ON A HIGHER-DIMENSIONAL WORM DOMAIN AND ITS GEOMETRIC PROPERTIES

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Abstract. We construct new 3-dimensional variants of the classical Diederich-Fornæss worm domain. We show that they are smoothly bounded, pseudoconvex, and have nontrivial Nebenhülle. We also show that their Bergman projections do not preserve the Sobolev space for sufficiently large Sobolev indices.

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1. Introduction

In the paper [DF77], Diederich and Fornæss constructed a class of smoothly bounded, pseudoconvex domains in $\mathbb{C}^2$, denoted here as Ω_μ for $\mu > 0$, having nontrivial Nebenhülle, i.e., such that the intersection of all pseudoconvex domains containing the closure $\overline{\Omega}_\mu$ properly contains Ω_μ. Such a domain Ω_μ in the two complex variables z_1, z_2, now called (smooth) *worm domain*, is a union of disks in the z_1-plane, each centered at the unitary point $e^{i\log|z_2|^2}$. In other words, Ω_μ is a union of disks winding about the origin in the z_1-plane according to a rotation angle $\log|z_2|^2$. The radii of these disks are 1 when $\log|z_2|^2$ belongs to the interval $I_\mu = (-\mu, \mu)$, and fade to 0 as $|\log|z_2|^2|$ ranges from μ to some finite $\mu' > \mu$. This ground-breaking idea of Diederich and Fornæss makes every defining function for Ω_μ not globally plurisubharmonic. Moreover, the boundary $\partial\Omega_\mu$ is strongly Levi-pseudoconvex at every point, apart from the 1-dimensional exceptional set $\{0\} \times A$, where A is an annulus centered at the origin in the z_2-plane, determined by the condition $\log|z_2|^2 \in I_\mu$. Thanks to these features, the domain Ω_μ also serves as a counterexample to a number of important geometric phenomena. We mention in particular the article [Bar92] where, for $\mu > \frac{\pi}{2}$, Barrett showed that the Bergman projection on Ω_μ fails to preserve the Sobolev space $W^{r,2}(\Omega_\mu)$ when $r \geq \nu = \frac{\pi}{2\mu}$. Based on Barrett's result, with a clever and far-reaching argument, Christ [Chr96] proved that the Bergman projection and the Neumann operator on Ω_μ fail to be globally regular. These properties are well described also in the monographs [CS01, Str10]. Besides the non-trivial Nebenhülle and the already mentioned questions of global regularity, the worm domains are "testing ground" for several other problems. Here we just mention some of the most recent papers on these topics, [ČŞ18, DM23, KPS16, KPS19, KMPS23, Liu19] and references therein.

It is a matter of great interest to fully understand the worm, and to put it in a more general context. For instance, the recent article [ADF23] (referring to [Bar98]) highlights a crucial geometric feature of the worm domain and uses it to generalize the construction of Diederich and Fornæss to a wider class of 2-dimensional domains. It is also interesting to investigate analogous constructions in higher dimensions. The article [BŞ12] constructed analogs of the worm domain within $\mathbb{C}^n$. There, the boundary is strongly Levi-pseudoconvex at every point apart from a 1-dimensional exceptional set $\{\mathbf{0}\} \times A$, where $\mathbf{0}$ is the origin in $\mathbb{C}^{n-1}$ and A is an annulus centered at the origin in the z_n-plane.

In the present work, we construct for every $\mu > 0$ a new class $\mathscr{C}_\mu$ of smoothly bounded, pseudoconvex domains in $\mathbb{C}^3$. Every $\Omega \in \mathscr{C}_\mu$ is, again, a union of disks winding about the origin in the z_1-plane; but now the rotation angle is $\log|z_2 z_3|^2$, where (z_2, z_3) is confined to a bounded neighborhood of the product of annuli $A \times A$ determined by the condition $(\log|z_2|^2, \log|z_3|^2) \in I_\mu \times I_\mu$. The boundary $\partial\Omega$ turns out to be weakly Levi-pseudoconvex at every point $z = (z_1, z_2, z_3) \in \partial\Omega$ with $(z_2, z_3) \in A \times A$. Moreover, $\partial\Omega$ includes the 2-dimensional complex manifold $\{0\} \times A \times A$. We prove that Ω, just like the original worm domain of [DF77], has nontrivial Nebenhülle when $\mu > \frac{\pi}{2}$. We also prove, following the lines of [Bar92], that the Bergman projection of Ω does not preserve the Sobolev space $W^{r,2}(\Omega)$ when $\mu > \frac{\pi}{2}, r \geq \nu = \frac{\pi}{2\mu}$.

The paper is structured as follows:

Section 2 constructs a wide family of domains $\mathcal{W}_\eta$ in $\mathbb{C}^3$, varying with the choice of a function η. Two conditions on η are determined, to make the boundary $\partial\mathcal{W}_\eta$ smooth except at (possible) boundary points z with $z_2 z_3 = 0$. A third condition on η is then added, to make $\mathcal{W}_\eta$ smoothly bounded.

