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MATHEMATICS AND MECHANICS OF COMPLEX SYSTEMS
Vol. (2023), No. , p. 101

DOI: 10.2140/memocs.2023..101

DOUBLE LAYER CONCENTRATION AND HOMOGENIZATION FOR THE HEAT DIFFUSION IN A COMPOSITE MATERIAL

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We investigate the possibility of deriving "nonstandard" transmission conditions, across a sharp contact interface, for a heat equation (in its static, i.e., elliptic, counterpart), by means of a concentration approach performed on a composite "thick" interface separating two thermally conductive media. Subsequently, a homogenization limit is performed via two-scale asymptotic expansions on the system of equations thus obtained.

1. Introduction

In the literature, many models describing the evolution of the temperature across an interface have been studied. For example, if the contact between the two media is deemed to be "thermally perfect", we have that the temperature u and its flux $A\nabla u \cdot \nu$ are continuous across the interface Γ (where A is the conductivity and ν is the unit normal vector to Γ). On the other hand, if the thermal contact is "imperfect", we have the well-known Newton boundary conditions, in which the heat flux is continuous across Γ while the temperature u is not. Namely we have

$$[A\nabla u \cdot \nu] = A^{\text{out}}\nabla u^{\text{out}} \cdot \nu - A^{\text{int}}\nabla u^{\text{int}} \cdot \nu = 0 \qquad \text{on } \Gamma,$$
$$[u] = u^{\text{out}} - u^{\text{int}} = A^{\text{out}}\nabla u^{\text{out}} \cdot \nu = A^{\text{int}}\nabla u^{\text{int}} \cdot \nu \quad \text{on } \Gamma,$$

where $[\,\cdot\,]$ denotes the jump across Γ.

Similar models appear in the context of electrical conduction where, denoting by u the potential, the current j (the flux of u) is continuous across a capacitor, while the time derivative of the potential jump is proportional to the flux (see, for instance, [5; 6; 7; 8; 10; 13; 15; 26]); namely

$$[A\nabla u \cdot \nu] = 0 \quad \text{and} \quad \frac{\partial [u]}{\partial t} = A^{\text{out}}\nabla u^{\text{out}} \cdot \nu \qquad \text{on } \Gamma.$$

Communicated by Emilio Barchiesi.
MSC2020: 35B27, 35C20, 35J47.
Keywords: homogenization, asymptotic expansions, concentration, elliptic partial differential equations.

Going back to heat conduction, we see that, in recent years, different models to describe the thermal behavior of the interface Γ have appeared ([19; 20; 22; 24; 25; 29]). In particular, in [2; 3] the temperature is continuous across Γ but the flux is not. This is due to the fact that the jump of the flux $[A\nabla u \cdot \nu]$, i.e., the "missing energy", is transported along the interface which is assumed to be "highly conductive" and, therefore, it appears as a source term for a differential equation, satisfied on Γ, governed by the Laplace–Beltrami operator.

Previous models, where the interface represents the mathematical approximation of a physical membrane having a very small thickness, are usually obtained via concentration or suitable Taylor expansions. Similar techniques were used by some of the authors in [4; 6; 9] to derive models to describe the behavior of active interfaces for electrical conduction and heat diffusion. In particular, in [6], it has been proven that a thick interface in which diffusion occurs with transversal diffusivity vanishing as the thickness η of the interface goes to zero leads to a model for which the flux is continuous but the unknown u has a jump across Γ. On the other hand, in [4], it is assumed that the transversal diffusivity remains stable while the tangential one tends to infinity, as η tends to zero, leading to a model where the heat flux is discontinuous, while u is continuous on Γ and satisfies a diffusion equation whose source term is just the jump of the flux.

It is, therefore, natural to investigate if it is possible to conceive particular structures for a "fat" (N-dimensional) interface that, via concentration (letting the thickness of the interface go to zero), give rise to a new set of interface conditions, in which both the temperature and its flux are discontinuous across Γ (see, for example, [27]).

Motivated by the previous considerations, we study, in this paper, the concentration limit of a problem in which two different media are separated by a composite thick interface made of two materials with dissimilar physical properties. In this structure, the two materials are disposed in such a way that one of them is encapsulated in the other and, as the thickness goes to zero, in the internal material the tangential diffusivity stays stable and the transversal one goes to zero, while in the external material the transversal diffusivity remains stable and the tangential one goes to infinity. In the concentration limit, we obtain a problem in which both the state variable u and its flux are discontinuous across the limiting sharp interface Γ. However, besides the jump operator $[\cdot]$, the new average operator $\{\cdot\}$, where $\{f\} = f^{\text{int}} + f^{\text{out}}$ (with the usual meanings of the superscripts), appears and plays a relevant role in this new model. In fact, on Γ the jump of the flux $[A\nabla u \cdot \nu]$ is the source term of a Laplace–Beltrami equation for $\{u\}$, while $[u]$ solves a diffusion equation still involving the Laplace–Beltrami operator having as a source term $\{A\nabla u \cdot \nu\}$.

In this framework, it is worthwhile to recall [11], in which the concentration is performed for a similarly layered interface, with the difference that the roles of the external and the internal layer are interchanged. It turns out that, as expected, in the spatially one-dimensional case (i.e., when the interface is just a point), our concentrated model reduces to the well-known Newton model for imperfect thermal contact.

The system of PDEs associated to the quoted model is well posed (see Section 3) and it is possible to find a reasonable energy estimate for it, thus providing a further physical validation of our problem.

We stress the fact that our concentration limit is performed under the simplifying assumption that the thick interface, as well as the sharp one, is flat. Once we have obtained the concentrated model, we consider a composite medium Ω having a microstructure constituted by an array of periodic cells, with a very small characteristic dimension ε, made up of two different materials $\Omega_\varepsilon^{\text{int}}$ and $\Omega_\varepsilon^{\text{out}}$ separated by an $(N-1)$-dimensional interface Γ_ε acting according to the concentration limit equations found before. Therefore, in the spirit of homogenization, we proceed to perform the limit $\varepsilon \to 0$ for this new set of equations. The homogenization limit will be done via formal two-scale expansions, following the technique devised by Bensoussan, Lions and Papanicolau in [14]. As done in other papers (see, for instance, [6; 13; 15; 16]), we consider a hierarchy of different scalings of the physical constants present in our system of equations. Essentially, we will study three different scalings $m = -1, 0, 1$ (see 3-1), which, in accordance with the previous literature, are the main ones, and we will prove that, only in the case $m = -1$, the macroscopic model preserves memory of the complete physical structure of the interface. This seems to be the correct physical scaling leading to the more relevant model, which describes the problem under investigation.

The problem here studied is, up to our knowledge, mathematically new and quite interesting, since it provides a new mechanism by which heat is transmitted through an $(N-1)$-dimensional interface. However, in the engineering literature, similar problems, involving simultaneously the jump, the average and the Laplace–Beltrami operators acting both on the solution and on the flux, have already appeared (see, for instance, [18; 27; 28]). With respect to such models, not fully studied from a rigorous mathematical point of view, the concentration procedure formally provided in this paper (as well as the homogenization one) can be made rigorous (see [11; 12]). This fact gives the novelty and the relevance of the present paper and also hints at possible applications of the models contained therein to real-life problems.

The paper is organized as follows. In Section 2, we recall the main properties of the tangential operators, state our geometrical settings and introduce the proper functional spaces needed in the sequel. Section 3 is devoted to the well-posedness of the microscopic problems considered in this paper. In Section 4, in a simpler two-dimensional flat geometry, we formally derive the microscopic model via a concentration procedure. Finally, in Section 5, by means of the two-scale expansion technique, we perform the homogenization limit of our model for three different scalings.

2. Notation and preliminaries

Notation. We will assume that $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded open set with smooth boundary $\partial\Omega$.

The sets $C_c^k(\Omega)$, with $k \in \mathbb{N}$ ($C_c^\infty(\Omega)$, respectively), will denote the subset of the functions belonging to the standard space $C^k(\Omega)$ ($C^\infty(\Omega)$, respectively) with compact support in Ω.

Also, $H^1(\Omega)$, $H_0^1(\Omega)$ and $H^1_{\mathrm{loc}}(\Omega)$ will denote the usual Sobolev spaces.

Since, in the sequel, we will also deal with periodic functions, we recall here the main associated functional spaces. Let $Y = (0, 1)^N$ be the reference unit cell in $\mathbb{R}^N$. We will denote by $C^k_{\mathrm{per}}(Y)$ the set of the Y-periodic functions in $C^k(\mathbb{R}^N)$, by $L^p_{\mathrm{per}}(Y)$ the set of the Y-periodic functions in $L^p_{\mathrm{loc}}(\mathbb{R}^N)$ and by $H^1_{\mathrm{per}}(Y)$ the set of the Y-periodic functions in $H^1_{\mathrm{loc}}(\mathbb{R}^N)$.

Finally, C will be a strictly positive constant, which may vary from line to line.

Tangential differential operators. We recall that for a function $\phi \in C^1(\Omega)$ and a smooth surface $S \subset \Omega \subseteq \mathbb{R}^N$, the tangential gradient $\nabla^B \phi$ on S is the projection of $\nabla\phi$ on the tangent hyperplane to S, that is,

$$\nabla^B \phi := \nabla\phi - (n \cdot \nabla\phi)n, \tag{2-1}$$

where n is the normal unit vector to S and ∇ is the classical gradient.

For a vector-valued function $\Phi \in C^1(\Omega)$, the tangential divergence of Φ on S is defined as

$$\begin{aligned}\operatorname{div}^B \Phi &:= \operatorname{div}(\Phi - (n \cdot \Phi)n) \\ &= \operatorname{div} \Phi - (n \cdot \nabla\Phi_i)n_i - (\operatorname{div} n)(n \cdot \Phi),\end{aligned}$$

where, taking into account the smoothness of S, the normal vector n can be naturally defined in a small neighborhood of S as $\nabla d/|\nabla d|$, where d is the signed distance from S.

For a scalar function $\phi \in C^2(\Omega)$, the Laplace–Beltrami operator $\Delta^B\phi$ is defined as

$$\begin{aligned}\Delta^B \phi &:= \operatorname{div}^B(\nabla^B \phi) = \Delta\phi - n^t \nabla^2\phi\, n - (n \cdot \nabla\phi) \operatorname{div} n \\ &= (\delta_{ij} - n_i n_j)\partial^2_{ij}\phi - (n_i \partial_i \phi)(\partial_j n_j),\end{aligned} \tag{2-2}$$

where δ_{ij} is the Kronecker delta and, as usual, we sum with respect to repeated indices. Here, $\nabla^2\phi$ denotes the Hessian matrix of ϕ.

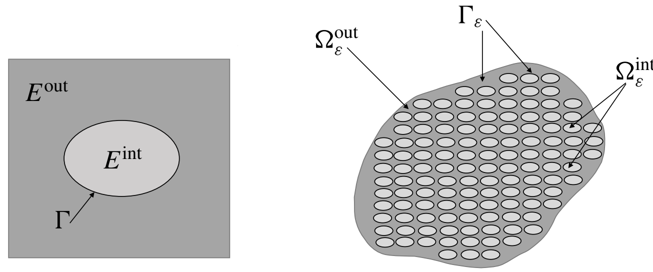

Figure 1. Micro- and macroscopic view of the periodic structure in the connected-disconnected geometrical settings.

We recall that, if S is a regular surface with no boundary, i.e., $\partial S = \varnothing$, we have

$$\int_S \operatorname{div}^B \Phi \, d\sigma = 0. \tag{2-3}$$

Geometrical settings. The typical periodic geometrical settings are displayed in Figures 1 and 2 and, in this section, we give their detailed formal definitions. Assume that E is a periodic open subset of $\mathbb{R}^N$ such that $E + z = E$ for all $z \in \mathbb{Z}^N$. For the sake of simplicity, assume that the boundaries of Ω and E are of class C^∞. Set

$$E^{\text{int}} := E \cap Y, \quad E^{\text{out}} := Y \setminus \bar{E}, \quad \Gamma := \partial E \cap Y,$$

so that Y is the union of the two disjoint open subsets E^{int} and E^{out} and the common boundary Γ.

