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CR-Submanifolds of Almost Hermitian Manifolds.

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In memory of Franco Tricerri

Sunto. – *In questo lavoro si studia la geometria delle CR-sottovarietà di varietà quasi-hermitiane. In particolare, si studiano accuratamente le CR-sottovarietà a curvatura di Levi identicamente nulla.*

Introduction.

Suppose S is a real submanifold of an almost hermitian manifold (M,g). We say that S is a *CR-submanifold* of M if the dimension of the *CR-tangent space* $\mathcal{H}_p(S) = T_p(S) \cap JT_p(S)$ is constant, for all $p \in S$. Note that the hypersurfaces and the J-invariant submanifolds are particular CR-submanifolds. Our definition of CR-submanifold is classical in complex analysis and does not make use of the metric g of M. Another definition of CR-submanifold due to Bejancu ([B]) makes explicit use of the metric tensor g: it is less general than our definition; however also in this case hypersurfaces and J-invariant submanifolds are considered as CR-submanifolds.

CR-submanifolds are studied enough in the case that (M,g) is a kähler manifold. See, for instance, [P1], [P2], [CH2] for the study of CR-submanifolds in our sense and [B], [CH1], [D] for the study of CR-submanifolds in the sense of Bejancu. In this paper (M,g), in general, is not kähler. We examine the case of some classes of almost hermitian manifolds, which generalize the kähler case. In particular the more interesting results are obtained for CR-submanifolds of quasi kähler, semi-kähler, $\mathcal{G}_1$ and $W_1 \oplus W_2 \oplus W_4$-manifolds. (See [G-H] for the definitions.)

After a first introductive section, in section 2 we express the Levi form of a CR-submanifold in terms of second fundamental form of S and covariant derivative of the kähler form of (M,g), and we give, in some particular cases, integrability conditions for the Levi distribution $p \to \mathcal{H}_p(S)$. In this setting we generalize a theorem due to A. Gray. In [G2] he proved that, in the nearly kähler manifold S^6, there exist no J-invariant submanifolds of real dimension 4. We obtain the same result for all manifolds of class $\mathcal{G}_1$, whose torsion number is 2 (cf. Proposition 8). In section 3 we define the *Levi curvature* of a CR-submanifold and we study especially those CR-submanifolds whose Levi curvature vanishes, as, for instance, J-invariant submanifolds or, more in general, Levi-flat CR-submanifolds. At last in section 4 we give some inequalities for the scalar curvature of a CR-submanifold.

1. – Preliminaries.

1. In this paper we shall consider an almost hermitian manifold (M, g) of real dimension $2n \geqslant 4$, having J as almost complex structure. We shall denote by $\mathcal{X}(M)$ the $C^\infty(M)$-module of the tangent vector fields on M, and by N the torsion of J. We recall that, for every $X, Y \in \mathcal{X}(M)$, we have

$$N(X,Y) = [JX, JY] - [X,Y] - J[X,JY] - J[JX,Y].$$

We know that N is an antisymmetric tensor and that

$$N(JX,Y) = N(X,JY) = -JN(X,Y), \qquad \forall X,Y \in \mathcal{X}(M).$$

From this relationship, if $p \in M$ and Π_p is a J-invariant subspace of $T_p(M)$ of real dimension 2, we obtain

$$N|_{\Pi_p \times \Pi_p} = 0. \tag{1}$$

A famous theorem of Newlander-Nirenberg asserts that J is a complex structure (i.e. (M, g) is an hermitian manifold) if and only if $N \equiv 0$.

We shall denote by ∇ the Levi-Civita connection of (M, g) and by Φ the kähler form defined by the following formula:

$$\Phi(X,Y) = \frac{1}{2} g(JX,Y), \qquad \forall X,Y \in \mathcal{X}(M).$$

The following equalities hold true:

$$g(\nabla J(X,Y),Z) = \tag{2}$$

$$3d\Phi(X,Y,Z) - 3d\Phi(X,JY,JZ) + \frac{1}{2} g(N(Y,Z),JX) = 2\nabla\Phi(X,Y,Z),$$

$\forall X, Y, Z \in \mathcal{X}(M)$;

(3) $3d\Phi(X,Y,Z) = \underset{X,Y,Z}{\sigma} \nabla\Phi(X,Y,Z), \ \forall X,Y,Z \in \mathfrak{X}(M)$

(where σ denotes the cyclic sum).

We recall that g is said to be a kähler metric (and (M, g) a kähler manifold) if $N = 0$ and $d\Phi = 0$, or, equivalently by (2) and (3), if $\nabla J = 0$ or $\nabla\Phi = 0$.

