\beta = 1$, $\mu_T = 2$, $\gamma = 2$, $\nu_L = 0.5$, $\Delta t = 0.01$, $t = 1$. Note that the vertical axis range differs between panels (a) and (b), being adapted to the magnitude of each component.

Table 1

Sup-norm of the divergence of the displacement field for different numbers N of Gauss–Lobatto points and various fiber orientations θ .

N	$\theta = 0$	$\theta = \pi/4$	$\theta = \pi/3$	$\theta = \pi/2$
20	9.13×10^{-5}	1.17×10^{-4}	2.19×10^{-4}	7.92×10^{-5}
30	2.92×10^{-5}	5.12×10^{-5}	8.87×10^{-5}	2.35×10^{-5}
40	7.38×10^{-6}	2.23×10^{-5}	4.46×10^{-5}	4.47×10^{-6}

The second step is called the Poisson step, as we get the following Poisson problem for an auxiliary pressure p^{n+1} , that is,

$$\begin{cases} \Delta p^{(n+1)} = \frac{\text{div}(\bar{\mathbf{u}})}{(\Delta t)^2}, \\ p_x^{(n+1)} = 0 \quad \text{at } x = \pm 1, \\ p^{(n+1)} = 1, \quad \text{at } y = \pm 1, \end{cases} \quad (34)$$

where we have imposed Neumann and Dirichlet boundary conditions on inlet/outlet and upper/lower boundaries, respectively, and Δ represents the Laplace operator $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$. We emphasize that the above-mentioned pressure is an auxiliary pressure rather than the Lagrange multiplier associated with the incompressibility constraint.

The third (and last) step accounts for the incompressibility and is usually referred to as the projection (or correction) step as its purpose is to project the solution onto the divergence-free space (Guermond et al., 2006). Indeed, it consists of building a correction of the intermediate solution of (33), i.e.

$$\mathbf{u}^{(n+1)} = \bar{\mathbf{u}} - (\Delta t)^2 \nabla p^{(n+1)}, \quad (35)$$

which is the divergence-free solution of the IBVP (11), (14), and (16). Once having determined the displacement solution, the Lagrangian multiplier p can be obtained by integration of the governing equation.

5.2. Results

In this section, we present the results of the numerical approach outlined in the previous subsection for different orientations of the preferred direction $\mathbf{M}$ and various dimensionless parameters: the dispersion parameter of the ground matrix ν_L , the stiffness of the ground matrix μ_T , the dispersion parameter of the fibers γ , and the stiffness of the fibers β . In what follows, the normal stress at the upper boundary has been taken $\sigma = 1$.

As an example, Fig. 5 displays the components of the displacement field at time $t = 1$ in the case where the preferred direction is oriented along the x -axis, $\beta = 1$, $\gamma = 2$, $\mu_T = 2$ and $\nu_L = 0.5$.

To quantify the incompressibility error, in Table 1 we report the sup-norm of the divergence of the displacement field at time $t = 1$,

$$\|\text{div}(\mathbf{u})\|_\infty = \max_{(x,y) \in [-1,1]^2} |\text{div}(\mathbf{u}(x, y, 1))|,$$

computed using the projection algorithm for different numbers N of Gauss–Lobatto points and various fiber orientations θ .

