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Cognitive roles in cooperative problem solving at university level

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This paper addresses the didactic issue of promoting the construction of the problem-solving competency for undergraduate students. We report the design of a learning activity, engaging students in collaborative problem solving in topology, each of them playing as a cognitive function coming into play when a mathematician faces a problem. To shed light on the impact of the roles played by the students with respect to the solving-problem process, we analyze some student's protocols from which we carry out for each role some actions as well as some benefits recognized by the students.

Keywords: mathematics education, university, problem solving, cognitive functions, role-playing.

Introduction and conceptual background

Problem solving has a fundamental role in mathematics learning and its educational importance is stated at each school level and in every field of mathematics, as it provides experience of those key processes in mathematics education that are involved in exploring, conjecturing, proving, constructing examples and counterexamples, representing, monitoring etc. (Polya, 1945; Schoenfeld, 1992; Carlson & Bloom, 2005). An important problem-solving activity, particularly efficient for the construction of new concepts (Dahlberg & Housman, 1997) is the students' generation of examples satisfying particular constraints. In fact, the students' production of examples can be a very complex activity, also for university students, that promotes the activation of fundamental cognitive processes (Antonini, 2011). At university level, a rich and deep field of mathematics is topology, a basic working tool of mathematicians in a variety of fields that can offer the undergraduate mathematics students the experience of engagement in a real problem-solving process. In this paper we are interested in promoting the construction of problem solving competency for undergraduate students, attending a topology course within Bachelor of Mathematics. The study of introductory general topology topics requires activating the significant cognitive functions, that are the mental processes (Kiely, 2014) that come into play during the problem-solving process. Polya (1945) highlighted that often the teacher poses questions and suggestions useful to the problem solver and recognizes them as indicators of mental operations typically useful for the solution of problems. In this strand, Albano, Coppola and Dello Iacono (2021), looking at how mathematicians behave when solving a problem, individuated some mental processes they usually activate (e.g. looking for paths, questioning herself, organizing herself, systematizing the findings, …), identifying them as roles/cognitive functions that a problem-solver should activate. These roles are: 1) Boss: she manages the work from every point of view (organizes actions, calls to the task, requires participation); 2) Promoter: she gives insights to promote a path that, starting from and manipulating prior mathematical knowledge (concepts and propositions), leads to the construction of examples, conjectures and outlines a solving strategy. In case of trouble, she asks for the teacher's help; 3) Critical mind: she questions the truth of the arguments and the validity of the answers proposed by the group, with the aim of corroborating their findings; 4) Blogger: she collects and rearranges everything that emerges from the discussion, to draw up a document containing all the arguments, notes, doubts, questions, answers.

We face the issue of promoting the development of students' ability to solve problems by offering them structured opportunities which allow the students to imitate and practice (Polya, 1945) and to become aware of their cognitive processes, so being able to monitor and coordinate them (Schoenfeld, 1992). The assignment of roles expands the potential of students as it gives them the opportunity to improve their problem solving skills at various levels: cognitive, metacognitive, affective. Exploiting and extending the results of a national project[1], we used the metaphor of storytelling to characterize the problem solving process at two levels: the process itself is seen as a mathematical story and the cognitive functions coming into play during the process become characters of such a story. Students are engaged in collaborative problem solving, where each student is required to play the role of a character of the story, that is she personifies one of the envisaged cognitive functions (Albano, Coppola & Dello Iacono, 2021). During the problem solving process, the students are expected to develop a mathematical narrative, consisting of their co-constructed notes on a digital board and a final collective report on the problems' solution. In this paper we report the design of a learning activity (Podolskiy, 2012), engaging undergraduate students in collaborative problem solving tasks. We discuss the first outcomes of the analysis concerning the metacognitive aspects, that is the students' declarative knowledge and awareness of the cognitive functions they experienced by playing the corresponding roles during the activity.