Let $\varepsilon \in (0, 1]$ be the small parameter accounting for the micro length-scale, which will converge to zero. We define

- $\Omega_\varepsilon^{\text{int}} := \Omega \cap \varepsilon E$ to be the the inner conductive phase;
- $\Omega_\varepsilon^{\text{out}} := \Omega \setminus \varepsilon \bar{E} = \Omega \setminus \overline{\Omega_\varepsilon^{\text{int}}}$ to be the the outer conductive phase;
- $\Gamma_\varepsilon := \partial \Omega_\varepsilon^{\text{int}} \cap \Omega = \partial \Omega_\varepsilon^{\text{out}} \cap \Omega$ to be the the interface between the two conductive phases,

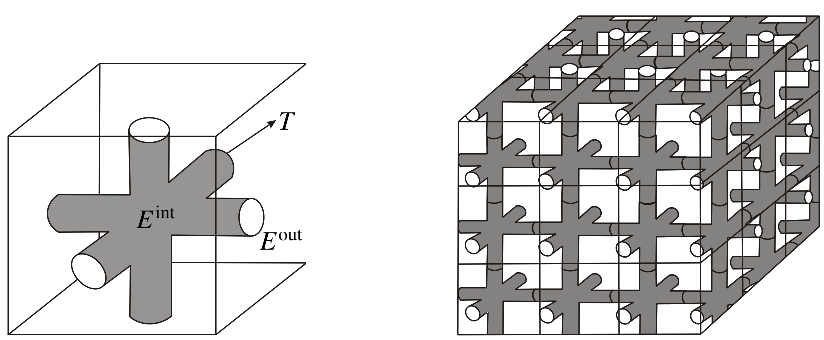

Figure 2. Micro- and macroscopic view of the periodic structure in the connected-connected geometrical settings.

so that

$$\Omega = \Omega_\varepsilon^{\mathrm{int}} \cup \Omega_\varepsilon^{\mathrm{out}} \cup \Gamma_\varepsilon.$$

We assume also that $\Omega_\varepsilon^{\mathrm{out}}$ is connected at each step $\varepsilon > 0$, whereas $\Omega_\varepsilon^{\mathrm{int}}$ will be connected or disconnected. Indeed, we will consider the following two situations.

Connected-disconnected case: We assume that $\Gamma \cap \partial Y = \varnothing$, that is, the boundary of E does not touch the boundary of the unit cell Y (see Figure 1). Here, the domain Ω is the union of the connected domain $\Omega_\varepsilon^{\mathrm{out}}$, the disconnected domain $\Omega_\varepsilon^{\mathrm{int}}$ and the common boundary Γ_ε. We also assume that the cells intersecting the boundary $\partial\Omega$ do not contain any inclusion, so that we have $\mathrm{dist}(\Gamma_\varepsilon, \partial\Omega) \geq C_0\,\varepsilon$, for some suitable constant $C_0 > 0$ independent of ε.

Connected-connected case: We assume that $\partial E \cap Y \neq \varnothing$, but $|\partial E \cap Y|_{N-1} = 0$ (where $|\cdot|_{N-1}$ denotes the $(N-1)$-dimensional Hausdorff measure). In this situation, we stipulate that E^{int}, E^{out}, $\Omega_\varepsilon^{\mathrm{int}}$, and $\Omega_\varepsilon^{\mathrm{out}}$ are connected and, without any loss of generality, that they have Lipschitz continuous boundary (at least for a suitable choice of a subsequence $\varepsilon_n \to 0$). In this case, at each level $\varepsilon > 0$, we have that both $\partial\Omega \cap \partial\Omega_\varepsilon^{\mathrm{int}}$ and $\partial\Omega \cap \partial\Omega_\varepsilon^{\mathrm{out}}$ are nonempty (see Figure 2).

Finally, let ν be the normal unit vector to Γ pointing into E^{out}, extended by periodicity to the whole of $\mathbb{R}^N$, so that $\nu_\varepsilon(x) = \nu\big(\frac{x}{\varepsilon}\big)$ denotes the normal unit vector to Γ_ε pointing into $\Omega_\varepsilon^{\mathrm{out}}$.

The space $\widehat{\mathcal{H}}^\varepsilon_{0,m}(\Omega)$. We introduce here the proper functional setting for the ε-microscopic problem we will analyze. For this purpose, given a function u defined in Ω, we denote by u^{int} and u^{out} the restriction of u to $\Omega_\varepsilon^{\mathrm{int}}$ and $\Omega_\varepsilon^{\mathrm{out}}$, respectively, and, with abuse of notation, we use the same symbols also for the corresponding traces on Γ_ε. We denote by $[u]$ the jump of u across the interface Γ_ε, i.e.,

$$[u] = u^{\mathrm{out}} - u^{\mathrm{int}}, \tag{2-4}$$

and, similarly, $\{u\}$ denotes the sum of the two potentials u^{int} and u^{out} at the interface Γ_ε, i.e.,

$$\{u\} = u^{\mathrm{out}} + u^{\mathrm{int}}. \tag{2-5}$$

The same notation will be used for other quantities. Let us remark, for later use, that

$$u^{\mathrm{out}} = \tfrac{1}{2}(\{u\} + [u]) \quad \text{and} \quad u^{\mathrm{int}} = \tfrac{1}{2}(\{u\} - [u]).$$

Definition 2.1. For a given $\varepsilon \in (0, 1]$ and $m = -1, 0, 1$, let us define

$$\widehat{\mathcal{H}}^\varepsilon_{0,m}(\Omega) := \big\{u = (u^{\mathrm{int}}, u^{\mathrm{out}}) : u^{\mathrm{int}} \in H^1(\Omega_\varepsilon^{\mathrm{int}}),\, u^{\mathrm{out}} \in H^1(\Omega_\varepsilon^{\mathrm{out}}),\\ [u] \in L^2(\Gamma_\varepsilon),\, \nabla^B[u] \in L^2(\Gamma_\varepsilon),\, \{u\} \in L^2(\Gamma_\varepsilon),\\ \nabla^B\{u\} \in L^2(\Gamma_\varepsilon),\, u = 0 \text{ on } \partial\Omega\big\} \tag{2-6}$$

endowed with the norm

$$\|u\|^2_{\widehat{\mathcal{H}}^{\varepsilon}_{0,m}(\Omega)} := \|\nabla u\|^2_{L^2(\Omega^{\rm int}_\varepsilon)} + \|\nabla u\|^2_{L^2(\Omega^{\rm out}_\varepsilon)} + \frac{\varepsilon^m}{2}\|[u]\|^2_{L^2(\Gamma_\varepsilon)} + \frac{\varepsilon^{m+2}}{2}\|\nabla^B[u]\|^2_{L^2(\Gamma_\varepsilon)} + \frac{\varepsilon^{m+2}}{2}\|\nabla^B\{u\}\|^2_{L^2(\Gamma_\varepsilon)}. \tag{2-7}$$

Clearly, (2-7) defines a norm. Indeed, the positive 1-homogeneity and the triangle inequality are straightforward. On the other hand, when $||u||_{\widehat{\mathcal{H}}^{\varepsilon}_{0,m}(\Omega)} = 0$, it follows that $\nabla u = 0$ and, then, u is constant in $\Omega^{\rm int}_\varepsilon$ and $\Omega^{\rm out}_\varepsilon$. However, from $[u] = 0$ on Γ and $u = 0$ on $\partial\Omega$, we conclude that $u = 0$ in the whole of Ω.

The norm defined above is associated with the scalar product given by

$$(u, v)_{\widehat{\mathcal{H}}^{\varepsilon}_{0,m}(\Omega)} = \int_{\Omega^{\rm int}_\varepsilon} \nabla u\cdot\nabla v\,dx + \int_{\Omega^{\rm out}_\varepsilon} \nabla u\cdot\nabla v\,dx + \frac{\varepsilon^m}{2}\int_{\Gamma_\varepsilon}[u]\,[v]\,d\sigma + \frac{\varepsilon^{m+2}}{2}\int_{\Gamma_\varepsilon}\nabla^B[u]\cdot\nabla^B[v]\,d\sigma + \frac{\varepsilon^{m+2}}{2}\int_{\Gamma_\varepsilon}\nabla^B\{u\}\cdot\nabla^B\{v\}\,d\sigma, \tag{2-8}$$

for any $u, v \in \widehat{\mathcal{H}}^{\varepsilon}_{0,m}(\Omega)$.

Notice that the space $\widehat{\mathcal{H}}^{\varepsilon}_{0,m}(\Omega)$ coincides with the space of the piecewise H^1-functions in $\Omega^{\rm int}_\varepsilon$ and $\Omega^{\rm out}_\varepsilon$, with zero boundary value, whose traces on Γ_ε from $\Omega^{\rm int}_\varepsilon$ and $\Omega^{\rm out}_\varepsilon$ belong to the space $H^1(\Gamma_\varepsilon)$, where

$$H^1(\Gamma_\varepsilon) = \{v \in L^2(\Gamma_\varepsilon) : \nabla^B v \in L^2(\Gamma_\varepsilon)\}.$$

We recall also that, for $u \in \widehat{\mathcal{H}}^{\varepsilon}_{0,m}(\Omega)$, the following Poincaré inequality holds (see [23, Lemma 6 complemented with Lemma 4] for the connected-disconnected case, and [1, Lemma A.4] for the connected-connected case):

$$\|u\|^2_{L^2(\Omega)} \le C(\|\nabla u\|^2_{L^2(\Omega^{\rm out}_\varepsilon)} + \|\nabla u\|^2_{L^2(\Omega^{\rm int}_\varepsilon)} + \varepsilon\|[u]\|^2_{L^2(\Gamma_\varepsilon)}) \le C\,\|u\|^2_{\widehat{\mathcal{H}}^{\varepsilon}_{0,m}(\Omega)}, \tag{2-9}$$

where the constant C is independent of ε.

Lemma 2.2. *For any fixed $\varepsilon \in (0, 1]$ and $m = -1, 0, 1$, the space $\widehat{\mathcal{H}}^{\varepsilon}_{0,m}(\Omega)$ is a Banach space.*

Proof. Consider an arbitrary Cauchy sequence (u_n) in $\widehat{\mathcal{H}}^{\varepsilon}_{0,m}(\Omega)$. From the inequality (2-9), it follows that $(u^{\rm int}_n)$ and $(u^{\rm out}_n)$ are Cauchy sequences in $H^1(\Omega^{\rm int}_\varepsilon)$ and $H^1(\Omega^{\rm out}_\varepsilon)$, respectively. Thus, they converge in such spaces to $u^{*\rm int}$ and $u^{*\rm out}$, respectively, and, hence, $u^* = (u^{*\rm int}, u^{*\rm out})$ has null trace on $\partial\Omega$, as a consequence of the fact that the sequence (u_n) satisfies the same property.

In particular, $u^{\rm int}_n \to u^{*\rm int}$ and $u^{\rm out}_n \to u^{*\rm out}$ strongly in $L^2(\Gamma_\varepsilon)$; thus, $[u_n] \to [u^*]$ and $\{u_n\} \to \{u^*\}$ strongly in $L^2(\Gamma_\varepsilon)$.

Also, the sequences $(\nabla^B u^{\rm int}_n)$ and $(\nabla^B u^{\rm out}_n)$ are Cauchy sequences in $L^2(\Gamma_\varepsilon)$; hence, there exist two limit vectors $\zeta^{\rm int}$ and $\zeta^{\rm out}$ such that

$$\nabla^B u^{\rm int}_n \to \zeta^{\rm int} \quad\text{and}\quad \nabla^B u^{\rm out}_n \to \zeta^{\rm out} \qquad \text{strongly in } L^2(\Gamma_\varepsilon). \tag{2-10}$$

It remains only to prove that $\zeta^{\text{int}} = \nabla^B u^{*\text{int}}$ and $\zeta^{\text{out}} = \nabla^B u^{*\text{out}}$, respectively. For every vector function $\Phi \in C_c^\infty(\Omega)$, we have

$$\int_{\Gamma_\varepsilon} \zeta^{\text{int}} \cdot \Phi \, d\sigma \leftarrow \int_{\Gamma_\varepsilon} \nabla^B u_n^{\text{int}} \cdot \Phi \, d\sigma = -\int_{\Gamma_\varepsilon} u_n^{\text{int}} \operatorname{div}^B \Phi \, d\sigma \to -\int_{\Gamma_\varepsilon} u^{*\text{int}} \operatorname{div}^B \Phi \, d\sigma.$$

This implies that $u^{*\text{int}} \in H^1(\Gamma_\varepsilon)$ and $\zeta^{\text{int}} = \nabla^B u^{*\text{int}}$. Clearly, the same holds for ζ^{out} and $\nabla^B u^{*\text{out}}$. In particular, we obtain $\nabla^B [u_n] \to \nabla^B [u^*]$ and $\nabla^B \{u_n\} \to \nabla^B \{u^*\}$, which completes the proof. □

For later use, let us also define the periodic version of the previous space as

$$\widehat{\mathcal{H}}_{\text{per}}(Y) := \big\{ u = (u^{\text{int}}, u^{\text{out}}) : u^{\text{int}} \in H^1_{\text{per}}(E^{\text{int}}),\ u^{\text{out}} \in H^1_{\text{per}}(E^{\text{out}}),\ [u] \in H^1_{\text{per}}(\Gamma),\ \{u\} \in H^1_{\text{per}}(\Gamma) \big\}. \tag{2-11}$$

Here and in the following $H^1_{\text{per}}(E^{\text{int}})$ ($H^1_{\text{per}}(E^{\text{out}})$ and $H^1_{\text{per}}(\Gamma)$, respectively) denotes the space of the Y-periodic functions belonging to $H^1_{\text{loc}}(E)$ ($H^1_{\text{loc}}(\mathbb{R}^N \setminus \bar{E})$ and $H^1_{\text{loc}}(\partial E)$, respectively).