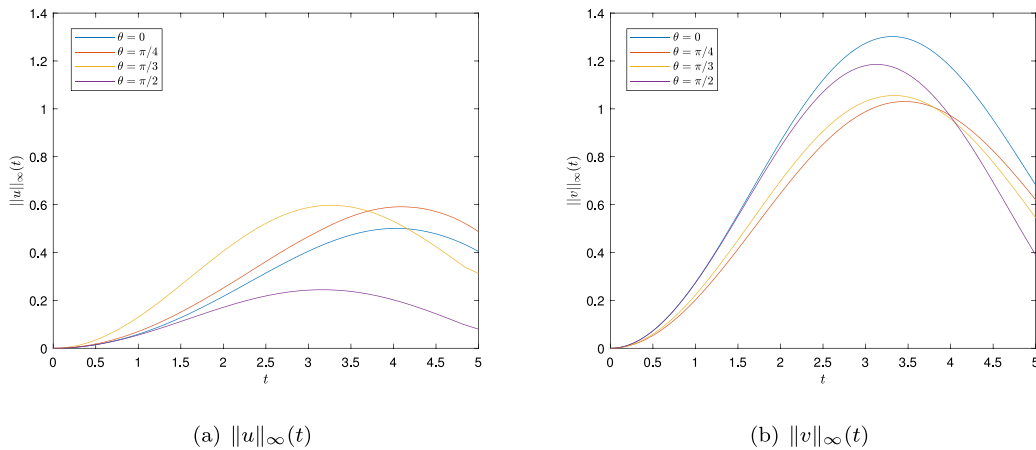


Fig. 6. Time evolution of the spatial sup-norms of the displacement components, defined in (36), for the fiber orientations $\theta = 0$, $\theta = \pi/4$, $\theta = \pi/3$, and $\theta = \pi/2$: (a) axial component $\|u\|_{\infty}(t)$; (b) transverse component $\|v\|_{\infty}(t)$. Parameters are set to $\beta = 1$, $\mu_T = 2$, $\gamma = 2$, $\nu_L = 0.5$, $\Delta t = 0.0025$, and $T = 5$.

To investigate the time evolution of the two-dimensional motion, we now consider, at each fixed time, the sup-norms of the displacement components with respect to the spatial variables only. More precisely, for every $t \in [0, T]$, we define

$$\|u\|_{\infty}(t) = \max_{(x,y) \in [-1,1]^2} |u(x, y, t)|, \quad \|v\|_{\infty}(t) = \max_{(x,y) \in [-1,1]^2} |v(x, y, t)|. \quad (36)$$

Unlike the space-time sup-norm introduced in (32), the quantities in (36) measure the maximum amplitude of each displacement component over the spatial domain at a given time.

Fig. 6 display the time evolution of the spatial sup-norms of the axial and transverse displacement components, respectively, for $\theta = 0, \pi/4, \pi/3$, and $\pi/2$. For every fiber orientation, both components start from the undeformed state, increase after the application of the normal traction, attain a maximum, and subsequently decrease over the simulated time interval. Thus, the response is non-monotone and strongly time dependent.

The temporal evolution is markedly affected by the fiber orientation. For the parameter values $\beta = 1$, $\mu_T = 2$, $\gamma = 2$, and $\nu_L = 0.5$, the transverse component reaches its largest amplitude for the longitudinal and transversal fiber orientations, namely $\theta = 0$ and $\theta = \pi/2$, respectively, while smaller maxima are obtained for the oblique orientations $\theta = \pi/4$ and $\theta = \pi/3$. Thus, the dependence of the transverse response on fiber orientation is non-monotone. This behavior indicates that the response cannot be explained solely in terms of the alignment of the fibers with the direction of the imposed normal traction. Instead, it results from the coupled anisotropic elastic and dispersive contributions entering the two-dimensional governing equations.

The axial component displays a different dependence on orientation: its largest maximum is attained for $\theta = \pi/3$, with a comparable value for $\theta = \pi/4$, whereas its smallest maximum is obtained for $\theta = \pi/2$. The orientation of the fibers therefore affects both the relative magnitudes of the displacement components and their temporal evolution. In particular, the maxima of the transverse component occur between $t \approx 3.13$ and $t \approx 3.46$, while those of the axial component are attained between $t \approx 3.16$ and $t \approx 4.09$. Since $t = 1$ lies in the initial growth phase for all the configurations considered, the snapshots at that time should be regarded as representative spatial configurations within the transient response.

To highlight how anisotropy influences the motion, in Figs. 7 and 8 we show the sup-norm of the components of the displacement field at time $t = 1$ as functions of the angle θ for different values of the stiffness of the ground matrix μ_T and of the dispersion material parameter of the ground matrix ν_L , respectively. We observe that the sup-norm of the transversal component of the displacement v is essentially insensitive to the stiffness of the ground matrix μ_T (Fig. 7b), whereas it decreases considerably as the dispersion parameter of the ground matrix ν_L increases, for every fiber orientation (Fig. 8b). The sup-norm of the axial component u , by contrast, is only mildly affected by μ_T and exhibits a more pronounced, non-monotonic dependence on ν_L (Figs. 7a and 8a).

