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A bifurcation phenomenon for the critical Laplace and p-Laplace equation in the ball

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A R T I C L E I N F O

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A B S T R A C T

We consider radial positive solutions for a class of quasilinear differential equations ruled by the p-Laplace differential operator with a critical weighted nonlinearity. We show that the problem undergoes a bifurcation phenomenon. We provide a new multiplicity result, even in the classical Laplace case. The proofs use the Fowler transformation and dynamical systems tools.

1. Introduction

We focus on positive radial solutions for the generalized scalar curvature equation

$$\Delta_p u + \mathcal{K}(|x|)\, u|u|^{q-2} = 0, \tag{1}$$

where $\Delta_p u = \mathrm{div}(\nabla u|\nabla u|^{p-2})$ denotes the p-Laplace operator, $x \in \mathbb{R}^n$, $2\,\frac{n}{2+n} \le p \le 2$, $1 < p < n$, and q is the Sobolev critical exponent

$$q = p^* = \frac{np}{n-p}. \tag{2}$$

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The function $\mathcal{K}\colon [0,+\infty[\to]0,+\infty[$ is assumed to be C^1, for simplicity. We are interested in *crossing* solutions, which means solutions of the problem

$$\begin{cases} \Delta_p u(x) + \mathcal{K}(|x|)\, u(x)\, |u(x)|^{q-2} = 0\,, & \\ u(x) > 0\,, & |x| < \mathcal{R}\,, \\ u(x) = 0\,, & |x| = \mathcal{R}. \end{cases} \tag{3}$$

In particular, we show that the existence of radial solutions of (3) depends on the behavior of $\mathcal{K}$ in a neighborhood of zero, and on the length of the radius $\mathcal{R}$.

Since we just consider radial solutions, we can reduce equation (1) to the singular ordinary differential equation

$$(r^{n-1}\, u'(r)|u'(r)|^{p-2})' + r^{n-1}\, \mathcal{K}(r)\, u|u|^{q-2} = 0\,, \tag{4}$$

where $'$ denotes the differentiation with respect to $r = |x|$, and, with a slight abuse of notation, $u(r) = u(x)$.

We say that a solution $u(r)$ of (4) is *regular* if and only if $\lim_{r\to 0} u(r) = d > 0$ for a suitable $d > 0$: in this case it will be denoted by $u(r; d)$. It is well known that $u(r; d)$ exists and it is unique for any $d > 0$, see, e.g., [21,22,28,29]. Note that $u'(0; d) = 0$.

As an alternative, from a standard scaling argument, we can find a counterpart for the *eigenvalue* equation where we fix $\mathcal{R} = 1$ for definiteness and we multiply the potential $\mathcal{K}$ by a parameter λ

$$\begin{cases} \Delta_p u(x) + \lambda\mathcal{K}(|x|)\, u(x)\, |u(x)|^{q-2} = 0\,, & \\ u(x) > 0\,, & |x| < 1\,, \\ u(x) = 0\,, & |x| = 1. \end{cases} \tag{5}$$

Indeed, if we set

$$w(s) = u(r), \qquad s = \frac{r}{\lambda^{1/p}}, \tag{6}$$

and $\mathscr{K}(s) = \mathcal{K}(s\lambda^{1/p})$, then u solves equation (4) if and only if w solves

$$(s^{n-1}\, w'(s)|w'(s)|^{p-2})' + \lambda s^{n-1}\, \mathscr{K}(s)\, w|w|^{q-2} = 0.$$

Moreover, $w(1) = 0$ if and only if $u(\mathcal{R}) = 0$, where $\mathcal{R} := \lambda^{1/p}$.

Hence, the number of solutions u of (3) as $\mathcal{R}$ varies coincides with the number of solutions u of (5) as λ varies.

The scalar curvature equation (1) has been extensively studied in the literature due to its significance in a broad variety of applications, such as Riemannian geometry, astrophysics, quantum mechanic, chemistry, theory of non-Newtonian fluids and elasticity (cf. [17] for more detailed references on applications of p-Laplace equations and, e.g., [5,6] for applications to Riemannian geometry in the case $p = 2$). Further, positivity of solutions has a physical relevance in many phenomena.

A key role in our analysis is played by the following hypothesis.

($\mathbf{H}_{\boldsymbol{\ell}}$) There are some constants $A, B, \ell > 0$ such that

$$\mathcal{K}(r) = A + Br^{\ell} + h(r) \qquad \text{and} \qquad \lim_{r\to 0} \frac{|h(r)| + r|h'(r)|}{r^{\ell}} = 0\,.$$

The existence and the multiplicity of the solutions of (3) and (5) depend crucially on ℓ, which, in literature, is often referred to as the *order of flatness* of the function $\mathcal{K}$ at $r = 0$.

Problem (3) subject to condition $(\mathbf{H}_{\boldsymbol{\ell}})$ has been already investigated in the early 1990s. In particular, Bianchi and Egnell in [3], Lin and Lin in [34] determined the existence of the critical value $\ell_2^* = n - 2$ in the Laplacian setting $p = 2$, and proved the following result.

Theorem A *([34]). Assume that* $\mathcal{K}$ *satisfies* $(\mathbf{H}_{\boldsymbol{\ell}})$ *and consider* $\mathbf{p} = \mathbf{2}$ *in* (3).

(i) If $\ell < n - 2$*, then problem* (3) *admits a radial solution for every* $\mathcal{R} > 0$.
(ii) If $\ell \geq n - 2$*, then there exists a sufficiently small constant* $r_0 > 0$ *such that problem* (3) *does not admit any radial solution when* $\mathcal{R} < r_0$.
(iii) If $n - 2 \leq \ell < n$*, then there exists a sufficiently large constant* $R_0 > 0$ *such that problem* (3) *admits a radial solution for every* $\mathcal{R} \geq R_0$.

In fact, Bianchi and Egnell in [3] focused on the case $R = 1$. In particular, following a shooting approach based on ordinary differential equations, they constructed and glued together two regular solutions of (4), one shooting from the zero initial condition and the other shooting from infinity.

The restriction $\ell < n$ has often been adopted in the Laplacian literature to ensure the existence of a solution to problem (3). For instance, it appears in [32, Theorem 0.19], where the scalar curvature $\mathcal{K}$ is required to be strictly decreasing in a left neighborhood of $\mathcal{R}$. The upper constraint $\ell < n$ can be also found in the recent work [33] dealing with the σ_k-Nirenberg problem on the standard sphere $\mathbb{S}^n$ for $2 \leq k < n/2$. We refer to the very recent papers [8,37], among the very few examples of existence results for the Laplacian scalar curvature problem in the absence of upper bound conditions on ℓ. In these articles hypothesis $(\mathbf{H}_{\boldsymbol{\ell}})$ is combined with a (not so easily verifiable) topological global index formula on the critical points of $\mathcal{K}$.

Note that in the p-Laplace context, the condition $\ell < n$ generalizes to $\ell < \frac{n}{p-1}$, cf. [18].

We emphasize that Theorem A, besides its intrinsic interest, has been a key starting point in proving the existence of ground states with fast decay, i.e., solutions $u(r)$ positive for any $r > 0$ and decaying as $r^{-(n-2)}$ at infinity.

In fact, it can be shown that if $\mathcal{K}$ is increasing close to $r = 0$ and decreasing close to $r = \infty$, we might expect to find ground states with fast decay: in the 1990s there was a flourishing of papers giving sufficient conditions for existence and non-existence of these solutions. A possible strategy is indeed to combine Theorem A, or similar results, with the use of Kelvin inversion, which transfers the information on regular solutions to fast decay solutions, see, e.g., [3,10]. Roughly speaking, one can expect that if $\mathcal{K}$ is steep enough at 0 (i.e., $\ell < n$), and at infinity $\mathcal{K}(r) \sim a + br^{-\ell}$ with $\ell < n$, $a > 0$, $b > 0$, then there is a ground state with fast decay. Conversely, if these conditions are violated one can construct a counterexample to Theorem A, see [3, Theorem 0.3]. We also refer to [18] for a generalization of results of this type to the p-Laplace setting: this paper follows a different strategy since the Kelvin inversion is not available in such a context.

We think it is worthwhile to point out that, if $A = 0$ in $(\mathbf{H}_{\boldsymbol{\ell}})$, then the problem becomes easier: roughly speaking, its solutions behave as the solutions of the $A > 0$, ℓ-subcritical case, i.e., (3) admits a radial solution for any $\mathcal{R} > 0$ (or, equivalently, (5) admits a radial solution for any $\lambda > 0$), for any $\ell > 0$. Again, combining this result with Kelvin inversion one might obtain the existence of ground states with fast decay. This idea was extensively used in the 1990s in many papers to handle the Laplacian problem, starting, probably, from [7,30,38]. Afterwards, it was developed and adapted to related problems, such as existence of ground states with fast decay with a prescribed number of sign changes, see, e.g., [28] and [39], dealing with the Laplacian and p-Laplacian setting, respectively.

The aim of this paper is to improve Theorem A in three main directions.

Firstly, we extend the results to the p-Laplace context, and in particular we find the formula for the critical order of flatness, i.e.,

$$\ell_p^* = \frac{n-p}{p-1},$$

which coincides with $\ell_2^* = n-2$ of Theorem A for $p = 2$. As far as we are aware, these are the first results in this direction, although we have to require the probably technical condition $\frac{2n}{2+n} \le p \le 2$. The possibility to remove this restriction will be the object of further investigations.

Secondly, we are able to remove the condition $\ell \le n$ (i.e., $\ell \le \frac{n}{p-1}$ in the p-Laplace context), by requiring that $\mathcal{K}(r)$ is increasing for any $r > 0$.

Thirdly, and probably more importantly, we are able to prove the existence of a second solution when $\ell > n-2$ (i.e., $\ell > \ell_p^*$ in the p-Laplace context), thus completing the bifurcation diagram even in the case $p = 2$.

Let us state our assumptions and the main results of the paper.

($\mathbf{H}_\uparrow$) The function $\mathcal{K}(r)$ is increasing for any $r \ge 0$, strictly in some interval.

($\mathbf{W}_s$) The function $\mathcal{K}(e^t)$ is uniformly continuous in $[0, +\infty[$ and there are $\overline{K} > \underline{K} > 0$ such that $\underline{K} < \mathcal{K}(r) < \overline{K}$ for any $r > 0$.

Note that, if $r\mathcal{K}'(r)$ is bounded in $[1, +\infty[$, then $\mathcal{K}(e^t)$ is uniformly continuous in $[0, +\infty[$.

Theorem 1. *Assume that $\mathcal{K}$ satisfies* ($\mathbf{H}_{\boldsymbol{\ell}}$).

If $\ell < \ell_p^*$, then problem (3) admits a radial solution for every $\mathcal{R} > 0$.

If $\ell \ge \ell_p^*$, then there exists $\mathcal{R}_0 > 0$ such that problem (3) does not admit any radial solution when $\mathcal{R} < \mathcal{R}_0$.

Theorem 2. *Assume that $\mathcal{K}$ satisfies* ($\mathbf{H}_{\boldsymbol{\ell}}$) *and* ($\mathbf{H}_\uparrow$).

If $\ell = \ell_p^*$, then there exists $\mathcal{R}_0 > 0$ such that problem (3)

- *does not admit any radial solution when $\mathcal{R} < \mathcal{R}_0$;*
- *admits at least a radial solution when $\mathcal{R} > \mathcal{R}_0$.*

If $\ell > \ell_p^*$, then there exists $\mathcal{R}_0 > 0$ such that problem (3)

- *does not admit any radial solution when $\mathcal{R} < \mathcal{R}_0$;*
- *admits at least a radial solution when $\mathcal{R} = \mathcal{R}_0$;*
- *admits at least 2 radial solutions when $\mathcal{R} > \mathcal{R}_0$.*

In fact, Theorem 2 holds also if we drop the global assumption ($\mathbf{H}_\uparrow$), but we strengthen the requirement on $\mathcal{K}'(0)$ by introducing the upper bound on the order of flatness $\ell \le \frac{n}{p-1}$ at zero, in the spirit of Theorem A. Unfortunately, we need to ask for some further very weak technical conditions on $\mathcal{K}$ for r large.

Theorem 3. *Assume that $\mathcal{K}$ satisfies* ($\mathbf{H}_{\boldsymbol{\ell}}$) *with $\ell_p^* \le \ell \le \frac{n}{p-1}$. Assume further that either* ($\mathbf{W}_s$) *holds or there is $\rho_1 > 0$ such that $\mathcal{K}(r) \ge \mathcal{K}(\rho_1)$ when $r \ge \rho_1$, then we get the same conclusion as in Theorem 2.*

Using the change of variable (6), we can rewrite Theorems 1, 2, and 3 as follows.

Corollary 4. *Assume that $\mathcal{K}$ satisfies* ($\mathbf{H}_{\boldsymbol{\ell}}$).

If $\ell < \ell_p^*$, then problem (5) admits a radial solution for every $\lambda > 0$.

If $\ell \geq \ell_p^*$, then there exists $\lambda_0 > 0$ such that problem (5) *does not admit any radial solution when $\lambda < \lambda_0$.*

Corollary 5. *Assume that $\mathcal{K}$ satisfies $(\mathbf{H}_{\boldsymbol{\ell}})$ and $(\mathbf{H}_{\uparrow})$.*

If $\ell = \ell_p^*$, then there exists $\lambda_0 > 0$ such that problem (5)

- *does not admit any radial solution when $\lambda < \lambda_0$;*
- *admits at least a radial solution when $\lambda > \lambda_0$.*

If $\ell > \ell_p^*$, then there exists $\lambda_0 > 0$ such that problem (5)

- *does not admit any radial solution when $\lambda < \lambda_0$;*
- *admits at least a radial solution when $\lambda = \lambda_0$;*
- *admits at least two radial solutions when $\lambda > \lambda_0$.*

Again, according to Theorem 3, we can drop the global condition $(\mathbf{H}_{\uparrow})$ and get the same result by restricting the interval in which ℓ varies and by imposing some further weak asymptotic conditions on $\mathcal{K}(r)$ for r large.

Corollary 6. *Assume that $\mathcal{K}$ satisfies $(\mathbf{H}_{\boldsymbol{\ell}})$ with $\ell_p^* \leq \ell \leq \frac{n}{p-1}$; assume further that either $(\mathbf{W}_{\boldsymbol{s}})$ holds or there is $\rho_1 > 0$ such that $\mathcal{K}(r) \geq \mathcal{K}(\rho_1)$ for any $r \geq \rho_1$. Then, we get the same conclusions as in Corollary 5.*

In fact, via Theorem 3 and Corollary 6, we are also able to extend Theorem A to the case $\ell = \frac{n}{p-1}$, which was not covered by [3,34], with the addition of a very weak technical condition, which, roughly speaking, is satisfied unless $\mathcal{K}(r)$ is subject to wild oscillations for r large, or converges to 0 for r large.

Moreover, our methods are considerably different from those of Lin and Lin in [34]. Our proofs are based on the Fowler transformation, which discloses the geometrical aspect of our problem by converting the singular ordinary differential equation (4) into an equivalent dynamical system, cf. (10), following the way paved by [25–27] and later on by [1,2,19]. Then, we develop a detailed phase plane analysis, involving invariant manifold theory for non-autonomous systems, energy estimates, comparisons of the non-autonomous planar system with suitable autonomous ones and a Grönwall's argument.

1.1. On the proofs of the theorems

We briefly sketch the plan of our proof. Let

$$J = \{d > 0 \mid u(r; d) \ \text{ is a crossing solution}\}, \tag{7}$$

and denote by $R(d)$ the *first zero* of $u(r; d)$ when $d \in J$. Our argument relies on a study of the properties of the function $R\colon J \to \,]0, +\infty[$.

Using some classical results (see [12,31] for the Laplacian case, and [22,29] for p-Laplacian extensions), we know that $J = \,]0, +\infty[$ under assumption $(\mathbf{H}_{\uparrow})$. Then, using a standard transversality argument, it is straightforwardly proved that $R(d)$ is continuous, see Proposition 22 below.

Moreover, it is not difficult to show that

$$\lim_{d \to 0} R(d) = +\infty\,,$$

see, e.g., [28, Proposition 2.4].

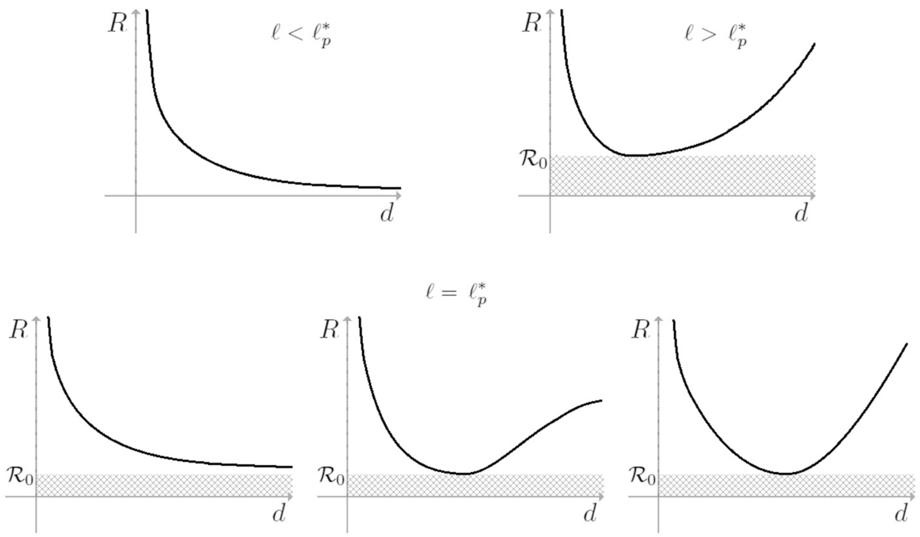

Fig. 1. A sketch of the graph of the function $R(d)$, when $(\mathbf{H}_{\boldsymbol{\ell}})$ and $(\mathbf{H}_{\uparrow})$ are assumed. In the critical case $\ell = \ell_p^*$, we have three possible alternatives.

The focus and the original part of our study consist in analyzing the asymptotic behavior of the solutions with large initial data d, which is determined by the parameter ℓ in $(\mathbf{H}_{\boldsymbol{\ell}})$. In particular, we have the following bifurcation phenomenon, when $(\mathbf{H}_{\uparrow})$ is assumed

$$\begin{cases} \lim_{d\to+\infty} R(d) = 0\,, & \text{if } 0 < \ell < \ell_p^*\,, \\ \liminf_{d\to+\infty} R(d) > 0\,, & \text{if } \ell = \ell_p^*\,, \\ \lim_{d\to+\infty} R(d) = +\infty\,, & \text{if } \ell > \ell_p^*\,, \end{cases} \tag{8}$$

see Propositions 37 and 43 below. This asymptotic analysis allows us to draw the diagrams in Fig. 1.

Since the number of solutions of problem (3) is the number of points of the preimage $R^{-1}(\mathcal{R})$, the proof of Theorem 2 is immediately given.

Actually, using a truncation argument, see Remark 25, we are able to get the first estimate in (8), also when assumption $(\mathbf{H}_{\uparrow})$ is dropped, cf. Proposition 37. Hence, the proof of Theorem 1 in the subcritical case follows.

In fact, in the case $p = 2$, a part of (8) has been already shown in [34, Theorem 1.6], see also [5, Remark 4.1]:

$$\begin{cases} \lim_{d\to+\infty} R(d) = 0 & \text{if } \ \ell < \ell_2^*\,, \\ \exists\, r_0 > 0 \,:\, R(d) \geq r_0\,,\, \forall\, d > 0 & \text{if } \ \ell \geq \ell_2^*. \end{cases}$$

Finally, we adapt our analysis to the context where the global requirement $(\mathbf{H}_{\uparrow})$ is replaced by the local requirement $\ell \leq \frac{n}{p-1}$, and we reprove the existence of the second solution in the supercritical case. As shown in Lemma 45 below, the restriction $\ell \leq \frac{n}{p-1}$ and the technical requirement that either $(\mathbf{W}_{\boldsymbol{s}})$ holds or there is $\rho_1 > 0$ such that $\mathcal{K}(r) \geq \mathcal{K}(\rho_1)$ when $r \geq \rho_1$ are needed just in order to ensure that there is $\hat{D} \geq 0$ such that $]\hat{D}, +\infty[\, \subset J$, with J as in (7).