It is easy to check that, $\forall X,Y,Z \in \mathfrak{X}(M)$, we have:

(4) $$\nabla\Phi(X,Y,Z) = -\nabla\Phi(X,Z,Y),$$

(5) $$\nabla\Phi(X,Y,Z) + \nabla\Phi(X,JY,JZ) = 0,$$

(6) $$\nabla\Phi(X,Y,Z) - \nabla\Phi(JX,JY,Z) = \nabla\Phi(Y,X,Z) - \nabla\Phi(JY,JX,Z) + \frac{1}{2}g(Z,JN(X,Y)).$$

In particular, from the above relationships, we deduce that (M, g) is an hermitian manifold if and only if $\nabla\Phi(X,Y,Z) = \nabla\Phi(JX,JY,Z)$, for every $X,Y,Z \in \mathfrak{X}(M)$.

We define the *Lee form* ω of (M, g) by the formula

$$\omega(Z) = -\frac{1}{n-1}\delta\Phi(JZ), \qquad \forall Z \in \mathfrak{X}(M),$$

where δ is the codifferential operator. We recall that

$$\delta\Phi(X) = -2\sum_{1}^{2n}{}_i \nabla\Phi(E_i,E_i,X), \qquad \forall X \in \mathfrak{X}(M),$$

where $\{E_1,\dots,E_{2n}\}$ is an orthonormal frame.

At last the *Lee field B of (M, g)* is defined by the equality

$$\omega(Z) = g(B,Z), \qquad \forall Z \in \mathfrak{X}(M).$$

2. In [G-H] Gray and Hervella have introduced, in a natural way, sixteen classes of almost hermitian manifolds. We recall the definitions of those classes which we shall use in the sequel. Thus, if we denote by X, Y, Z generic vector fields on M, an almost hermitian manifold (M, g) is said to be

nearly kähler if $\nabla\Phi(X,X,Y) = 0$ (or equivalently if $d\Phi = \nabla\Phi$),

quasi kähler if $\nabla\Phi(X,Y,Z) + \nabla\Phi(JX,JY,Z) = 0$,

semi kähler if $\delta\Phi = 0$,

of class W_4 if

$$\nabla\Phi(X,Y,Z) = -\frac{1}{4(n-1)}\{g(X,Y)\,\delta\Phi(Z) - g(X,Z)\,\delta\Phi(Y) - g(X,JY)\,\delta\Phi(JZ) + g(X,JZ)\,\delta\Phi(JY)\},$$

of class $W_1 \oplus W_2 \oplus W_4$ if

$$\nabla\Phi(X,Y,Z) + \nabla\Phi(JX,JY,Z) = -\frac{1}{2(n-1)}\cdot \{g(X,Y)\,\delta\Phi(Z) - g(X,Z)\,\delta\Phi(Y) - g(X,JY)\,\delta\Phi(JZ) + g(X,JZ)\,\delta\Phi(JY)\},$$

of class $\mathcal{G}_1$ if $\nabla\Phi(X,X,Y) = \nabla\Phi(JX,JX,Y)$ (or equivalently if $g(N(X,Y),X) = 0$).

It is easy to check, for instance, that nearly kähler $\Rightarrow$ quasi kähler, quasi kähler $\Rightarrow$ semi kähler, quasi kähler $\Rightarrow$ $W_1 \oplus W_2 \oplus W_4$, $W_4 \Rightarrow W_1 \oplus W_2 \oplus W_4$, hermitian $\Rightarrow \mathcal{G}_1$, nearly kähler $\Rightarrow \mathcal{G}_1$.

Moreover it is well known that, if $n=2$, $W_4 =$ class of hermitian manifolds, whereas if $n \geq 3$, we have $W_4 =$ class of locally conformal kähler manifolds.

3. Let us consider now a CR-submanifold S (in the sense of complex analysis) of an almost hermitian manifold (M, g). Then, at every point p of the real submanifold S, it is defined the CR-tangent space $\mathcal{H}_p(S) = T_p(S) \cap JT_p(S)$ as the maximal J-invariant subspace of the tangent space $T_p(S)$. By definition of CR-submanifold of M, the complex dimension of $\mathcal{H}_p(S)$ is constant, for all $p \in S$, and it is called the CR-dimension of S. It is easy to see that every hypersurface of (M, g) is a CR-submanifold of CR-dimension $n-1$. If, for all $p \in S$, $\mathcal{H}_p(S) = T_p(S)$, S is called a *J-invariant submanifold* of M. In this paper the CR-dimension of S will be always denoted by r, whereas the real dimension of S will be denoted by m. The distribution $p \to \mathcal{H}_p(S)$ is called the *Levi distribution* of S. Moreover we denote by h the second fundamental form of S in M, by R ($\bar{R}$) the Riemann curvature tensor of S (resp. M), by τ ($\bar{\tau}$) the scalar curvature of S (resp. M), by K ($\bar{K}$) the riemannian curvature of the two-plans of S (resp. M), by $\bar{\rho}$ the Ricci tensor of M and by H the mean curvature of S in M.

It is clear that we have the following decomposition:

$$T(M)|_S = \mathcal{H}(S) \oplus O(S) \oplus T(S)^{\perp},$$

where $\mathcal{H}(S)$ is the *CR-tangent bundle of S*, $O(S)$ is the subbundle of $T(S)$ orthogonal to $\mathcal{H}(S)$, and $T(S)^{\perp}$ is the normal bundle to S.