3. Position and well-posedness of the problem $\mathcal{B}_\varepsilon$

The ε-microscopic model, which we are interested in, is given by

$$\mathcal{B}_\varepsilon : \begin{cases} -\operatorname{div}(A_\varepsilon \nabla u_\varepsilon^{\text{int}}) = f & \text{in } \Omega_\varepsilon^{\text{int}}, & \text{(3-1a)} \\ -\operatorname{div}(A_\varepsilon \nabla u_\varepsilon^{\text{out}}) = f & \text{in } \Omega_\varepsilon^{\text{out}}, & \text{(3-1b)} \\ -\gamma \varepsilon^{m+2} \Delta^B \{u_\varepsilon\} = [A_\varepsilon \nabla u_\varepsilon \cdot \nu_\varepsilon] & \text{on } \Gamma_\varepsilon, & \text{(3-1c)} \\ \alpha \varepsilon^m [u_\varepsilon] - \beta \varepsilon^{m+2} \Delta^B [u_\varepsilon] = \{A \nabla u_\varepsilon \cdot \nu_\varepsilon\} & \text{on } \Gamma_\varepsilon, & \text{(3-1d)} \\ u_\varepsilon = 0 & \text{on } \partial\Omega, & \text{(3-1e)} \end{cases}$$

where α, β, γ are strictly positive constants. The source term $f \in L^2(\Omega)$ and the diffusivity matrix A_ε is given by $A_\varepsilon(x) = A\big(\frac{x}{\varepsilon}\big)$ where A is a measurable, Y-periodic symmetric matrix satisfying

$$\lambda |\zeta|^2 \le (A(y)\zeta, \zeta) \le \Lambda |\zeta|^2 \quad \text{for a.e. } y \in Y \text{ and any } \zeta \in \mathbb{R}^N, \tag{3-2}$$

where $0 < \lambda < \Lambda < +\infty$ are suitable constants. The physical meaning of the constants γ, β, α is inherited from the one of the corresponding constants in problem (4-1), describing the heat distribution in the case where a thick membrane is present and such quantities represent the tangential (β and γ) and the transversal (α) diffusivities, which can be experimentally measurable (see, for instance, [17; 21]).

On the other hand, the mathematical description of our problem is given by an elliptic equation in each phase $\Omega_\varepsilon^{\text{int}}$ and $\Omega_\varepsilon^{\text{out}}$, complemented with a homogenous Dirichlet boundary condition on $\partial\Omega$. The thermal potentials $u_\varepsilon^{\text{int}}$ and $u_\varepsilon^{\text{out}}$ of the two phases are coupled by means of two interface conditions: the jump of the flux of the solution u_ε is assumed to be proportional to the Laplace–Beltrami of the sum $\{u_\varepsilon\}$ of the traces of the two potentials at the interface, whereas the jump $[u_\varepsilon]$ of the solution across the interface is governed by an equation involving the Laplace–Beltrami operator and having as a source the sum of the two fluxes from the external and from the internal phase. Note that, if $\beta = \gamma = 0$, conditions (3-1c) and (3-1d) reduce to the well-known Newton boundary conditions, describing the imperfect thermal contact (see, for instance, [5; 23; 26]). On the other hand, if $\alpha \to +\infty$, we get the static counterpart of the problem studied in [3].

In the following, we will consider the problem $\mathcal{B}_\varepsilon$ for different scalings of the parameter ε, by taking into account the exponents $m = -1, 0, 1$. This is consistent with what has been done, for instance, in [6; 7; 8; 10; 13; 15; 16], where it has been proved that the only relevant cases, from the point of view of the homogenization, appear when $m \in [-1, 1]$. In this situation, only three different regimes are possible, which are precisely $m = -1$, $m \in (-1, 1)$ (that is, $m = 0$, in our case where we adopt the two-scale expansion technique) and $m = 1$. The different scalings in conditions (3-1c)–(3-1d) are due to a homogeneity argument, since the Laplace–Beltrami is a second-order operator, while the jump $[\,\cdot\,]$ is a zero-order one.

Since problem (3-1a)–(3-1e) is not standard, at the end of this section we will state and prove an existence and uniqueness result, starting from its weak formulation.

Definition 3.1 (weak solution of $\mathcal{B}_\varepsilon$). We say that $u_\varepsilon \in \widehat{\mathcal{H}}^\varepsilon_{0,m}(\Omega)$ is a weak solution of the problem $\mathcal{B}_\varepsilon$, given by (3-1a)–(3-1e), if

$$\int_{\Omega^{\rm int}_\varepsilon} A_\varepsilon \nabla u^{\rm int}_\varepsilon \cdot \nabla\varphi\, dx + \int_{\Omega^{\rm out}_\varepsilon} A_\varepsilon \nabla u^{\rm out}_\varepsilon \cdot \nabla\varphi\, dx + \alpha\frac{\varepsilon^m}{2}\int_{\Gamma_\varepsilon}[u_\varepsilon]\,[\varphi]\, d\sigma$$
$$+ \beta\frac{\varepsilon^{m+2}}{2}\int_{\Gamma_\varepsilon}\nabla^B[u_\varepsilon]\cdot\nabla^B[\varphi]\, d\sigma + \gamma\frac{\varepsilon^{m+2}}{2}\int_{\Gamma_\varepsilon}\nabla^B\{u_\varepsilon\}\cdot\nabla^B\{\varphi\}\, d\sigma = \int_\Omega f\varphi\, dx, \tag{3-3}$$

for every test function $\varphi \in \widehat{\mathcal{H}}^\varepsilon_{0,m}(\Omega)$. □

Proposition 3.2. *If $u_\varepsilon \in \widehat{\mathcal{H}}^\varepsilon_{0,m}(\Omega)$ is a weak solution of problem (3-1), then there exists a constant C (independent of ε) such that*

$$\|u_\varepsilon\|^2_{\widehat{\mathcal{H}}^\varepsilon_{0,m}(\Omega)} \le C\,\|f\|^2_{L^2(\Omega)}.$$

Proof. By choosing $\varphi = u_\varepsilon$ in the weak formulation (3-3), recalling (3-2), using Young's inequality and the Poincaré inequality (2-9), we get

$$\begin{aligned}
\lambda\,\|\nabla u_\varepsilon\|^2_{L^2(\Omega^{\rm int}_\varepsilon\cup\Omega^{\rm out}_\varepsilon)} &+ \alpha\frac{\varepsilon^m}{2}\|[u_\varepsilon]\|^2_{L^2(\Gamma_\varepsilon)} \\
&\qquad + \beta\frac{\varepsilon^{m+2}}{2}\|\nabla^B[u_\varepsilon]\|^2_{L^2(\Gamma_\varepsilon)} + \gamma\frac{\varepsilon^{m+2}}{2}\|\nabla^B\{u_\varepsilon\}\|^2_{L^2(\Gamma_\varepsilon)} \\
\le \int_{\Omega^{\rm int}_\varepsilon\cup\Omega^{\rm out}_\varepsilon} & A_\varepsilon\nabla u_\varepsilon\cdot\nabla u_\varepsilon\, dx + \alpha\frac{\varepsilon^m}{2}\int_{\Gamma_\varepsilon}|[u_\varepsilon]|^2\, d\sigma \\
&\qquad + \beta\frac{\varepsilon^{m+2}}{2}\int_{\Gamma_\varepsilon}|\nabla^B[u_\varepsilon]|^2\, d\sigma + \gamma\frac{\varepsilon^{m+2}}{2}\int_{\Gamma_\varepsilon}|\nabla^B\{u_\varepsilon\}|^2\, d\sigma
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{\delta}\,\|f\|^2_{L^2(\Omega)} + \delta\,\|u_\varepsilon\|^2_{L^2(\Omega)}\\
&\le \frac{1}{\delta}\,\|f\|^2_{L^2(\Omega)} + C\delta\left(\|\nabla u_\varepsilon\|^2_{L^2(\Omega^{\rm int}_\varepsilon\cup\Omega^{\rm out}_\varepsilon)} + \varepsilon\,\|[u_\varepsilon]\|^2_{L^2(\Gamma_\varepsilon)}\right)\\
&\le \frac{1}{\delta}\,\|f\|^2_{L^2(\Omega)} + C\delta\left(\|\nabla u_\varepsilon\|^2_{L^2(\Omega^{\rm int}_\varepsilon\cup\Omega^{\rm out}_\varepsilon)} + \varepsilon^m\,\|[u_\varepsilon]\|^2_{L^2(\Gamma_\varepsilon)}\right),
\end{aligned}
$$

where $\delta > 0$ can be chosen arbitrarily. Thus, by choosing $C\delta = \min\left(\frac{\lambda}{2}, \frac{\alpha}{4}\right)$, we obtain

$$
\begin{aligned}
\|u_\varepsilon\|^2_{\mathcal{H}^\varepsilon_{0,m}(\Omega)} = \|\nabla u_\varepsilon\|^2_{L^2(\Omega^{\rm int}_\varepsilon\cup\Omega^{\rm out}_\varepsilon)} &+ \frac{\varepsilon^m}{2}\|[u_\varepsilon]\|^2_{L^2(\Gamma_\varepsilon)}\\
&+ \frac{\varepsilon^{m+2}}{2}\|\nabla^B[u_\varepsilon]\|^2_{L^2(\Gamma_\varepsilon)} + \frac{\varepsilon^{m+2}}{2}\|\nabla^B\{u_\varepsilon\}\|^2_{L^2(\Gamma_\varepsilon)}\\
\le C\,\|f\|^2_{L^2(\Omega)}, \qquad\qquad &\qquad\qquad\qquad\qquad\qquad\qquad \text{(3-4)}
\end{aligned}
$$

where C is independent of ε. This completes the proof. $\square$

Remark 3.3. By taking into account (3-4) and the Poincaré inequality (2-9), it follows that there exists a function $u_0 \in L^2(\Omega)$, such that, up to a subsequence, $u_\varepsilon \rightharpoonup u_0$ weakly in $L^2(\Omega)$, as $\varepsilon \to 0$. Our interest will be the characterization of such a limit u_0 as the solution of a suitable differential problem.

Equation (3-4) also implies that there exists a constant $C \ge 0$, independent of ε, such that

$$
\begin{gathered}
\|u^{\rm int}_\varepsilon\|_{H^1(\Omega^{\rm int}_\varepsilon)} \le C, \quad \|u^{\rm out}_\varepsilon\|_{H^1(\Omega^{\rm out}_\varepsilon)} \le C, \quad \|[u_\varepsilon]\|_{L^2(\Gamma_\varepsilon)} \le C\varepsilon^{-\frac{m}{2}},\\
\|\nabla^B[u_\varepsilon]\|_{L^2(\Gamma_\varepsilon)} \le C\varepsilon^{-\frac{m+2}{2}}, \quad \|\nabla^B\{u_\varepsilon\}\|_{L^2(\Gamma_\varepsilon)} \le C\varepsilon^{-\frac{m+2}{2}}.
\end{gathered}
$$

$\square$

The last part of this section will be devoted to prove the existence and uniqueness of the solution $u_\varepsilon \in \widehat{\mathcal{H}}^\varepsilon_{0,m}(\Omega)$ of the problem (3-1) for any fixed ε. To this end, let us take $\varepsilon = 1$ and rewrite the problem $\mathcal{B}_\varepsilon$ as

$$
\mathcal{B}:\begin{cases}
-\operatorname{div}(A\nabla u^{\rm int}) = f & \text{in } \Omega^{\rm int}, \qquad \text{(3-5a)}\\
-\operatorname{div}(A\nabla u^{\rm out}) = f & \text{in } \Omega^{\rm out}, \qquad \text{(3-5b)}\\
-\gamma\triangle^B\{u\} = [A\nabla u\cdot\nu] + g & \text{on } \Gamma, \qquad \text{(3-5c)}\\
\alpha[u] - \beta\triangle^B[u] = \{A\nabla u\cdot\nu\} + h & \text{on } \Gamma, \qquad \text{(3-5d)}\\
u = 0 & \text{on } \partial\Omega. \qquad \text{(3-5e)}
\end{cases}
$$

Here, $\Omega^{\rm int}$ and $\Omega^{\rm out}$ denote the two phases and Γ is the interface between them; $g, h \in L^2(\Gamma)$ are given source terms (in problem (3-1) they are assumed to be identically zero).