In Figs. 9 and 10 we show how increasing values of the stiffness (β) and the dispersion parameter (γ) of the fibers, respectively, affect the maximum of the components of the displacement field during motion for different orientations of the fibers. We see that increasing the stiffness of the fibers may reduce or enhance the norm of u , depending on the angle θ , while the norm of v decreases as the fiber stiffness β increases irrespective of the orientation of the fibers (Fig. 9). Finally, we notice that, at sufficiently small values of the dispersion parameter of the fibers γ , both norms are sensitive to variations of the fiber orientation, while for larger values of γ , the norms tend to some constant values (Fig. 10).

6. Conclusions

We have studied one- and two-dimensional motions of a transversely isotropic dispersive material induced by applications of shear or normal stress on part of the boundary. In particular, we have considered the linearized version of the model proposed

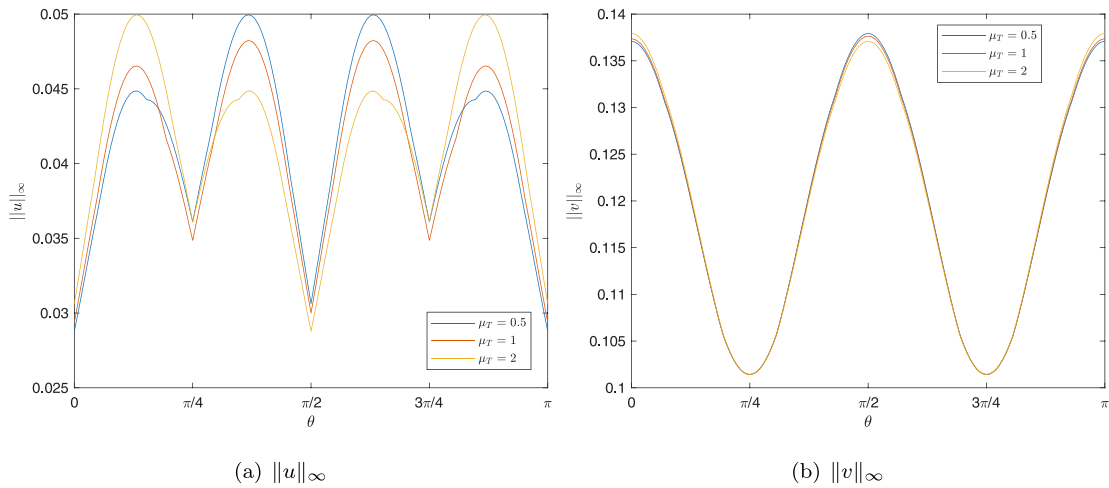


Fig. 7. Plots of the sup-norm of the components of the displacement field as functions of θ at different values of μ_T . Parameters are set to $\beta = 1$, $\gamma = 2$, $\nu_L = 0.5$, $\sigma = 1$, $\Delta t = 0.01$, $t = 1$.

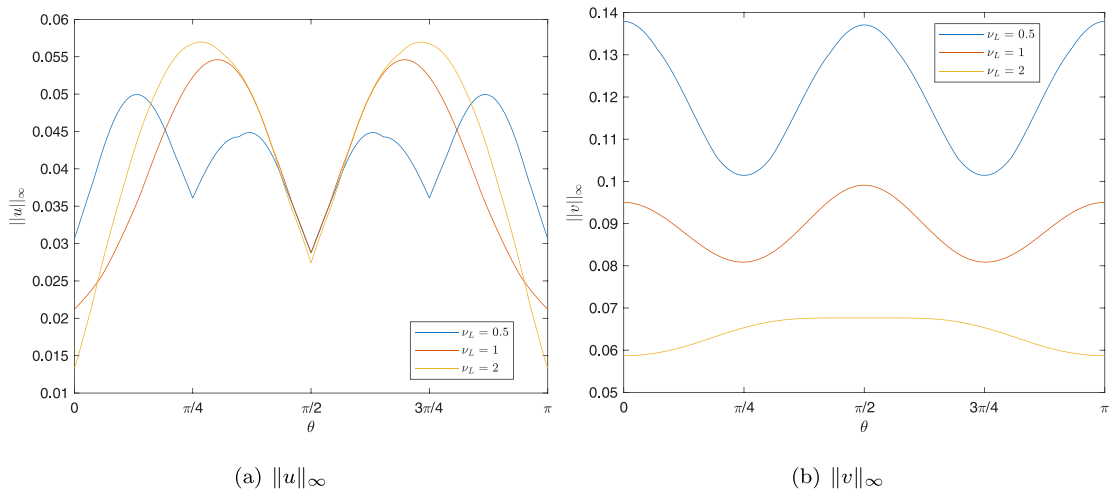


Fig. 8. Plots of the sup-norm of the components of the displacement field as functions of θ at different values of ν_L . Parameters are set to $\beta = 1$, $\mu_T = 2$, $\gamma = 2$, $\sigma = 1$, $\Delta t = 0.01$, $t = 1$.