Thus, if V is a section of $T(M)|_S$, we have $V = V^{\mathbb{C}} + V^0 + V^{\perp}$, where $V^{\mathbb{C}} \in \mathcal{H}(S)$, $V^0 \in O(S)$, $V^{\perp} \in T(S)^{\perp}$.

We set

$$\mathcal{X}(S) = \{\text{tangent vector fields of class } C^{\infty} \text{ on S}\},$$

$$\mathcal{X}^{\mathbb{C}}(S) = \{X \in \mathcal{X}(S)\colon X_p \in \mathcal{H}_p(S),\ \ \forall p \in S\},$$

$$\mathcal{X}^{0}(S) = \{X \in \mathcal{X}(S)\colon X_p \in O_p(S),\ \ \forall p \in S\},$$

$$\mathcal{X}^{\perp}(S) = \{\text{normal vector fields on } S\}.$$

If $X \in \mathcal{X}(S)$ and $\xi \in \mathcal{X}^{\perp}(S)$ we set $JX = PX + FX$, $J\xi = t\xi + f\xi$, where PX, $t\xi \in \mathcal{X}(S)$, and FX, $f\xi \in \mathcal{X}^{\perp}(S)$.

It is easy to check that $PO(S) \subseteq O(S)$, $O(S) \cap JO(S) = \{0\}$, $t(T(S)^{\perp}) = O(S)$.

Moreover

$$F|_{O(S)}\colon O(S) \to FO(S) \quad \text{and} \quad t|_{FO(S)}\colon FO(S) \to O(S) \text{ are isomorphisms}. \tag{7}$$

2. – The Levi form of a *CR*-submanifold.

1. Let S be a *CR*-submanifold of an almost hermitian manifold (M, g) and let $p \in S$. In accordance with [P2] we define the *Levi form* of S at p, $L_p\colon \mathcal{H}_p(S) \times \mathcal{H}_p(S) \to O_p(S) \otimes_{\mathbb{R}} \mathbb{C}$, by the formula

$$L_p(X_p, Y_p) = \operatorname{Re} L_p(X_p, Y_p) + i \operatorname{Im} L_p(X_p, Y_p) = [X, -JY]_p^0 + i[X, Y]_p^0, \quad \text{where } X, Y \in \mathcal{X}^{\mathbb{C}}(S).$$

Clearly L_p is well defined. In order to simplify the notations, in the sequel we shall omit the point p. We remark that, if we denote by π the natural projection of $T(M)$ onto $T(M)/T(S)$, the form $G = \pi \circ F \circ \operatorname{Re} L$ coincides with the Levi form introduced by Hermann in [H]. From (7), it is clear that G and Re L are equivalent. It is straightforward to check that L is a $\mathbb{R}$-bilinear map, $\mathbb{C}$-antilinear in the second argument, whose imaginary part Im L is antisymmetric, but, in general, L is not a sesquilinear hermitian form on $\mathcal{H}(S)$.

In fact the following holds true:

PROPOSITION 1. – *The following statements are equivalent:*

a) L *is a sesquilinear hermitian form,*

b) $\operatorname{Re} L$ *is symmetric,*

c) $N(X,Y)^{\perp} = 0$, *for all* $X, Y \in \mathcal{X}^{C}(S)$.

PROOF. – The equivalence of *a)* and *b)* is trivial. We have

$$N(X,Y)^{\perp} = 0 \Leftrightarrow J([X,JY] + [JX,Y]) \in \mathcal{X}(S) \Leftrightarrow$$

$$[X,JY] + [JX,Y] \in \mathcal{X}^{C}(S) \Leftrightarrow \operatorname{Re} L(Y,X) = \operatorname{Re} L(X,Y)$$

and this proves the equivalence of *b)* and *c)*. q.e.d.

Taking into account the above proposition and (1) we get

PROPOSITION 2. – *The Levi form L of S is always a sesquilinear hermitian form if one of these statements holds true:*

a) (M,g) *is an hermitian manifold;*

b) S *is a J-invariant submanifold of* M;

c) $CR\text{-}\dim S = 1$.

The CR-submanifold S is said to be *Levi-flat* if $L \equiv 0$ (at every point), or equivalently if the Levi distribution of S is integrable. It is clear that S is Levi-flat if and only if L is a sesquilinear hermitian form such that $L(X,X) = 0$, for every $X \in \mathcal{X}^{C}(S)$. Obviously every J-invariant submanifold is Levi-flat.

Instead the CR-submanifold S is said to be *strictly Levi-convex* if $L(X,X) = 0 \Leftrightarrow X = 0$ (at every point of S). Note that this definition is weaker than strongly pseudoconvexity introduced in [H], even though they coincide for $\dim O(S) = 1$ or $CR\text{-}\dim S = 1$.