Afterwards, using classical arguments, we see that if $\hat{D} \notin J$, then

$$\lim_{d\to\hat{D}} R(d) = +\infty.$$

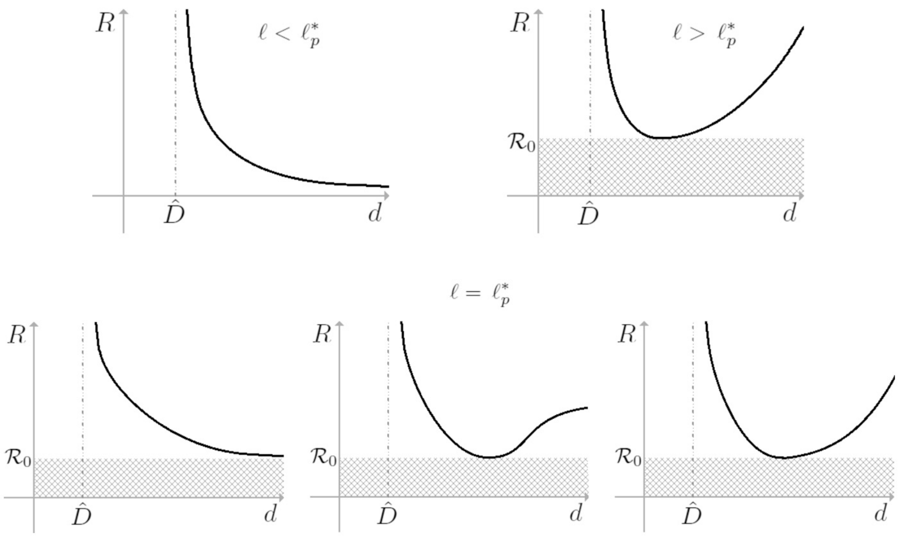

Fig. 2. A sketch of the graph of the function $R(d)$, in the setting of Theorem 3. In the critical case $\ell = \ell_p^*$, we have three possible alternatives.

Moreover, we prove that $R(d)$ satisfies (8) if $\ell \leq \frac{n}{p-1}$ by adapting the computation performed when $(\mathbf{H}_\uparrow)$ holds, see Propositions 44 and 46 below. As a consequence, we are able to draw the diagrams for $R(d)$ in Fig. 2 from which Theorem 3 follows.

Notice that if $\hat{D} > 0$, then $u(r; \hat{D})$ is a ground state, that is a solution of (4), positive for any $r > 0$.

The paper is organized as follows. In Section 2 we introduce the Fowler transformation, i.e., the change of variables (9), which turns (4) into the planar non-autonomous dynamical system (10). Then we recall some basic tools in this context. In particular, we review some aspects of invariant manifold theory for non-autonomous systems, and we introduce the unstable leaves $W^u(\tau)$ of (10) which correspond to regular solutions of (4). In Section 3 we establish some standard properties of the function $R(d)$, such as continuity and asymptotic properties close to $d = 0$, and to $d = \hat{D}$ whenever $u(r; \hat{D})$ is a ground state. The core of our argument is in Section 4, where we prove the asymptotic properties of $R(d)$ as d tends to infinity. In Section 4.1 we focus on the subcritical case $\ell < \ell_p^*$. In Section 4.2 we consider the critical and supercritical setting $\ell \geq \ell_p^*$. In Section 5 we briefly conclude the proofs of the main results of the paper.

2. Fowler transformation, basic notation and preliminaries

Let us introduce a change of variables known as Fowler transformation, which allows to transform (4) into a two-dimensional dynamical system. We define

$$\alpha = \frac{n-p}{p}, \qquad \beta = \frac{n(p-1)}{p},$$
$$x = u(r) r^{\alpha}, \qquad y = u'(r)|u'(r)|^{p-2} r^{\beta}, \qquad r = e^t. \tag{9}$$

This change of variable is known from the 1930s, see [13], and it has been generalized to the p-Laplacian case by Bidaut-Véron [4] and some years later (independently) by Franca, see, e.g., [14–16,18,19].

According to (9), we can rewrite (4) as the following dynamical system:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} y\,|y|^{\frac{2-p}{p-1}} \\ -K(t)\,x|x|^{q-2} \end{pmatrix}, \tag{10}$$

where $K(t) = \mathcal{K}(e^t)$, $q = p^* = \frac{np}{n-p}$ as in (2), and "·" denotes the differentiation with respect to t.

Remark 7. Note that system (10) is C^1 if and only if $\frac{2n}{2+n} \le p \le 2$.

Given the initial data $\tau \in \mathbb{R}$ and $\boldsymbol{Q} \in \mathbb{R}^2$, we denote by

$$\boldsymbol{\phi}(t;\tau,\boldsymbol{Q}) = \big(x(t;\tau,\boldsymbol{Q}), y(t;\tau,\boldsymbol{Q})\big)$$

the trajectory of (10) such that $\boldsymbol{\phi}(\tau;\tau,\boldsymbol{Q}) = \boldsymbol{Q}$.
Define the energy function

$$\mathcal{H}(x,y;t) := \alpha xy + \frac{p-1}{p}\,|y|^{\frac{p}{p-1}} + K(t)\,\frac{|x|^q}{q}\,. \tag{11}$$

If we evaluate $\mathcal{H}$ along a solution $(x(t),y(t))$ of (10), we obtain the associated Pohozaev type energy $\mathcal{H}(x(t),y(t);t)$, whose derivative with respect to t satisfies

$$\frac{d}{dt}\mathcal{H}\big(x(t),y(t);t\big) = \dot{K}(t)\,\frac{|x(t)|^q}{q}\,. \tag{12}$$

We also need to consider the autonomous system obtained by *freezing* the t-dependence of $K(\cdot)$. In particular, fixed $\tau \in \mathbb{R}\cup\{\pm\infty\}$, we consider the *frozen* autonomous system:

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}\alpha & 0\\ 0 & -\alpha\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} + \begin{pmatrix} y\,|y|^{\frac{2-p}{p-1}}\\ -K(\tau)\,x|x|^{q-2}\end{pmatrix}, \tag{13}$$

where $\mathcal{K}$ is assumed to have finite limit whenever $\tau = \pm\infty$.

If we differentiate the energy $\mathcal{H}(\cdot;\tau)$ of system (13) along a solution $(x(t),y(t))$ of the dynamical system (10), we get

$$\frac{d}{dt}\mathcal{H}\big(x(t),y(t);\tau\big) = \big(K(\tau)-K(t)\big)\,x(t)|x(t)|^{q-2}\,\dot{x}(t)\,. \tag{14}$$

Let us fix $\tau \in \mathbb{R}\cup\{\pm\infty\}$ in (13), let $\tau_0 \in \mathbb{R}$ and $\boldsymbol{Q} \in \mathbb{R}^2$; we denote by

$$\boldsymbol{\phi_\tau}(t;\tau_0,\boldsymbol{Q}) = \big(x_\tau(t;\tau_0,\boldsymbol{Q}), y_\tau(t;\tau_0,\boldsymbol{Q})\big)$$

the trajectory of (13) such that $\boldsymbol{\phi_\tau}(\tau_0;\tau_0,\boldsymbol{Q}) = \boldsymbol{Q}$.

System (13) exhibits an equilibrium point in $\{x>0\,, y<0\}$ of coordinates

$$\boldsymbol{E}(\tau) = (E_x(\tau),E_y(\tau)) = \left(\left(\frac{\alpha^p}{K(\tau)}\right)^{\frac{1}{q-p}}, -\left(\frac{\alpha^q}{K(\tau)}\right)^{\frac{p-1}{q-p}}\right). \tag{15}$$

It is also well known (see, among others, [18]) that for any fixed $\tau \in \mathbb{R}\cup\{\pm\infty\}$ system (13) admits a homoclinic orbit (see Fig. 3)

$$\boldsymbol{\Gamma_\tau} = \{(x,y) \mid \mathcal{H}(x,y;\tau) = 0\,, x>0\}\,, \tag{16}$$

recalling that the case $\tau = +\infty$ requires a boundedness restriction on $\mathcal{K}$. Taking into account that the flows of (10) and (13) are ruled by their linear part near the origin, we easily deduce that the origin is a saddle-type critical point for (10) and (13). Finally, notice that $\boldsymbol{E}(\tau)$ is a center and it lies in the interior of the region enclosed by $\boldsymbol{\Gamma_\tau}$.

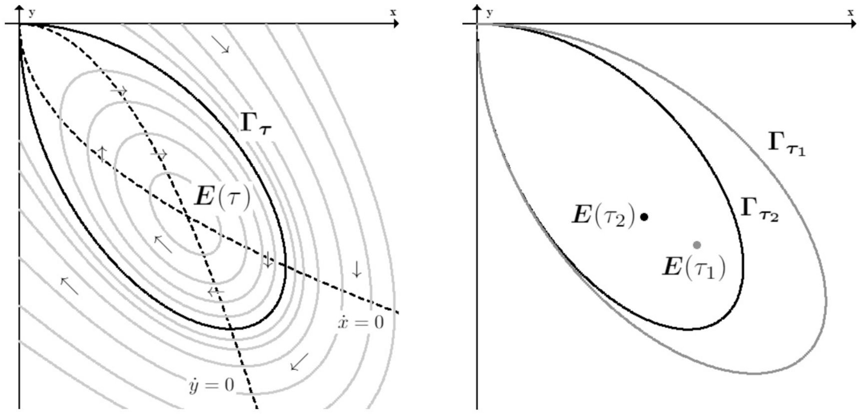

Fig. 3. On the left, the energy levels of $\mathcal{H}$ at fixed time τ, with the homoclinic orbit $\boldsymbol{\Gamma}_{\boldsymbol{\tau}}$. On the right, the position of the set $\boldsymbol{\Gamma}_{\boldsymbol{\tau_2}}$ with respect to the set $\boldsymbol{\Gamma}_{\boldsymbol{\tau_1}}$ in the case $K(\tau_2) > K(\tau_1)$.

According to [19,23], we also know the exact expression of the homoclinic trajectories $\boldsymbol{\phi}^* = (\boldsymbol{x}^*, \boldsymbol{y}^*)$ of (13). In particular, the one corresponding to the regular ground state u^* such that $u^*(0) = d$ satisfies

$$x^*(t) = d\left[\mathrm{e}^{-t} + C(d)\,\mathrm{e}^{\frac{t}{p-1}}\right]^{-\frac{n-p}{p}}, \tag{17}$$

where $C(d) > 0$ is a computable constant, see [23].

Moreover, fixed $\boldsymbol{P} \in \boldsymbol{\Gamma}_{-\infty}$, there exists a constant $c_{\boldsymbol{P}} > 0$ such that

$$\sup\{\|\boldsymbol{\phi}_{-\infty}(\theta+\tau;\tau,\boldsymbol{Q})\|\mathrm{e}^{-\alpha\theta} \mid \theta \le 0,\ \tau \le 0,\ \boldsymbol{Q} \in \widetilde{\boldsymbol{\Gamma}}^{\boldsymbol{P}}_{-\infty}\} \le c_{\boldsymbol{P}}, \tag{18}$$

where $\widetilde{\boldsymbol{\Gamma}}^{\boldsymbol{P}}_{-\infty} := \boldsymbol{\phi}_{-\infty}(]-\infty,0];0,\boldsymbol{P})$.

Let us now list some immediate consequences of assumption $(\mathbf{H}_\uparrow)$.

Remark 8. Assume $K(\tau_2) > K(\tau_1)$, then the homoclinic orbit $\boldsymbol{\Gamma}_{\boldsymbol{\tau_2}}$ belongs to the region enclosed by $\boldsymbol{\Gamma}_{\boldsymbol{\tau_1}}$, cf. Fig. 3.

Lemma 9. *Assume* $(\mathbf{H}_\uparrow)$. *If* $u(r;d)$ *is a regular solution, then the corresponding trajectory* $\boldsymbol{\phi}(\cdot;d) = (x(\cdot;d), y(\cdot;d))$ *of system* (10) *satisfies*

$$\lim_{t\to-\infty} \mathcal{H}(\boldsymbol{\phi}(t;d);t) = 0$$

and

$$\mathcal{H}(\boldsymbol{\phi}(t;d);t) \ge 0 \quad \text{for every } t \le T(d)$$

(the equality occurs only when $K(t) = K(-\infty)$ *holds), where* $T(d) := \sup\{\tau \in \mathbb{R} : x(t;d) > 0, \forall t \le \tau\}$ *is the first zero of* $x(\cdot;d)$. *Moreover,*

$$\mathcal{H}(\boldsymbol{\phi}(t;d);\tau) \ge \mathcal{H}(\boldsymbol{\phi}(t;d);t) \ge 0 \quad \text{for every } t < \min\{\tau, T(d)\}$$

(the first equality occurs only when $K(t) = K(\tau)$ *holds).*

Proof. The result immediately follows from $(\mathbf{H}_\uparrow)$, (12), and (11). □

As an immediate consequence, we have the following.

Lemma 10. *Assume* $(\mathbf{H}_\uparrow)$ *and fix* $d > 0$. *Then, for every* $t \leq T(d)$, *the trajectory* $\boldsymbol{\phi}(t; d)$ *of system* (10) *either lies in the exterior of the region enclosed by* $\boldsymbol{\Gamma_t}$, *or it lies on* $\boldsymbol{\Gamma_t}$ *if* $K(t) = K(-\infty)$.

The next part of the section is devoted to explore the dynamical system (10) through an invariant manifold approach. From [19,24,26] and [9, Section 13.4] we know that the existence of the unstable manifold is ensured by the following condition:

$(\mathbf{W_u})$ The function K is bounded and uniformly continuous in $]-\infty, 0]$ and $\lim_{t\to-\infty} K(t) = K(-\infty) \geq 0$ is finite.

Notice that $(\mathbf{W_u})$ is always satisfied when $(\mathbf{H_\ell})$ holds.

Correspondingly, the existence of the stable manifold is guaranteed by $(\mathbf{W_s})$. The unstable manifold plays a key role in this paper, since we will see via Remark 17 that regular solutions $u(r; d)$ correspond to trajectories of (10) converging to the origin as $t \to -\infty$, i.e., leaving from the unstable manifold.

Let $B(\mathbf{0}, \delta)$ denote the open ball of radius δ, centered at the origin. Following [26, Theorem 2.1], which is in fact a rewording of [24, Theorem 2.25], and its reformulation in [20, Appendix], we find the next result.

Theorem 11. *Assume* $(\mathbf{W_u})$, *then there is* $\delta > 0$ *such that for any* $\tau \leq 0$ *the set*

$$M^u_{loc}(\tau) := \{\boldsymbol{Q} \mid \boldsymbol{\phi}(t; \tau, \boldsymbol{Q}) \in B(\mathbf{0}, \delta) \text{ for any } t \leq \tau\}$$

is a C^1 *embedded manifold tangent to the* x*-axis at the origin if* $\frac{2n}{2+n} < p \leq 2$ *and to the line* $y = -\frac{K(-\infty)}{n} x$ *if* $p = \frac{2n}{2+n}$. *Further, let* $\mathcal{L}$ *be a segment transversal to* $M^u_{loc}(\tau)$, *then* $M^u_{loc}(\tau) \cap \mathcal{L} = \{\boldsymbol{Q^u}(\tau)\}$ *is a singleton, and* $\boldsymbol{Q^u}(\tau)$ *is uniformly continuous in* τ.

Assume $(\mathbf{W_s})$, *then there is* $\delta > 0$ *such that for any* $\tau \geq 0$ *the set*

$$M^s_{loc}(\tau) := \{\boldsymbol{Q} \mid \boldsymbol{\phi}(t; \tau, \boldsymbol{Q}) \in B(\mathbf{0}, \delta) \text{ for any } t \geq \tau\}$$

is a C^1 *embedded manifold tangent to the* y*-axis at the origin if* $\frac{2n}{2+n} \leq p < 2$ *and to the line* $y = -(n-2)x$ *if* $p = 2$. *Further, let* $\mathcal{L}$ *be a segment transversal to* $M^s_{loc}(\tau)$, *then* $M^s_{loc}(\tau) \cap \mathcal{L} = \{\boldsymbol{Q^s}(\tau)\}$ *is a singleton, and* $\boldsymbol{Q^s}(\tau)$ *is uniformly continuous in* τ.

This result may also be obtained by using the simpler approach developed in [9, Section 13.4], with the exception of the part concerning the uniform continuity of $\boldsymbol{Q^u}(\cdot)$ and $\boldsymbol{Q^s}(\cdot)$. Actually, [26, Theorem 2.1] ensures that $\boldsymbol{Q^u}(\tau)$ and $\boldsymbol{Q^s}(\tau)$ are C^1 if $K(t)$ is C^1.

We notice that $M^u_{\mathrm{loc}}(\tau)$ is split by the origin in two connected components. Since we are just interested in positive solutions, we denote by $W^u_{\mathrm{loc}}(\tau)$ and $W^{u,-}_{\mathrm{loc}}(\tau)$ the components lying respectively in $x > 0$ and $x < 0$. Similarly, $M^s_{\mathrm{loc}}(\tau)$ is split by the origin in two connected components, say $W^s_{\mathrm{loc}}(\tau)$ and $W^{s,-}_{\mathrm{loc}}(\tau)$, lying in $x > 0$ and $x < 0$, respectively.

Following again either [24, Theorem 2.25] or [9, Section 13.4] (and in particular [9, Theorem 4.5]), we deduce some crucial properties concerning the uniform exponential asymptotic behavior of the trajectories intersecting the manifolds $W^u_{\mathrm{loc}}(\tau)$ and $W^s_{\mathrm{loc}}(\tau)$.

Theorem 12. *Assume* $(\mathbf{W_u})$ *and* $(\mathbf{W_s})$, *then the sets*

$$W^u_{loc}(\tau_1) := \{\boldsymbol{Q} \in M^u_{loc}(\tau_1) \mid x(t; \tau_1, \boldsymbol{Q}) > 0 \text{ for any } t \leq \tau_1\} \cup \{(0,0)\},$$
$$W^s_{loc}(\tau_2) := \{\boldsymbol{Q} \in M^s_{loc}(\tau_2) \mid x(t; \tau_2, \boldsymbol{Q}) > 0 \text{ for any } t \geq \tau_2\} \cup \{(0,0)\}$$

are C^1 embedded manifolds for any $\tau_1 \le 0 \le \tau_2$. Furthermore, if $\boldsymbol{Q^u} \in W^u_{loc}(\tau_1)$ and $\boldsymbol{Q^s} \in W^s_{loc}(\tau_2)$, then the limits

$$\lim_{t\to-\infty} \|\boldsymbol{\phi}(t;\tau_1,\boldsymbol{Q^u})\|\mathrm{e}^{-\alpha t}, \qquad \lim_{t\to+\infty} \|\boldsymbol{\phi}(t;\tau_2,\boldsymbol{Q^s})\|\mathrm{e}^{\alpha t}$$

are positive and finite. Further, there is $c_0 = c_0(\delta)$ such that

$$\begin{aligned}
\sup\{\|\boldsymbol{\phi}(\theta+\tau_1;\tau_1,\boldsymbol{Q^u})\|\mathrm{e}^{-\alpha\theta} \mid \theta \le 0,\ \tau_1 \le 0,\ \boldsymbol{Q^u} \in W^u_{loc}(\tau_1)\} \le c_0(\delta),\\
\sup\{\|\boldsymbol{\phi}(\theta+\tau_2;\tau_2,\boldsymbol{Q^s})\|\,\mathrm{e}^{\alpha\theta} \mid \theta \ge 0,\ \tau_2 \ge 0,\ \boldsymbol{Q^s} \in W^s_{loc}(\tau_2)\} \le c_0(\delta).
\end{aligned}$$

Using the flow of (10), with a standard argument, we pass from the local manifolds $W^u_{\mathrm{loc}}(\tau_1)$ and $W^s_{\mathrm{loc}}(\tau_2)$, defined for $\tau_1 \le 0 \le \tau_2$, to the global manifolds $W^u(\tau)$ and $W^s(\tau)$ defined for any $\tau \in \mathbb{R}$. In fact, rephrasing [20, Appendix], we set

$$\begin{aligned}
W^u(\tau) &:= \bigcup_{T\le 0} \{\boldsymbol{\phi}(\tau;T,\boldsymbol{Q}) \mid \boldsymbol{Q} \in W^u_{\mathrm{loc}}(T)\},\\
W^s(\tau) &:= \bigcup_{T\ge 0} \{\boldsymbol{\phi}(\tau;T,\boldsymbol{Q}) \mid \boldsymbol{Q} \in W^s_{\mathrm{loc}}(T)\}.
\end{aligned} \tag{19}$$

We observe that $W^u(\tau)$ and $W^s(\tau)$, the unstable and stable leaves respectively, are C^1 immersed manifolds which can be characterized as follows

$$\begin{aligned}
W^u(\tau) &:= \Big\{\boldsymbol{Q} \mid \lim_{t\to-\infty} \boldsymbol{\phi}(t;\tau,\boldsymbol{Q}) = (0,0),\ x(t;\tau,\boldsymbol{Q}) \ge 0 \text{ when } t \ll 0\Big\},\\
W^s(\tau) &:= \Big\{\boldsymbol{Q} \mid \lim_{t\to+\infty} \boldsymbol{\phi}(t;\tau,\boldsymbol{Q}) = (0,0),\ x(t;\tau,\boldsymbol{Q}) \ge 0 \text{ when } t \gg 0\Big\}.
\end{aligned} \tag{20}$$

By construction, if $\boldsymbol{Q} \in W^u(\tau)$, then $\boldsymbol{\phi}(t;\tau,\boldsymbol{Q}) \in W^u(t)$ for any $t \in \mathbb{R}$.