by [Amendola et al. \(2024\)](#) for transversely isotropic dispersive bodies, in which both elastic and dispersive properties of the material depend on the orientation of the fibers. After having enforced the governing equations with the appropriate boundary and initial conditions, we have determined analytically the one-dimensional motions induced by the action of shear stress at the upper boundary and numerically the two-dimensional motions caused by normal stress acting at the upper boundary by using the projection algorithm based on spectral methods. We have quantified how the orientation of the fibers influences the motions and discussed how material anisotropy affects the sup-norm of the components of the displacement field.

In the one-dimensional shearing problem, the fiber orientation θ affects the sup-norm of the displacement field only through terms that are periodic functions of $\sin^2(2\theta)$, so that the response is maximum when the fibers are aligned with the direction of shear (i.e. $\theta = 0$) and minimum when $\theta = \pi/4$. We have also shown that, in this one-dimensional problem, the stiffness of the ground matrix μ_T has no influence at all on the displacement field, since it does not enter the governing equation of the shearing motion and only affects the extra normal stresses. The dispersion parameter of the ground matrix ν_L , on the other hand, affects the displacement field only through the temporal frequencies of the normal modes composing the solution: for moderate observation times, the sup-norm of the displacement increases monotonically with ν_L for every fiber orientation, whereas for longer observation times this monotonic relation no longer holds uniformly, the case $\nu_L = 0.5$ turning out to be the most sensitive to the length of the observation window.

In the two-dimensional problem, we have found that the axial and transverse displacement components respond differently to the material parameters. The transversal component is essentially insensitive to the stiffness of the ground matrix μ_T , but decreases

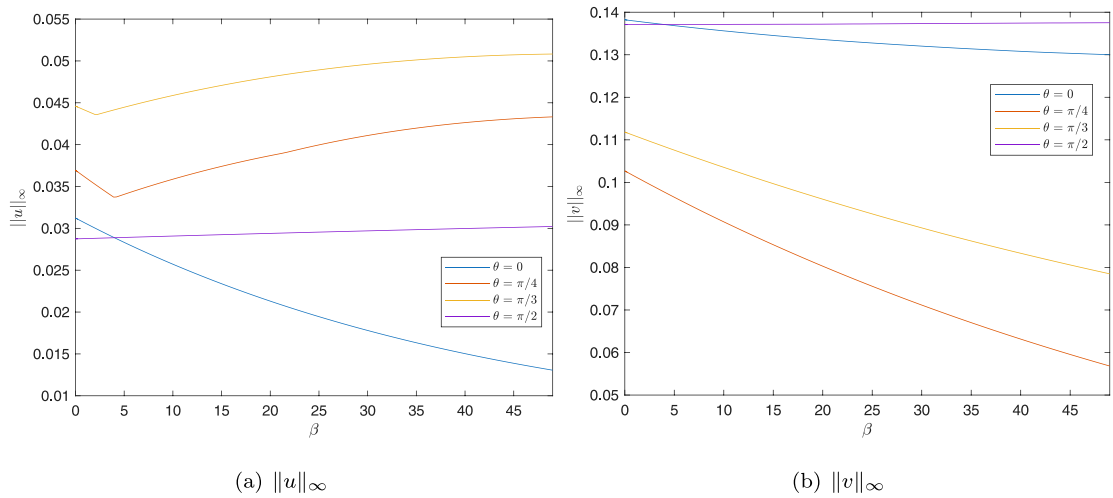


Fig. 9. Plots of the sup-norm of the components of the displacement field as functions of β for different orientations of the fibers. Parameters are set to $\mu_T = 2$, $\gamma = 2$, $\nu_L = 0.5$, $\Delta t = 0.01$, $t = 1$, and $\sigma = 1$.