2. We will prove now the following (cf. [H], [P1], [P2])

PROPOSITION 3. – *Let* $\bar{\nabla}$ *be any symmetric affine connection on the almost hermitian manifold* (M,g). *If we denote by* $\bar{h}$ *the second fundamental form of S in M with respect to* $\bar{\nabla}$, *we have*

$$(8) \qquad F \operatorname{Re} L(X,Y) = [\bar{\nabla}J(X,JY) - \bar{\nabla}J(JY,X)]^{\perp} + \bar{h}(JX,JY) + \bar{h}(X,Y),$$

(9) $F\,\mathrm{Im}\,L(X,Y) =$

$$[\bar\nabla J(Y,X) - \bar\nabla J(X,Y)]^{\perp} + \bar h(X,JY) - \bar h(JX,Y), \quad \forall X,Y \in \mathfrak{X}^{C}(S).$$

PROOF. – Let $X, Y \in \mathfrak{X}^{C}(S)$. If we denote by ∇' the connection on S induced by $\bar\nabla$, from the Gauss formula we get

$$\bar\nabla J(X,Y) = \bar\nabla_X JY - J\bar\nabla_X Y = \nabla'_X JY - J\nabla'_X Y + \bar h(X,JY) - J\bar h(X,Y).$$

Taking into account that also ∇' is symmetric, we obtain, by equalizing the normal components of the above equality:

$$(10) \qquad [\bar\nabla J(X,Y)]^{\perp} = \bar h(X,JY) - f\bar h(X,Y) - F\nabla'_Y X - F[X,Y].$$

Because $\bar h$ is a symmetric bilinear form, we have

$$F\nabla'_Y X = [J\bar\nabla_Y X]^{\perp} - [J\bar h(X,Y)]^{\perp} = [\bar\nabla_Y JX - \bar\nabla J(Y,X)]^{\perp} - f\bar h(X,Y) =$$

$$\bar h(Y,JX) - [\bar\nabla J(Y,X)]^{\perp} - f\bar h(X,Y).$$

By substituting in (10) we obtain

$$(11) \quad [\bar\nabla J(X,Y) - \bar\nabla J(Y,X)]^{\perp} = \bar h(X,JY) - \bar h(JX,Y) - F[X,Y].$$

Because $F[X,Y] = F\,\mathrm{Im}\,L(X,Y)$, we get (9).

Instead if we replace Y with $-JY$ in (11), we get (8).

q.e.d.

The formulas (8) and (9) are simplified if $\bar\nabla J$ is a symmetric tensor, but in this case it is necessary for (M, g) to be hermitian. In fact the following formula holds true:

$$(12) \quad N(X,Y) = \bar\nabla J(X,JY) - \bar\nabla J(JY,X) +$$

$$\bar\nabla J(JX,Y) - \bar\nabla J(Y,JX), \qquad \forall X,Y \in \mathfrak{X}(M),$$

for any symmetric affine connection $\bar\nabla$ on M.

3. Henceforward we shall consider only the Levi-Civita connection ∇ on the almost hermitian manifold (M, g). From (8) then we obtain

(13) $F\,\mathrm{Re}\,L(X,Y) =$

$$[\nabla J(X,JY) - \nabla J(JY,X)]^{\perp} + h(JX,JY) + h(X,Y), \qquad \forall X,Y \in \mathfrak{X}^{C}(S).$$

Thus if $Z \in \mathcal{X}^{\perp}(S)$, from (2) and (5), we get

$$g(\nabla J(X,JY) - \nabla J(JY,X),Z) = 2\{\nabla\Phi(X,Y,JZ) + \nabla\Phi(JY,JX,JZ)\}$$

and then, from (13), we obtain

PROPOSITION 4. – *Let S be any CR-submanifold of an almost hermitian manifold (M,g). We have*

$$(14) \quad g(F\,\mathrm{Re}\,L(X,Y),Z) = g(h(JX,JY) + h(X,Y),Z) + 2\{\nabla\Phi(X,Y,JZ) + \nabla\Phi(JY,JX,JZ)\}, \quad \forall X,Y \in \mathcal{X}^{C}(S),\ \forall Z \in \mathcal{X}^{\perp}(S).$$

In particular S is Levi-flat if and only if $N(X,Y)^{\perp} = 0$, $\forall X,Y \in \mathcal{X}^{C}(S)$ and

$$(15) \quad g(h(JX,JX) + h(X,X),Z) + 2\{\nabla\Phi(X,X,JZ) + \nabla\Phi(JX,JX,JZ)\} = 0, \quad \forall X \in \mathcal{X}^{C}(S),\ \forall Z \in \mathcal{X}^{\perp}(S).$$

Suppose now that (M,g) is quasi kähler. Since $\nabla\Phi(JY,JX,JZ) = -\nabla\Phi(Y,X,JZ)$, $\forall X,Y,Z \in \mathcal{X}(M)$, from Proposition 4 we obtain

PROPOSITION 5. – *Let* S *be any* CR- *submanifold of a quasi kähler manifold. We have, $\forall X,Y \in \mathcal{X}^{C}(S)$, $\forall Z \in \mathcal{X}^{\perp}(S)$,*

$$(16) \quad g(F\,\mathrm{Re}\,L(X,Y),Z) = g(h(JX,JY) + h(X,Y),Z) + 2\{\nabla\Phi(X,Y,JZ) - \nabla\Phi(Y,X,JZ)\}.$$

In particular S is Levi-flat if and only if $N(X,Y)^{\perp} = 0$, $\forall X,Y \in \mathcal{X}^{C}(S)$ and

$$(17) \quad h(JX,JX) + h(X,X) = 0, \quad \forall X \in \mathcal{X}^{C}(S).$$

The formula (17) permits to extend to quasi kähler case some results obtained in the kähler case (cf. [P1], [P2]).