From Theorem 11 and the smooth dependence of the flow of (10) on initial data, we get the smoothness property of the unstable and stable manifold.

Remark 13. Assume $\mathbf{(W_{\boldsymbol{u}})}$, then $W^u(\tau)$ depends continuously on τ. Namely, let $\mathcal{L}$ be a segment which intersects $W^u(\tau_0)$ transversely in a point $\boldsymbol{Q}(\tau_0)$, then there is a neighborhood I of τ_0 such that $W^u(\tau)$ intersects $\mathcal{L}$ in a point $\boldsymbol{Q}(\tau)$ for any $\tau \in I$, and $\boldsymbol{Q}(\cdot)$ is continuous. Actually, it has the same regularity as (10), so it is C^1 if (10) is C^1 in t. Analogously, if $\mathbf{(W_{\boldsymbol{s}})}$ holds, then $W^s(\tau)$ depends continuously on τ, and smoothly if (10) is smooth in t.

Let us denote by $W^u(-\infty)$ the unstable manifold $\boldsymbol{\Gamma_{-\infty}}$, defined in (16). According to [1] and [20, Section 2.2], the smoothness property of $W^u(\tau)$ observed in Remark 13 can be extended to $\tau_0 = -\infty$.

Remark 14. Assume $\mathbf{(W_{\boldsymbol{u}})}$, then $W^u(\tau)$ depends smoothly on $\tau \in \mathbb{R} \cup \{-\infty\}$. In particular, let $\mathcal{L}$ be a segment transversal to $\boldsymbol{\Gamma_{-\infty}}$ and let $\boldsymbol{Q}_{\mathcal{L}}(-\infty)$ be the intersection point between $\mathcal{L}$ and $\boldsymbol{\Gamma_{-\infty}}$; follow $W^u(\tau)$ from the origin towards $x > 0$, then it intersects $\mathcal{L}$ transversely in a point, say $\boldsymbol{Q}_{\mathcal{L}}(\tau)$, for any $\tau \le -N$, for a suitable sufficiently large $N = N(\mathcal{L}) > 0$. Furthermore, the function $\boldsymbol{Q}_{\mathcal{L}}(\cdot)$ is C^1 and $\lim_{\tau\to-\infty} \boldsymbol{Q}_{\mathcal{L}}(\tau) = \boldsymbol{Q}_{\mathcal{L}}(-\infty)$.

We stress that $\boldsymbol{Q}_{\mathcal{L}}(\tau)$ is uniquely defined as the first intersection between $W^u(\tau)$ and $\mathcal{L}$, although it might not be the unique intersection, especially if $\mathcal{L}$ is too large.

Let $\mathcal{L}$ be a segment transversal to $\boldsymbol{\Gamma_{-\infty}}$, take $\tau \le -N$, and denote by $\tilde{W}^u_{\mathcal{L}}(\tau)$ the compact and connected branch of $W^u(\tau)$ between the origin and $\boldsymbol{Q}_{\mathcal{L}}(\tau)$. Analogously, denote by $\tilde{W}^u_{\mathcal{L}}(-\infty)$ the compact connected branch of $\boldsymbol{\Gamma_{-\infty}} = W^u(-\infty)$ between the origin and $\boldsymbol{Q}_{\mathcal{L}}(-\infty)$.

Lemma 15. *Assume* $(\mathbf{W}_{\boldsymbol{u}})$. *Let* $\mathcal{L}$ *and* N *be as in Remark 14, then there is* $c = c(\mathcal{L})$ *such that*

$$\sup\{\|\boldsymbol{\phi}(\theta+\tau;\tau,\boldsymbol{Q})\|\mathrm{e}^{-\alpha\theta} \mid \boldsymbol{Q} \in \tilde{W}^u_{\mathcal{L}}(\tau),\ \theta \le 0,\ \tau \le -N\} \le c(\mathcal{L}). \tag{21}$$

Proof. Let $\delta > 0$ be the fixed constant defined in Theorem 11. If $\mathcal{L}$ is close enough to the origin so that $\|\boldsymbol{Q}_{\mathcal{L}}(-\infty)\| < \delta$, then by construction $\tilde{W}^u_{\mathcal{L}}(\tau) \subset W^u_{\mathrm{loc}}(\tau)$ for every $\tau \le -N$, see (19), and (21) follows straightforwardly from Theorem 12.

Now, let $\mathcal{L}$ be a generic segment transversal to $\boldsymbol{\Gamma}_{-\infty}$ satisfying $\|\boldsymbol{Q}_{\mathcal{L}}(-\infty)\| \ge \delta$, and let $M = M(\delta,\mathcal{L}) > 0$ be such that

$$\|\boldsymbol{\phi}_{-\infty}(-M;0,\boldsymbol{Q}_{\mathcal{L}}(-\infty))\| = \delta/2\,.$$

Fix a point $\hat{\boldsymbol{Q}}$ belonging to $\tilde{W}^u_{\mathcal{L}}(-\infty)\backslash B(\mathbf{0},\frac{\delta}{2})$, then there is $t_{\hat{\boldsymbol{Q}}} \in [-M,0]$ such that $\boldsymbol{\phi}_{-\infty}(t_{\hat{\boldsymbol{Q}}};0,\boldsymbol{Q}_{\mathcal{L}}(-\infty)) = \hat{\boldsymbol{Q}}$. By construction, $\|\boldsymbol{\phi}_{-\infty}(\tau-M;\tau,\hat{\boldsymbol{Q}})\| \le \frac{\delta}{2}$ for any $\tau < 0$, since $\boldsymbol{\phi}_{-\infty}(\tau-M;\tau,\hat{\boldsymbol{Q}}) = \boldsymbol{\phi}_{-\infty}(-M;0,\hat{\boldsymbol{Q}}) = \boldsymbol{\phi}_{-\infty}(-M+t_{\hat{\boldsymbol{Q}}};0,\boldsymbol{Q}_{\mathcal{L}}(-\infty))$.

Using Remark 14 combined with the compactness of $\tilde{W}^u_{\mathcal{L}}(\tau) \setminus W^u_{\mathrm{loc}}(\tau)$, continuous dependence on initial data and parameters of the flow of (10), and possibly choosing a larger N, we can assume that $\boldsymbol{R} = \boldsymbol{\phi}(\tau-M;\tau,\boldsymbol{Q})$ is such that

$$\|\boldsymbol{R}\| < \delta, \qquad \text{so that} \quad \boldsymbol{R} \in W^u_{\mathrm{loc}}(\tau-M) \tag{22}$$

whenever $\tau < -N$ and $\boldsymbol{Q} \in \tilde{W}^u_{\mathcal{L}}(\tau) \setminus W^u_{\mathrm{loc}}(\tau)$. Notice that M does not depend on $\boldsymbol{Q}$ and τ, but depends on $\mathcal{L}$. The existence of $\boldsymbol{R}$ as in (22) is trivial for any $\boldsymbol{Q} \in W^u_{\mathrm{loc}}(\tau)$, so (22) holds for any $\boldsymbol{Q} \in \tilde{W}^u_{\mathcal{L}}(\tau)$.

Thus, Theorem 12 ensures the existence of $c_0 = c_0(\delta) > 0$ such that

$$\sup\{\|\boldsymbol{\phi}(s+\tau-M;\tau,\boldsymbol{Q})\|\mathrm{e}^{-\alpha s} \mid \boldsymbol{Q} \in \tilde{W}^u_{\mathcal{L}}(\tau),\ s \le 0,\ \tau \le -N\} \le c_0.$$

Hence, $\|\boldsymbol{\phi}(\theta+\tau;\tau,\boldsymbol{Q})\|\mathrm{e}^{-\alpha\theta} \le c_0\mathrm{e}^{\alpha M}$ for any $\theta \le -M$ and $\boldsymbol{Q} \in \tilde{W}^u_{\mathcal{L}}(\tau)$, which proves (21) when $\theta \le -M$.

To prove (21) for $-M \le \theta \le 0$, it is enough to recall that $\tilde{W}^u_{\mathcal{L}}(\tau)$ is compact, $M > 0$ is fixed and the flow of (10) is continuous; then the lemma follows by choosing some $c(\mathcal{L}) \ge c_0\mathrm{e}^{\alpha M}$. □

The following lemma describes better the behavior of the solutions $\boldsymbol{\phi}(\cdot;\tau,\boldsymbol{Q})$ departing from a point $\boldsymbol{Q} \in W^u(\tau)$, as $\tau \to -\infty$. Roughly speaking, we can say that such trajectories mime the autonomous dynamical system (13) frozen at $\tau = -\infty$.

Lemma 16. *Assume* $(\mathbf{W}_{\boldsymbol{u}})$. *Let* $\mathcal{L}$ *and* $\boldsymbol{Q}_{\mathcal{L}}(-\infty)$ *be as in Remark 14. Then,*

$$\lim_{\tau\to-\infty}\sup_{\theta\le 0}\left\|\boldsymbol{\phi}\big(\theta+\tau;\tau,\boldsymbol{Q}_{\mathcal{L}}(\tau)\big) - \boldsymbol{\phi}_{-\infty}\big(\theta;0,\boldsymbol{Q}_{\mathcal{L}}(-\infty)\big)\right\| = 0\,. \tag{23}$$

Proof. For brevity, we set

$$\boldsymbol{\xi}_{\boldsymbol{\tau}}(\theta) = \boldsymbol{\phi}(\theta+\tau;\tau,\boldsymbol{Q}_{\mathcal{L}}(\tau)) \quad \text{and} \quad \boldsymbol{\xi}^*(\theta) = \boldsymbol{\phi}_{-\infty}(\theta;0,\boldsymbol{Q}_{\mathcal{L}}(-\infty))\,.$$

We argue by contradiction, and assume that there exist $\varepsilon > 0$ and two sequences $(\theta_n)_n$, $(\tau_n)_n \subset\,]-\infty,0]$ such that $\tau_n \to -\infty$ satisfying

$$\|\boldsymbol{\xi}_{\boldsymbol{\tau_n}}(\theta_n) - \boldsymbol{\xi}^*(\theta_n)\| > \varepsilon\,. \tag{24}$$

Combining Lemma 15 with (18), we easily find that

$$\|\boldsymbol{\xi}_{\boldsymbol{\tau_n}}(\theta_n) - \boldsymbol{\xi}^*(\theta_n)\| \leq \|\boldsymbol{\xi}_{\boldsymbol{\tau_n}}(\theta_n)\| + \|\boldsymbol{\xi}^*(\theta_n)\| \leq 2c(\mathcal{L})\mathrm{e}^{\alpha\theta_n}, \quad \forall \tau_n \leq -N.$$

If $(\theta_n)_n$ is unbounded, then we easily get a contradiction with (24).

Thus, we can assume that there is $M > 0$ such that $(\theta_n)_n \subset [-M, 0]$. Using continuous dependence on initial data and parameters, we see that for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that if $\|\boldsymbol{Q}_{\mathcal{L}}(\tau) - \boldsymbol{Q}_{\mathcal{L}}(-\infty)\| < \delta$, then $\|\boldsymbol{\xi}_{\boldsymbol{\tau}}(\theta) - \boldsymbol{\xi}^*(\theta)\| < \varepsilon$ for any $\theta \in [-M, 0]$ whenever $\tau < -N$.

Afterwards, from Remark 14, we find $N_1 = N_1(\delta) > N > 0$ large enough so that $\|\boldsymbol{Q}_{\mathcal{L}}(\tau) - \boldsymbol{Q}_{\mathcal{L}}(-\infty)\| < \delta$ for any $\tau < -N_1$, thus eventually we get $\|\boldsymbol{\xi}_{\boldsymbol{\tau}}(\theta) - \boldsymbol{\xi}^*(\theta)\| < \varepsilon$ for any $\theta \in [-M, 0]$ whenever $\tau < -N_1$. However, since we are assuming that $(\theta_n)_n \subset [-M, 0]$ and $\tau_n \to -\infty$, this is in contradiction with (24), thus concluding the proof. □

We denote by

$$\boldsymbol{\phi}(t; d) = (x(t; d), y(t; d))$$

the trajectory of (10) corresponding to the regular solution $u(r; d)$ of (4).

According to [17,18], all the regular solutions correspond to trajectories in the unstable leaf.

Remark 17. Assume that $\mathcal{K} \in C^1$ and $(\mathbf{W_u})$ holds. Then,

$$u(r; d) \text{ is a regular solution} \quad \Longleftrightarrow \quad \boldsymbol{\phi}(\tau; d) \in W^u(\tau) \text{ for every } \tau \in \mathbb{R}.$$

Further, fixed $\tau_0 \in \mathbb{R}$, the function $\boldsymbol{\mathcal{Q}}_{\boldsymbol{\tau_0}} \colon [0, +\infty[\to W^u(\tau_0)$ defined by $\boldsymbol{\mathcal{Q}}_{\boldsymbol{\tau_0}}(d) := \boldsymbol{\phi}(\tau_0; d)$ is a continuous (bijective) parametrization of $W^u(\tau_0)$. In particular, $\boldsymbol{\mathcal{Q}}_{\boldsymbol{\tau_0}}(0) = (0, 0)$. Therefore, for any $\boldsymbol{Q} \in W^u(\tau_0)$ there is a unique $d(\boldsymbol{Q}) \geq 0$ such that $\boldsymbol{\phi}(t; d(\boldsymbol{Q})) \equiv \boldsymbol{\phi}(t; \tau_0, \boldsymbol{Q})$ for any $t \in \mathbb{R}$.

In fact, the dependence of $\boldsymbol{\mathcal{Q}}_{\boldsymbol{\tau_0}}$ with respect to the parameter τ_0 is C^1.

The existence of a bijective parametrization of $W^u(\tau)$ can be obtained extending to the p-Laplacian setting the argument developed in the proof of [10, Lemma 2.10], written in the case $p = 2$. The smoothness of $\boldsymbol{\mathcal{Q}}_{\boldsymbol{\tau_0}}(d) = \boldsymbol{\phi}(\tau_0; d)$ with respect to τ_0 follows from the smoothness of the flow of (10).

According to [35, Lemma 3.7], we can easily prove the monotonicity property of regular solutions.

Remark 18. Any regular solution $u(r; d)$ of (4) is decreasing until its first zero.

Fix $\mathcal{L}$ as in Remark 14, so that $\boldsymbol{Q}_{\mathcal{L}}(\tau)$ is well-defined for any $\tau < -N(\mathcal{L})$. From Remark 17, we can define the function $d_{\mathcal{L}} \colon]-\infty, -N(\mathcal{L})[\to]0, +\infty[$ by setting $d_{\mathcal{L}}(\tau) := d(\boldsymbol{Q}_{\mathcal{L}}(\tau))$ for any $\tau < -N(\mathcal{L})$. In particular, $\boldsymbol{\phi}(t; d_{\mathcal{L}}(\tau)) \equiv \boldsymbol{\phi}(t; \tau, \boldsymbol{Q}_{\mathcal{L}}(\tau))$.

Now, we need the following weak version of the implicit function theorem, cf. [11, Theorem 15.1].

Theorem 19. *Let $A(x, y) \colon \mathbb{R}^2 \to \mathbb{R}$ be a function continuous along with its partial derivative $\frac{\partial A}{\partial x}(x, y)$. Let $A(x_0, y_0) = 0$ and $\frac{\partial A}{\partial x}(x_0, y_0) \neq 0$.*

Then, there exist $\delta > 0$ and exactly one continuous function $x(y) \colon [y_0 - \delta, y_0 + \delta] \to \mathbb{R}$ such that $x(y_0) = x_0$ and $A(x(y), y) = 0$ for any $y \in [y_0 - \delta, y_0 + \delta]$.

Our next aim consists in showing the invertibility and monotonicity of $d_{\mathcal{L}}$.

Lemma 20. *Assume $(\mathbf{W_u})$. Let $\mathcal{L}$, $N = N(\mathcal{L})$ be as in Remark 14. Then, there is $D = D(\mathcal{L}, N)$ such that the function $d_{\mathcal{L}} \colon]-\infty, -N[\to]D, +\infty[$, defined by the property $\boldsymbol{\phi}(t; d_{\mathcal{L}}(\tau)) \equiv \boldsymbol{\phi}(t; \tau, \boldsymbol{Q}_{\mathcal{L}}(\tau))$, is continuous, bijective, monotone decreasing, and its inverse $\tau_{\mathcal{L}} \colon]D, +\infty[\to]-\infty, -N[$ is continuous.*

Furthermore, $\lim\limits_{\tau \to -\infty} d_{\mathcal{L}}(\tau) = +\infty$, and so $\lim\limits_{d \to +\infty} \tau_{\mathcal{L}}(d) = -\infty$.

Proof. As noticed in Remark 14, the function $d_{\mathcal{L}} \colon]-\infty,-N[\, \to]0,+\infty[$ is well-defined for $\tau < -N$, due to the transversality of the first crossing between $W^u(\tau)$ and $\mathcal{L}$. Now, we show that $d_{\mathcal{L}}$ admits a continuous inverse by constructing it via Theorem 19.

Denote by $\bar{\mathcal{L}}$ the straight line containing the segment $\mathcal{L}$, and let $\mathcal{D}(\boldsymbol{Q})$ be the smooth function which evaluates the directed distance from $\bar{\mathcal{L}}$ to $\boldsymbol{Q}$, so that $\mathcal{D}(\boldsymbol{Q}) = 0$ if and only if $\boldsymbol{Q} \in \bar{\mathcal{L}}$. Recalling the parametrization $\boldsymbol{\mathcal{Q}_\tau}$ of $W^u(\tau)$ introduced in Remark 17, we define $A(\tau, d) := \mathcal{D}(\boldsymbol{\mathcal{Q}_\tau}(d))$. Let us consider a couple (τ_0, d_0) with $\tau_0 < -N$ and $d_0 = d_{\mathcal{L}}(\tau_0)$. In particular, $\boldsymbol{\mathcal{Q}_{\tau_0}}(d_0) = \boldsymbol{Q}_{\mathcal{L}}(\tau_0)$ and so $A(\tau_0, d_0) = \mathcal{D}(\boldsymbol{Q}_{\mathcal{L}}(\tau_0)) = 0$. From the smoothness property of $W^u(\tau)$ given in Remark 13, we can compute

$$\frac{\partial A}{\partial \tau}(\tau_0, d_0) = \left\langle \nabla \mathcal{D}(\boldsymbol{\mathcal{Q}_{\tau_0}}(d_0))\,,\, \frac{\partial}{\partial \tau} \boldsymbol{\mathcal{Q}_{\tau_0}}(d_0) \right\rangle =$$

$$= \left\langle \nabla \mathcal{D}(\boldsymbol{\mathcal{Q}_{\tau_0}}(d_0))\,, \dot{\boldsymbol{\phi}}(\tau_0; \tau_0, \boldsymbol{\mathcal{Q}_{\tau_0}}(d_0)) \right\rangle \neq 0,$$

since $\nabla \mathcal{D}(\boldsymbol{Q})$ is orthogonal to $\bar{\mathcal{L}}$ and $\dot{\boldsymbol{\phi}}(\tau_0; \tau_0, \boldsymbol{\mathcal{Q}_{\tau_0}}(d_0))$ is transversal to $\mathcal{L}$.