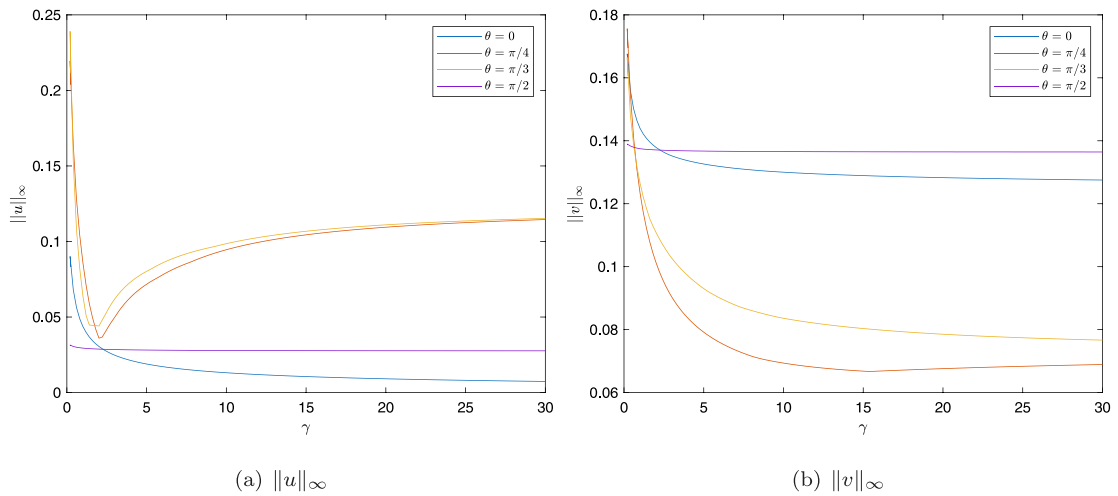


Fig. 10. Plots of the sup-norm of the components of the displacement field as functions of γ for different orientations of the fibers. Parameters are set to $\beta = 1$, $\mu_T = 2$, $\nu_L = 0.5$, $\Delta t = 0.01$, $t = 1$, and $\sigma = 1$.

considerably as the dispersion parameter of the ground matrix ν_L increases, for every fiber orientation; the axial component, by contrast, is only mildly affected by μ_T and depends on ν_L in a more pronounced, non-monotonic way. Similarly, an increase in the stiffness of the fibers reduces the sup-norm of the transversal component of the displacement irrespective of the fiber orientation, whereas its effect on the longitudinal component depends on the orientation of the fibers; finally, for small values of the dispersion parameter of the fibers the displacement field is markedly sensitive to the fiber orientation, an effect that fades as the fiber dispersion parameter increases.

The additional time-dependent analysis of the two-dimensional problem shows that the response to the imposed normal traction is transient and non-monotone. For all the fiber orientations considered, the spatial sup-norms of both displacement components initially increase, reach a maximum at later times, and subsequently decrease over the observation interval. Fiber orientation affects not only the spatial profiles of the displacement field at a fixed time, but also the amplitude and temporal development of the motion. In particular, the axial component attains its largest maximum for $\theta = \pi/3$, whereas the transverse component has its largest amplitudes for the longitudinal and transversal fiber orientations $\theta = 0$ and $\theta = \pi/2$, respectively, with reduced responses for the oblique orientations. This non-monotone dependence highlights the role of the combined anisotropic elastic and dispersive contributions in determining the transient dynamics.

CRediT authorship contribution statement

Rebecca Tozzi: Writing – review & editing, Writing – original draft, Software, Methodology, Investigation, Formal analysis, Conceptualization. **Júlia de Castro Motta:** Writing – review & editing, Writing – original draft, Methodology, Investigation, Formal analysis, Conceptualization. **Lorenzo Fusi:** Writing – review & editing, Writing – original draft, Supervision, Software, Methodology, Investigation, Formal analysis, Conceptualization. **Luigi Vergori:** Writing – review & editing, Writing – original draft, Supervision, Methodology, Investigation, Formal analysis, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgments

L.F. and R.T. have been funded by the NextGenerationEU PRIN 2022 research project “Mathematical Modelling of Heterogeneous Systems” (grant n. 2022MKB7MM). L.V. and J.d.C.M have been partially supported by the PRIN 2022 research project (grant no. 2022P5R22A) ‘The Mathematics and Mechanics of Nonlinear Wave Propagation in Solids’. J.d.C.M. and R.T. also acknowledge the financial support received from the GNFM of Italian INDAM, under the project “Dispersive Wave Mechanics in Transversely Isotropic Soft Tissues: Modeling and Computation” (grant n. E5324001950001). All the Authors performed this study under the auspices of the GNFM of Italian INDAM.