Among others, we can prove

PROPOSITION 6. – *Let (M,g) denote a quasi kähler manifold.*

a) Let S be a Levi-flat CR-submanifold of M. Then, if π is a J-invariant 2*-section of* S *we have $K(\pi) \leq \bar{K}(\pi)$ and the equality holds if and only if $h|_{\pi\times\pi} \equiv 0$.*

b) Let S be a CR-submanifold of M such that $K > \bar{K}$, for every J*-invariant* 2*-section of S. Then S is strictly Levi-convex.*

c) Let S be a Levi-flat CR-submanifold of M such that $m \leq$

$2r+1$. *If* $\bar{K} \leq 0$ *we have* $\tau \leq 0$. *If* S *is any hypersurface of* M *and* $\bar{\rho}$ *is positive (semi) definite, we have* $\tau < \bar{\tau}$ *(resp.* $\tau \leq \bar{\tau}$*).*

d) If $r=1$ *and* $m=3$*, any CR-submanifold* S *of* M *is strictly Levi-convex if* $\bar{K} \leq 0$ *and* $\tau > 0$*, or, in the case* $n=2$*, if* $\bar{\rho}$ *is positive (semi) definite and* $\tau \geq \bar{\tau}$ *(resp.* $\tau > \bar{\tau}$*).*

4. Suppose now that S is *CR*-submanifold of a manifold (M, g) of class $\mathcal{G}_1$.

If *CR*-dim $S=2$, we observe that

$$N(X,Y)^{\mathbb{C}} = 0, \qquad \forall X, Y \in \mathfrak{X}^{\mathbb{C}}(S). \tag{18}$$

If X and Y are $\mathbb{C}$-dependent vectors this follows from (1). On the contrary if X, Y, JX, JY generate $\mathcal{H}(S)$, we have, since (M, g) is of class $\mathcal{G}_1$,

$$g(N(X,Y),X) = g(N(X,Y),Y) = 0;$$

moreover

$$g(N(X,Y),JX) = g(N(X,JY),X) = 0$$

and also $g(N(X,Y),JY) = 0$. Thus we obtain $N(X,Y)^{\mathbb{C}} = 0$.
From (15) and (18) it follows easily

PROPOSITION 7. – *A CR-submanifold of CR-dimension* 2 *of a manifold of class* $\mathcal{G}_1$ *is Levi-flat if and only if we have* $N(X,Y)=0$, $\forall X, Y \in \mathfrak{X}^{\mathbb{C}}(S)$ *and*

$$g(h(JX,JX) + h(X,X),Z) + 4\nabla\Phi(X,X,JZ) = 0\,, \qquad \forall X \in \mathfrak{X}^{\mathbb{C}}(S) \text{ and } \forall Z \in \mathfrak{X}^{\perp}(S).$$

Let $p \in M$ and $X_p \in T_p(M)$. We set $t(X_p) = \dim_{\mathbb{R}} \{Y_p \in T_p(M): N(X_p, Y_p) = 0\}$. $t(X_p)$ is even and ≥ 2. Moreover if (M, g) is hermitian we have $t(X_p) = 2n$.
We define the *torsion number of* (M, g) *at* p setting

$$t_p = \sup\{t(X_p): X_p \in T_p(M) - \{0\}\}\,,$$

and the *torsion number* of (M,g) setting

$$t = \sup\{t_p: p \in M\}\,.$$

From Proposition 7 it follows easily

PROPOSITION 8. – *Let (M,g) be an almost hermitian manifold of class $\mathcal{G}_1$. If, for some $p \in M$, we have $t_p = 2$, then there exist no Levi-flat CR-submanifolds through p of CR-dimension 2. In particular if $t = 2$, there exist no Levi-flat CR-submanifolds of (M, g) of CR-dimension 2.*

5. It is well-known (cf. [CA] and [G1]) that, using the Cayley numbers, we may define a vector product $\times$ in $\mathbb{R}^7$ and that, by this vector product, we may introduce an almost complex structure J on every orientable hypersurface M of $\mathbb{R}^7$. These hypersurfaces, endowed with the metric induced by euclidean metric of $\mathbb{R}^7$, are always semi kähler ([G3]). Moreover the sphere S^6 is a nearly kähler manifold (and then also of class $\mathcal{G}_1$).