Experimental design

The design of the learning activity foresees that the students face a problem-solving task, consisting of three problems. According to the engagement model of Albano, Coppola and Dello Iacono (2021), the students work in groups of four or five people, each of them playing the role of one cognitive function. The groups are engaged in problem solving at different levels. One group, called Solver group, is devoted to collectively solve the three problems, and each of the students in the group acts according to one of the cognitive roles described in the previous section. The remaining groups, called Onlooker groups, are required to observe how the Solver group is working. Each student is guided to reflect both on how a specific one Solver member acts with respect to her role and on how the mathematical process is carried out by the entire Solver group. Therefore, personifying an Onlooker role stimulates a critical reflection not only at the cognitive level, as it allows a student's engagement in the mathematical problem, but also at a metacognitive level, as it fosters student's monitoring skills related to a role to play in a subsequent activity. A further level of reflection is added with respect to the previously cited engagement model of Albano et al. Indeed, we assume an incremental goals' structure of the problem-solving task: each of the three problems aims at a specific sub-goal, going from routinary employing mathematical concepts, facts, procedures and reasoning to creating new

[1] PRIN "Digital Interactive Storytelling in Mathematics: a competence-based social approach", PRIN 2015, Prot. 20155NPRA5, national project funded by the Italian Ministry of Education, University and Research.

mathematical knowledge. We envisage as many Onlooker groups as there are problems, so that each of them is focused on one specific problem. Each student is required to draw up a personal Logbook, containing some guidelines to reflect on the role she assumed, both as Solver and as Onlooker. Furthermore, at the end of each activity, every group is required to draw up a collective Logbook, reporting the solution of the problem and the process made to reach it (detailing the experience, how the construction of the answers took place, paying particular attention to the arguments they adduced in solving the three sub-problems). In order to foster the awareness of all the cognitive functions, students change roles with each new activity, also changing Solver with an Onlooker group and permuting the Onlooker groups (so changing the problem to be focused on).

Methodology

The experience involved about fifty students attending the course of *Geometry III* at the second year of a degree course in mathematics. The course Geometry III aims to introduce students to the fundamental concepts of general topology and to stimulate them to be able to use 'topological eyes', often far from the Euclidean ones. Besides acquiring content knowledge, the main educational goal is to construct students' mathematical reasoning capability, by means of analyzing and exploring problems, with an efficient use of topology concepts and results.

According to the design, each activity has an assignment on which students work in groups of 16-20, divided into 4 subgroups corresponding to the Solver group and the three Onlooker groups. In our experiment, all the participants have been split into three groups, named WG1, WG2, WG3, each of them consisting of four subgroups WSGi.1, WSGi.2, WSGi.3, WSGi.4 (i=1,2,3). Each student has been associated with a role-pair (subgroup role, individual role). Each subgroup acted as Solver or as Onlooker, so the corresponding value of the variable 'subgroup role' could be S or Oi, where Oi means that the subgroup acted as onlooker on the i-th sub-problem. The values assumed by 'individual role' corresponds to the cognitive functions played.

Along the course, students have been involved in three activities CWi (i=1,2,3). Both individual and collective roles changed as the activity CWi changed. More precisely, the assignment of roles in the passage from one activity to another can be described by a double permutation, one corresponding to the subgroup role and the other to the individual role. As an example, a student who has been assigned the role-pair (O1, Promoter) could take on the role-pair (S, Critical mind) in the next activity. This means that, while in the first activity the student belonged to the Onlooker group focused on the first problem and she was required to observe the work of the Promoter in the Solver group, with whom she confronted, during the second activity, she belonged to the Solver subgroup and acted a Critical Mind. In every activity CWi, the problem-solving task consisted of three problems: the first two concerned basic concepts introduced during the lectures and required the construction of examples of topological spaces or subspaces under given constraints; the third one was less routine (for instance, students may be asked to provide some characterization related to the property that is being investigated). Figure 1 gives a flavor of the kind of problems the students were asked to face.

PROBLEM n.1

1.a Construct a topological space (S, τ), where S is a non-empty set and τ a topology on S, different from those studied in class (invent it !!), in such that S contains two proper non-empty subsets X_1 e X_2 such that $Fr(X_1) = \emptyset$ e $Fr(X_2) \neq \emptyset$ in (S, τ), where $Fr(X_1)$ e $Fr(X_2)$ are the boundaries of X_1 and X_2 in (S, τ).