Following (3-3), the rigorous weak formulation of (3-5a)–(3-5e) is

$$\int_{\Omega^{\rm int}} A\nabla u^{\rm int}\cdot\nabla v\,dx+\int_{\Omega^{\rm out}} A\nabla u^{\rm out}\cdot\nabla v\,dx+\frac{\alpha}{2}\int_\Gamma [u]\,[v]\,d\sigma +\frac{\beta}{2}\int_\Gamma \nabla^B[u]\cdot\nabla^B[v]\,d\sigma+\frac{\gamma}{2}\int_\Gamma \nabla^B\{u\}\cdot\nabla^B\{v\}\,d\sigma = \int_\Omega f\,v\,dx+\frac12\int_\Gamma g\{v\}\,d\sigma+\frac12\int_\Gamma h[v]\,d\sigma, \tag{3-6}$$

for every test function $v\in\widehat{\mathcal{H}}_0(\Omega)$. Here, the space $\widehat{\mathcal{H}}_0(\Omega)$ is defined analogously to (2-6) as

$$\widehat{\mathcal{H}}_0(\Omega):=\big\{u=(u^{\rm int},u^{\rm out})\mid u^{\rm int}\in H^1(\Omega^{\rm int}),\ u^{\rm out}\in H^1(\Omega^{\rm out}),\ [u]\in L^2(\Gamma),\ \nabla^B[u]\in L^2(\Gamma),\ \{u\}\in L^2(\Gamma),\ \nabla^B\{u\}\in L^2(\Gamma),\ u=0 \text{ on } \partial\Omega\big\}. \tag{3-7}$$

Let us define the bilinear form $b:\widehat{\mathcal{H}}_0(\Omega)\times\widehat{\mathcal{H}}_0(\Omega)\to\mathbb{R}$ as

$$b(u,v):=\int_{\Omega^{\rm int}} A\nabla u^{\rm int}\cdot\nabla v^{\rm int}\,dx+\int_{\Omega^{\rm out}} A\nabla u^{\rm out}\cdot\nabla v^{\rm out}\,dx+\frac{\alpha}{2}\int_\Gamma [u]\,[v]\,d\sigma +\frac{\beta}{2}\int_\Gamma \nabla^B[u]\cdot\nabla^B[v]\,d\sigma+\frac{\gamma}{2}\int_\Gamma \nabla^B\{u\}\cdot\nabla^B\{v\}\,d\sigma, \tag{3-8}$$

where α, β, γ and A are defined above and (3-2) is in force. It is not difficult to see that the bilinear form $b(u,v)$ is symmetric, continuous and coercive.

Theorem 3.4 (existence and uniqueness result for problem (3-5)). *Let $A\in L^\infty(\Omega)$ be a symmetric matrix satisfying (3-2), $\alpha,\beta,\gamma>0$, $f\in L^2(\Omega)$ and $g,h\in L^2(\Gamma)$. Then, problem (3-6) admits a unique solution $u\in\widehat{\mathcal{H}}_0(\Omega)$.*

Proof. Clearly, the weak formulation (3-6) can be written as

$$b\,(u,v)=\langle F,v\rangle\quad\text{for all }\ v\in\widehat{\mathcal{H}}_0(\Omega), \tag{3-9}$$

where $b(u,v)$ is the symmetric, continuous and coercive bilinear form defined in (3-8) and

$$\langle F,v\rangle:=\int_\Omega f\,v\,dx+\frac12\int_\Gamma g\{v\}\,d\sigma+\frac12\int_\Gamma h[v]\,d\sigma\quad\text{for all }\ v\in\widehat{\mathcal{H}}_0(\Omega)$$

is a linear and continuous functional. Hence, existence and uniqueness of a solution $u\in\widehat{\mathcal{H}}_0(\Omega)$ of problem (3-6) is a direct consequence of the Lax–Milgram lemma. □

4. Concentration

We will formally derive problem (3-1) through a concentration limit of a more realistic problem in which the two different diffusive phases are separated by a fat membrane having its own diffusion properties. More precisely, the fat membrane is assumed to be made of an internal material of thickness 2δ, having a stable diffusivity in the tangential direction and a very low diffusivity (of order δ) in the

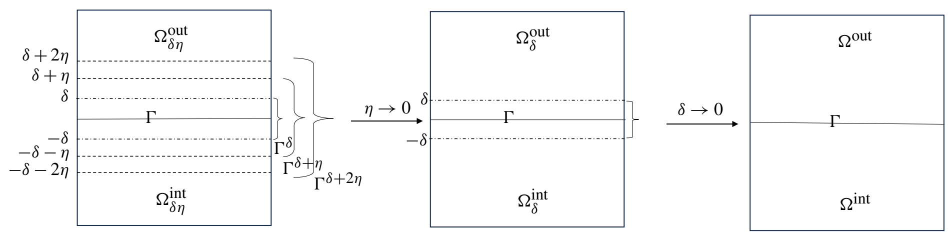

Figure 3. Concentration: first $\eta \to 0$, then $\delta \to 0$.

orthogonal direction, which is surrounded by an external material of thickness 2η, having a stable diffusivity in the orthogonal direction and a very high diffusivity (of order $1/\eta$) in the tangential direction. Since the ratio δ, η between the characteristic dimension of the separating membrane and the characteristic dimension of the microstructure is very small, we will let it go to zero, in order to approximate the physical model with a simpler mathematical description, in which the membrane is replaced by a thin interface.

To this end, let us introduce the following geometrical setting. In order to avoid technical difficulties, we assume to be in a bidimensional flat geometry, where $\Omega = (0,1)\times(-1,1)$ and we have three different physical phases: $\Omega^{int}_{\delta\eta}$, $\Omega^{out}_{\delta\eta}$ which, in the limit $\eta, \delta \to 0$, become Ω^{int} and Ω^{out}, respectively, and $\Gamma^{\delta+\eta}$ which, in the limit $\eta, \delta \to 0$, becomes Γ. More precisely,

$$\Omega = \Omega^{int}_{\delta\eta} \cup \Omega^{out}_{\delta\eta} \cup \Gamma^{\delta+\eta} \cup (\partial\Gamma^{\delta+\eta} \cap \Omega),$$

where

$$\begin{aligned} \Omega^{out}_{\delta\eta} &= (0,1)\times(\delta+\eta, 1), \\ \Omega^{int}_{\delta\eta} &= (0,1)\times(-1, -\delta-\eta), \\ \Gamma^{\delta+\eta} &= (0,1)\times(-\delta-\eta, \delta+\eta) = \Gamma\times(-\delta-\eta, \delta+\eta), \end{aligned}$$

with $\Gamma = (0,1)\times\{0\}$. We also set $\Omega^{int}_{\delta} = (0,1)\times(-1,-\delta)$, $\Omega^{out}_{\delta} = (0,1)\times(\delta,1)$, $\Gamma^{\delta} = (0,1)\times(-\delta,\delta) = \Gamma\times(-\delta,\delta)$, $\Gamma^{\delta+2\eta} = (0,1)\times(-\delta-2\eta, \delta+2\eta) = \Gamma\times(-\delta-2\eta,\delta+2\eta)$ and $\widetilde{\Gamma}^{\delta\eta} := \Gamma^{\delta+2\eta}\setminus\overline{\Gamma^{\delta+\eta}}$ (see Figure 3). Finally, we define

- $\Gamma^{\delta+\eta}_{+} = \Gamma\times(0, \delta+\eta)$,
- $\Gamma^{\delta+\eta}_{-} = \Gamma\times(-\delta-\eta, 0)$,
- $\widetilde{\Gamma}^{\delta\eta}_{+} = \Gamma\times(\delta+\eta, \delta+2\eta)$,
- $\widetilde{\Gamma}^{\delta\eta}_{-} = \Gamma\times(-\delta-2\eta, -\delta-\eta)$,
- $\partial\Omega^{t,b} = ((0,1)\times\{1\})\cup((0,1)\times\{-1\})$,
- $\partial\Omega^{l,r} = (\{0\}\times(-1,1))\cup(\{1\}\times(-1,1))$.

Let us consider, as a model case, the elliptic problem

$$\begin{cases} -\mathrm{div}(A^{\eta,\delta}\nabla u^{\eta,\delta}) = f & \text{in } \Omega, \\ u^{\eta,\delta} = 0 & \text{on } \partial\Omega^{t,b}, \\ u^{\eta,\delta} \text{ periodic} & \text{on } \partial\Omega^{l,r}, \end{cases} \tag{4-1}$$

where $f \in L^2(\Omega)$ and

$$A^{\eta,\delta} = \begin{cases} A^{\eta} & \text{in } \Gamma^{\delta+\eta} \setminus \overline{\Gamma^{\delta}}, \\ A^{\delta} & \text{in } \Gamma^{\delta}, \\ A & \text{in } \Omega^{\text{int}}_{\delta\eta} \cup \Omega^{\text{out}}_{\delta\eta}, \end{cases} \tag{4-2}$$

with A given in (3-2),

$$A^{\eta} = \begin{pmatrix} \beta/\eta & 0 \\ 0 & \alpha \end{pmatrix} \quad \text{and} \quad A^{\delta} = \begin{pmatrix} \beta & 0 \\ 0 & \alpha\delta \end{pmatrix}, \tag{4-3}$$

and $\alpha, \beta > 0$.

Clearly, for η, δ fixed, due to the ellipticity of the matrix $A^{\eta,\delta}$, problem (4-1) admits existence and uniqueness of a solution $u^{\eta,\delta} \in H^1(\Omega)$, complemented with the required boundary conditions. Its weak formulation is given by

$$\begin{aligned} &\int_{\Omega^{\text{int}}_{\delta\eta} \cup \Omega^{\text{out}}_{\delta\eta}} A\nabla u^{\eta,\delta} \cdot \nabla\varphi \, dx\, dy + \frac{\beta}{\eta}\int_{\Gamma^{\delta+\eta}\setminus\overline{\Gamma^{\delta}}} u^{\eta,\delta}_x \varphi_x \, dx\, dy \\ &\quad + \alpha \int_{\Gamma^{\delta+\eta}\setminus\overline{\Gamma^{\delta}}} u^{\eta,\delta}_y \varphi_y \, dx\, dy + \beta\int_{\Gamma^{\delta}} u^{\eta,\delta}_x \varphi_x \, dx\, dy + \alpha\delta \int_{\Gamma^{\delta}} u^{\eta,\delta}_y \varphi_y \, dx\, dy \\ &\qquad = \int_{\Omega} f\varphi \, dx\, dy, \end{aligned} \tag{4-4}$$

for every $\varphi \in H^1(\Omega)$, periodic in the horizontal direction and with null trace on $\partial\Omega^{t,b}$.

In order to pass to the limit, first for $\eta \to 0$ and then for $\delta \to 0$, we consider the following test function $\varphi(x, y) = \phi(x)\psi(y)$, where $\phi \in C^1(0, 1)$ is a periodic function and ψ is defined as

$$\psi(y) = \begin{cases} \psi_1(y) & \text{in } \Omega^{\text{int}}_{\delta\eta} \setminus \widetilde{\Gamma}^{\delta\eta}_{-}, \\ (\psi_1(-\delta) - \psi_1(-\delta-2\eta))\frac{y+(\delta+\eta)}{\eta} + \psi_1(-\delta) & \text{in } \widetilde{\Gamma}^{\delta\eta}_{-}, \\ \psi_1(-\delta) & \text{in } \Gamma^{\delta+\eta}_{-} \setminus \Gamma^{\delta}, \\ (\psi_2(\delta) - \psi_1(-\delta))\frac{y}{2\delta} + \frac{\psi_2(\delta)+\psi_1(-\delta)}{2} & \text{in } \Gamma^{\delta}, \\ \psi_2(\delta) & \text{in } \Gamma^{\delta+\eta}_{+} \setminus \Gamma^{\delta}, \\ (\psi_2(\delta+2\eta) - \psi_2(\delta))\frac{y-(\delta+\eta)}{\eta} + \psi_2(\delta) & \text{in } \widetilde{\Gamma}^{\delta\eta}_{+}, \\ \psi_2(y) & \text{in } \Omega^{\text{out}}_{\delta\eta} \setminus \widetilde{\Gamma}^{\delta\eta}_{+}, \end{cases} \tag{4-5}$$

with $\psi_1 \in H^1(-1, 0)$, $\psi_2 \in H^1(0, 1)$ and $\psi_1(-1) = \psi_2(1) = 0$.