PROPOSITION 9. – *The torsion number t of S^6 is 2.*

PROOF. – Let $p \in S^6$ and $X_p \in T_p(S^6) - \{0\}$. We shall prove that $t(X_p) = 2$. Suppose that $N(X_p, Y_p) = 0$. From (6) and formulas which are proved in [G1] we obtain $(1/4) JN(X,Y) =$ tangential part of $X \times Y$, $\forall X, Y \in \mathcal{X}(S^6)$.

Then $X_p \times Y_p = \lambda \nu_p$, where $\lambda \in \mathbb{R}$ and ν is a normal versor of S^6. From the table of multiplication of $\times$ (cf. [CA]) we obtain easily that Y_p is a linear combination of X_p and JX_p. Thus $t(X_p) = 2$.

q.e.d.

From Propositions 8 and 9 we deduce a Gray's theorem ([G2]), which we express in the following way

PROPOSITION 10. – *In S^6 there exist no Levi-flat CR-submanifolds of CR-dimension 2.*

6. Suppose now that (M, g) is of class W_4. In particular (M, g) is hermitian. We obtain if $X, Y \in \mathcal{X}^{\mathbb{C}}(S)$ and $Z \in \mathcal{X}^{\perp}(S)$,

$$2\{\nabla\Phi(X,Y,JZ) + \nabla\Phi(Y,X,JZ)\} =$$

$$-\frac{1}{n-1} g(X,Y)\, \delta\Phi(JZ) = g(X,Y)\, g(B,Z).$$

From this equality and (14) we have

PROPOSITION 11. – *Let (M,g) be of class W_4 and let* S *be a* CR-*sub-*

manifold of M. We have, $\forall X, Y \in \mathcal{X}^{C}(S)$,

$$F\,\mathrm{Re}\,L(X,Y) = h(JX,JY) + h(X,Y) + g(X,Y)B^{\perp}\,. \tag{19}$$

In the same way, we can prove that if (M, g) is of class $W_1 \oplus W_2 \oplus W_4$ only, formula (19) holds for $X = Y$.

3. – The Levi curvature of a *CR*-submanifold.

1. Now and later on we shall denote by $V_1, \ldots, V_r, JV_1, \ldots, JV_r$ an orthonormal frame of $\mathcal{X}^{C}(S)$ and by $\mu_1, \ldots, \mu_{m-2r}$ an orthonormal frame of $\mathcal{X}^{0}(S)$.

We define now the *complex mean curvature* H^{C} and the *orthogonal mean curvature* H^{0} of the *CR*-submanifold S setting

$$H^{C} = \frac{1}{m}\left\{\sum_{1}^{r}{}_i\, h(V_i, V_i) + h(JV_i, JV_i)\right\},$$

$$H^{0} = \frac{1}{m}\sum_{1}^{m-2r}{}_i\, h(\mu_i \mu_i)\,.$$

It is easy to check that H^{C} and H^{0} are well defined. Moreover we have $H = H^{C} + H^{0}$.

We say that S is $\mathbb{C}$*-minimal* if $H^{C} = 0$.

2. Let us define a normal vector field β on S, setting, for all $W \in \mathcal{X}^{\perp}(S)$,

$$g(\beta, W) = \frac{2}{r}\left\{\sum_{1}^{r}{}_i\, \nabla\Phi(V_i, V_i, JW) + \nabla\Phi(JV_i, JV_i, JW)\right\}.$$

We can easily verify that β is well defined.

We observe that, if (M, g) is quasi kähler, we have $\beta = 0$, for every *CR*-submanifold of M. The normal field β is related to the Lee field B of M, as it is shown by the following Propositions.

PROPOSITION 12. – *If (M,g) is of class $W_1 \oplus W_2 \oplus W_4$, then for every CR-submanifold S of M we have* $\beta = B^{\perp}$.

PROOF. – We have, if $W \in \mathcal{X}^{\perp}(S)$

$$g(\beta, W) = -\frac{1}{(n-1)}\delta\Phi(JW) = g(B, W) \quad \text{and then } \beta = B^{\perp}. \qquad \text{q.e.d.}$$

PROPOSITION 13. – *Let (M,g) be an almost hermitian manifold of dimension $2n$. If S is a CR-submanifold of M of CR-dimension $n-1$, we have $\beta = B^{\perp}$. In particular if (M,g) is semi kähler we have $\beta = 0$.*

PROOF. – Let $W \in \mathfrak{X}^{\perp}(S)$, $W \neq 0$. Since $r = n-1$, $V_1, \dots, V_{n-1}, W/\|W\|, JV_1, \dots, JV_{n-1}, J(W/\|W\|)$ is an orthonormal frame of $T(M)|_S$. Then

$$g(B,W) = -\frac{1}{n-1}\delta\Phi(JW) = g(\beta, W) +$$

$$\frac{2}{n-1}\left\{\nabla\Phi\left(\frac{W}{\|W\|}, \frac{W}{\|W\|}, JW\right) + \nabla\Phi\left(J\frac{W}{\|W\|}, J\frac{W}{\|W\|}, JW\right)\right\} = g(\beta, W),$$

from (4) and (5). Then we deduce that $\beta = B^{\perp}$. The second part of the Proposition follows from the fact that $B = 0$ in every semi kähler manifold. q.e.d.