1.b Consider the set S introduced in point $1.a$. Denoted by τ_1 the topology τ and by τ_2, τ_3, τ_4, respectively, the trivial topology, the discrete topology, a topology of your choice distinct from the previous ones on S, determine

1.b.1 the frontier of X_1 in (S, τ_i), $Fr(X_1)^{\tau_i}$, for $i = 2, 3, 4$.

1.b.2 the frontier of X_2 in (S, τ_i), $Fr(X_2)^{\tau_i}$, for $i = 2, 3, 4$.

The boundary of a set can vary as the topology varies. Explain why.

PROBLEM n.2 Consider the topological space you defined in the previous problem (point 1.a). Construct, if they exist, three proper non-empty subsets of S, X_3, X_4, X_5, each distinct from X_1 and X_2, so that

2.a X_3 has an empty boundary with respect to all topologies.

2.b X_4 has a non-empty boundary with respect to all topologies.

2.c X_5 has empty boundary only with respect to some topology.

PROBLEM n.3

3.a Choose one of the topological spaces (S, τ_i) with $i \neq 2$ among those considered in problems 1 and 2 and investigate how a subset with an empty boundary can be constructed. In other words, characterize, if possible, the subsets with empty boundaries.

3.b Do the considerations made in the previous point or the statements you have reached also apply to the other topologies? Consider at least one of the two topologies τ_j on S with $j \notin \{i, 2\}$ and determine if the results found in (S, τ_i) continue to hold in (S, τ_j).

Figure 1: The task for the students

Each working group WGi had at their disposal a digital environment, consisting of various tools: a) a collaborative board Miro, enabling to collectively brainstorm in order to solve the problems, by adding post-it, importing images, drawing and connecting ideas, exchanges comments by chat; b) a collaborative document (shared Google-doc), used to report the solution product and process; c) a personal document (private Google-doc), used to report the reflections concerning the roles played during the experience.

All the data concerning the learning activity have been digitally stored, by means of the used tools (Miro and Google-doc). At the end, a questionnaire was submitted for investigating the students' perceptions. Here, we focus the qualitative analysis of the roles, looking at the students' answers in:

- the personal Logbook: *Describe how you played your role (as Solver and Onlooker) and what your contribution was. Do you think that the interventions related to your role were useful to achieve the objective? Why? Would you have done something differently? Why?*
- the questionnaire: *In the activities you played a role as Solver or Onlooker. For each role, tell us from your point of view how it contributes to solving the problem. If you think about when you are solving a problem on your own, do you recognize any of these roles in what you do during the solving process? If so, which ones more frequently? Are there any other roles you identify? Tell us.*

In particular, we are looking for excerpts describing the characteristics of the roles and the benefits of playing such roles, as perceived by the students.

Preliminary findings and discussion

We analyzed the Logbook and the questionnaires of 24 students. The answers to the above questions shed light on the impact of the roles played by the students with respect to the problem-solving process and to their usual own experience of problem solving.

Concerning the role of the Boss, the student St1 says:

St1: I immediately noticed her attention to details. Thinking of doing something useful for those of us who might have some deficiency on the tools required to face the task, she wrote in a frame [on Miro] all the useful definitions, such as boundary, interior, closure [of a set]. This has created a sort of "safe-spot", a point where you can have everything you need ready for use. She acted as a leader.

This excerpt seems to highlight the Boss' characteristics of a leader, implemented as someone who makes the group "safe" with respect to the problem-solving process. This means that she takes charge of supplying the group with mathematical recall notes connected to the given problem. The student St6 reports a similar work of the Boss (recall definitions), highlighting that such work let all the students be more comfortable with the needed background mathematical knowledge.

The role of the Boss is a disputed one. On the one hand, it is considered as a key role:

St1: The only real distinction was therefore that of boss / rest of the solver group, but without this negatively affecting the results.