Using such a test function in (4-4) and letting first $\eta \to 0$ and then $\delta \to 0$, we get

(1)
$$
\begin{aligned}
&\int_{(\Omega^{\text{int}}_{\delta\eta}\setminus\widetilde{\Gamma}^{\delta\eta}_{-})\cup(\Omega^{\text{out}}_{\delta\eta}\setminus\widetilde{\Gamma}^{\delta\eta}_{+})} A\nabla u^{\eta,\delta}\cdot\nabla\varphi\,dx\,dy\\
&\quad\xrightarrow{\eta\to 0}\int_{(\Omega^{\text{int}}_{\delta}\cup\Omega^{\text{out}}_{\delta})\setminus\overline{\Gamma^{\delta}}} A\nabla u^{\delta}\cdot\nabla\varphi\,dx\,dy\\
&\quad\quad\xrightarrow{\delta\to 0}\int_{\Omega^{\text{int}}\cup\Omega^{\text{out}}} A\nabla u\cdot\nabla\varphi\,dx\,dy\\
&\quad\quad\quad=\int_{\Omega^{\text{int}}} A\nabla u^{\text{int}}\cdot\nabla(\phi\psi_1)\,dx\,dy+\int_{\Omega^{\text{out}}} A\nabla u^{\text{out}}\cdot\nabla(\phi\psi_2)\,dx\,dy;
\end{aligned}
$$

(2)
$$
\int_{\widetilde{\Gamma}^{\delta\eta}_{-}\cup\widetilde{\Gamma}^{\delta\eta}_{+}} A\nabla u^{\eta,\delta}\cdot\nabla\varphi\,dx\,dy\xrightarrow{\eta\to 0}0,
$$

since $|\widetilde{\Gamma}^{\delta\eta}_{-}|=|\widetilde{\Gamma}^{\delta\eta}_{+}|\sim\eta$, $\psi_1(-\delta)-\psi_1(-\delta-2\eta)\to 0$ and $\psi_2(\delta+2\eta)-\psi_2(\delta)\to 0$, while $\nabla u^{\eta,\delta}$ is bounded in $L^2(\Omega^{\text{int}}_{\delta\eta}\cup\Omega^{\text{out}}_{\delta\eta})$ because of a standard energy estimate;

(3)
$$
\begin{aligned}
\frac{\beta}{\eta}\int_{\Gamma^{\delta+\eta}_{-}\setminus\overline{\Gamma^{\delta}}} u^{\eta,\delta}_x\varphi_x\,dx\,dy &\xrightarrow{\eta\to 0}\beta\int_{\Gamma\times\{-\delta\}} u^{\delta}_x(x,-\delta)\phi_x\psi_1(-\delta)\,d\sigma\\
&\xrightarrow{\delta\to 0}\beta\int_{\Gamma} u^{\text{int}}_x\phi_x\psi_1(0)\,d\sigma,\\
\frac{\beta}{\eta}\int_{\Gamma^{\delta+\eta}_{+}\setminus\overline{\Gamma^{\delta}}} u^{\eta,\delta}_x\varphi_x\,dx\,dy &\xrightarrow{\eta\to 0}\beta\int_{\Gamma\times\{\delta\}} u^{\delta}_x(x,\delta)\phi_x\psi_2(\delta)\,d\sigma\\
&\xrightarrow{\delta\to 0}\beta\int_{\Gamma} u^{\text{out}}_x\phi_x\psi_2(0)\,d\sigma,
\end{aligned}
$$

(4)
$$
\alpha\int_{\Gamma^{\delta+\eta}\setminus\overline{\Gamma^{\delta}}} u^{\eta,\delta}_y\varphi_y\,dx\,dy=0,
$$

since φ is independent of y;

(5)
$$
\begin{aligned}
&\beta\int_{\Gamma^{\delta}} u^{\eta,\delta}_x\varphi_x\,dx\,dy\\
&\quad=\beta\int_{\Gamma^{\delta}} u^{\eta,\delta}_x\phi_x\Big((\psi_2(\delta)-\psi_1(-\delta))\frac{y}{2\delta}+\frac{\psi_2(\delta)+\psi_1(-\delta)}{2}\Big)\xrightarrow{\eta\to 0}O(\sqrt{\delta})\xrightarrow{\delta\to 0}0;
\end{aligned}
$$

(6)
$$
\begin{aligned}
\alpha\delta\int_{\Gamma^{\delta}} u^{\eta,\delta}_y\varphi_y\,dx\,dy &\xrightarrow{\eta\to 0}\alpha\delta\int_{\Gamma^{\delta}} u^{\delta}_y\phi\frac{(\psi_2(\delta)-\psi_1(-\delta))}{2\delta}\,dx\,dy\\
&=\frac{\alpha}{2}\int_{\Gamma}\phi(x)(\psi_2(\delta)-\psi_1(-\delta))\Big(\int_{-\delta}^{\delta} u^{\delta}_y\,dy\Big)\,d\sigma\\
&\xrightarrow{\delta\to 0}\frac{\alpha}{2}\int_{\Gamma}\phi(x)(\psi_2(0)-\psi_1(0))[u]\,d\sigma;
\end{aligned}
$$

(7)
$$
\int_{\Omega} f\varphi\,dx\,dy\xrightarrow{\eta,\delta\to 0}\int_{\Omega^{\text{int}}} f\phi\psi_1\,dx\,dy+\int_{\Omega^{\text{out}}} f\phi\psi_2\,dx\,dy.
$$

Combining the previous results, we arrive at

$$
\begin{aligned}
&\int_{\Omega^{\text{int}}} A\nabla u^{\text{int}}\cdot\nabla\varphi_1\,dx\,dy+\int_{\Omega^{\text{out}}} A\nabla u^{\text{out}}\cdot\nabla\varphi_2\,dx\,dy\\
&\qquad+\beta\int_{\Gamma} u^{\text{int}}_x\partial_x\varphi_1\,d\sigma+\beta\int_{\Gamma} u^{\text{out}}_x\partial_x\varphi_2\,d\sigma+\frac{\alpha}{2}\int_{\Gamma}[u]\,[\varphi]\,d\sigma
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{\Omega^{\text{int}}} A\nabla u^{\text{int}} \cdot \nabla\varphi_1 \, dx\, dy + \int_{\Omega^{\text{out}}} A\nabla u^{\text{out}} \cdot \nabla\varphi_2 \, dx\, dy \\
&\qquad + \beta \int_\Gamma u_x^{\text{int}} \frac{\partial_x\{\varphi\} - \partial_x[\varphi]}{2} \, d\sigma + \beta \int_\Gamma u_x^{\text{out}} \frac{\partial_x\{\varphi\} + \partial_x[\varphi]}{2} \, d\sigma + \frac{\alpha}{2} \int_\Gamma [u]\,[\varphi] \, d\sigma \\
&= \int_{\Omega^{\text{int}}} A\nabla u^{\text{int}} \cdot \nabla\varphi_1 \, dx\, dy + \int_{\Omega^{\text{out}}} A\nabla u^{\text{out}} \cdot \nabla\varphi_2 \, dx\, dy \\
&\qquad + \frac{\beta}{2} \int_\Gamma \nabla^B[u] \cdot \nabla^B[\varphi] \, d\sigma + \frac{\beta}{2} \int_\Gamma \nabla^B\{u\} \cdot \nabla^B\{\varphi\} \, d\sigma + \frac{\alpha}{2} \int_\Gamma [u]\,[\varphi] \, d\sigma \\
&= \int_{\Omega^{\text{int}}} f\varphi_1 \, dx\, dy + \int_{\Omega^{\text{out}}} f\varphi_2 \, dx\, dy,
\end{aligned}
\tag{4-6}
$$

where we define $\varphi_i = \phi\psi_i$, $i = 1, 2$. A standard localization procedure leads from (4-6) to problem (3-5), with $\gamma = \beta$ and $g = h = 0$.

5. Homogenization of the problem $\mathcal{B}_\varepsilon$

Following the ideas in [14], we will study the homogenization limit for the problem $\mathcal{B}_\varepsilon$ introduced in Section 3, by using the formal two-scale asymptotic expansion technique. As $\varepsilon \to 0$, we will find three different macroscopic models, corresponding to the different scalings, which can be compared, for instance, with the models obtained in [6; 7; 8; 10; 13; 15; 16; 26] in the framework of electrical conduction or heat diffusion. More precisely, we have three cases:

- $m = -1$: the macroscopic model consists of a monodomain governed by an elliptic equation whose diffusion matrix keeps memory of the geometry and all the physical properties of the microscopic structure; i.e., the two phases E^{int} and E^{out} and the interface Γ (through the transversal diffusion coefficients α and the tangential diffusion coefficients β and γ) play an active role in the limit model.

- $m = 0$: the macroscopic model consists of a monodomain governed by an elliptic equation whose homogenized diffusion matrix does not keep any memory of the physical properties of the interface Γ and, in the connected-disconnected geometrical setting, not even of the phase E^{int}. Hence, in the homogenization of problem (5-21) below, only the geometry of the microscopic structure plays a role, i.e., the presence of the interface affects the cell function, which is not continuous across Γ, but α, β and γ are not involved.

- $m = 1$: the macroscopic model is a bidomain system, where, in the limit, only the geometry and the transversal diffusion α play a role, while the tangential diffusion coefficients β and γ do not.

Following the standard technique, we set

$$
u_\varepsilon(x, y) = u_0(x, y) + \varepsilon u_1(x, y) + \varepsilon^2 u_2(x, y) + \cdots, \tag{5-1}
$$

where u_i, $i = 0, 1, 2, \dots$, are assumed to be Y-periodic with respect to the second variable y. Analogously, we have

$$[u_\varepsilon] = [u_0] + \varepsilon[u_1] + \varepsilon^2[u_2] + \cdots \quad \text{and} \quad \{u_\varepsilon\} = \{u_0\} + \varepsilon\{u_1\} + \varepsilon^2\{u_2\} + \cdots . \tag{5-2}$$

As a consequence, the total spatial derivatives become

$$\nabla = \nabla_x + \frac{1}{\varepsilon}\nabla_y, \quad \operatorname{div} = \operatorname{div}_x + \frac{1}{\varepsilon}\operatorname{div}_y, \tag{5-3}$$

$$\text{and} \quad \triangle^B = \frac{1}{\varepsilon^2}\triangle_y^B + \frac{1}{\varepsilon}(\operatorname{div}_y^B \nabla_x^B + \operatorname{div}_x^B \nabla_y^B) + \triangle_x^B. \tag{5-4}$$

Inserting the previous expansion in (3-1) and matching the corresponding powers of ε, we arrive at the expansions for the problems corresponding to $m = -1, 0, 1$.

Case $m = -1$. We consider the problem (3-1) for $m = -1$, namely

$$\mathcal{Q}_\varepsilon : \begin{cases} -\operatorname{div}(A_\varepsilon \nabla u_\varepsilon) = f & \text{in } \Omega_\varepsilon^{\text{int}}, & (5\text{-}5a) \\ -\operatorname{div}(A_\varepsilon \nabla u_\varepsilon) = f & \text{in } \Omega_\varepsilon^{\text{out}}, & (5\text{-}5b) \\ -\gamma\varepsilon\triangle^B\{u_\varepsilon\} = [A_\varepsilon \nabla u_\varepsilon \cdot \nu_\varepsilon] & \text{on } \Gamma_\varepsilon, & (5\text{-}5c) \\ \frac{\alpha}{\varepsilon}[u_\varepsilon] - \beta\varepsilon\triangle^B[u_\varepsilon] = \{A_\varepsilon \nabla u_\varepsilon \cdot \nu_\varepsilon\} & \text{on } \Gamma_\varepsilon, & (5\text{-}5d) \\ u_\varepsilon = 0 & \text{on } \partial\Omega. & (5\text{-}5e) \end{cases}$$

Terms of order ε^{-2}. By comparing the corresponding coefficients of the terms of order ε^{-2}, from the asymptotic expansion of (5-5a) and (5-5b) and, similarly, the coefficients of order ε^{-1} in (5-5c) and (5-5d), we get

$$\mathcal{Q}_0 : \begin{cases} -\operatorname{div}_y(A\nabla_y u_0^{\text{int}}) = 0 & \text{in } E^{\text{int}}, & (5\text{-}6a) \\ -\operatorname{div}_y(A\nabla_y u_0^{\text{out}}) = 0 & \text{in } E^{\text{out}}, & (5\text{-}6b) \\ -\gamma\triangle_y^B\{u_0\} = [A\nabla_y u_0 \cdot \nu] & \text{on } \Gamma, & (5\text{-}6c) \\ \alpha[u_0] - \beta\triangle_y^B[u_0] = \{A\nabla_y u_0 \cdot \nu\} & \text{on } \Gamma. & (5\text{-}6d) \end{cases}$$

By a standard energy estimate for problem (5-6), we get

$$u_0(x, y) = u_0(x) \quad \text{for a.e. } (x, y) \in \Omega \times Y. \tag{5-7}$$

Terms of order ε^{-1}. By comparing the coefficients of ε^{-1} in (5-5a) and (5-5b) and of ε^0 in (5-5c) and (5-5d), and using (5-7), we obtain

$$\mathcal{Q}_1 : \begin{cases} -\operatorname{div}_y(A\nabla_y u_1^{\text{int}}) = \operatorname{div}_y(A\nabla_x u_0) & \text{in } E^{\text{int}}, & (5\text{-}8a) \\ -\operatorname{div}_y(A\nabla_y u_1^{\text{out}}) = \operatorname{div}_y(A\nabla_x u_0) & \text{in } E^{\text{out}}, & (5\text{-}8b) \\ -\gamma(\triangle_y^B\{u_1\} + \operatorname{div}_y^B \nabla_x^B\{u_0\}) = [A(\nabla_y u_1 + \nabla_x u_0) \cdot \nu] & \text{on } \Gamma, & (5\text{-}8c) \\ \alpha[u_1] - \beta\triangle_y^B[u_1] = \{A(\nabla_y u_1 + \nabla_x u_0) \cdot \nu\} & \text{on } \Gamma. & (5\text{-}8d) \end{cases}$$