From the previous formula, we also find that $\frac{\partial A}{\partial \tau}$ is continuous. Hence, we can apply Theorem 19 to find a local continuous function $\tau_{\mathcal{L}}(d)$ such that $A(\tau_{\mathcal{L}}(d), d) \equiv 0$; so by construction $\tau_{\mathcal{L}}(d)$ is the local inverse of $d_{\mathcal{L}}$. Since the previous argument can be performed for every couple (τ_0, d_0) satisfying $d_0 = d_{\mathcal{L}}(\tau_0)$ with $\tau_0 \in]-\infty, -N[$, we can conclude that the image of $d_{\mathcal{L}}$ is an interval $\mathcal{U}$ and $\tau_{\mathcal{L}}(d)$ is a global inverse.

We now show that $\lim_{\tau \to -\infty} d_{\mathcal{L}}(\tau) = +\infty$.

Let $c > 0$ be such that $\mathcal{L} \subset \{(x, y) \mid x > c\}$. Then, for every $\tau < -N$ the point $\boldsymbol{Q}_{\mathcal{L}}(\tau) = (Q_x(\tau), Q_y(\tau))$ is such that $Q_x(\tau) > c$. Let us consider the solution $u(r; d_{\mathcal{L}}(\tau))$ corresponding to the trajectory $\boldsymbol{\phi}(t; d_{\mathcal{L}}(\tau)) \equiv \boldsymbol{\phi}(t; \tau, \boldsymbol{Q}_{\mathcal{L}}(\tau))$. According to Lemma 16, $u(r; d_{\mathcal{L}}(\tau))$ is positive for $r \in [0, \mathrm{e}^\tau]$, and from Remark 18 we deduce that

$$d_{\mathcal{L}}(\tau) = u(0; d_{\mathcal{L}}(\tau)) \geq u(\mathrm{e}^\tau; d_{\mathcal{L}}(\tau)) = Q_x(\tau)\, \mathrm{e}^{-\alpha\tau} > c\, \mathrm{e}^{-\alpha\tau} \to +\infty$$

as $\tau \to -\infty$. Finally, we find that $d_{\mathcal{L}}$ is decreasing, and there exists $D = D(\mathcal{L}, N)$ such that $d_{\mathcal{L}}(]-\infty, -N[) =]D, +\infty[$. $\square$

3. Basic properties of the function $R(d)$

Using the Pohozaev identity, see, e.g., [16,17,29,36], it is possible to obtain the following classical result.

Proposition 21. *Assume* $(\mathbf{H}_\uparrow)$, *then all the regular solutions* $u(r; d)$ *of* (4) *have a zero at* $r = R(d)$. *Hence, all the corresponding trajectories* $\boldsymbol{\phi}(t; d)$ *of* (10) *are such that* $y(t; d) < 0 < x(t; d)$ *when* $t < T(d) := \ln R(d)$ *and* $x(T(d); d) = 0 > y(T(d); d)$.

For the proof, we refer to [12,31] for the Laplacian operator and to [22,29] for p-Laplacian extensions. Then, following [17, Theorem 4.2], we can prove the continuity of the function $R(d)$.

Proposition 22. *Assume* $(\mathbf{W}_{\boldsymbol{u}})$, *then the set* J *introduced in* (7) *is open and the function* $R \colon J \to \mathbb{R}$ *is continuous.*

We need two further asymptotic results, which can already be found in literature in slightly different contexts.

Proposition 23. *Assume* $(\mathbf{W}_{\boldsymbol{u}})$ *and* $\inf J = 0$, *then* $\lim_{d \to 0^+} R(d) = +\infty$.

We refer to [28, Proposition 2.4] for the proof. With some effort, we can generalize the previous proposition a bit.

Proposition 24. *Assume* $(\mathbf{W}_{\boldsymbol{u}})$ *and the existence of* $\tilde{D} > 0$*, with* $\tilde{D} \notin J$ *which is an accumulation point for* J*. Then,*

$$\lim_{\substack{d \to \tilde{D} \\ d \in J}} R(d) = +\infty.$$

Proof. According to Remark 18, $u(r; \tilde{D})$ is positive and decreasing for any $r \geq 0$, since $\tilde{D} \notin J$. Let $\delta > 0$ be as in Theorem 11. Choose $\tau_0 \ll 0$ such that

$$\tilde{W}^u(\tau_0) := \{\boldsymbol{\mathcal{Q}}_{\boldsymbol{\tau_0}}(d) \mid d \in [0, \tilde{D} + 1]\} \subset W^u_{\text{loc}}(\tau_0) \subset B(\mathbf{0}, \delta),$$

where $\boldsymbol{\mathcal{Q}}_{\boldsymbol{\tau_0}}(d)$ is as in Remark 17. By construction, $R(d) > \mathrm{e}^{\tau_0}$ for any $0 < d \leq \tilde{D} + 1$.

Let us assume that the thesis is false and look for a contradiction. We allow the existence of a sequence $(d_n)_n \subset J$, with $d_n \to \tilde{D}$, and of $M \in \mathbb{R}$ such that $R(d_n) \leq \mathrm{e}^M$. Let us denote by $\varepsilon = \frac{1}{2} \min\{x(t; \tilde{D}) \mid t \in [\tau_0, M]\}$.

From the continuity of the parametrization $\boldsymbol{\mathcal{Q}}_{\boldsymbol{\tau_0}}(\cdot)$, for any $\sigma > 0$ we can find $\bar{n} = \bar{n}(\sigma) > 0$ such that $\|\boldsymbol{\mathcal{Q}}_{\boldsymbol{\tau_0}}(d_n) - \boldsymbol{\mathcal{Q}}_{\boldsymbol{\tau_0}}(\tilde{D})\| < \sigma$ when $n > \bar{n}$. Then, using the continuity of the flow of (10), we can choose $\sigma = \sigma(\varepsilon) > 0$ small enough (and $\bar{n} > 0$ large enough) such that

$$\|\boldsymbol{\phi}(t; d_n) - \boldsymbol{\phi}(t; \tilde{D})\| = \|\boldsymbol{\phi}(t; \tau_0, \boldsymbol{\mathcal{Q}}_{\boldsymbol{\tau_0}}(d_n)) - \boldsymbol{\phi}(t; \tau_0, \boldsymbol{\mathcal{Q}}_{\boldsymbol{\tau_0}}(\tilde{D}))\| < \varepsilon$$

for any $t \in [\tau_0, M]$, whenever $n > \bar{n}$. Hence, we get

$$x(t; d_n) \geq x(t; \tilde{D}) - |x(t; d_n) - x(t; \tilde{D})| > 2\varepsilon - \varepsilon = \varepsilon$$

for any $t \in [\tau_0, M]$ and $n > \bar{n}$. Hence, $x(t; d_n) > 0$ for any $t \leq M$ and, consequently, $R(d_n) > \mathrm{e}^M$ for any $n > \bar{n}$: a contradiction. □

Let us observe also that $u(r; \tilde{D})$ is a ground state and converges to 0 as $r \to +\infty$.

4. The asymptotic estimate of $R(d)$ for d large

We proceed to study the behavior of $R(d)$ for d large. We emphasize that the asymptotic estimates of the first zero for large initial data are mainly based on the crucial assumption $(\mathbf{H}_{\boldsymbol{\ell}})$.

Our first step is to locate $\boldsymbol{Q}_{\mathcal{L}}(\tau)$ through the function $\mathcal{H}$, see Proposition 26 and Remark 27. Then, we use a Grönwall's argument to get a lower and an upper bound of the time $T(\tau, \mathcal{L})$ taken by $\boldsymbol{\phi}(t; \tau, \boldsymbol{Q}_{\mathcal{L}}(\tau))$ to cross the negative y-semiaxis, see Lemmas 31 and 33. Finally, we prove the asymptotic estimates of $R(d)$ in Proposition 35 (for the case $\ell < \ell_p^*$) and in Propositions 43 and 46 (for the case $\ell \geq \ell_p^*$).

We emphasize that assumption $(\mathbf{H}_{\boldsymbol{\ell}})$ implies that the function $\mathcal{K}$ is strictly increasing in a neighborhood of $r = 0$, so we can use the following *truncation argument*.

Remark 25. From $(\mathbf{H}_{\boldsymbol{\ell}})$ we see that there exists $\hat{T}_0 < 0$ such that $K(\hat{T}_0) < A + 1$ and

$$0 < \frac{1}{2} B\ell \mathrm{e}^{\ell t} < \dot{K}(t) < 2B\ell \mathrm{e}^{\ell t} \text{ for every } t \leq \hat{T}_0. \tag{25}$$

So, we can find a function $\hat{K} \colon \mathbb{R} \to \mathbb{R}$ of class C^1 satisfying $\hat{K} = K$ in the interval $]-\infty, \hat{T}_0]$, $\frac{d\hat{K}}{dt}(t) > 0$ for every $t \in \mathbb{R}$, and

$$A + 1 = \hat{K}(+\infty) := \lim_{t \to +\infty} \hat{K}(t) \in \mathbb{R}\,.$$

In particular, $\hat{K}$ satisfies $(\mathbf{H}_{\boldsymbol{\ell}})$, $(\mathbf{H}_{\uparrow})$ and $(\mathbf{W}_{\boldsymbol{s}})$. Such a truncation argument permits us to assume implicitly the validity of the previous hypotheses, when we look for properties of system (10) in a neighborhood of $t = -\infty$ in the presence of the only hypothesis $(\mathbf{H}_{\boldsymbol{\ell}})$.

Now, we provide some estimates of the energy $\mathcal{H}(\boldsymbol{Q}_{\mathcal{L}}(\tau);\tau)$, where $\boldsymbol{Q}_{\mathcal{L}}(\tau)$ is as in Remark 14.

Proposition 26. *Assume* $(\mathbf{H}_{\boldsymbol{\ell}})$. *Let* $\mathcal{L}$ *be a small enough segment transversal to* $\boldsymbol{\Gamma}_{-\infty}$, *and consider the point* $\boldsymbol{Q}_{\mathcal{L}}(\tau)$ *with* $\tau < -N(\mathcal{L})$. *Then, there is a constant* $\tilde{c}(\mathcal{L}) > 0$ *such that*

$$\mathcal{H}(\boldsymbol{Q}_{\mathcal{L}}(\tau);\tau) = \tilde{c}(\mathcal{L})\,\mathrm{e}^{\ell\tau} + o(\mathrm{e}^{\ell\tau}) \qquad \text{as } \tau \to -\infty.$$

Proof. We write for brevity $\boldsymbol{\xi}_{\boldsymbol{\tau}}(t) = (x_\tau(t), y_\tau(t)) = \boldsymbol{\phi}(t+\tau;\tau,\boldsymbol{Q}_{\mathcal{L}}(\tau))$ and $\boldsymbol{\xi}^*(t) = (x^*(t), y^*(t)) = \boldsymbol{\phi}_{-\infty}(t;0,\boldsymbol{Q}_{\mathcal{L}}(-\infty))$. Notice that, according to Lemma 16, x_τ is positive in $]-\infty,0]$.

Since $\boldsymbol{Q}_{\mathcal{L}}(\tau) \in W^u(\tau)$, from Lemma 9 combined with Remark 17, we know that $\lim_{t\to-\infty} \mathcal{H}(\boldsymbol{\xi}_{\boldsymbol{\tau}}(t); t+\tau) = 0$. Hence, in the spirit of [18, p. 357], by (12) and Remark 25, we find

$$\begin{aligned}\mathcal{H}(\boldsymbol{Q}_{\mathcal{L}}(\tau);\tau) &= \int_{-\infty}^{0} \dot{K}(t+\tau)\frac{x_\tau(t)^q}{q}\,dt = \\ &= \int_{-\infty}^{0} \left[B\ell \mathrm{e}^{\ell(t+\tau)} + h'(\mathrm{e}^{t+\tau})\mathrm{e}^{t+\tau}\right]\frac{x_\tau(t)^q}{q}\,dt = \mathrm{e}^{\ell\tau}\,(I_1 + I_2)\,,\end{aligned} \tag{26}$$

where

$$I_1 := \frac{B\ell}{q}\int_{-\infty}^{0} \mathrm{e}^{\ell t} x_\tau(t)^q\,dt \quad \text{and} \quad I_2 := \frac{1}{q}\int_{-\infty}^{0} \frac{h'(\mathrm{e}^{t+\tau})\mathrm{e}^{t+\tau}}{\mathrm{e}^{\ell(t+\tau)}}\,\mathrm{e}^{\ell t} x_\tau(t)^q\,dt.$$

We can rewrite I_1 in the following equivalent form:

$$I_1 = \frac{B\ell}{q}\int_{-\infty}^{0} \mathrm{e}^{\ell t} x^*(t)^q\,dt + \frac{B\ell}{q}\int_{-\infty}^{0} \mathrm{e}^{\ell t}[x_\tau(t)^q - x^*(t)^q]\,dt.$$

We recall (23) and the fact that both $x_\tau(t)$, $x^*(t)$ converge to 0 exponentially as $t \to -\infty$, as observed in Lemma 15 and in (17). We apply the Lebesgue theorem to deduce the existence of $\omega_1(t)$ such that

$$I_1 = \tilde{c}(\mathcal{L}) + \omega_1(\tau) \quad \text{with} \quad \lim_{\tau\to-\infty} \omega_1(\tau) = 0\,, \tag{27}$$

where $\tilde{c}(\mathcal{L}) := \frac{B\ell}{q}\int_{-\infty}^{0} \mathrm{e}^{\ell t} x^*(t)^q\,dt > 0$ is finite due to (17).

Analogously, from $(\mathbf{H}_{\boldsymbol{\ell}})$, we find $\omega_2(t)$ such that

$$I_2 = \omega_2(\tau) \quad \text{with} \quad \lim_{\tau\to-\infty} \omega_2(\tau) = 0\,. \tag{28}$$

The statement follows by substituting (27) and (28) into (26). $\square$

For a fixed $a > 0$, let $\mathcal{L}(a)$ denote the segment of $y = -a$ lying between the negative y-semiaxis and the isocline $\dot{x} = 0$ of system (10), i.e.,

$$\mathcal{L}(a) = \{(s, -a) \mid 0 \le s \le \tilde{X}(a)\}, \quad \text{where } \tilde{X}(a) := \tfrac{1}{\alpha}\, a^{\frac{1}{p-1}}\,. \tag{29}$$

Recalling the definition of equilibrium point $\boldsymbol{E}(-\infty)$ given in (15), we observe that the homoclinic orbit $\boldsymbol{\Gamma}_{-\infty}$ intersects $\mathcal{L}(a)$ transversely, for any $0 < a < |E_y(-\infty)|$. From Lemma 16 and Proposition 26, we get the following asymptotic result.

Remark 27. Assume $(\mathbf{H}_{\boldsymbol{\ell}})$. Follow $W^u(\tau)$ from the origin towards $x > 0$. Then, for any $0 < a < |E_y(-\infty)|$, we can find $N(a) > 0$ such that $W^u(\tau)$ intersects $\mathcal{L}(a)$ transversely in a point, say $\boldsymbol{Q}(\tau, a) := \boldsymbol{Q}_{\mathcal{L}(a)}(\tau) = (Q_x(\tau, a), -a)$, whenever $\tau < -N(a)$. From (29), we see that

$$\dot{y}(\tau; \tau, \boldsymbol{Q}(\tau, a)) > 0 > \dot{x}(\tau; \tau, \boldsymbol{Q}(\tau, a)) \quad \text{for any } \tau \le -N(a).$$

Moreover, there is a constant $c(a) > 0$ such that

$$\mathcal{H}(\boldsymbol{Q}(\tau, a); \tau) = c(a)\, \mathrm{e}^{\ell\tau} + o(\mathrm{e}^{\ell\tau}) \qquad \text{as } \tau \to -\infty\,.$$

In the next part of this section we study the asymptotic behavior of the solutions, under the more restrictive assumptions $(\mathbf{H}_{\uparrow})$, $(\mathbf{H}_{\boldsymbol{\ell}})$ and $(\mathbf{W}_{\boldsymbol{s}})$. Finally, $(\mathbf{H}_{\uparrow})$ and $(\mathbf{W}_{\boldsymbol{s}})$ will be removed using the truncation argument suggested by Remark 25.

Lemma 28. *Assume* $(\mathbf{H}_{\uparrow})$. *Let $u(r; d)$ be a regular solution of (4) and let $\boldsymbol{\phi}(t; d)$ be the corresponding trajectory of (10). Then, there are $T_x(d) < T(d)$ such that $x(t; d) > 0$ when $t < T(d)$, and it becomes null at $t = T(d)$. Furthermore, $\dot{x}(t; d) > 0$ when $t < T_x(d)$, and $\dot{x}(t; d) < 0$ when $T_x(d) < t \le T(d)$.*

Proof. The existence of $T(d)$ follows from Proposition 21. From Lemma 9, we know that $\mathcal{H}(\boldsymbol{\phi}(t; d); t) \ge 0$ for any $t \le T(d)$. By a simple calculation, $\mathcal{H}(\boldsymbol{Q}; t) < 0$ for every $\boldsymbol{Q} = (Q_x, Q_y)$ in the isocline $\dot{x} = 0$ with $0 < Q_x \le E_x(t)$, so we easily deduce the existence of $T_x(d) < T(d)$ such that $\dot{x}(T_x(d); d) = 0$ and $x(T_x(d); d) > E_x(T_x(d))$. □

Assume $(\mathbf{H}_{\boldsymbol{\ell}})$ and $(\mathbf{H}_{\uparrow})$. Fixed $0 < a < |E_y(-\infty)|$, consider $\mathcal{L}(a)$ and $N(a) > 0$ as in Remark 27 so that $\boldsymbol{Q}(\tau, a) \in W^u(\tau) \cap \mathcal{L}(a)$ is well-defined for any $\tau < -N(a)$.

From Proposition 21, there exists $T(\tau, a) > \tau$ such that $\boldsymbol{\phi}(t; \tau, \boldsymbol{Q}(\tau, a))$ crosses the negative y-semiaxis at $t = T(\tau, a)$ in a point, say (see Fig. 4)

$$\boldsymbol{R}(\tau, a) = (0, R_y(\tau, a)) = \boldsymbol{\phi}(T(\tau, a); \tau, \boldsymbol{Q}(\tau, a))\,. \tag{30}$$

According to Remark 25, it is not restrictive to assume that $\dot{K}(t) > 0$ for any $t \le \tau \le -N(a)$ (provided that we choose $-N(a) < \hat{T}_0$ in Remark 27). So, from Lemma 9 we deduce that $\mathcal{H}(\boldsymbol{Q}(\tau, a); \tau) > 0$.

Our next purpose is to provide suitable estimates from above and from below of $T(\tau, a)$. From Lemma 28, $\boldsymbol{\phi}(t; \tau, \boldsymbol{Q}(\tau, a))$ is a graph on the x-axis when $t \in [\tau, T(\tau, a)]$. In particular, we can find a function $\psi\colon [0, Q_x(\tau, a)] \to\,]-\infty, 0[$ such that the image of $\boldsymbol{\phi}(\cdot; \tau, \boldsymbol{Q}(\tau, a))\colon [\tau, T(\tau, a)] \to \mathbb{R}^2$ can be parametrized as $(x, \psi(x))$ for $x \in [0, Q_x(\tau, a)]$.