St4: The boss of the solver group is the main role because he coordinates the entire group and manages to make the onlookers understand the work done.

On the other hand, sometimes, it can be perceived/played as a rote-role, without any added value, as shown by the following excerpts.

St15: I think it is the least useful role, since its role is only to check if all the work is progressing correctly, but does not interact in any way with the other students.

Actually, it seems that the experience of student St15, playing as 'boss' or onlooking 'boss', leads her to perceive the role differently from the designers of the activity. In fact, the cognitive function corresponding to 'boss' envisages someone who coordinates and takes care that everyone participates in the solving process. It is not her business to check the correctness of the work.

The role of Critical Mind seems to have promoted the action of questioning, and it has been recognized how much such action fosters a broader view of the problem solving:

St7: This work allowed her to ask more questions about the topics useful for solving the problem, and to have a broader vision of how a given exercise could be solved with different methods and observations. The role of Critical mind and, at the same time, that of the onlooker of the critical mind, proved useful in stimulating new proposals and new points of view, questioning the choices that are made each time and rattling off the strategies adopted as much as possible.

It is worthwhile to note that the role of Critical Mind has stimulated the student to pose questions to herself and to promote a critical attitude:

St17: It helps me to understand the mistakes I make, saying to myself: "stop, think: why did you carry out this calculation like this? Would it be simpler in another way? Is there any theorem or statement that can help me solve this exercise more quickly? And I find all of this very useful for the smooth running of a problem".

The student S12 highlight the importance of this role at individual and group level:

St12: In addition to an individual utility, I believe that the role of critical mind was very important for the whole group because often in the resolution of an exercise it can happen to make mistakes or inaccuracies without realizing it and the critical mind, insinuating doubts, manages to bring to attention the steps on which we need to work better.

The student St5 points out the role of the Promoter as someone who sketches a working outline, to engage the whole group in developing a solution and activating a thinking process:

St5: My idea as a promoter is precisely this, to propose your ideas despite some small inaccuracies in order to encourage the intervention of others and reach the final solution.

St13: I think about how to develop a certain problem.

It is interesting to note that the Promoters' suggestions are constructed on previous knowledge:

St14: She manipulated known propositions and concepts to build examples and strategies that led to the resolution of the problems

However, the role of Promoter does not always find the same appreciation and this is probably due to different values and personal beliefs about the roles:

St22: I also appreciated the promoter but to a lesser extent than Boss and Critical mind only because I personally think that finding an idea is less interesting than stressing it.

Concerning the role of the Blogger, it seems to impact on the cognitive level (e.g. reasoning) as well as on the affective one (e.g. engagement of the group), as shown in the following:

St11: The student I observed, that is the one who played the role of Blogger, played his role well, collecting and tidying up the board, highlighting some definitions and observations useful for carrying out the exercises, through the use of arrows and schemes that made the key concepts clear. He exploited and used the comments made in the chats by the other members of the group as well.

St13: His interventions were therefore also useful for taking stock of the situation from time to time, reorganizing ideas and, asking to repeat some of the concepts, also giving the possibility to those who may have remained behind to pick up the thread of the discussion.

As for the Boss, also the Blogger is a disputed role. On the one hand, it is considered pivotal:

St12: I believe that the role of blogger was fundamental for managing space and order on the board. Working in an orderly environment, in my opinion, favors concentration, allows you to easily identify the elements necessary to conduct a certain reasoning.

On the other hand, it can be played at surface level, just looking at aesthetic aspects:

St15: It is a secondary role in my opinion, since in most cases he was only concerned with the stylistic point of view of the board, caring little or nothing about the content.

St22: …the blogger did not impress me as I found it marginal with respect to solving the problem.

Table 1 shows a synoptic picture of the performed analysis. The first column shows some excerpts from the personal Logbooks that the students filled along with the activity on the basis of the roles they played (steered by the questions reported in the 'methodology' section). The second column shows excerpts from the questionnaires, submitted at the end of all the activities.