Cell function. By following the classical approach, the solution of (5-8) can be explicitly factorized in terms of $\nabla_x u_0$ and the cell function $\chi_Q = (\chi_Q^1, \dots, \chi_Q^N) \in \widehat{\mathcal{H}}_{\text{per}}(Y)$ as

$$u_1(x, y) = -\chi_Q(y) \cdot \nabla_x u_0(x) + \widetilde{u}(x) = -\chi_Q^j(y) \frac{\partial u_0(x)}{\partial x_j} + \widetilde{u}(x), \tag{5-9}$$

where, as usual, without loss of generality, we can assume $\widetilde{u}(x) = 0$. Here, $\chi_Q = (\chi_Q^{\text{int}}, \chi_Q^{\text{out}})$ is a vector function, having components $\chi_Q^j = (\chi_Q^{j,\text{int}}, \chi_Q^{j,\text{out}})$, with null mean average over Y and, for $j = 1, 2, \dots, N$, satisfying the cell problem

$$\begin{cases} -\text{div}_y(A\nabla_y(\chi_Q^{j,\text{int}} - y_j)) = 0 & \text{in } E^{\text{int}}, \quad (5\text{-}10a) \\ -\text{div}_y(A\nabla_y(\chi_Q^{j,\text{out}} - y_j)) = 0 & \text{in } E^{\text{out}}, \quad (5\text{-}10b) \\ -\gamma \triangle_y^B\{\chi_Q^j - y_j\} = [A\nabla_y(\chi_Q^j - y_j) \cdot \nu] & \text{on } \Gamma, \quad (5\text{-}10c) \\ \alpha[\chi_Q^j] - \beta \triangle_y^B[\chi_Q^j] = \{A\nabla_y(\chi_Q^j - y_j) \cdot \nu\} & \text{on } \Gamma. \quad (5\text{-}10d) \end{cases}$$

In (5-10c), we use (5-13) and (5-14) below. The well-posedness of problem (5-10) can be easily obtained as done in Theorem 3.4, but in a periodic setting.

Terms of order ε^0. By comparing the coefficients of order ε^0 in (5-5a), (5-5b) and of order ε in (5-5c), (5-5d), we obtain

$$Q_2 : \begin{cases} -\text{div}_y(A\nabla_y u_2^{\text{int}}) = f + \text{div}_y(A\nabla_x u_1^{\text{int}}) \\ \qquad\qquad + \text{div}_x(A\nabla_y u_1^{\text{int}}) + \text{div}_x(A\nabla_x u_0) & \text{in } E^{\text{int}}, \quad (5\text{-}11a) \\ -\text{div}_y(A\nabla_y u_2^{\text{out}}) = f + \text{div}_y(A\nabla_x u_1^{\text{out}}) \\ \qquad\qquad + \text{div}_x(A\nabla_y u_1^{\text{out}}) + \text{div}_x(A\nabla_x u_0) & \text{in } E^{\text{out}}, \quad (5\text{-}11b) \\ -\gamma\big(\triangle_y^B\{u_2\} + \text{div}_x^B \nabla_y^B\{u_1\} + \text{div}_y^B \nabla_x^B\{u_1\} + \triangle_x^B\{u_0\}\big) \\ \qquad\qquad = [A(\nabla_y u_2 + \nabla_x u_1) \cdot \nu] & \text{on } \Gamma, \quad (5\text{-}11c) \\ \alpha[u_2] - \beta\big(\triangle_y^B[u_2] + \text{div}_x^B \nabla_y^B[u_1] + \text{div}_y^B \nabla_x^B[u_1]\big) \\ \qquad\qquad = \{A(\nabla_y u_2 + \nabla_x u_1) \cdot \nu\} & \text{on } \Gamma, \quad (5\text{-}11d) \end{cases}$$

where we have taken into account (5-7): $[u_0] = 0$ and, then, $\triangle_x^B[u_0] = 0$.

Derivation of the homogenized equation. We will attain the limiting equation for u_0 as a compatibility condition for (5-11). To begin, let us integrate (5-11a), (5-11b) by parts, respectively, in E^{int} and in E^{out}. By adding the two contributions, we find

$$\int_\Gamma [A\nabla_y u_2 \cdot \nu]\, d\sigma = \int_Y f\, dy + \int_{E^{\text{int}}} \text{div}_y(A\nabla_x u_1^{\text{int}})\, dy \\ + \int_{E^{\text{out}}} \text{div}_y(A\nabla_x u_1^{\text{out}})\, dy + \int_{E^{\text{int}}} \text{div}_x(A\nabla_y u_1^{\text{int}})\, dy \\ + \int_{E^{\text{out}}} \text{div}_x(A\nabla_y u_1^{\text{out}})\, dy + \int_{E^{\text{int}} \cup E^{\text{out}}} \text{div}_x(A\nabla_x u_0)\, dy$$

$$= \int_Y f\,dy - \int_\Gamma [A\nabla_y u_1 \cdot \nu]\,d\sigma + \int_{E^{\rm int}\cup E^{\rm out}} {\rm div}_x(A\nabla_y u_1)\,dy + \int_{E^{\rm int}\cup E^{\rm out}} {\rm div}_x(A\nabla_x u_0)\,dy.$$

Then, using (5-11c), we obtain

$$-{\rm div}_x\Big(\int_{E^{\rm int}\cup E^{\rm out}} A(\nabla_x u_0 + \nabla_y u_1)\Big)\,dy - \gamma\int_\Gamma \triangle_y^B\{u_2\}\,d\sigma - \gamma\int_\Gamma {\rm div}_x^B\,\nabla_y^B\{u_1\}\,d\sigma - \gamma\int_\Gamma {\rm div}_y^B\,\nabla_x^B\{u_1\}\,d\sigma - \gamma\int_\Gamma \triangle_x^B\{u_0\}\,d\sigma = f,$$

which becomes the homogenized equation

$$-{\rm div}_x\Big(\int_{E^{\rm int}\cup E^{\rm out}} A(\nabla_x u_0+\nabla_y u_1)\,dy+\gamma\int_\Gamma (\nabla_y^B\{u_1\}+\nabla_x^B\{u_0\})\,d\sigma\Big)=f, \tag{5-12}$$

after having taken into account that

$$\int_\Gamma \triangle_y^B\{u_2\}\,d\sigma = 0 \quad \text{and} \quad \int_\Gamma {\rm div}_y^B\,\nabla_x^B\{u_1\}\,d\sigma = 0.$$

Inserting now, in the homogenized equation (5-12) above, the factorization for u_1 given in (5-9) and recalling that

$$\begin{aligned}\nabla_y^B\{u_1\} &= \nabla_y\{u_1\} - (\nu\cdot\nabla_y\{u_1\})\nu\\ &= -\nabla_y\{\chi_Q\}\nabla_x u_0 + \big(\nu\cdot(\nabla_y\{\chi_Q\}\nabla_x u_0)\big)\nu\\ &= -(I-\nu\otimes\nu)\nabla_y\{\chi_Q\}\nabla_x u_0 = -\nabla_y^B\{\chi_Q\}\nabla_x u_0,\end{aligned} \tag{5-13}$$

$$\begin{aligned}\nabla_x^B\{u_0\} &= 2\nabla_x^B u_0\\ &= 2(\nabla_x u_0 - (\nu\cdot\nabla_x u_0)\nu)\\ &= 2(I-\nu\otimes\nu)\nabla_x u_0 = \nabla_y^B\{y\}\nabla_x u_0,\end{aligned} \tag{5-14}$$

we arrive at

$$\begin{aligned} f &= -{\rm div}\Big(\Big(\int_{E^{\rm int}\cup E^{\rm out}} A(I-\nabla_y\chi_Q)\,dy + \gamma\int_\Gamma \nabla_y^B\{y-\chi_Q\}\,d\sigma\Big)\nabla u_0\Big)\\ &= -{\rm div}(A_Q\nabla u_0).\end{aligned}$$

Here, the homogenized matrix is given by

$$A_Q = \int_{E^{\rm int}\cup E^{\rm out}} A\nabla_y(y-\chi_Q)\,dy + \gamma\int_\Gamma \nabla_y^B\{y-\chi_Q\}\,d\sigma. \tag{5-15}$$

From (5-15) and the definition of the cell function χ_Q in (5-10), we obtain that the homogenized matrix A_Q depends on the geometry and the whole physical properties of the microstructure, described by A, α, β, γ.

Theorem 5.1. *The homogenized matrix A_Q is symmetric and positive definite.*

Proof. First, we obtain

$$\int_{E^{\text{int}}\cup E^{\text{out}}} A\nabla_y(\chi_Q^j - y_j)\cdot\nabla_y y_i\,dy = \int_{E^{\text{int}}\cup E^{\text{out}}} (A\nabla_y(y_j-\chi_Q^j))_i\,dy \tag{5-16}$$

and

$$\begin{aligned}\int_\Gamma \nabla_y^B\{\chi_Q^j - y_j\}\cdot\nabla_y^B\{y_i\}\,d\sigma &= 2\int_\Gamma \nabla_y^B\{\chi_Q^j - y_j\}\cdot\nabla_y^B y_i\,d\sigma\\ &= 2\int_\Gamma \nabla_y^B\{\chi_Q^j - y_j\}\cdot(e_i - \nu_i\nu)\,d\sigma\\ &= 2\int_\Gamma (\nabla_y^B\{\chi_Q^j - y_j\})_i\,d\sigma,\end{aligned} \tag{5-17}$$

after having taken into account that the tangential gradient and the normal ν have null scalar product. Hence, we can write $(A_Q)_{ij}$ as

$$(A_Q)_{ij} = -\int_{E^{\text{int}}\cup E^{\text{out}}} A\nabla_y(\chi_Q^j - y_j)\cdot\nabla_y y_i\,dy - \frac{\gamma}{2}\int_\Gamma \nabla_y^B\{\chi_Q^j - y_j\}\cdot\nabla_y^B\{y_i\}\,d\sigma. \tag{5-18}$$

By taking $\chi_Q^{i,\text{int}}$ and $\chi_Q^{i,\text{out}}$ as test functions in (5-10a) and (5-10b), respectively, integrating by parts, summing the resulting equations and using (5-10c) and (5-10d), we get

$$\begin{aligned}0 = \int_{E^{\text{int}}\cup E^{\text{out}}} A\nabla_y(\chi_Q^j - y_j)\cdot\nabla_y\chi_Q^i\,dy + \frac{\alpha}{2}\int_\Gamma [\chi_Q^j][\chi_Q^i]\,d\sigma\\ + \frac{\beta}{2}\int_\Gamma \nabla_y^B[\chi_Q^j]\cdot\nabla_y^B[\chi_Q^i]\,d\sigma + \frac{\gamma}{2}\int_\Gamma \nabla_y^B\{\chi_Q^j - y_j\}\cdot\nabla_y^B\{\chi_Q^i\}\,d\sigma.\end{aligned} \tag{5-19}$$

By adding (5-18) and (5-19), we get

$$\begin{aligned}(A_Q)_{ij} = \int_{E^{\text{int}}\cup E^{\text{out}}} A\nabla_y(\chi_Q^j - y_j)\cdot\nabla_y(\chi_Q^i - y_i)\,dy\\ + \frac{\alpha}{2}\int_\Gamma [\chi_Q^j][\chi_Q^i]\,d\sigma + \frac{\beta}{2}\int_\Gamma \nabla_y^B[\chi_Q^j]\cdot\nabla_y^B[\chi_Q^i]\,d\sigma\\ + \frac{\gamma}{2}\int_\Gamma \nabla_y^B\{\chi_Q^j - y_j\}\cdot\nabla_y^B\{\chi_Q^i - y_i\}\,d\sigma,\end{aligned} \tag{5-20}$$

which immediately gives the symmetry of A_Q. To prove the ellipticity, we consider

$$\begin{aligned}\sum_{i,j=1}^N (A_Q)_{ij}\xi_i\xi_j = \int_{E^{\text{int}}\cup E^{\text{out}}} \sum_{i,j=1}^N A\nabla_y(\xi_i\chi_Q^i - \xi_i y_i)\cdot\nabla_y(\xi_j\chi_Q^j - \xi_j y_j)\,dy\\ + \frac{\alpha}{2}\int_\Gamma \sum_{i,j=1}^N [\xi_i\chi_Q^i][\xi_j\chi_Q^j]\,d\sigma\\ + \frac{\beta}{2}\int_\Gamma \sum_{i,j=1}^N \nabla_y^B[\xi_i\chi_Q^i]\cdot\nabla_y^B[\xi_j\chi_Q^j]\,d\sigma\\ + \frac{\gamma}{2}\int_\Gamma \sum_{i,j=1}^N \nabla_y^B\{\xi_i\chi_Q^i - \xi_i y_i\}\cdot\nabla_y^B\{\xi_j\chi_Q^j - \xi_j y_j\}\,d\sigma\end{aligned}$$

$$
\begin{aligned}
&\geq \lambda \int_{E^{\rm int}\cup E^{\rm out}} \Big| \sum_{i=1}^N (\nabla_y \chi_Q^i \xi_i - e_i\xi_i)\Big|^2 dy + \frac{\alpha}{2}\int_\Gamma \Big|\sum_{i=1}^N \xi_i [\chi_Q^i]\Big|^2 d\sigma \\
&\qquad + \frac{\beta}{2}\int_\Gamma \Big|\sum_{i=1}^N \xi_i \nabla_y^B [\chi_Q^i]\Big|^2 d\sigma + \frac{\gamma}{2}\int_\Gamma \Big|\sum_{i=1}^N \xi_i \nabla_y^B \{\chi_Q^i - y_i\}\Big|^2 d\sigma \\
&\geq \lambda \Big(\int_{E^{\rm int}} \Big| \sum_{i=1}^N (\nabla_y \chi_Q^i \xi_i - e_i\xi_i)\Big|^2 dy + \int_{E^{\rm out}} \Big| \sum_{i=1}^N (\nabla_y \chi_Q^i \xi_i - e_i\xi_i)\Big|^2 dy \Big) \\
&\geq 0.
\end{aligned}
$$