Let us now consider the trajectory $\boldsymbol{\phi}_{\boldsymbol{\tau}}(t; \tau, \boldsymbol{Q}(\tau, a))$ of the system (13) frozen at $t = \tau$. Notice that its graph is contained in the level set

$$\{(x, y) \mid \mathcal{H}(x, y; \tau) = \mathcal{H}(\boldsymbol{Q}(\tau, a); \tau) > 0\}, \tag{31}$$

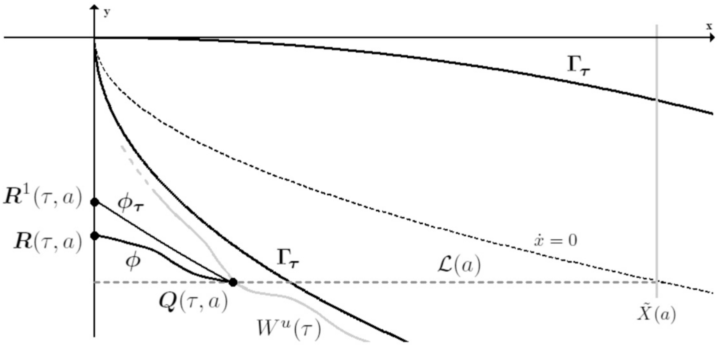

Fig. 4. The position of the points $\boldsymbol{Q}(\tau,a)$, $\boldsymbol{R}(\tau,a)$, and $\boldsymbol{R}^1(\tau,a)$.

which lies in the exterior of the homoclinic orbit $\boldsymbol{\Gamma_\tau}$ defined in (16).

Hence, there exists $T^1(\tau,a) > \tau$ such that $\boldsymbol{\phi_\tau}(\cdot;\tau,\boldsymbol{Q}(\tau,a))$ lies in the 4^{th} quadrant when $t \in [\tau, T^1(\tau,a)[$, and it crosses transversely the negative y-semiaxis at $t = T^1(\tau,a)$ in a point, say (see Fig. 4)

$$\boldsymbol{R}^1(\tau,a) = (0, R^1_y(\tau,a)) = \boldsymbol{\phi_\tau}(T^1(\tau,a);\tau,\boldsymbol{Q}(\tau,a))\,. \tag{32}$$

Remark 29. Assume $(\mathbf{W_u})$, then the functions $T(\tau,a)$ and $T^1(\tau,a)$ are continuous in their domain, since the flow on the negative y-semiaxis is transversal.

From the study of the trajectories of the autonomous system (13), we can deduce that

$$\dot{x}_\tau(t;\tau,\boldsymbol{Q}(\tau,a)) < 0 < \dot{y}_\tau(t;\tau,\boldsymbol{Q}(\tau,a)) \quad \text{for any } t \in [\tau, T^1(\tau,a)]. \tag{33}$$

Consequently, we can find a strictly decreasing function $\psi_\tau\colon [0, Q_x(\tau,a)] \to [-a, R^1_y(\tau,a)]$ such that the image of $\boldsymbol{\phi_\tau}(\cdot;\tau,\boldsymbol{Q}(\tau,a))\colon [\tau, T^1(\tau,a)] \to \mathbb{R}^2$ can be parametrized as $(x, \psi_\tau(x))$ for $x \in [0, Q_x(\tau,a)]$.

Lemma 30. *Assume* $(\mathbf{H_\ell})$ *and that* $\dot{K}(t) > 0$ *for any* $t \in \mathbb{R}$. *Fix* $0 < a < |E_y(-\infty)|$, *and let* $N(a) > 0$ *be as in Remark 27. Then,*

$$T(\tau,a) < T^1(\tau,a) \qquad \textit{for any } \tau < -N(a).$$

Proof. For brevity, let us write $\boldsymbol{\phi}(t) = (x(t),y(t)) = \boldsymbol{\phi}(t;\tau,\boldsymbol{Q}(\tau,a))$ and $\boldsymbol{\phi_\tau}(t) = (x_\tau(t),y_\tau(t)) = \boldsymbol{\phi_\tau}(t;\tau,\boldsymbol{Q}(\tau,a))$.

From (14) combined with Lemma 28, we see that $\mathcal{H}(\boldsymbol{\phi}(\cdot);\tau)$ is strictly increasing in the interval $[\tau, T(\tau,a)]$. In particular, recalling (31), we have

$$\mathcal{H}(\boldsymbol{\phi}(t);\tau) > \mathcal{H}(\boldsymbol{Q}(\tau,a);\tau) = \mathcal{H}(\boldsymbol{\phi_\tau}(s);\tau)$$

for every $t \in\,]\tau, T(\tau,a)]$ and $s \in [\tau, T^1(\tau,a)]$. Thus, we get

$$\psi(x) < \psi_\tau(x) \qquad \text{for any } 0 \le x < Q_x(\tau,a)\,, \tag{34}$$

and, as an immediate consequence,

$$R_y(\tau,a) < R^1_y(\tau,a)\,.$$

The statement of the Lemma holds if we prove that $x(t) < x_\tau(t)$ for any $t \in\,]\tau, T(\tau,a)]$. Observe first that $\boldsymbol{\phi}(\tau) = \boldsymbol{\phi_\tau}(\tau)$. Then, from (10) and (13) we find that $x(\tau)$ equals $x_\tau(\tau)$ along with its first and second derivatives, however

$$\frac{d^3}{dt^3}[x(\tau)-x_\tau(\tau)] = \frac{\ddot{y}(\tau)-\ddot{y}_\tau(\tau)}{p-1}|y(\tau)|^{\frac{2-p}{p-1}} = -\dot{K}(\tau)\frac{x(\tau)^{q-1}}{p-1}|y(\tau)|^{\frac{2-p}{p-1}} < 0\,.$$

Consequently, $x(t) < x_\tau(t)$ when t is in a suitable right neighborhood of τ. Let

$$\hat{T} := \sup\{t > \tau \mid x(s) < x_\tau(s)\,,\ \forall s \in\,]\tau, t]\,\}\,.$$

If $\hat{T} > T(\tau,a)$, the lemma is proved. So, we assume by contradiction that $\tau < \hat{T} \le T(\tau,a)$. Then, we have $\bar{x} = x(\hat{T}) = x_\tau(\hat{T})$, $\boldsymbol{\phi}(\hat{T}) = (\bar{x}, \psi(\bar{x}))$ and $\boldsymbol{\phi_\tau}(\hat{T}) = (\bar{x}, \psi_\tau(\bar{x}))$. Hence, recalling (34),

$$\begin{aligned}\dot{x}(\hat{T}) = \alpha x(\hat{T}) - |y(\hat{T})|^{\frac{1}{p-1}} = \alpha\bar{x} - |\psi(\bar{x})|^{\frac{1}{p-1}} <\\ < \alpha\bar{x} - |\psi_\tau(\bar{x})|^{\frac{1}{p-1}} = \alpha x_\tau(\hat{T}) - |y_\tau(\hat{T})|^{\frac{1}{p-1}} = \dot{x}_\tau(\hat{T}).\end{aligned}$$

So $x > x_\tau$ in a left neighborhood of $\hat{T}$, giving a contradiction. □

In the following lemma we assume K bounded. This assumption will be removed via Remark 25.

Lemma 31. *Assume* $(\mathbf{H}_{\boldsymbol{\ell}})$, $(\mathbf{H}_{\uparrow})$ *and* $(\mathbf{W}_{\boldsymbol{s}})$. *For any* $\varepsilon > 0$ *we define*

$$a = a(\varepsilon) := \left(\frac{\alpha^q \varepsilon}{(1+\varepsilon)\overline{K}}\right)^{\frac{p-1}{q-p}} < |E_y(+\infty)| = \left(\frac{\alpha^q}{K(+\infty)}\right)^{\frac{p-1}{q-p}}. \tag{35}$$

Then, there is $\mathcal{T}(\varepsilon)$ *such that, for any* $\tau < \mathcal{T}(\varepsilon)$,

$$T(\tau,a) > \tau + \frac{1}{\alpha}\ln\left(\frac{a}{|R_y(\tau,a)|}\right), \tag{36}$$

$$T^1(\tau,a) < \tau + \frac{(1+\varepsilon)}{\alpha}\ln\left(\frac{a}{|R^1_y(\tau,a)|}\right). \tag{37}$$

Proof. Let $\mathcal{T}(\varepsilon) = -N(a(\varepsilon))$ be the value provided by Remark 27, and fix $\tau < \mathcal{T}(\varepsilon)$. We consider again the trajectories $\boldsymbol{\phi}(t) = (x(t), y(t)) = \boldsymbol{\phi}(t; \tau, \boldsymbol{Q}(\tau,a))$ and $\boldsymbol{\phi_\tau}(t) = (x_\tau(t), y_\tau(t)) = \boldsymbol{\phi_\tau}(t; \tau, \boldsymbol{Q}(\tau,a))$.

Observe that $\dot{y} > 0$ along the horizontal segment $\mathcal{L}(a)$, as well as in the interior of the bounded set enclosed by $\mathcal{L}(a)$, $\boldsymbol{\Gamma_\tau}$ and the negative y-semiaxis. Using this fact, according to Lemma 28 and (33), we see that

$$\begin{aligned}&\dot{x}(t) < 0 < \dot{y}(t)\,, &&\dot{x}_\tau(s) < 0 < \dot{y}_\tau(s)\,,\\ &-a \le y(t) < 0\,, &&-a \le y_\tau(s) < 0\,,\end{aligned}$$

for any $t \in [\tau, T(\tau,a)]$ and any $s \in [\tau, T^1(\tau,a)]$. Moreover, for both the trajectories, we get

$$\dot{x} < 0 \quad\Rightarrow\quad \alpha x < |y|^{\frac{1}{p-1}} \quad\Rightarrow\quad -x^{q-1} > -\frac{|y|^{\frac{q-1}{p-1}}}{\alpha^{q-1}}\,. \tag{38}$$

Then, since $\boldsymbol{\phi}(t) = (x(t), y(t))$ solves (10), we find

$$-\alpha y(t) - \frac{K(t)}{\alpha^{q-1}}|y(t)|^{\frac{q-1}{p-1}} < \dot{y}(t) = -\alpha y(t) - K(t)x(t)^{q-1} < -\alpha y(t) \tag{39}$$

for any $t \in [\tau, T(\tau,a)[$. Analogously, since $\boldsymbol{\phi_\tau}(t) = (x_\tau(t), y_\tau(t))$ solves (13),

$$-\alpha y_\tau(t) - \frac{K(\tau)}{\alpha^{q-1}}|y_\tau(t)|^{\frac{q-1}{p-1}} < \dot{y}_\tau(t) = -\alpha y_\tau(t) - K(\tau)x_\tau(t)^{q-1} < -\alpha y_\tau(t) \tag{40}$$

for any $t \in [\tau, T^1(\tau,a)[$.

Note that (35) guarantees that

$$\frac{\overline{K}}{\alpha^q}|y|^{\frac{q-p}{p-1}} \le \frac{\varepsilon}{1+\varepsilon} \qquad \text{for every } y \in [-a, 0].$$

Thus, we get

$$-\alpha y - \frac{K(t)}{\alpha^{q-1}}|y|^{\frac{q-1}{p-1}} \ge -\alpha y\left(1 - \frac{\overline{K}}{\alpha^q}|y|^{\frac{q-p}{p-1}}\right) \ge -\frac{\alpha}{1+\varepsilon}y \tag{41}$$

for every $t \in \mathbb{R}$ and $y \in [-a, 0]$. In particular, from (39), (40) and (41), we find

$$-\frac{\alpha}{1+\varepsilon}y(t) < \dot{y}(t) < -\alpha y(t)\,, \qquad -\frac{\alpha}{1+\varepsilon}y_\tau(s) < \dot{y}_\tau(s) < -\alpha y_\tau(s) \tag{42}$$

for every $t \in [\tau, T(\tau,a)[$ and $s \in [\tau, T^1(\tau,a)[$.

Consequently, $\frac{\dot{y}(t)}{-\alpha y(t)} < 1$ for any $t \in [\tau, T(\tau,a)[$; hence, recalling the definition of $\boldsymbol{R}$ in (30), we obtain

$$\begin{aligned} T(\tau,a) - \tau &= \int_\tau^{T(\tau,a)} dt > \int_\tau^{T(\tau,a)} \frac{\dot{y}(t)}{-\alpha y(t)}dt = \\ &= -\frac{1}{\alpha}\ln\left(\frac{y(T(\tau,a))}{y(\tau)}\right) = \frac{1}{\alpha}\ln\left(\frac{a}{|R_y(\tau,a)|}\right). \end{aligned}$$

Analogously, from (42), we also find $\frac{(1+\varepsilon)\dot{y}_\tau(s)}{-\alpha y_\tau(s)} > 1$ for any $s \in [\tau; T^1(\tau,a)[$. Hence, recalling the definition of $\boldsymbol{R}^1$ in (32), we get

$$\begin{aligned} T^1(\tau,a) - \tau &= \int_\tau^{T^1(\tau,a)} ds < \int_\tau^{T^1(\tau,a)} \frac{(1+\varepsilon)\dot{y}_\tau(s)}{-\alpha y_\tau(s)}ds = \\ &= -\frac{1+\varepsilon}{\alpha}\ln\left(\frac{y_\tau(T^1(\tau,a))}{y_\tau(\tau)}\right) = \frac{1+\varepsilon}{\alpha}\ln\left(\frac{a}{|R^1_y(\tau,a)|}\right). \quad \square \end{aligned}$$

By a simple integration of (42), we obtain a crucial inequality.

Remark 32. Assume $(\mathbf{H}_{\boldsymbol{\ell}})$, $(\mathbf{H}_{\uparrow})$, and $(\mathbf{W}_{\boldsymbol{s}})$. Then, for every $\varepsilon > 0$, there is $\mathcal{T}(\varepsilon)$ such that, for any $\tau < \mathcal{T}(\varepsilon)$, the trajectory $\boldsymbol{\phi}(t;\tau,\boldsymbol{Q}(\tau,a)) = (x(t), y(t))$ satisfies

$$a\mathrm{e}^{-\alpha(t-\tau)} < |y(t)| < a\,\mathrm{e}^{-\frac{\alpha}{1+\varepsilon}(t-\tau)} \qquad \text{for every } t \in]\tau, T(\tau,a)[\,. \tag{43}$$

Lemma 33. *Assume* $(\mathbf{H}_{\boldsymbol{\ell}})$*,* $(\mathbf{H}_{\uparrow})$*, and* $(\mathbf{W}_{\boldsymbol{s}})$*. Let* $\varepsilon > 0$ *and* $a = a(\varepsilon) > 0$ *be as in Lemma 31. Then, there are* $\mathcal{T}(\varepsilon) < 0$ *and a constant* $C = C(a(\varepsilon), \varepsilon)$ *such that*

$$T^1(\tau, a(\varepsilon)) < C(a(\varepsilon),\varepsilon) + |\tau|\left((1+\varepsilon)\frac{\ell}{\ell_p^*} - 1\right) \qquad \textit{for any } \tau < \mathcal{T}(\varepsilon).$$

Proof. From Remark 27, (11) and (31), we deduce the following estimates on the y-coordinate of the point $\boldsymbol{R}^1(\tau, a)$ introduced in (32):

$$\frac{p-1}{p}|R^1_y(\tau,a)|^{\frac{p}{p-1}} = \mathcal{H}(\boldsymbol{R}^1(\tau,a);\tau) = \mathcal{H}(\boldsymbol{Q}(\tau,a);\tau) = c(a)\mathrm{e}^{\ell\tau} + o\left(\mathrm{e}^{\ell\tau}\right)$$

for any $\tau < \mathcal{T}(\varepsilon)$, possibly choosing a larger $|\mathcal{T}(\varepsilon)|$. So, there is $c^1(a) = \left(c(a)\frac{p}{p-1}\right)^{\frac{p-1}{p}} > 0$, which is independent of τ, such that

$$|R^1_y(\tau,a)| = c^1(a)\mathrm{e}^{\frac{p-1}{p}\ell\tau} + o\left(\mathrm{e}^{\frac{p-1}{p}\ell\tau}\right) \qquad \text{for any } \tau < \mathcal{T}(\varepsilon).$$

Then, by (37), setting $C(a,\varepsilon) = \frac{1+\varepsilon}{\alpha}\ln\left(\frac{2a}{c^1(a)}\right)$, we find

$$\begin{aligned} T^1(\tau,a) &< \tau + \frac{1+\varepsilon}{\alpha}\ln\left(\frac{2a}{c^1(a)\mathrm{e}^{\frac{p-1}{p}\ell\tau}}\right) = \\ &= \tau + C(a,\varepsilon) + |\tau|\ \ell\ \frac{1+\varepsilon}{\alpha}\ \frac{p-1}{p} = \\ &= C(a,\varepsilon) + |\tau|\left(\ell\ \frac{1+\varepsilon}{\alpha}\ \frac{p-1}{p} - 1\right) \end{aligned}$$

for any $\tau < \mathcal{T}(\varepsilon)$. Since

$$\ell^*_p = \frac{n-p}{p-1} = \frac{\alpha p}{p-1}\,, \tag{44}$$

we conclude. □

4.1. The case $\boldsymbol{\ell}<\boldsymbol{\ell^*_p}$

Proposition 34. *Assume* $(\mathbf{H}_{\boldsymbol{\ell}})$, $(\mathbf{W}_{\boldsymbol{s}})$ *and that* $\dot{K}(t) > 0$ *for any* $t \in \mathbb{R}$. *If* $\ell < \ell^*_p$, *then there exists* $a > 0$ *such that*

$$\lim_{\tau\to-\infty} T(\tau,a) = -\infty\,.$$

Proof. We simply need to choose $\varepsilon > 0$ in Lemma 33 satisfying the assumption $\ell(1+\varepsilon) < \ell^*_p$. Then, we set $a = a(\varepsilon)$ as in (35) and recall the inequality $T(\tau,a) < T_1(\tau,a)$ provided by Lemma 30. The estimate in Lemma 33 allows us to conclude the proof. □

Proposition 35. *Assume* $(\mathbf{H}_{\boldsymbol{\ell}})$, $(\mathbf{W}_{\boldsymbol{s}})$ *and that* $\dot{K}(t) > 0$ *for any* $t \in \mathbb{R}$. *If* $\ell < \ell^*_p$, *then there exists* $D > 0$ *such that* $]D,+\infty[\, \subset J$ *and* $\lim_{d\to+\infty} R(d) = 0$.

Proof. Let $a = a(\varepsilon) > 0$ be as in Proposition 34. Then, we can recover the constant $N(a) > 0$ provided by Remark 27 and the function $d_{\mathcal{L}(a)}$ defined in Lemma 20 such that $\boldsymbol{\phi}(t; d_{\mathcal{L}(a)}(\tau)) := \boldsymbol{\phi}(t;\tau,\boldsymbol{Q}(\tau,a))$. From Lemma 20, there exists $D = D(a)$ such that the inverse function $\tau_{\mathcal{L}(a)} :]D,+\infty[\, \to\,]-\infty,-N(a)[$ is continuous and satisfies $\lim_{d\to+\infty}\tau_{\mathcal{L}(a)}(d) = -\infty$. In particular, $]D,+\infty[\, \subset J$.

Then, from Proposition 34 and Remark 29, we find

$$\lim_{d\to+\infty} T(\tau_{\mathcal{L}(a)}(d),a) = \lim_{\tau\to-\infty} T(\tau,a) = -\infty.$$

Therefore, the first zero $R(d)$ of $u(r;d)$ satisfies

$$\lim_{d\to+\infty} R(d) = \lim_{d\to+\infty} \mathrm{e}^{T(\tau_{\mathcal{L}(a)}(d),a)} = 0\,,$$

thus concluding the proof. □

Since in the previous results we analyze the dynamics in a neighborhood of $\tau = -\infty$, recalling Remark 25, we can remove the hypothesis $\mathbf{(W_s)}$ and the monotonicity assumption on K.

Proposition 36. *Assume* $\mathbf{(H_\ell)}$. *If* $\ell < \ell_p^*$, *then there exists* $a > 0$ *such that*

$$\lim_{\tau\to-\infty} T(\tau, a) = -\infty\,.$$

Now, repeating the argument of the proof of Proposition 35 combined with the truncation argument of the proof of Proposition 36, we obtain the asymptotic behavior of $R(d)$ for large values of d. Therefore, we can prove the part of Theorem 1 concerning the case $\ell < \ell_p^*$.

Proposition 37. *Let assumption* $\mathbf{(H_\ell)}$ *hold with* $\ell < \ell_p^*$. *Then, there exists* $\hat{D} > 0$ *such that* $]\hat{D}, +\infty[\, \subset J$ *and* $\lim_{d\to+\infty} R(d) = 0$.