3. We define now the *Levi curvature* $\mathcal{L}$ of the CR-submanifold S setting

$$\mathcal{L} = \sum_{1}^{r}{}_i L(V_i, V_i) = \sum_{1}^{r}{}_i \operatorname{Re} L(V_i, V_i).$$

From (14) we obtain

PROPOSITION 14. – *For any CR-submanifold S of (M,g) we have*

$$F\mathcal{L} = mH^{\mathbb{C}} + r\beta. \tag{20}$$

From (7) and (20) we deduce that the definition of $\mathcal{L}$ is independent of the orthonormal frame $V_1, \dots, V_r$ since $H^{\mathbb{C}}$ and β are well defined. Moreover from (20) we obtain that $\mathcal{L}$ vanishes identically on S if and only if $H^{\mathbb{C}} = -(r/m)\beta$.

A non-trivial example of CR-submanifold with vanishing Levi curvature is the following: consider a totally geodesic hypersurface N of an oriented even dimensional riemannian manifold (M, g); then the bundle $Z|_N$ over N obtained by restricting to N the Twistor Space Z of (M, g) (endowed with its standard almost hermitian metric) is a hypersurface of Z, whose Levi curvature vanishes identically, even though it is not Levi-flat. Note that, in general, Z is not hermitian. For instance, since S^3 is a totally geodesic hypersurface of S^4 with canonical metric, we have that $Z(S^4)|_{S^3}$ is a compact hypersurface of $\mathbb{CP}^3 = Z(S^4)$ (endowed with the Fubini-Study metric) with vanishing Levi curvature.

4. From Propositions 12 and 13 we deduce

PROPOSITION 15. – *Let (M,g) be an almost hermitian manifold of dimension $2n$ and suppose that S is a CR-submanifold of M of CR-dimension $n-1$ whose Levi curvature vanishes identically. Then S is $\mathbb{C}$-minimal if and only if it is tangent to the Lee field of M. In particular a J-invariant submanifold of dimension $2n-2$ is minimal if and only if it is tangent to the Lee field of M.*

COROLLARY. – *Every CR-submanifold of CR-dimension $n-1$ of a semi kähler manifold (M,g) of dimension $2n$, whose Levi curvature vanishes identically, is $\mathbb{C}$-minimal. In particular every J-invariant submanifold of (M,g) of dimension $2n-2$ is minimal.*

PROPOSITION 16. – *Let (M,g) be of class $W_1 \oplus W_2 \oplus W_4$ and suppose that S is a CR-submanifold of M, whose Levi curvature vanishes identically. Then S is $\mathbb{C}$-minimal if and only if it is tangent to the Lee field of M. In particular a J-invariant submanifold of (M,g) is minimal if and only if it is tangent to the Lee field of M* (cf. [D], [C-M]).

COROLLARY. – *A CR-submanifold of a quasi kähler manifold (M,g), whose Levi curvature vanishes identically, is $\mathbb{C}$-minimal. In particular every J-invariant submanifold of (M,g) is minimal* (cf. [G1]).

5. Suppose now that (M, g) is a complete, simply connected almost hermitian manifold, with sectional curvature $\bar{K} \leqslant 0$. From a theorem due to O'Neill [O], in M there exist no compact $\mathbb{C}$-minimal CR-submanifolds of CR-dimension >0. Thus, from (20), we obtain

PROPOSITION 17. – *Let (M, g) a complete, simply connected $2n$-dimensional almost hermitian manifold with sectional curvature $\bar{K} \leqslant 0$. Then*

a) In M there exist no compact CR-submanifolds of CR-dimension $n-1$ tangent to the Lee field of M, whose Levi curvature vanishes identically. In particular there exist no compact J-invariant

$(2n-2)$-dimensional submanifolds tangent to the Lee field of M.

b) If (M,g) is of class $W_1 \oplus W_2 \oplus W_4$, in M there exist no compact CR-submanifolds of CR-dimension >0, tangent to the Lee field of M, whose Levi curvature vanishes identically. In particular there exist no compact J-invariant submanifolds (of dimension >0) tangent to the Lee field of M.

4. - The scalar curvature of a *CR*-submanifold.