In order to conclude, we exploit the periodicity of χ_Q, which implies that the last inequality is actually strict for any $\xi \in \mathbb{R}^N$ with $|\xi| = 1$. Then, the thesis is achieved. □

Case m = 0. We consider the problem (3-1) for $m = 0$, namely

$$
\mathcal{R}_\varepsilon : \begin{cases}
-{\rm div}(A_\varepsilon \nabla u_\varepsilon) = f & \text{in } \Omega_\varepsilon^{\rm int}, \quad (5\text{-}21a)\\
-{\rm div}(A_\varepsilon \nabla u_\varepsilon) = f & \text{in } \Omega_\varepsilon^{\rm out}, \quad (5\text{-}21b)\\
-\gamma\varepsilon^2 \Delta^B \{u_\varepsilon\} = [A_\varepsilon \nabla u_\varepsilon \cdot \nu_\varepsilon] & \text{on } \Gamma_\varepsilon, \quad (5\text{-}21c)\\
\alpha [u_\varepsilon] - \beta\varepsilon^2 \Delta^B [u_\varepsilon] = \{A_\varepsilon \nabla u_\varepsilon \cdot \nu_\varepsilon\} & \text{on } \Gamma_\varepsilon, \quad (5\text{-}21d)\\
u_\varepsilon = 0 & \text{on } \partial\Omega, \quad (5\text{-}21e)
\end{cases}
$$

and proceed as in case $m = -1$. Thus, for the terms of order ε^{-2}, we get

$$
\mathcal{R}_0 : \begin{cases}
-{\rm div}_y (A\nabla_y u_0^{\rm int}) = 0 & \text{in } E^{\rm int}, \quad (5\text{-}22a)\\
-{\rm div}_y (A\nabla_y u_0^{\rm out}) = 0 & \text{in } E^{\rm out}, \quad (5\text{-}22b)\\
\big[A\nabla_y u_0 \cdot \nu\big] = 0 & \text{on } \Gamma, \quad (5\text{-}22c)\\
\{A\nabla_y u_0 \cdot \nu\} = 0 & \text{on } \Gamma, \quad (5\text{-}22d)
\end{cases}
$$

which corresponds to two independent homogenous Neumann problems in $E^{\rm int}$ and $E^{\rm out}$, respectively, with periodic boundary condition on $\partial E^{\rm int} \cap \partial Y$ and $\partial E^{\rm out} \cap \partial Y$, so that

$$
u_0(x,y) = \begin{cases}
u_0^{\rm int}(x) & \text{a.e. in } \Omega \times E^{\rm int}, \quad (5\text{-}23a)\\
u_0^{\rm out}(x) & \text{a.e. in } \Omega \times E^{\rm out}. \quad (5\text{-}23b)
\end{cases}
$$

On the other hand, for the terms of order ε^{-1} we obtain

$$
\mathcal{R}_1 : \begin{cases}
-{\rm div}_y (A\nabla_y u_1^{\rm int}) = {\rm div}_y (A\nabla_x u_0^{\rm int}) & \text{in } E^{\rm int}, \quad (5\text{-}24a)\\
-{\rm div}_y (A\nabla_y u_1^{\rm out}) = {\rm div}_y (A\nabla_x u_0^{\rm out}) & \text{in } E^{\rm out}, \quad (5\text{-}24b)\\
[A(\nabla_y u_1 + \nabla_x u_0)\cdot \nu] = 0 & \text{on } \Gamma, \quad (5\text{-}24c)\\
\alpha [u_0] = \{A(\nabla_y u_1 + \nabla_x u_0)\cdot \nu\} & \text{on } \Gamma. \quad (5\text{-}24d)
\end{cases}
$$

By integrating (5-24b) in E^{out} and taking into account (5-24c) and (5-24d), we have

$$0=\int_\Gamma (A\nabla_y u_1\cdot\nu)^{\text{out}}\,d\sigma+\int_\Gamma (A\nabla_x u_0\cdot\nu)^{\text{out}}\,d\sigma=\frac{\alpha}{2}\int_\Gamma [u_0]\,d\sigma,$$

which proves that $[u_0]=0$. Thus, we get $u_0^{\text{out}}(x)=u_0^{\text{int}}(x)=u_0(x)=u_0(x,y)$ a.e. in $\Omega\times Y$. It follows that (5-24d) can be rewritten as

$$\{A(\nabla_y u_1+\nabla_x u_0)\cdot\nu\}=0\quad\text{on }\Gamma. \tag{5-25}$$

Then, the system (5-24a)–(5-24c) and (5-25) turns out to be a decoupled pair of standard Neumann problems. Therefore, we can factorize u_1 (up to an irrelevant additive function of x, which will, therefore, be taken equal to 0) as

$$u_1(x,y)=-\chi_{\mathcal{M}}(y)\cdot\nabla_x u_0(x)=-\chi_{\mathcal{M}}^j(y)\frac{\partial u_0(x)}{\partial x_j}, \tag{5-26}$$

where $\chi_{\mathcal{M}}=(\chi_{\mathcal{M}}^{\text{int}},\chi_{\mathcal{M}}^{\text{out}}):Y\to\mathbb{R}^N$ is a Y-periodic vector function such that $\chi_{\mathcal{M}}^{\text{int}}$ and $\chi_{\mathcal{M}}^{\text{out}}$ have null mean average in E^{int} and E^{out}, respectively, and satisfy the cell problem

$$\begin{cases}-\operatorname{div}_y(A\nabla_y(\chi_{\mathcal{M}}^{j,\text{int}}-y_j))=0 & \text{in } E^{\text{int}}, \quad (5\text{-}27\text{a})\\ -\operatorname{div}_y(A\nabla_y(\chi_{\mathcal{M}}^{j,\text{out}}-y_j))=0 & \text{in } E^{\text{out}}, \quad (5\text{-}27\text{b})\\ (A\nabla_y(\chi_{\mathcal{M}}^{j}-y_j)\cdot\nu)^{\text{int}}=0 & \text{on } \Gamma, \quad (5\text{-}27\text{c})\\ (A\nabla_y(\chi_{\mathcal{M}}^{j}-y_j)\cdot\nu)^{\text{out}}=0 & \text{on } \Gamma. \quad (5\text{-}27\text{d})\end{cases}$$

Notice that, again, (5-27) is a system of two decoupled Neumann problems in E^{int} and E^{out}, respectively; therefore, its well posedness is a classical matter.

Finally, for the terms of order ε^0, we obtain

$$\mathcal{R}_2:\begin{cases}-\operatorname{div}_y(A\nabla_y u_2^{\text{int}})=f+\operatorname{div}_y(A\nabla_x u_1^{\text{int}}) \\ \qquad\qquad +\operatorname{div}_x(A\nabla_y u_1^{\text{int}})+\operatorname{div}_x(A\nabla_x u_0) & \text{in } E^{\text{int}}, \quad (5\text{-}28\text{a})\\ -\operatorname{div}_y(A\nabla_y u_2^{\text{out}})=f+\operatorname{div}_y(A\nabla_x u_1^{\text{out}}) \\ \qquad\qquad +\operatorname{div}_x(A\nabla_y u_1^{\text{out}})+\operatorname{div}_x(A\nabla_x u_0) & \text{in } E^{\text{out}}, \quad (5\text{-}28\text{b})\\ -\gamma(\triangle_y^B\{u_1\}+\operatorname{div}_y^B\nabla_x^B\{u_0\})=[A(\nabla_y u_2+\nabla_x u_1)\cdot\nu] & \text{on } \Gamma, \quad (5\text{-}28\text{c})\\ \alpha[u_1]-\beta(\triangle_y^B[u_1]+\operatorname{div}_y^B\nabla_x^B[u_0]) \\ \qquad\qquad =\{A(\nabla_y u_2+\nabla_x u_1)\cdot\nu\} & \text{on } \Gamma. \quad (5\text{-}28\text{d})\end{cases}$$

As above, the limiting equation is obtained integrating (5-28a), (5-28b) by parts in E^{int} and in E^{out}, respectively, and by adding the two contributions. Thus, we find

$$\begin{aligned}\int_\Gamma [A\nabla_y u_2\cdot\nu]\,d\sigma &= \int_Y f\,dy + \int_{E^{\mathrm{int}}\cup E^{\mathrm{out}}} \operatorname{div}_y(A\nabla_x u_1)\,dy \\ &\qquad + \int_{E^{\mathrm{int}}\cup E^{\mathrm{out}}} \operatorname{div}_x(A\nabla_y u_1)\,dy + \int_{E^{\mathrm{int}}\cup E^{\mathrm{out}}} \operatorname{div}_x(A\nabla_x u_0)\,dy \\ &= \int_Y f\,dy - \int_\Gamma [A\nabla_x u_1\cdot\nu]\,d\sigma \\ &\qquad + \int_{E^{\mathrm{int}}\cup E^{\mathrm{out}}} \operatorname{div}_x(A\nabla_y u_1)\,dy + \int_{E^{\mathrm{int}}\cup E^{\mathrm{out}}} \operatorname{div}_x(A\nabla_x u_0)\,dy.\end{aligned}$$

By taking into account condition (5-28c), the previous equation turns into the homogenized one

$$f = -\operatorname{div}_x\Big(\int_{E^{\mathrm{int}}\cup E^{\mathrm{out}}} A(\nabla_y u_1 + \nabla_x u_0)\,dy\Big), \tag{5-29}$$

where we have used that

$$\int_\Gamma \triangle_y^B\{u_1\}\,d\sigma = 0 \quad \text{and} \quad \int_\Gamma \operatorname{div}_y^B \nabla_x^B\{u_0\}\,d\sigma = 0.$$

Finally, inserting in (5-29) the factorization of u_1 given in (5-26), we arrive at

$$f = -\operatorname{div}\Big(\Big(\int_{E^{\mathrm{int}}\cup E^{\mathrm{out}}} A(I - \nabla_y\chi_{\mathcal{M}})\,dy\Big)\nabla u_0\Big) = -\operatorname{div}(A_{\mathcal{M}}\nabla u_0),$$

where $A_{\mathcal{M}}$ is given by

$$A_{\mathcal{M}} = \int_{E^{\mathrm{int}}\cup E^{\mathrm{out}}} A(I - \nabla_y\chi_{\mathcal{M}}(y))\,dy. \tag{5-30}$$

As in the previous section, $A_{\mathcal{M}}$ can be rewritten in the more meaningful form

$$(A_{\mathcal{M}})_{ij} = \int_{E^{\mathrm{int}}\cup E^{\mathrm{out}}} A\nabla_y(\chi_{\mathcal{M}}^j - y_j)\cdot\nabla_y(\chi_{\mathcal{M}}^i - y_i)\,dy, \tag{5-31}$$

which immediately gives the symmetry. The positive definiteness of $A_{\mathcal{M}}$ is a standard matter in the literature (see, e.g., [1, Section 1] and [9, Proof of Lemma 4.7.], for the same idea applied in a different framework).

Remark 5.2. Notice that, as remarked at the beginning of Section 5, the homogenized matrix $A_{\mathcal{M}}$ does not depend on the physical properties of the interface Γ (it does not involve the coefficients α, β, γ). The presence of the interface has only the effect to produce a discontinuity across Γ of the cell function $\chi_{\mathcal{M}}$, i.e., only the geometry of Γ has an influence on the limiting equation.