4.2. The case $\boldsymbol{\ell \geq \ell_p^*}$

In this subsection we focus our attention on the opposite case $\ell \geq \ell_p^*$. We invite the reader to take in mind Lemma 20, i.e., the intersection time $\tau_{\mathcal{L}}(d) \to -\infty$ if and only if $d \to +\infty$.

Remark 38. Recalling the definition of α in (9), we see that if $\varepsilon < 1/\alpha$ then

$$\ell_p^* < \frac{1}{1+\varepsilon}\,\frac{q\alpha}{p-1}\,. \tag{45}$$

Lemma 39. *Let assumptions* $\mathbf{(H_\ell)}$, $\mathbf{(H_\uparrow)}$ *and* $\mathbf{(W_s)}$ *hold with* $\ell \geq \ell_p^*$. *Let us fix* $\varepsilon < 1/\alpha$ *and define* $a = a(\varepsilon)$ *as in* (35). *Assume that there exists a sequence* $(\tau_n)_n$ *with* $\tau_n \to -\infty$ *satisfying* $\lim_{n\to+\infty} T(\tau_n, a) = -\infty$. *Then, if* n *is sufficiently large,*

$$\mathcal{H}(\boldsymbol{R}(\tau_n, a); T(\tau_n, a)) \leq 2\mathcal{H}(\boldsymbol{Q}(\tau_n, a); \tau_n)\,.$$

Proof. Since both τ_n and $T(\tau_n, a)$ converge to $-\infty$, according to Remark 25, we can set $\dot{K}(t) < 2B\ell \mathrm{e}^{\ell t} \leq 2B\ell \mathrm{e}^{\ell_p^* t}$ for every $t \in [\tau_n, T(\tau_n, a)]$. Then, using (12), (38), Remark 32 and (45), we get

$$\begin{aligned}
\mathcal{H}(\boldsymbol{R}(\tau_n, a); T(\tau_n, a)) &- \mathcal{H}(\boldsymbol{Q}(\tau_n, a); \tau_n) = \\
&= \int_{\tau_n}^{T(\tau_n,a)} \dot{K}(t)\frac{|x(t)|^q}{q}\,dt < \\
&< \frac{2B\ell}{q\alpha^q}\int_{\tau_n}^{T(\tau_n,a)} \mathrm{e}^{\ell_p^* t}|y(t)|^{\frac{q}{p-1}}\,dt < \\
&< \frac{2B\ell}{q\alpha^q} a^{\frac{q}{p-1}} \mathrm{e}^{\ell_p^*\tau_n} \int_0^{T(\tau_n,a)-\tau_n} \mathrm{e}^{\left(\ell_p^* - \frac{\alpha}{1+\varepsilon}\,\frac{q}{p-1}\right)s}\,ds <
\end{aligned}$$

$$< \frac{2B\ell}{q\alpha^q} a^{\frac{q}{p-1}} \mathrm{e}^{\ell_p^* \tau_n} \int_0^{+\infty} \mathrm{e}^{\left(\ell_p^* - \frac{1}{1+\varepsilon} \frac{q\alpha}{p-1}\right)s} \, ds \; =$$

$$= \frac{2B\ell}{q\alpha^q} a^{\frac{q}{p-1}} \frac{1}{\frac{1}{1+\varepsilon} \frac{q\alpha}{p-1} - \ell_p^*} \mathrm{e}^{\ell_p^* \tau_n} =: C_{\mathcal{H}}(\varepsilon) \, \mathrm{e}^{\ell_p^* \tau_n} \, ,$$

when n is sufficiently large.

We argue by contradiction, and we assume that there exists a subsequence of τ_n, still called τ_n for simplicity, which satisfies

$$\mathcal{H}(\boldsymbol{R}(\tau_n, a); T(\tau_n, a)) \; > 2\mathcal{H}(\boldsymbol{Q}(\tau_n, a); \tau_n) \, .$$

So, for sufficiently large n,

$$\begin{aligned} \mathcal{H}(\boldsymbol{R}(\tau_n, a); T(\tau_n, a)) \; &< 2 \left[\mathcal{H}(\boldsymbol{R}(\tau_n, a); T(\tau_n, a)) - \mathcal{H}(\boldsymbol{Q}(\tau_n, a); \tau_n)\right] \; \leq \\ &\leq 2C_{\mathcal{H}}(\varepsilon) \, \mathrm{e}^{\ell_p^* \tau_n} \, . \end{aligned}$$

Then, recalling the definition of $\mathcal{H}$ in (11) and the definition of $\boldsymbol{R}$ in (30), we get

$$|R_y(\tau_n, a)| < \hat{C}_{\mathcal{H}}(\varepsilon) \, \mathrm{e}^{\ell_p^* \frac{p-1}{p} \tau_n} \, ,$$

where $\hat{C}_{\mathcal{H}}(\varepsilon) = \left[2C_{\mathcal{H}}(\varepsilon) \, \frac{p}{p-1}\right]^{\frac{p-1}{p}}$. Hence, setting $\widetilde{C}_{\mathcal{H}}(\varepsilon) := \frac{1}{\alpha} \ln\left(\frac{a}{\hat{C}_{\mathcal{H}}(\varepsilon)}\right)$, from (36) and (44) we deduce

$$\begin{aligned} T(\tau_n, a) \; &> \tau_n + \frac{1}{\alpha} \ln\left(\frac{a}{\hat{C}_{\mathcal{H}}(\varepsilon)} \mathrm{e}^{-\ell_p^* \frac{p-1}{p} \tau_n}\right) \; = \\ &= \widetilde{C}_{\mathcal{H}}(\varepsilon) + \tau_n \left(1 - \ell_p^* \frac{p-1}{p\alpha}\right) = \widetilde{C}_{\mathcal{H}}(\varepsilon) \, , \end{aligned}$$

which contradicts our assumption $\lim_{n \to +\infty} T(\tau_n, a) = -\infty$. $\square$

Lemma 40. *Let assumption* $(\mathbf{H}_{\boldsymbol{\ell}})$ *hold with* $\ell \geq \ell_p^*$. *Fix* $\varepsilon < 1/\alpha$ *and define*

$$a = a(\varepsilon) := \left(\frac{\alpha^q \varepsilon}{[K(-\infty) + 1](1 + \varepsilon)}\right)^{\frac{p-1}{q-p}} \, . \tag{46}$$

Let us assume that there exists a sequence $(\tau_n)_n$ *with* $\tau_n \to -\infty$ *such that* $\lim_{n \to +\infty} T(\tau_n, a) = -\infty$. *Then, if* n *is sufficiently large,*

$$\mathcal{H}(\boldsymbol{R}(\tau_n, a); T(\tau_n, a)) \leq 2\mathcal{H}(\boldsymbol{Q}(\tau_n, a); \tau_n) \, .$$

Proof. Arguing as in Remark 25, we can modify system (10), replacing K with $\hat{K}$ and notice that we can apply Lemma 39 in this case. From the hypothesis, we deduce that, for n large, we have $\tau_n < T(\tau_n, a) < \hat{T}_0$. Since $\hat{K}$ and the original K coincide on $]-\infty, \hat{T}_0]$, we easily conclude. $\square$

Proposition 41. *Assume* $(\mathbf{H}_{\boldsymbol{\ell}})$ *with* $\ell \geq \ell_p^*$. *Fix* $\varepsilon < 1/\alpha$, *and define* $a = a(\varepsilon)$ *as in* (46). *Then,*

$$\liminf_{\tau \to -\infty} T(\tau, a) > -\infty \, ,$$

where we set $T(\tau, a) := +\infty$ *if the corresponding trajectory* $\boldsymbol{\phi}(t; \tau, \boldsymbol{Q}(\tau, a))$ *does not cross the negative* y*-semiaxis.*

Proof. We argue by contradiction assuming the existence of a sequence $(\tau_n)_n$, with $\tau_n \to -\infty$ and satisfying $\lim_{n\to+\infty} T(\tau_n, a) = -\infty$.

Since we are focusing our attention on a neighborhood of $t = -\infty$, we can again suitably modify K in the function $\hat{K}$ as suggested in Remark 25, in order to ensure the validity of the hypotheses of Lemma 39.

Hence, without loss of generality, we can assume that $(\mathbf{H}_{\uparrow})$ and $(\mathbf{W}_{\boldsymbol{s}})$ also hold. So, according to (12), the energy $\mathcal{H}$ increases along the trajectories. As a consequence, from Lemma 40 and Remark 27, we get

$$\begin{aligned} \tfrac{1}{2}c(a)\mathrm{e}^{\ell\tau_n} \le \mathcal{H}(\boldsymbol{Q}(\tau_n, a); \tau_n) \le \mathcal{H}(\boldsymbol{R}(\tau_n, a); T(\tau_n, a)) \le \\ \le 2\mathcal{H}(\boldsymbol{Q}(\tau_n, a); \tau_n) \le 3c(a)\mathrm{e}^{\ell\tau_n} \end{aligned}$$

for n sufficiently large. So, recalling the definition of $\mathcal{H}$ in (11) and the definition of $\boldsymbol{R}$ in (30), we can find a positive constant $\check{c}_1(a)$ such that

$$|R_y(\tau_n, a)| \le \check{c}_1(a)\mathrm{e}^{\frac{p-1}{p}\,\ell\tau_n}\,.$$

Moreover, we can use the estimate in Remark 32 to get

$$a\mathrm{e}^{-\alpha(T(\tau_n,a)-\tau_n)} \le |R_y(\tau_n, a)|$$

for n sufficiently large. Hence, we have

$$a\mathrm{e}^{-\alpha(T(\tau_n,a)-\tau_n)} \le \check{c}_1(a)\mathrm{e}^{\frac{p-1}{p}\,\ell\tau_n}\,,$$

leading to

$$\begin{aligned} T(\tau_n, a) &\ge \frac{1}{\alpha}\ln\left(\frac{a}{\check{c}_1(a)}\right) + \tau_n\left[1 - \frac{1}{\alpha}\,\frac{p-1}{p}\,\ell\right] = \\ &= \check{C}_1(\varepsilon) + \tau_n\left[1 - \frac{\ell}{\ell_p^*}\right] \ge \check{C}_1(\varepsilon)\,, \end{aligned}$$

which contradicts $\lim_{n\to+\infty} T(\tau_n, a) = -\infty$. □

When $\ell > \ell_p^*$ (with the strict inequality), we find a more precise estimate.

Proposition 42. *Assume* $(\mathbf{H}_{\boldsymbol{\ell}})$ *with* $\ell > \ell_p^*$. *Fix* $\varepsilon > 0$ *with* $\varepsilon < 1/\alpha$ *and define* $a = a(\varepsilon)$ *as in* (46).

Assume that there is $\widetilde{N}(a) > 0$ *such that, for every* $\tau < -\widetilde{N}(a)$, *there exists a time* $T(\tau, a) \in \mathbb{R}$ *at which the trajectory* $\boldsymbol{\phi}(t; \tau, \boldsymbol{Q}(\tau, a))$ *crosses the negative* y*-semiaxis and* $\dot{x}(t; \tau, \boldsymbol{Q}(\tau, a)) < 0$ *for any* $\tau \le t \le T(\tau, a)$. *Then*

$$\lim_{\tau\to-\infty} T(\tau, a) = +\infty\,.$$

Proof. Let $\check{T} := \liminf_{\tau\to-\infty} T(\tau, a) \in \mathbb{R} \cup \{+\infty\}$, given by Proposition 41. Without loss of generality, we choose the value $\hat{T}_0$ provided by Remark 25 so that $\hat{T}_0 \le \check{T}$. Recalling Remark 38, we first fix $\delta \in\,]0, 1[$ satisfying

$$\ell_p^* < \frac{1-\delta}{1+\varepsilon}\,\frac{q\alpha}{p-1},$$

and then $\overline{\tau}_\delta < -\widetilde{N}(a)$ such that

$$\tau < \delta\tau < \hat{T}_0 - 1 \le \check{T} - 1 < T(\tau, a) \quad \text{for any } \tau < \overline{\tau}_\delta .$$

Since $\ell > \ell_p^*$, we can introduce $\ell_1 \le \ell$ such that

$$\ell_p^* < \ell_1 \le \frac{1-\delta}{1+\varepsilon}\,\frac{q\alpha}{p-1} < \frac{1}{1+\varepsilon}\,\frac{q\alpha}{p-1} . \tag{47}$$

Consider the trajectory of system (10) departing from $\boldsymbol{Q}(\tau, a)$ at the time τ and the points

$$\boldsymbol{S}(\tau, a) = \boldsymbol{\phi}(\delta\tau; \tau, \boldsymbol{Q}(\tau, a)) , \qquad \boldsymbol{R}(\tau, a) = \boldsymbol{\phi}(T(\tau, a); \tau, \boldsymbol{Q}(\tau, a)) .$$

We now focus our attention on the trajectory $\boldsymbol{\phi}(\cdot; \tau, \boldsymbol{Q}(\tau, a))$ restricted to the interval $[\tau, \delta\tau] \subset]-\infty, \hat{T}_0[$. Hence, recalling the truncation argument in Remark 25, we can assume that both $(\mathbf{H}_\uparrow)$ and $(\mathbf{W}_{\boldsymbol{s}})$ hold. So, we can argue as in the proof of Lemma 39 to obtain $\mathcal{T}(\varepsilon, \delta) \le \overline{\tau}_\delta$ such that

$$\begin{aligned} \mathcal{H}(\boldsymbol{S}(\tau, a); \delta\tau) \; - \mathcal{H}(\boldsymbol{Q}(\tau, a); \tau) = & \\ = \int_\tau^{\delta\tau} \dot{K}(t) \frac{|x(t)|^q}{q} \, dt \le & \\ \le \frac{2B\ell}{q\alpha^q} \int_\tau^{\delta\tau} \mathrm{e}^{\ell_1 t} |y(t)|^{\frac{q}{p-1}} \, dt \le & \\ \le \frac{2B\ell}{q\alpha^q} a^{\frac{q}{p-1}} \mathrm{e}^{\ell_1 \tau} \int_0^{+\infty} \mathrm{e}^{\left(\ell_1 - \frac{\alpha}{1+\varepsilon}\,\frac{q}{p-1}\right)s} \, ds = C_{\mathcal{H}}(\varepsilon)\, \mathrm{e}^{\ell_1 \tau} & \end{aligned} \tag{48}$$

for any $\tau < \mathcal{T}(\varepsilon, \delta)$. Moreover, from (43) and $\dot{x}(\delta\tau) < 0$, we have

$$x(\delta\tau) < \frac{1}{\alpha} |y(\delta\tau)|^{\frac{1}{p-1}} < \frac{1}{\alpha}\, a^{\frac{1}{p-1}}\, \mathrm{e}^{\frac{1-\delta}{1+\varepsilon}\,\frac{\alpha}{p-1}\tau} \quad \text{for any } \tau < \mathcal{T}(\varepsilon, \delta). \tag{49}$$

Assume by contradiction that there are $M > 0$ and a sequence $(\tau_n)_n$ such that $\tau_n \to -\infty$ and $T(\tau_n, a) \le M$.

Let us now focus our attention on $[\delta\tau_n, T(\tau_n, a)] \subset]-\infty, M]$. We remark that the validity of $(\mathbf{H}_\uparrow)$ and $(\mathbf{W}_{\boldsymbol{s}})$ is not guaranteed anymore in $[\delta\tau_n, T(\tau_n, a)]$; however $\dot{x} < 0$, so $x(t) < x(\delta\tau_n)$ in this interval. Hence, we can compute

$$\begin{aligned} \mathcal{H}(\boldsymbol{R}(\tau_n, a); T(\tau_n, a)) \; - \mathcal{H}(\boldsymbol{S}(\tau_n, a); \delta\tau_n) = \int_{\delta\tau_n}^{T(\tau_n, a)} \dot{K}(t) \frac{|x(t)|^q}{q} \, dt \le & \\ \le \frac{1}{q} \int_{\delta\tau_n}^{T(\tau_n, a)} \max\{\dot{K}(t), 0\} |x(\delta\tau_n)|^q \, dt := \frac{\Delta(\tau_n)}{q\alpha^q} . & \end{aligned} \tag{50}$$

Then, using (49), we find

$$\begin{aligned} \Delta(\tau_n) \le \left[\int_{-\infty}^{M} \max\{\dot{K}(t), 0\} \, dt \right] a^{\frac{q}{p-1}}\, \mathrm{e}^{\frac{1-\delta}{1+\varepsilon}\,\frac{q\alpha}{p-1}\tau_n} = & \\ := \mathcal{I}_K(M)\, a^{\frac{q}{p-1}}\, \mathrm{e}^{\frac{1-\delta}{1+\varepsilon}\,\frac{q\alpha}{p-1}\tau_n} . & \end{aligned}$$

Substituting this last inequality into (50) and using (47), we obtain

$$\mathcal{H}(\boldsymbol{R}(\tau_n,a);T(\tau_n,a)) - \mathcal{H}(\boldsymbol{S}(\tau_n,a);\delta\tau_n) \le \mathcal{I}_K(M)a^{\frac{q}{p-1}}\mathrm{e}^{\ell_1\tau_n} \tag{51}$$

for n sufficiently large.

Since $\ell \ge \ell_1$, summing (48) and (51) with the estimate in Remark 27, we get the existence of a constant $\widetilde{C}_{\mathcal{H}}(\varepsilon) > 0$ satisfying

$$\frac{p-1}{p}|R_y(\tau_n,a)|^{\frac{p}{p-1}} = \mathcal{H}(\boldsymbol{R}(\tau_n,a),T(\tau_n,a)) \le \tilde{C}_{\mathcal{H}}(\varepsilon)\mathrm{e}^{\ell_1\tau_n}\,.$$

Since $\dot{y} < -\alpha y$ in $[\tau_n, T(\tau_n,a)[$, we deduce that $-\frac{1}{\alpha}\frac{\dot{y}}{y} < 1$ in this interval. Hence, for a certain constant $\widetilde{C}_H(\varepsilon)$, we obtain

$$\begin{aligned} T(\tau_n,a) &> \tau_n - \frac{1}{\alpha}\int_{\tau_n}^{T(\tau_n,a)} \frac{\dot{y}(s)}{y(s)}\,ds = \\ &= \tau_n - \frac{1}{\alpha}\int_a^{|R_y(\tau_n,a)|}\frac{dy}{y} = \tau_n + \frac{1}{\alpha}\ln\left(\frac{a}{|R_y(\tau_n,a)|}\right) \ge \\ &\ge \widetilde{C}_H(\varepsilon) + \tau_n\left(1 - \frac{\ell_1}{\ell_p^*}\right) \end{aligned}$$

when n is large enough. Recalling (47), we get $\lim_{n\to+\infty} T(\tau_n,a) = +\infty$, giving a contradiction. □

Proposition 43. *Assume* $(\mathbf{H}_{\boldsymbol{\ell}})$ *and* $(\mathbf{H}_{\uparrow})$. *Then,* $J =]0,+\infty[$ *and*

- *if* $\ell = \ell_p^*$, *then* $\liminf\limits_{d\to+\infty} R(d) > 0$,
- *if* $\ell > \ell_p^*$, *then* $\lim\limits_{d\to+\infty} R(d) = +\infty$.

Proof. As stated in Proposition 21, $J =]0,+\infty[$ is a well-known consequence of assumption $(\mathbf{H}_{\uparrow})$.

Let $\varepsilon < \frac{1}{\alpha}$, define $a = a(\varepsilon)$ as in (46), and set $N = N(a)$ as in Remark 27. So, $W^u(\tau)$ crosses transversely $\mathcal{L}(a)$ in $\boldsymbol{Q}(\tau,a)$ for any $\tau < -N$.

From Proposition 21 and Lemma 28, we see that for every $\tau < -N$ there is $T(\tau,a)$ such that $\dot{x}(t;\tau,\boldsymbol{Q}(\tau,a)) < 0$ for any $\tau \le t \le T(\tau,a)$, and $\boldsymbol{\phi}(t;\tau,\boldsymbol{Q}(\tau,a))$ crosses the negative y-semiaxis at $t = T(\tau,a)$.