Data availability

No data was used for the research described in the article.

References

- Amendola, A., de Castro Motta, J., Saccomandi, G., & Vergori, L. (2024). A constitutive model for transversely isotropic dispersive materials. *Proceedings of the Royal Society A*, 480(2281), Article 20230374.
- Amendola, A., de Castro Motta, J., & Vergori, L. (2025). Wave propagation in compressible hyperelastic solids in the presence of powerless internal forces. *International Journal of Engineering Science*, 214, Article 104294.
- Annio, G., Holm, S., Mangin, G., Penney, J., Bacquét, R., Mustapha, R., Darwish, O., Wittgenstein, A. S., Schregel, K., Vilgrain, V., et al. (2024). Making sense of scattering: Seeing microstructure through shear waves. *Science Advances*, 10(31), eadp3363.
- Bell, J. B., Colella, P., & Glaz, H. M. (1989). A second-order projection method for the incompressible Navier-Stokes equations. *Journal of Computational Physics*, 85(2), 257–283.
- Bell, J., Howell, L., & Colella, P. (1991). An efficient second-order projection method for viscous incompressible flow. In *10th computational fluid dynamics conference* (p. 1560).
- Destrade, M., & Saccomandi, G. (2006). Solitary and compact-like shear waves in the bulk of solids. *Physical Review E*, 73, Article 065604.
- Destrade, M., & Saccomandi, G. (2008). Nonlinear transverse waves in deformed dispersive solids. *Wave Motion*, 45, 325–336.
- Feng, K.-A., Teng, C.-H., & Chen, M.-H. (2007). A pseudospectral penalty scheme for 2D isotropic elastic wave computations. *Journal of Scientific Computing*, 33(3), 313–348.
- Fusi, L., & Giovinetto, A. (2024). A spectral collocation scheme for the two dimensional flow of a regularized viscoplastic fluid: numerical results and comparison with analytical solution. *Mathematics and Computers in Simulation*, 225, 1237–1256.
- Fusi, L., Giovinetto, A., & Tozzi, R. (2025). Unsteady couette flow of a class of fluids described by non-monotone models. *European Journal of Mechanics. B. Fluids*, 112, 1–9.
- Fusi, L., Tozzi, R., Farina, A., & Rajagopal, K. R. (2023). Flow of a limited stress fluid between plates rotating about different axes. *Acta Mechanica*, 234(12), 6691–6703.
- Guermond, J.-L., Mineev, P., & Shen, J. (2006). An overview of projection methods for incompressible flows. *Computer Methods in Applied Mechanics and Engineering*, 195(44–47), 6011–6045.
- Klein, C., & Saut, J.-C. (2021). Nonlinear dispersive equations. *Inverse Scattering and PDE Methods*.
- Linares, F., & Ponce, G. (2014). *Introduction to nonlinear dispersive equations*. Springer.
- Qiang, B., Brigham, J. C., McGough, R. J., Greenleaf, J. F., & Urban, M. W. (2017). Mapped Chebyshev pseudo-spectral method for simulating the shear wave propagation in the plane of symmetry of a transversely isotropic viscoelastic medium. *Medical & Biological Engineering & Computing*, 55(3), 389–401.
- Saccomandi, G., & Vergori, L. (2023). Waves in isotropic dispersive elastic solids. *Wave Motion*, 116, Article 103066.
- Saccomandi, G., & Vergori, L. (2024). Solitary waves in slightly dispersive quasi-incompressible hyperelastic materials. *International Journal of Solids and Structures*, Article 112861.
- Talbot, C., & Crampton, A. (2006). Pseudo spectral methods applied to problems in elasticity. *Journal of Scientific Computing*, 27(1), 443–454.
- Timmermans, L. J., Mineev, P. D., & Van De Vosse, F. N. (1996). An approximate projection scheme for incompressible flow using spectral elements. *International Journal for Numerical Methods in Fluids*, 22(7), 673–688.
- W. Urban, M., Chen, S., & Fatemi, M. (2012). A review of shearwave dispersion ultrasound vibrometry (SDUV) and its applications. *Current Medical Imaging*, 8(1), 27–36.