1. In this section we will extimate the scalar curvature τ of a *CR*-submanifold (cf. [P1] and [P2]). We have, taking the Gauss formula into account,

$$\tau = \sum_{1}^{r}{}_{i,k}\{R(V_i,V_k,V_i,V_k) + 2R(V_i,JV_k,V_i,JV_k) + R(JV_i,JV_k,JV_i,JV_k)\} +$$

$$2\sum_{1}^{r}{}_{i}\sum_{1}^{m-2r}{}_{s}\{R(V_i,\mu_s,V_i,\mu_s) + R(JV_i,\mu_s,JV_i,\mu_s)\} + \sum_{1}^{m-2r}{}_{s,l}R(\mu_s,\mu_l,\mu_s,\mu_l) =$$

$$\sum_{1}^{r}{}_{i,k}\{\bar R(V_i,V_k,V_i,V_k) + 2\bar R(V_i,JV_k,V_i,JV_k) + \bar R(JV_i,JV_k,JV_i,JV_k)\} +$$

$$2\sum_{1}^{r}{}_{i}\sum_{1}^{m-2r}{}_{s}\{\bar R(V_i,\mu_s,V_i,\mu_s) + \bar R(JV_i,\mu_s,JV_i,\mu_s)\} +$$

$$\sum_{1}^{m-2r}{}_{s,l}\bar R(\mu_s,\mu_l,\mu_s,\mu_l) + m^2\|H\|^2 - \sum_{1}^{m-2r}{}_{s,l}\|h(\mu_s,\mu_l)\|^2 -$$

$$\sum_{1}^{r}{}_{i,k}\{\|h(V_i,V_k)\|^2 + 2\|h(V_i,JV_k)\|^2 + \|h(JV_i,JV_k)\|^2\} -$$

$$2\sum_{1}^{r}{}_{i}\sum_{1}^{m-2r}{}_{s}\{\|h(V_i,\mu_s)\|^2 + \|h(JV_i,\mu_s)\|^2\}.$$

If S is any hypersurface of (M,g), denoting by ν a normal versor of S in M, the above equality becomes

$$\tau = \bar\tau - 2\bar\rho(\nu,\nu) + (2n-1)^2(\|H\|^2 - \|H^0\|^2) -$$

$$\left\{\sum_{1}^{r}{}_{i,k}\|h(V_i,V_k)\|^2 + 2\|h(V_i,JV_k)\|^2 + \right.$$

$$\left.\|h(JV_i,JV_k)\|^2 + 2\sum_{1}^{r}{}_{i}\|h(V_i,\mu_1)\|^2 + \|h(JV_i,\mu_1)\|^2\right\}.$$

Since $\|H\|^2 - \|H^0\|^2 = \|H^C\|^2 + 2g(H^C, H^0) = 2g(H^C, H) - \|H^C\|^2$, and $(2n-1)H^C = F\mathcal{L} - (n-1)B^\perp$ (from (20) and Proposition 13), we get

$$\begin{aligned}(21)\quad \tau = \bar{\tau} - 2\bar{\rho}(\nu,\nu) + 2(2n-1)g(J\mathcal{L}, H) - \\ 2(2n-1)(n-1)g(B^\perp, H) - \|F\mathcal{L}\|^2 - \\ (n-1)^2\|B^\perp\|^2 + 2(n-1)g(J\mathcal{L}, B^\perp) - \\ \Big\{\sum_{i,k=1}^{r} \|h(V_i, V_k)\|^2 + 2\|h(V_i, JV_k)\|^2 + \\ \|h(JV_i, JV_k)\|^2 + 2\sum_{i=1}^{r} \|h(V_i, \mu_1)\|^2 + \|h(JV_i, \mu_1)\|^2\Big\}.\end{aligned}$$

From this relationship we get

PROPOSITION 18. – *Let S be any hypersurface of a 2n-dimensional almost hermitian manifold (M,g), whose Levi curvature vanishes. We have*

$$\tau \leq \bar{\tau} - 2\bar{\rho}(\nu,\nu) - 2(2n-1)(n-1)g(B^\perp, H) - (n-1)^2\|B^\perp\|^2$$

and the equality holds if and only if $h|_{\mathcal{H}(S)\times T(S)} \equiv 0$.

2. Suppose now that $m \leq 2r+1$ and that the sectional curvature $\bar{K}$ of (M,g) is $\leq A$. Then it is easy to check that

$$\tau \leq (m^2 - m)A + m^2(\|H\|^2 - \|H^0\|^2),$$

and then, from (20), as above we get

$$\begin{aligned}(22)\quad \tau \leq (m^2-m)A + 2mg(J\mathcal{L}, H) - \\ 2mrg(\beta, H) - \|F\mathcal{L}\|^2 - r^2\|\beta\|^2 + 2rg(J\mathcal{L}, \beta).\end{aligned}$$

It follows

PROPOSITION 19. – *Let S be a CR-submanifold of an almost hermitian manifold (M,g), whose Levi curvature vanishes. If* $m \leq 2r+1$ *and* $\bar{K} \leq A$, *we have*

$$\tau \leq (m^2-m)A - 2mrg(\beta, H) - r^2\|\beta\|^2.$$

In particular if (M,g) is of class $W_1 \oplus W_2 \oplus W_4$ or if $r = n-1$, we have

$$\tau \leqslant (m^2 - m)A - 2mrg(B^{\perp}, H) - r^2 \|B^{\perp}\|^2 .$$

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