Now, we consider the function $d_{\mathcal{L}(a)}\colon]-\infty,-N[\to]D,+\infty[$ introduced in Lemma 20. In particular, $u(r;d_{\mathcal{L}(a)}(\tau))$ is the solution of (4) corresponding to the trajectory $\boldsymbol{\phi}(t;\tau,\boldsymbol{Q}(\tau,a))$. Further, the inverse $\tau_{\mathcal{L}(a)}$ of $d_{\mathcal{L}(a)}$ satisfies $\lim_{d\to+\infty}\tau_{\mathcal{L}(a)}(d) = -\infty$.

If $\ell \ge \ell_p^*$, Proposition 41 gives $\liminf_{\tau\to-\infty} T(\tau,a) > -\infty$. Arguing as in the proof of Proposition 35, we have

$$\liminf_{d\to+\infty} R(d) = \liminf_{d\to+\infty} \mathrm{e}^{T(\tau_{\mathcal{L}(a)}(d),a)} = \liminf_{\tau\to-\infty}\mathrm{e}^{T(\tau,a)} > 0\,.$$

If $\ell > \ell_p^*$, we are able to apply Proposition 42 and get

$$\lim_{d\to+\infty} R(d) = \lim_{d\to+\infty}\mathrm{e}^{T(\tau_{\mathcal{L}(a)}(d),a)} = \lim_{\tau\to-\infty}\mathrm{e}^{T(\tau,a)} = +\infty\,. \quad \square \tag{52}$$

Theorem 2 follows from Proposition 43, cf. Section 5 for more details.

In order to prove the second part of Theorem 1, we need to remove the monotonicity assumption from Proposition 43. Note that if $\mathbf{(H_\uparrow)}$ does not hold, we cannot ensure the existence of points of J in a neighborhood of $+\infty$ when $\ell \geq \ell_p^*$. However, we can prove a weaker result.

Proposition 44. *Assume* $\mathbf{(H_\ell)}$*, and define*

$$\tilde{R}(d) = \begin{cases} R(d) & \text{if } d \in J\,, \\ +\infty & \text{if } d \notin J\,. \end{cases}$$

If $\ell \geq \ell_p^*$*, then there is* $\mathcal{R}_0 > 0$ *such that* $\tilde{R}(d) \geq \mathcal{R}_0$ *for any* $d \geq 0$.

Proof. Fix $0 < \varepsilon < 1/\alpha$, $a = a(\varepsilon)$ as in (46) and set $N(a)$ as in Remark 27. Let $\hat{T}_0$ be as in Remark 25. From Proposition 41 we deduce that

$$\text{there is } T' \leq \hat{T}_0 \text{ such that } T(\tau, a) \geq T' \text{ for any } \tau < -N(a)\,. \tag{53}$$

Now, let us fix $\tau_0 < -N(a)$ and denote by $\tilde{W}^u(\tau_0)$ the branch of the unstable manifold $W^u(\tau_0)$ between the origin and $\boldsymbol{Q}(\tau_0, a)$. From Remark 17, we see that there is $D^* > 0$ such that the function $\boldsymbol{\mathcal{Q}}_{\boldsymbol{\tau_0}}(d)$ restricted to $d \in [0, D^*]$ gives a parametrization of $\tilde{W}^u(\tau_0)$, i.e., $\boldsymbol{\mathcal{Q}}_{\boldsymbol{\tau_0}} \colon [0, D^*] \to \tilde{W}^u(\tau_0)$ is a continuous bijective function.

Using Remark 14 and Lemma 16, reducing τ_0 if necessary, we see that if $\boldsymbol{Q} \in \tilde{W}^u(\tau_0)$ then $\boldsymbol{\phi}(t; \tau_0, \boldsymbol{Q})$ is close to the corresponding trajectory in $W^u(-\infty)$ when $t \leq \tau_0$; in particular $x(t; \tau_0, \boldsymbol{Q}) > 0$ if $t \leq \tau_0$. Hence $\tilde{R}(d) \geq \mathrm{e}^{\tau_0}$ for any $d \leq D^*$.

Let now $d > D^*$. Recalling Remark 17, we have $\boldsymbol{\mathcal{Q}}_{\boldsymbol{\tau_0}}(d) \in W^u(\tau_0) \setminus \tilde{W}^u(\tau_0)$. Using Lemma 20, we see that there is a unique $\tau_{\mathcal{L}(a)}(d) < \tau_0$ such that the trajectory $\boldsymbol{\phi}(t; \tau_0, \boldsymbol{\mathcal{Q}}_{\boldsymbol{\tau_0}}(d))$ crosses transversely $\mathcal{L}(a)$ at $t = \tau_{\mathcal{L}(a)}(d) < \tau_0 < -N(a)$. Then, from (53) we deduce that

$$T(\tau_{\mathcal{L}(a)}(d), a) \geq T' \qquad \text{for any } d > D^*.$$

Hence $\tilde{R}(d) \geq \mathrm{e}^{T'}$ for any $d > D^*$.

Finally, setting $\mathcal{R}_0 = \min\{\mathrm{e}^{\tau_0}, \mathrm{e}^{T'}\}$, we conclude. □

Now, we reprove Proposition 43 by replacing the global assumption $\mathbf{(H_\uparrow)}$ with the local assumption $\ell \leq \frac{n}{p-1}$. Motivated by [3,5,34] and inspired by [14,18], we obtain the following result.

Lemma 45. *Assume* $\mathbf{(H_\ell)}$ *with* $\ell \leq \frac{n}{p-1}$*; assume further either* $\mathbf{(W_s)}$ *or that there is* $\rho_1 > 0$ *such that* $\mathcal{K}(r) \geq \mathcal{K}(\rho_1)$ *when* $r \geq \rho_1$*. Then, there is* $\hat{D} > 0$ *such that* $]\hat{D}, +\infty[\, \subset J$.

Assume $\mathbf{(H_\ell)}$ with $\ell \leq \frac{n}{p-1}$; notice that Lemma 45 can be applied either if $\lim_{r\to+\infty} \mathcal{K}(r)$ exists and it is positive (either bounded or unbounded), or if $K(t)$ is asymptotically periodic.

The proof of Lemma 45 is rather technical and it is postponed to Section 4.2.1, as well as the related proof of the following adapted version of Proposition 43.

Proposition 46. *Assume* $\mathbf{(H_\ell)}$ *with* $\ell_p^* < \ell \leq \frac{n}{p-1}$*; assume further either* $\mathbf{(W_s)}$ *or that there is* $\rho_1 > 0$ *such that* $\mathcal{K}(r) \geq \mathcal{K}(\rho_1)$ *when* $r \geq \rho_1$*. Then*

$$\lim_{d\to+\infty} R(d) = +\infty\,.$$

4.2.1. Proof of Lemma 45 and Proposition 46

We start by proving Lemma 45 under assumption $(\mathbf{W_s})$. The alternative case, where $\mathcal{K}(r) \geq \mathcal{K}(\rho_1)$, is then obtained as a corollary. We begin with some preliminary arguments.

Let us assume $(\mathbf{W_s})$, so that we can construct the stable manifold $W^s(\tau)$, see Section 2 and, in particular, (20). Since we are dealing with system (10), we use the following shorthand notation: we write $\{\dot{x} \leq 0\}$ instead of $\{(x,y) \in \mathbb{R}^2 \mid \alpha x + y\,|y|^{\frac{2-p}{p-1}} \leq 0\}$. Similarly, we write $\{\dot{x} = 0\}$ and $\{\dot{x} < 0\}$.

Firstly, we need to define several sets, and we invite the reader to follow the argument on Fig. 5. Assume $(\mathbf{H_\ell})$ and $(\mathbf{W_s})$, then for any $\tau \in \mathbb{R}$ we set

$$\overline{\mathcal{H}}(x,y) = \alpha xy + \frac{p-1}{p}|y|^{\frac{p}{p-1}} + \overline{K}\,\frac{|x|^q}{q}\,,$$
$$\underline{\mathcal{H}}(x,y) = \alpha xy + \frac{p-1}{p}|y|^{\frac{p}{p-1}} + \underline{K}\,\frac{|x|^q}{q}\,.$$

We define $\overline{\Gamma} := \{(x,y) \mid \overline{\mathcal{H}}(x,y) = 0\,, x \geq 0\}$ and $\underline{\Gamma} := \{(x,y) \mid \underline{\mathcal{H}}(x,y) = 0\,, x \geq 0\}$. Notice that both $\overline{\Gamma}$ and $\underline{\Gamma}$ are the images of closed regular curves. We denote by $\overline{F}$ and by $\underline{F}$ the bounded sets enclosed by $\overline{\Gamma}$ and $\underline{\Gamma}$, respectively; notice that $\overline{F} \subset \underline{F}$. We denote by $\overline{\boldsymbol{G}} = (\overline{G}_x, \overline{G}_y)$ the (transversal) intersection between $\overline{\Gamma}$ and the isocline $\{\dot{x} = 0\}$ such that $\overline{G}_x > 0$. Then, we denote by $\underline{\boldsymbol{G}}$ the intersection between the line $x = \overline{G}_x$ and $\underline{\Gamma}$ contained in $\{\dot{x} < 0\}$ and by $\mathcal{G}$ the vertical segment between $\underline{\boldsymbol{G}}$ and $\overline{\boldsymbol{G}}$. Moreover, we denote by $\partial\mathcal{B}^{\uparrow}$ the branch of $\overline{\Gamma}$ between the origin and $\overline{\boldsymbol{G}}$ contained in $\{\dot{x} \leq 0\}$ and by $\partial\mathcal{B}^{\downarrow}$ the branch of $\underline{\Gamma}$ between the origin and $\underline{\boldsymbol{G}}$ contained in $\{\dot{x} \leq 0\}$. Finally, we denote by $\mathcal{B}$ the compact set enclosed by $\mathcal{G}$, $\partial\mathcal{B}^{\uparrow}$ and $\partial\mathcal{B}^{\downarrow}$, see Fig. 5.

We emphasize that if $\boldsymbol{\phi}(t) = (x(t), y(t))$ is a trajectory of (10), we find, according to (14),

$$\frac{d}{dt}\,\overline{\mathcal{H}}(\boldsymbol{\phi}(t)) = \big(\overline{K} - K(t)\big)\,x(t)\,|x(t)|^{q-2}\;\dot{x}(t)\,,$$
$$\frac{d}{dt}\,\underline{\mathcal{H}}(\boldsymbol{\phi}(t)) = \big(\underline{K} - K(t)\big)\,x(t)\,|x(t)|^{q-2}\;\dot{x}(t)\,.$$

Using this fact, we easily obtain the following crucial remark.

Remark 47. Assume $(\mathbf{H_\ell})$ and $(\mathbf{W_s})$, then the flow of (10) on $\breve{\mathcal{G}} := \mathcal{G} \setminus \{\overline{\boldsymbol{G}}, \underline{\boldsymbol{G}}\}$ aims towards the interior of $\mathcal{B}$ for any $t \in \mathbb{R}$, while on $(\partial\mathcal{B}^{\uparrow} \cup \partial\mathcal{B}^{\downarrow}) \setminus \{(0,0)\}$ aims towards the exterior of $\mathcal{B}$ for any $t \in \mathbb{R}$.

Let us now fix $\tau \in \mathbb{R}$. From Remark 47 and the construction in Section 2, we see that $W^s(\tau)$ intersects $\breve{\mathcal{G}}$ (not necessarily transversely), see [18, Lemma 2.8]. Follow $W^s(\tau)$ from the origin towards $x > 0$: denote by $\boldsymbol{Q^s}(\tau)$ the first intersection with $\breve{\mathcal{G}}$ and by $\tilde{W}^s(\tau)$ the connected branch of $W^s(\tau)$ between the origin and $\boldsymbol{Q^s}(\tau)$.

Moreover, we introduce the set

$$\hat{W}^s(\tau) := \{\boldsymbol{Q} \in W^s(\tau) \cap \mathcal{B} \mid \dot{x}(t;\tau,\boldsymbol{Q}) < 0 \;\text{ for any } t \geq \tau\} \supset \tilde{W}^s(\tau)\,.$$

In particular, using Remark 47, we have

$$\boldsymbol{Q} \in \hat{W}^s(\tau) \Rightarrow \boldsymbol{\phi}(t;\tau,\boldsymbol{Q}) \in \hat{W}^s(t) \text{ for every } t \geq \tau\,, \tag{54}$$
$$\boldsymbol{Q} \in \tilde{W}^s(\tau) \Rightarrow \boldsymbol{\phi}(t;\tau,\boldsymbol{Q}) \in \tilde{W}^s(t) \text{ for every } t \geq \tau\,.$$

In what follows, we recall the argument from [18, pp. 357–360]. Let us consider the autonomous systems (10), where $K(t)$ is replaced by $\overline{K}$, respectively $\underline{K}$, and we denote by

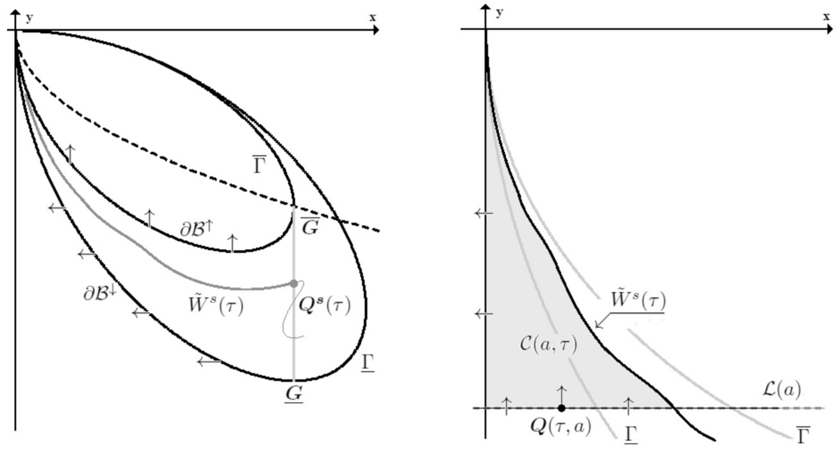

Fig. 5. The constructions needed in the proof of Lemma 45.

$$\overline{\boldsymbol{\phi}}(t;\tau,\boldsymbol{Q}) = (\overline{x}(t;\tau,\boldsymbol{Q}),\overline{y}(t;\tau,\boldsymbol{Q}))\,,\ \text{resp. } \underline{\boldsymbol{\phi}}(t;\tau,\boldsymbol{Q}) = (\underline{x}(t;\tau,\boldsymbol{Q}),\underline{y}(t;\tau,\boldsymbol{Q}))\,,$$

the trajectories of these systems starting at time τ from the point $\boldsymbol{Q}$. Recall that in the region $\{\dot{x}<0\}$ the branch of $\underline{\Gamma}$ lies under the corresponding branch of $\overline{\Gamma}$, see Fig. 5. So, from (17), we can find $C>c>0$ such that

$$c\,\mathrm{e}^{-\frac{n-p}{p(p-1)}t} \le \underline{x}(t;0,\underline{\boldsymbol{G}}) \le \overline{x}(t;0,\overline{\boldsymbol{G}}) \le C\,\mathrm{e}^{-\frac{n-p}{p(p-1)}t} \text{ for any } t\ge 0,$$

or, equivalently,

$$c\,\mathrm{e}^{-\frac{n-p}{p(p-1)}(t-\tau)} \le \underline{x}(t;\tau,\underline{\boldsymbol{G}}) \le \overline{x}(t;\tau,\overline{\boldsymbol{G}}) \le C\,\mathrm{e}^{-\frac{n-p}{p(p-1)}(t-\tau)} \text{ for any } t\ge\tau\,. \tag{55}$$

Such estimates permit us to provide analogous ones for the solutions of the non-autonomous system (10).

Lemma 48. *Assume* $(\mathbf{H}_{\boldsymbol{\ell}})$ *and* $(\mathbf{W}_{\boldsymbol{s}})$. *Let* $\tau\in\mathbb{R}$ *and* $\hat{\boldsymbol{Q}}\in\hat{W}^s(\tau)\cap\mathcal{G}$, *then*

$$c\,\mathrm{e}^{-\frac{n-p}{p(p-1)}(t-\tau)} \le x(t;\tau,\hat{\boldsymbol{Q}}) \le C\,\mathrm{e}^{-\frac{n-p}{p(p-1)}(t-\tau)} \qquad \textit{for any } t\ge\tau\,.$$

Proof. We will prove that

$$\underline{x}(t;\tau,\underline{\boldsymbol{G}}) \le x(t;\tau,\hat{\boldsymbol{Q}}) \le \overline{x}(t;\tau,\overline{\boldsymbol{G}}) \text{ for any } t\ge\tau\,.$$

Then, the conclusion follows from (55).

We present the proof of the first inequality, the other being similar.

Let $T=\sup\{t\ge\tau\,:\,\underline{x}(s;\tau,\underline{\boldsymbol{G}})\le x(s;\tau,\hat{\boldsymbol{Q}}) \text{ for any } s\in[\tau,t]\,\}$. If $T=+\infty$ we are done. Hence, assume that $T\in\mathbb{R}$. Then, $\underline{x}(T;\tau,\underline{\boldsymbol{G}})=x(T;\tau,\hat{\boldsymbol{Q}})$ and since $\hat{\boldsymbol{Q}}\in\hat{W}^s(\tau)$, from (54) and Remark 47, the trajectory remains in the interior of $\mathcal{B}$. In particular we get $\underline{y}(T;\tau,\underline{\boldsymbol{G}})<y(T;\tau,\hat{\boldsymbol{Q}})$. So, from (10) and (13), we deduce that $\dot{\underline{x}}(T;\tau,\underline{\boldsymbol{G}})<\dot{x}(T;\tau,\hat{\boldsymbol{Q}})$, providing $\underline{x}(t;\tau,\underline{\boldsymbol{G}})<x(t;\tau,\hat{\boldsymbol{Q}})$ in a right neighborhood of T, leading to a contradiction. □

From Lemma 48, we obtain the following result, which was proved in [18, Lemma 3.1], where the second author considered points in the smaller set $\boldsymbol{Q^s}(\tau)\in\tilde{W}^s(\tau)$. We repeat the argument here to correct some typos.

Lemma 49. *Assume* $(\mathbf{W}_{\boldsymbol{s}})$ *and* $(\mathbf{H}_{\boldsymbol{\ell}})$ *with* $\ell \leq \frac{n}{p-1}$. *Then, there is* $N_1 > 0$ *such that* $\mathcal{H}(\boldsymbol{Q};\tau) < 0$ *for any* $\boldsymbol{Q} \in \hat{W}^s(\tau)$ *and* $\tau < -N_1$.

Proof. Let $\hat{T}_0$ be as in Remark 25, so that (25) holds.

Take any $\tau \leq \hat{T}_0$ and consider a point $\boldsymbol{Q} \in \hat{W}^s(\tau)$. We denote, for brevity, the trajectory $\boldsymbol{\phi}(t;\tau,\boldsymbol{Q})$ as $\boldsymbol{\phi}(t) = (x(t), y(t))$.