It seems that the logbooks show descriptions of the roles according to the actions that are recognized as belonging to a certain role, while the questionnaires show the roles in terms of the characteristics that define them in a problem-solving process. Furthermore, the table shows that sometimes the roles are experienced by the students in a way that corresponds to the designers' model, while in other cases some students find them difficult or uninteresting or give them an interpretation different from that of the designers, as in the following examples.

Table 1: Students' excerpts

	Excerpts from the personal logbooks	Excerpts from questionnaires
Boss	– plans, arranges and coordinates the work – pays attention to certain details that can benefit all the group – briefly recalls the key starting concepts that could be useful for solving the problem – enforces the assigned roles – involves all members of his group in solving	– is a predominant role taking care of the group – puts all members of the group in the same starting conditions trying to eliminate all possible differences or prejudices about individual abilities – makes you break down shyness – keeps his group cohesive – is the least useful role
Critical mind	– raises questions, also concerning the validity of the arguments – allows questions to be asked – analyze the various procedures critically – stimulate new proposals and new points of view	– allows you to gain awareness and to understand your own mistakes – allows you to develop a broader view of how a problem can be solved – fosters reasoning and searching for alternative simpler ways of moving in the problem space
Promoter	– sketches ideas to be completed – manipulates known propositions and concepts to build examples and strategies – retrieves useful material (sources from books, lecture notes, ...) – communicates with the teacher for help	– allows you to start reasoning – allows you to repeat useful notions – allows you to think how to develop a problem – opens to change the way to solve the problem – promotes the engagement of peers – is less interesting than critical mind
Blogger	– rearranges the various information emerged from the discussion – clarifies the many ideas presented by peers – uses arrows and diagrams to make the key concepts clear – tidies up the board	– makes the solving process clear – promotes concentration – helps to keep the logical thread – helps to take stock of the situation from time to time – has a secondary role – it is marginal to the resolution of the problem

Sometimes roles didn't work out because students got 'too busy' with solving the problem. Here the roles functioned as a 'group', not fragmented:

St16: Personally, I didn't fully respect my role when I was in the solver group, as working all together, and taken up by problem solving, we didn't pay much attention to the roles we had to fill. …. I think this was the most difficult aspect to respect, everyone was aware of the roles, which were respected as a group but not by the individual.

What emerged and synthesized in the above table seems to confirm that the students actually grasped the characteristics of each cognitive function, recognizing their functional goal with respect to successful problem solving. Assigning a role serves to activate a specific cognitive function, to stimulate the activation of some cognitive processes, and the roles would seem to be tight especially to those students who usually activate all of them. Sometimes these kinds of students tried to cross over and join the group of solvers even if she was not part of it. We could compare the action of assigning a role with giving the student a piece of chalk, associating chalks of different colors to the different roles by which to write on the board Miro and to contribute to the construction of the story and the fabula. Thus, each group is engaged as in a *thinking classroom*, as "a space that is inhabited by thinking individuals as well as individuals thinking collectively, learning together and constructing knowledge and understanding through activity and discussion" (Liljedahl, 2016, p. 364). We could speak of “thinking groups” (Thinking Solver group and Thinking Onlooker group), where each member performs an active function to solve a problem, by means of an action that stimulates a cognitive thought process that could remain dormant and blocked.

We conclude by noting the educational importance of engaging students’ in experiencing all the roles, in order to recognize all of them as pivotal to be successful in problem solving, similarly to how a mathematician usually uses them all when facing a problem. However, it is not taken for granted that a student activates them all, and this can cause difficulties. There are problems in front of which some roles do not come out, maybe because students are not used to or are more comfortable only with some of them. The experiment allowed the students to share the profile of a mathematician and in some cases to recognize themselves as mathematicians. Generally speaking, the learning activity proposed aimed at promoting and developing processes involved in problem-solving competency and an attitude of a mathematician towards problems mathematics, not aimed at ‘re-producing’ theorems and proof learned in class, but to autonomously ‘produce’ something of their own, new and original. Further research will be needed to investigate the processes involved in playing specific roles in problem solving and in the personal development of such important competency.

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