In the connected-disconnected case, it can be easily seen that

$$\chi_{\mathcal{M}}^{\mathrm{int}}(y) = y - |E^{\mathrm{int}}|^{-1}\int_{E^{\mathrm{int}}} y\,dy,$$

so that the homogenized matrix $A_{\mathcal{M}}$ reduces to

$$A_{\mathcal{M}} = \int_{E^{\text{out}}} A\nabla_y(\chi_{\mathcal{M}}^{\text{out}} - y) \cdot \nabla_y(\chi_{\mathcal{M}}^{\text{out}} - y)\, dy;$$

that is, the physical properties of the inner phase (as well as the ones of the interface) do not play any role in the macroscopic model. □

Case m = 1. We consider the problem (3-1) for $m = 1$, namely

$$\mathcal{S}_\varepsilon : \begin{cases} -\text{div}(A_\varepsilon \nabla u_\varepsilon) = f & \text{in } \Omega_\varepsilon^{\text{int}}, & \text{(5-32a)} \\ -\text{div}(A_\varepsilon \nabla u_\varepsilon) = f & \text{in } \Omega_\varepsilon^{\text{out}}, & \text{(5-32b)} \\ -\gamma\varepsilon^3 \Delta^B \{u_\varepsilon\} = [A_\varepsilon \nabla u_\varepsilon \cdot \nu_\varepsilon] & \text{on } \Gamma_\varepsilon, & \text{(5-32c)} \\ \alpha\varepsilon[u_\varepsilon] - \beta\varepsilon^3 \Delta^B [u_\varepsilon] = \{A\nabla u_\varepsilon \cdot \nu_\varepsilon\} & \text{on } \Gamma_\varepsilon, & \text{(5-32d)} \\ u_\varepsilon = 0 & \text{on } \partial\Omega. & \text{(5-32e)} \end{cases}$$

Proceeding as in the previous cases $m = -1, 0$, we obtain, for the terms of order ε^{-2} the same problem (5-22), so that (5-23) is in force. On the other hand, for the term of order ε^{-1}, we obtain

$$\mathcal{S}_1 : \begin{cases} -\text{div}_y(A\nabla_y u_1^{\text{int}}) = \text{div}_y(A\nabla_x u_0^{\text{int}}) & \text{in } E^{\text{int}}, & \text{(5-33a)} \\ -\text{div}_y(A\nabla_y u_1^{\text{out}}) = \text{div}_y(A\nabla_x u_0^{\text{out}}) & \text{in } E^{\text{out}}, & \text{(5-33b)} \\ [A(\nabla_y u_1 + \nabla_x u_0) \cdot \nu] = 0 & \text{on } \Gamma, & \text{(5-33c)} \\ \{A(\nabla_y u_1 + \nabla_x u_0) \cdot \nu\} = 0 & \text{on } \Gamma. & \text{(5-33d)} \end{cases}$$

Again, the previous system is the same decoupled pair of standard Neumann problems given in (5-24a)–(5-24c) and (5-25). Therefore, (up to irrelevant additive functions of x, which will, therefore, be taken equal to 0) we can factor $u_1(x, y)$ as

$$u_1(x, y) = \begin{cases} -\chi_{\mathcal{M}}^{\text{int}}(y) \cdot \nabla_x u_0^{\text{int}}(x) & \text{in } E^{\text{int}}, & \text{(5-34a)} \\ -\chi_{\mathcal{M}}^{\text{out}}(y) \cdot \nabla_x u_0^{\text{out}}(x) & \text{in } E^{\text{out}}, & \text{(5-34b)} \end{cases}$$

where $\chi_{\mathcal{M}}$ is the Y-periodic solution of the cell problem (5-27) having null mean average in E^{int} and E^{out}, separately.

Terms of order ε^0: By comparing the coefficients of ε^0 in (5-32a), (5-32b) and of ε in (5-32c), (5-32d), we obtain

$$\mathcal{S}_2 : \begin{cases} -\text{div}_y(A\nabla_y u_2^{\text{int}}) = f + \text{div}_y(A\nabla_x u_1^{\text{int}}) \\ \qquad + \text{div}_x(A\nabla_y u_1^{\text{int}}) + \text{div}_x(A\nabla_x u_0^{\text{int}}) & \text{in } E^{\text{int}}, & \text{(5-35a)} \\ -\text{div}_y(A\nabla_y u_2^{\text{out}}) = f + \text{div}_y(A\nabla_x u_1^{\text{out}}) \\ \qquad + \text{div}_x(A\nabla_y u_1^{\text{out}}) + \text{div}_x(A\nabla_x u_0^{\text{out}}) & \text{in } E^{\text{out}}, & \text{(5-35b)} \\ [A(\nabla_y u_2 + \nabla_x u_1) \cdot \nu] = 0 & \text{on } \Gamma, & \text{(5-35c)} \\ \alpha[u_0] = \{A(\nabla_y u_2 + \nabla_x u_1) \cdot \nu\} & \text{on } \Gamma, & \text{(5-35d)} \end{cases}$$

where we have used (5-23), which implies $\triangle_y^B[u_0]=0$ and $\triangle_y^B\{u_0\}=0$. Taking into account that (5-35c) implies

$$\{A(\nabla_y u_2+\nabla_x u_1)\cdot\nu\}=2(A(\nabla_y u_2+\nabla_x u_1)\cdot\nu)^{\text{out}},$$

we can rewrite (5-35d) as

$$\frac{\alpha}{2}[u_0]=(A(\nabla_y u_2+\nabla_x u_1)\cdot\nu)^{\text{out}}\quad\text{on }\Gamma. \tag{5-36}$$

Formal derivation of the homogenized equation. To attain the limiting equation for u_0 as a compatibility condition for (5-35), let us integrate (5-35a), (5-35b) by parts in E^{int} and in E^{out}, separately. We find

$$\begin{aligned}-\int_\Gamma(A\nabla_y u_2\cdot\nu)^{\text{int}}\,d\sigma=\int_{E^{\text{int}}}f\,dy+\int_\Gamma(A\nabla_x u_1\cdot\nu)^{\text{int}}\,d\sigma\\ +\int_{E^{\text{int}}}\operatorname{div}_x(A\nabla_y u_1^{\text{int}})\,dy+\int_{E^{\text{int}}}\operatorname{div}_x(A\nabla_x u_0^{\text{int}})\,dy,\end{aligned}$$

and

$$\begin{aligned}\int_\Gamma(A\nabla_y u_2\cdot\nu)^{\text{out}}\,d\sigma=\int_{E^{\text{out}}}f\,dy-\int_\Gamma(A\nabla_x u_1\cdot\nu)^{\text{out}}\,d\sigma\\ +\int_{E^{\text{out}}}\operatorname{div}_x(A\nabla_y u_1^{\text{out}})\,dy+\int_{E^{\text{out}}}\operatorname{div}_x(A\nabla_x u_0^{\text{out}})\,dy.\end{aligned}$$

Hence, by taking into account the interface conditions (5-35c) and (5-36), we find

$$\begin{aligned}-\frac{\alpha}{2}\int_\Gamma[u_0]\,d\sigma&=|E^{\text{int}}|\,f+\operatorname{div}_x\Big(\int_{E^{\text{int}}}A(\nabla_y u_1^{\text{int}}+\nabla_x u_0^{\text{int}})\,dy\Big),\\ \frac{\alpha}{2}\int_\Gamma[u_0]\,d\sigma&=|E^{\text{out}}|\,f+\operatorname{div}_x\Big(\int_{E^{\text{out}}}A(\nabla_y u_1^{\text{out}}+\nabla_x u_0^{\text{out}})\,dy\Big).\end{aligned}$$

Finally, by inserting the factorization of u_1 given in (5-34), we get

$$\begin{aligned}f\,|E^{\text{int}}|&=-\operatorname{div}\Big(\Big(\int_{E^{\text{int}}}A(I-\nabla_y\chi_M^{\text{int}}(y))\,dy\Big)\nabla u_0^{\text{int}}\Big)-\frac{\alpha}{2}\,|\Gamma|\,[u_0]\\ &=-\operatorname{div}(A_{\mathcal{M}}^{\text{int}}\nabla u_0^{\text{int}})-\frac{\alpha}{2}\,|\Gamma|\,[u_0],\\ f\,|E^{\text{out}}|&=-\operatorname{div}\Big(\Big(\int_{E^{\text{out}}}A(I-\nabla_y\chi_M^{\text{out}}(y))\,dy\Big)\cdot\nabla u_0^{\text{out}}\Big)+\frac{\alpha}{2}\,|\Gamma|\,[u_0]\\ &=-\operatorname{div}(A_{\mathcal{M}}^{\text{out}}\nabla u_0^{\text{out}})+\frac{\alpha}{2}\,|\Gamma|\,[u_0],\end{aligned} \tag{5-37}$$

where the homogenized matrices are defined as

$$A_{\mathcal{M}}^{\text{int}}=\int_{E^{\text{int}}}A(I-\nabla_y\chi_{\mathcal{M}}^{\text{int}}(y))\,dy\quad\text{and}\quad A_{\mathcal{M}}^{\text{out}}=\int_{E^{\text{out}}}A(I-\nabla_y\chi_{\mathcal{M}}^{\text{out}}(y))\,dy. \tag{5-38}$$

Notice that the homogenized matrix $A_{\mathcal{M}}$ defined in (5-30) can be written as $A_{\mathcal{M}}=A_{\mathcal{M}}^{\text{int}}+A_{\mathcal{M}}^{\text{out}}$ and that $A_{\mathcal{M}}^{\text{int}}$, $A_{\mathcal{M}}^{\text{out}}$ can be written in the more meaningful form

$$(A_{\mathcal{M}}^{\text{out}})_{ij}=\int_{E^{\text{out}}}A\nabla_y(\chi_{\mathcal{M}}^{j,\text{out}}-y_j)\cdot\nabla_y(\chi_{\mathcal{M}}^{i,\text{out}}-y_i)\,dy, \tag{5-39}$$

and, in the connected-connected case,

$$(A^{\mathrm{int}}_{\mathcal{M}})_{ij} = \int_{E^{\mathrm{int}}} A\nabla_y(\chi^{j,\mathrm{int}}_{\mathcal{M}} - y_j)\cdot\nabla_y(\chi^{i,\mathrm{int}}_{\mathcal{M}} - y_i)\,dy, \tag{5-40}$$

which immediately give the symmetry (we recall that in the connected-disconnected case $A^{\mathrm{int}}_{\mathcal{M}} = 0$). The positive definiteness of $A^{\mathrm{out}}_{\mathcal{M}}$ and, in the connected-connected case, also of $A^{\mathrm{int}}_{\mathcal{M}}$, is a standard matter, as in the previous subsection.

Remark 5.3. The system (5-37) describes a bidomain model, where two overlapping macroscopic functions u^{int}_0 and u^{out}_0 appear. In the connected-connected case, such a bidomain model is described by a coupled system of two elliptic equations. On the other hand, in the connected-disconnected geometry, as recalled above, $A^{\mathrm{int}}_{\mathcal{M}} = 0$ and the system (5-37) describing the bidomain model becomes

$$\begin{aligned} -\frac{\alpha}{2}\,|\Gamma|\,[u_0] &= f\,|E^{\mathrm{int}}|,\\ -\mathrm{div}(A^{\mathrm{out}}_{\mathcal{M}}\nabla u^{\mathrm{out}}_0(x)) + \frac{\alpha}{2}\,|\Gamma|\,[u_0] &= f\,|E^{\mathrm{out}}|, \end{aligned}$$

which can be rewritten as

$$-\mathrm{div}(A^{\mathrm{out}}_{\mathcal{M}}\nabla u^{\mathrm{out}}_0(x)) = f, \tag{5-41}$$

$$u^{\mathrm{int}}_0 = u^{\mathrm{out}}_0 + \frac{2\,|E^{\mathrm{int}}|}{\alpha\,|\Gamma|}\,f. \tag{5-42}$$

More precisely, we obtain a decoupled system, where u^{out}_0 is determined by a standard elliptic equation (see (5-41)), in which the homogenized matrix depends only on the physical properties of the external phase, while u^{int}_0 is explicitly computed by means of u^{out}_0 in (5-42). Note that the internal phase is involved only through its measure, while its physical properties have no relevance in the macroscopic model.

Finally, we remark that in the connected-disconnected case, the solution u^{out}_0 of the leading phase of the bidomain system coincides with the homogenized limit u_0 obtained in the case $m = 0$, when the same geometrical setting is considered. □

Remark 5.4. Notice that, as in case $m = 0$, β and γ do not play any role in the homogenized limit. The only physical property of the interface which plays a role in the limit is α, i.e., the transversal heat diffusivity. □

Acknowledgements

Amar is a member of the *Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni* (GNAMPA) of the *Istituto Nazionale di Alta Matematica* (INdAM). Ayub wishes to thank *Dipartimento di Scienze di Base e Applicate per l'Ingegneria* for the warm hospitality and *Università "La Sapienza" of Rome* for the financial support, during her PhD program.

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Received 12 Apr 2023. Revised 14 Sep 2023. Accepted 3 Oct 2023.

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