We note that $\boldsymbol{\phi}$ must cross the line $\mathcal{G}$ at a time smaller than τ. So, let us denote by $T_{\mathcal{G}} \leq \tau$ the largest value with such a property. In particular, by construction, $x(T_{\mathcal{G}}) = \overline{G}_x$, $0 < x(t) < \overline{G}_x$ and $\dot{x}(t) < 0$ for every $t > T_{\mathcal{G}}$. Using Lemma 48, we have

$$c\,\mathrm{e}^{-\frac{n-p}{p(p-1)}(t-T_{\mathcal{G}})} \leq x(t) \leq C\,\mathrm{e}^{-\frac{n-p}{p(p-1)}(t-T_{\mathcal{G}})} \text{ for any } t \geq T_{\mathcal{G}}\,. \tag{56}$$

Since $T_{\mathcal{G}} \leq \tau \leq \hat{T}_0$, we see that x is decreasing in the interval $]\hat{T}_0, +\infty[\, \subset]T_{\mathcal{G}}, +\infty[$; so using (56),

$$\begin{aligned}
\int_{\hat{T}_0}^{+\infty} \dot{K}(t)\frac{x(t)^q}{q}\,dt &= -K(\hat{T}_0)\frac{x(\hat{T}_0)^q}{q} + \int_{\hat{T}_0}^{+\infty} K(t)\frac{d}{dt}\left[-\frac{x(t)^q}{q}\right] dt \geq \\
&\geq -K(\hat{T}_0)\frac{x(\hat{T}_0)^q}{q} \;\geq\; -\frac{C^q}{q}\,K(\hat{T}_0)\,\mathrm{e}^{-\frac{q(n-p)}{p(p-1)}(\hat{T}_0 - T_{\mathcal{G}})} = \\
&= -\frac{C^q}{q}\,K(\hat{T}_0)\,\mathrm{e}^{-\frac{n}{p-1}(\hat{T}_0 - T_{\mathcal{G}})}\,.
\end{aligned}$$

Moreover, using (25) and (56),

$$\begin{aligned}
\int_{\tau}^{\hat{T}_0} \dot{K}(t)\frac{x(t)^q}{q}\,dt &\geq \frac{B\ell c^q}{2q}\int_{\tau}^{\hat{T}_0} \mathrm{e}^{\ell t}\,\mathrm{e}^{-\frac{q(n-p)}{p(p-1)}(t-T_{\mathcal{G}})}\,dt = \\
&= \frac{B\ell c^q}{2q}\mathrm{e}^{\frac{n}{p-1}T_{\mathcal{G}}}\int_{\tau}^{\hat{T}_0} \mathrm{e}^{-\left(\frac{n}{p-1}-\ell\right)t}\,dt\,.
\end{aligned}$$

Summing up, setting $b_1 = \frac{C^q}{q}\,K(\hat{T}_0)\,\mathrm{e}^{-\frac{n}{p-1}\hat{T}_0}$ and $b_2 = \frac{B\ell c^q}{2q}$, from (12), we get

$$\begin{aligned}
\mathcal{H}(\boldsymbol{Q};\tau) &= -\int_{\tau}^{+\infty} \dot{K}(t)\frac{|x(t)|^q}{q}\,dt = \\
&= -\int_{\hat{T}_0}^{+\infty} \dot{K}(t)\frac{x(t)^q}{q}\,dt - \int_{\tau}^{\hat{T}_0} \dot{K}(t)\frac{x(t)^q}{q}\,dt \leq \\
&\leq \mathrm{e}^{\frac{n}{p-1}T_{\mathcal{G}}}\left[b_1 - b_2\int_{\tau}^{\hat{T}_0} \mathrm{e}^{-\left(\frac{n}{p-1}-\ell\right)t}\,dt\right].
\end{aligned}$$

So, since $\ell \leq \frac{n}{p-1}$, the integral diverges as $\tau \to -\infty$, thus giving the proof. □

Let us assume now $(\mathbf{H}_{\boldsymbol{\ell}})$ and $(\mathbf{W}_{\boldsymbol{s}})$. Recalling the definition of $\boldsymbol{E}$ given in (15), fix

$$0 < a < \left(\frac{\alpha^q}{\overline{\overline{K}}}\right)^{\frac{p-1}{q-p}} \leq \inf\{|E_y(t)| : t \in \mathbb{R}\}, \tag{57}$$

and consider the segment $\mathcal{L}(a)$ defined as in (29).

From Remark 27, there is $N(a) > 0$ such that $W^u(\tau)$ intersects transversely $\mathcal{L}(a)$ in a point denoted by $\boldsymbol{Q}(\tau, a)$ for every $\tau < -N(a)$. Moreover, since $a > \overline{G}_y$, a subsegment of $\mathcal{L}(a)$ joins $\partial\mathcal{B}^{\downarrow}$ with $\partial\mathcal{B}^{\uparrow}$ (transversely), and, consequently, $\tilde{W}^s(\tau)$ intersects $\mathcal{L}(a)$, too, for every $\tau \in \mathbb{R}$.

Lemma 50. *Assume* $(\mathbf{W}_{\boldsymbol{s}})$ *and* $(\mathbf{H}_{\boldsymbol{\ell}})$ *with* $\ell \leq \frac{n}{p-1}$. *Then, there is* $\hat{N}(a) > 0$ *such that for any* $\tau < -\hat{N}(a)$ *the trajectory* $\boldsymbol{\phi}(t; \tau, \boldsymbol{Q}(\tau, a))$ *corresponds to a crossing solution, i.e., there is* $T = T(\tau, a) > \tau$ *such that* $x(t; \tau, \boldsymbol{Q}(\tau, a)) > 0$ *for any* $t < T$, *and it becomes null at* $t = T$. *Further,* $\dot{x}(t; \tau, \boldsymbol{Q}(\tau, a)) < 0$ *for any* $\tau \leq t \leq T(\tau, a)$.

Proof. For any $\tau \in \mathbb{R}$, let us consider the compact set $\mathcal{C} = \mathcal{C}(a, \tau)$, delimited by $\tilde{W}^s(\tau)$, the negative y-semiaxis and the line $\mathcal{L}(a)$, see Fig. 5. Then, from Remark 27, we can find $\hat{N}(a) > N_1$, with N_1 provided by Lemma 49, such that

$$\mathcal{H}(\boldsymbol{Q}(\tau, a); \tau) > \frac{c(a)}{2} \mathrm{e}^{\ell\tau} > 0 \qquad \text{for every } \tau < -\hat{N}(a). \tag{58}$$

Hence, from Lemma 49, we get $\boldsymbol{Q}(\tau, a) \in \mathcal{C}(a, \tau)$ for any $\tau < -\hat{N}(a)$.

Fix $\tau < -\hat{N}(a)$; then by construction $\boldsymbol{\phi}(t; \tau, \boldsymbol{Q}(\tau, a)) \in \mathcal{C}(a, t)$, when t is in a sufficiently small right neighborhood of τ, see Remark 13.

So, there is $T(\tau, a) > \tau$ such that either $\boldsymbol{\phi}(t; \tau, \boldsymbol{Q}(\tau, a)) \in \mathcal{C}(a, t)$ for any $\tau \leq t \leq T(\tau, a)$ and it leaves $\mathcal{C}(a, t)$ when $t > T(\tau, a)$, or $\boldsymbol{\phi}(t; \tau, \boldsymbol{Q}(\tau, a)) \in \mathcal{C}(a, t)$ for any $t \geq \tau$ (i.e., $T(\tau, a) = +\infty$).

From an analysis of the phase portrait, we see that

$$\dot{x}(t; \tau, \boldsymbol{Q}(\tau, a)) < 0 < \dot{y}(t; \tau, \boldsymbol{Q}(\tau, a)) \quad \text{when } \tau \leq t \leq T(\tau, a)\,. \tag{59}$$

Assume first that $T(\tau, a) \in \mathbb{R}$, then either $\boldsymbol{\phi}(t; \tau, \boldsymbol{Q}(\tau, a))$ crosses the negative y-semiaxis at $t = T(\tau, a)$ and we are done, or $\boldsymbol{R} := \boldsymbol{\phi}(T(\tau, a); \tau, \boldsymbol{Q}(\tau, a)) \in \tilde{W}^s(T(\tau, a))$.

So, assume the latter and observe that $\boldsymbol{R} \in \hat{W}^s(T(\tau, a))$, thus $\boldsymbol{Q}(\tau, a) \in W^s(\tau)$. Then, $\dot{x}(t; T(\tau, a), \boldsymbol{R}) < 0$ for any $t > T(\tau, a)$, and by construction $\dot{x}(t; \tau, \boldsymbol{Q}(\tau, a)) < 0$ for any $\tau \leq t \leq T(\tau, a)$. So, since the trajectories $\boldsymbol{\phi}(\,\cdot\,; \tau, \boldsymbol{Q}(\tau, a))$ and $\boldsymbol{\phi}(\,\cdot\,; T(\tau, a), \boldsymbol{R})$ coincide, we conclude that $\dot{x}(t; \tau, \boldsymbol{Q}(\tau, a)) < 0$ for any $t \geq \tau$, thus giving us $\boldsymbol{Q}(\tau, a) \in \hat{W}^s(\tau)$.

Hence, we get a contradiction comparing (58) with Lemma 49.

We consider now the remaining case $T(\tau, a) = +\infty$. Recalling (59), the only reasonable conclusion is that $\boldsymbol{\phi}(t; \tau, \boldsymbol{Q}(\tau, a)) \to (0, 0)$ as $t \to +\infty$, i.e., $\boldsymbol{Q}(\tau, a) \in W^s(\tau)$. Then, from (59) we find $\boldsymbol{Q}(\tau, a) \in \hat{W}^s(\tau)$ and get again the contradiction comparing (58) with Lemma 49. □

We can extend Lemma 50 to a wider class of functions.

Lemma 51. *Assume* $(\mathbf{H}_{\boldsymbol{\ell}})$ *with* $\ell \leq \frac{n}{p-1}$ *and that there is* $\rho_1 > 0$ *such that* $\mathcal{K}(r) \geq \mathcal{K}(\rho_1)$ *when* $r \geq \rho_1$. *Then the same conclusion as in Lemma 50 holds.*

Proof. Recall that $K(t) := \mathcal{K}(\mathrm{e}^t)$. By assumption there is $\tau_1 \in \mathbb{R}$ such that $K(t) \geq K(\tau_1)$ when $t \geq \tau_1$. Let us define

$$K_m(t) := \begin{cases} K(t) & \text{if } t \le \tau_1, \\ K(\tau_1) & \text{if } t \ge \tau_1. \end{cases}$$

Let

$$\boldsymbol{\phi_m}(t;\tau,\boldsymbol{Q}(\tau,a)) = \big(x_m(t;\tau,\boldsymbol{Q}(\tau,a)), y_m(t;\tau,\boldsymbol{Q}(\tau,a))\big)$$

be the trajectory of the modified system (10) where $K(t)$ is replaced by $K_m(t)$. Observe that $K_m(t)$ satisfies $(\mathbf{W_s})$, and we can still apply Remark 13, but $W^s(\tau)$ depends continuously and not smoothly on τ.

Since the original system and the modified one coincide for $t \le \tau_1$, then their unstable manifolds are the same in this interval.

Concerning the modified system, let $\overline{K}_m > \sup_{t\in\mathbb{R}} K_m(t) = \sup_{t\le\tau_1} K(t)$ and select the point $\overline{\boldsymbol{G}}_{\boldsymbol{m}} = (\overline{G}_{x,m}, \overline{G}_{y,m})$ as above. Then, from Remark 27 we see that for any $0 < a < \left(\frac{\alpha^q}{\overline{K}_m}\right)^{\frac{p-1}{q-p}}$, cf. (57), there is $N = N(a) > |\tau_1| > 0$ such that $W^u(\tau)$ crosses transversely $\mathcal{L}(a)$ in $\boldsymbol{Q}(\tau,a)$ for any $\tau < -N$.

From Lemma 50 applied to the modified system, we can find $\hat{N}(a) \ge N(a)$ such that for any $\tau < -\hat{N}(a)$ there are $T_m(\tau,a) \in \mathbb{R}$ and $\boldsymbol{R_m} = (0, R_m)$ such that $\boldsymbol{\phi_m}(T_m(\tau,a);\tau,\boldsymbol{Q}(\tau,a)) = \boldsymbol{R_m}$ and both $x_m(t;\tau,\boldsymbol{Q}(\tau,a)) > 0$ and $\dot{x}_m(t;\tau,\boldsymbol{Q}(\tau,a)) < 0 < \dot{y}_m(t;\tau,\boldsymbol{Q}(\tau,a))$ hold for any $\tau \le t \le T_m(\tau,a)$, see (59).

Since $\boldsymbol{\phi_m}(t;\tau,\boldsymbol{Q}(\tau,a)) \equiv \boldsymbol{\phi}(t;\tau,\boldsymbol{Q}(\tau,a))$ for any $t \le \tau_1$, if $T_m(\tau,a) \le \tau_1$ the proof is complete. So, we assume now $T_m(\tau,a) > \tau_1$. Since K_m is constant, then also $\mathcal{H}(\boldsymbol{\phi_m}(t;\tau,\boldsymbol{Q}(\tau,a));\tau_1)$ is constant when $t \ge \tau_1$, see (14). Hence,

$$\mathcal{H}(\boldsymbol{\phi_m}(t;\tau,\boldsymbol{Q}(\tau,a));\tau_1) \equiv \mathcal{H}(\boldsymbol{R_m};\tau_1) = \tfrac{p}{p-1}|R_m|^{p/(p-1)} > 0$$

for any $t \in [\tau_1, T_m(\tau,a)]$.

Let us set $\boldsymbol{S} = (S_x, S_y) := \boldsymbol{\phi}(\tau_1;\tau,\boldsymbol{Q}(\tau,a)) = \boldsymbol{\phi_m}(\tau_1;\tau,\boldsymbol{Q}(\tau,a))$. Then, from the previous estimate, we get $\mathcal{H}(\boldsymbol{S};\tau_1) = \mathcal{H}(\boldsymbol{R_m};\tau_1) > 0$. Notice that the image of $\boldsymbol{\phi_m}(t;\tau,\boldsymbol{Q}(\tau,a))$ is a graph in x, i.e., there is a decreasing smooth function f_m such that

$$\mathcal{F} := \{\boldsymbol{\phi_m}(t;\tau,\boldsymbol{Q}(\tau,a)) \mid \tau_1 \le t \le T_m(\tau,a)\} = \{(x,y) \mid y = f_m(x),\ 0 \le x \le S_x\}.$$

Let us define

$$T(\tau,a) = \sup\{s \ge \tau_1 \mid \dot{x}(t;\tau,\boldsymbol{Q}(\tau,a)) < 0 < x(t;\tau,\boldsymbol{Q}(\tau,a)) \text{ for any } \tau_1 \le t \le s\}.$$

From (14) we have

$$\mathcal{H}(\boldsymbol{\phi}(t;\tau,\boldsymbol{Q}(\tau,a));\tau_1) \ge \mathcal{H}(\boldsymbol{S};\tau_1) = \mathcal{H}(\boldsymbol{R_m};\tau_1) > 0$$

for every $\tau_1 \le t \le T(\tau,a)$. Therefore, we see that the trajectory $\boldsymbol{\phi}(t;\tau,\boldsymbol{Q}(\tau,a))$ is forced to stay in the unbounded set

$$\begin{aligned}\Lambda =& \{(x,y) \mid 0 \le x \le S_x,\ y < 0,\ \mathcal{H}(x,y;\tau_1) \ge \mathcal{H}(S_x,S_y;\tau_1)\} \\ =& \{(x,y) \mid 0 \le x \le S_x\,;\, y \le f_m(x)\}\end{aligned}$$

whenever $\tau_1 \le t \le T(\tau,a)$. Observe now that the maximum of $\dot{x}(x,y) = \alpha x - |y|^{1/(p-1)}$ within Λ is obtained in $\mathcal{F}$, which is compact; further, it has to be negative, so there is $C > 0$ such that $\dot{x}(t;\tau,\boldsymbol{Q}(\tau,a)) \le -C$ when $\tau_1 \le t \le T(\tau,a)$. Then it is easy to check that $\boldsymbol{\phi}(t;\tau,\boldsymbol{Q}(\tau,a))$ is bounded when $\tau_1 \le t \le T(\tau,a)$. So, from elementary considerations, we see that $\boldsymbol{\phi}(t;\tau,\boldsymbol{Q}(\tau,a))$ crosses the negative y-semiaxis at $t = T(\tau,a) < \tau_1 + S_x/C$. □

The previous lemmas allow us to complete the proof of Lemma 45.

Proof of Lemma 45. If we assume $(\mathbf{W_s})$, we fix a as in (57), while if we assume the second alternative in the same formula we replace $\overline{K}$ with $\overline{K}_m$. Then, we consider the segment $\mathcal{L}(a)$ defined as in (29). From Remark 27, there is $N(a) > 0$ such that $W^u(\tau)$ intersects $\mathcal{L}(a)$ in $\boldsymbol{Q}(\tau, a)$ for every $\tau < -N(a)$.

Using Lemma 20, we consider the decreasing continuous function $d_{\mathcal{L}(a)} \colon]-\infty, -N(a)[\to]D, +\infty[$ such that the solution $u(r; d_{\mathcal{L}(a)}(\tau))$ of (4) corresponds to the trajectory $\boldsymbol{\phi}(t; \tau, \boldsymbol{Q}(\tau, a))$ of (10).

Now, we can apply either Lemma 50 or 51, in order to find the value $\hat{N}(a) \geq N(a)$ such that the solution $u(r; d_{\mathcal{L}(a)}(\tau))$ is a crossing solution for every $\tau < -\hat{N}(a)$. So, if we set $\hat{D}$ such that $d_{\mathcal{L}(a)}(]-\infty, -\hat{N}(a)[) =]\hat{D}, +\infty[$, we get $]\hat{D}, +\infty[\subset J$. □

Proof of Proposition 46. Take $\varepsilon < 1/\alpha$ such that the value $a = a(\varepsilon)$ introduced in (46) satisfies the inequality (57). Define the segment $\mathcal{L}(a)$ as in (29), and $\hat{N}(a)$, $T(\cdot, a)$ as in Lemma 50 or 51. Set $\hat{D}$ as in the proof of Lemma 45.

Recalling Lemmas 20 and 45, we have $\tau_{\mathcal{L}(a)}(]\hat{D}, +\infty[) =]-\infty, -\hat{N}(a)[$ and $\lim_{d \to +\infty} \tau_{\mathcal{L}(a)}(d) = -\infty$. Moreover, the function $T(\cdot, a)$ is well-defined in $]-\infty, -\hat{N}(a)[$ and it is continuous, cf. Remark 29.

Hence, we are able to apply Proposition 42 to infer (52). □

5. Proof of the theorems and final remarks

We conclude the paper by giving the explicit proof of the main theorems. Further, at the end of the section, we suggest some future developments of the results presented in this article.

Proof of Theorem 1. If $\ell < \ell_p^*$, from Proposition 37 there is $\hat{D} > 0$ such that $]\hat{D}, +\infty[\subset J$ and $\lim_{d \to +\infty} R(d) = 0$. So, there is $\tilde{D} \geq 0$ such that $\tilde{D} \notin J$ and $]\tilde{D}, +\infty[\subset J$. Hence, from Propositions 23 and 24, we have $\lim_{d \to \tilde{D}^+} R(d) = +\infty$. Further, since R is continuous in J, see Proposition 22, we find $R(J) =]0, +\infty[$, i.e., for every $\mathcal{R} > 0$ there is $d \in J$ such that $R(d) = \mathcal{R}$, which amounts to say that $u(r; d)$ solves problem (3).

The second assertion follows immediately from Proposition 44. □

Proof of Theorem 2. The proof takes advantage of Propositions 23 and 43.

If $\ell = \ell_p^*$, we distinguish two alternatives. Either $R(]0, +\infty[) = [\mathcal{R}_0, +\infty[$ with $\mathcal{R}_0$ internal minimum of R, or $R(]0, +\infty[) =]\mathcal{R}_0, +\infty[$ with $\mathcal{R}_0 = \liminf_{d \to +\infty} R(d) > 0$. Therefore, we obtain the required result in this case.

On the other hand, if $\ell > \ell_p^*$, from $\lim_{d \to 0} R(d) = \lim_{d \to +\infty} R(d) = +\infty$, we deduce that the function R has an internal minimum $\mathcal{R}_0$, and the pre-image $R^{-1}(\mathcal{R})$ has at least two elements for every $\mathcal{R} > \mathcal{R}_0$, thus giving the multiplicity result. □

Remark 52. We want to underline that, in the critical case $\ell = \ell_p^*$, we are not able to discern which of the alternatives analyzed in the proof of Theorem 2 holds, and, consequently, we are not able to say if there is a solution for $\mathcal{R} = \mathcal{R}_0$.

Proof of Theorem 3. From Proposition 44 and Lemma 45, the set J contains a nontrivial interval, and there exists $\mathcal{R}_0 := \inf_{d \in J} R(d) > 0$. Then, the proof follows the lines of the one of Theorem 2, profiting from Propositions 22, 23, 24 and 46. □

In this paper, we dealt with the differential equation (1) under the restrictive assumption $2\,\frac{n}{2+n} \leq p \leq 2$, in order to obtain a sufficiently smooth dynamical system (10). This allowed us to profit of classical techniques borrowed from invariant manifold theory for non-autonomous dynamical systems, cf. [9,24]. In order to remove this hypothesis, we could possibly rely on Wazewski's principle in the spirit of [14,15], or, alternatively, we could try to develop a degree theory approach. However, such a generalization could make the proofs even longer and more involved, so it will be the subject of a forthcoming paper.

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