Section 3 starts with a characterization of pseudoconvexity of $\mathcal{W}_\eta$ by means of two inequalities in the first and second derivatives of η. Some unbounded or nonsmooth examples are immediately provided. Then $\mathscr{C}_\mu$ is defined as the class of the $\mathcal{W}_\eta$ with η fulfilling all three conditions for boundedness and smoothness, as well as the inequalities for pseudoconvexity. Finally, $\mathscr{C}_\mu$ is proven not empty by constructing an explicit class of examples in Theorem 3.11, which is the main result of this work.

Section 4 is devoted to the study of any smoothly bounded, pseudoconvex domain Ω in the class $\mathscr{C}_\mu$. In particular, for the case when $\mu > \frac{\pi}{2}$: Theorem 4.5 shows that Ω has nontrivial Nebenhülle; Theorem 4.6 concerns the irregularity of the Bergman projection of Ω.

Section 5 comprises the proof of Theorem 4.6, as well as several tools used in the proof.

2. Basic construction

We define a family of domains $\mathcal{W}_\eta$ in dimension 3, varying with the choice of a function η. Making different assumptions on η, we will later prove different properties of $\mathcal{W}_\eta$. As customary, the word domain here means a nonempty connected open subset of $\mathbb{C}^3$.

Definition 2.1. Fix a real number $\mu > \pi$. Let $\eta : \mathbb{R}^2 \to [0, +\infty)$ be upper semicontinuous, with $\eta^{-1}(0)$ including the square $(-\mu, \mu) \times (-\mu, \mu)$ and with $\eta^{-1}([0, 1))$ path-connected. Set $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$ and

$$\mathcal{W} = \mathcal{W}_\eta = \left\{ (z_1, z_2, z_3) \in \mathbb{C} \times \mathbb{C}^* \times \mathbb{C}^* : \left| z_1 - e^{i \log|z_2 z_3|^2} \right|^2 < 1 - \eta(\log|z_2|^2, \log|z_3|^2) \right\}.$$

Equivalently, $\mathcal{W}$ is the subset of $\mathbb{C} \times \mathbb{C}^* \times \mathbb{C}^*$ having the upper semicontinuous function

$$\rho(z_1, z_2, z_3) = |z_1|^2 - 2\,\mathrm{Re}(z_1 e^{-i \log|z_2 z_3|^2}) + \eta(\log|z_2|^2, \log|z_3|^2)$$

as a defining function.

In what follows, we let $\triangle(P,r)$ denote the disk in the complex plane with center P and radius r. We also let $\triangle^*(P,r)$ denote the punctured disk. Finally, we let $A(P,r_1,r_2)$ denote the planar annulus with center P and radii $r_1 < r_2$.

The next remark decomposes $\mathcal{W}$ into a family of disks that rotate about the origin in $\mathbb{C}$.

Remark 2.2. If we set

$$R(z_2,z_3) := 1 - \eta(\log|z_2|^2, \log|z_3|^2)\,,$$

then

$$\mathcal{W} = \bigcup_{R(z_2,z_3)>0} \triangle\big(e^{i\log|z_2z_3|^2}, R(z_2,z_3)\big) \times \{(z_2,z_3)\}\,.$$

As a result, if we let π_j denote projection on the jth variable, we have $\pi_1(\mathcal{W}) = \triangle^*(0,2)$.

Remark 2.3. Let us show that $\mathcal{W}$ is a domain in $\mathbb{C}^3$.

The set $\mathcal{W}$ is an open subset of $\mathbb{C}\times\mathbb{C}^*\times\mathbb{C}^*$ (whence an open subset of $\mathbb{C}^3$) because its defining function $\rho : \mathbb{C}\times\mathbb{C}^*\times\mathbb{C}^* \to \mathbb{R}$ is upper semicontinuous.

Additionally, $\mathcal{W}$ is path-connected because for every $z=(z_1,z_2,z_3)\in\mathcal{W}$ we can find a path in $\mathcal{W}$ joining z to the point $(1,1,1)$. We begin by observing that, since $\eta^{-1}([0,1))$ is path-connected by hypothesis, there exists a path $\gamma=(\gamma_2,\gamma_3)$ from $(\log|z_2|^2,\log|z_3|^2)$ to $(0,0)$ in $\eta^{-1}([0,1))$. Now, $z=(z_1,z_2,z_3)$ can be joined by a line segment within $\mathcal{W}$ to the point $(e^{i\log|z_2z_3|^2}, z_2, z_3)$, which can be joined to the point $(1,\frac{z_2}{|z_2|},\frac{z_3}{|z_3|})$ by means of the path

$$[0,1]\ni s \mapsto \left(e^{i\gamma_2(s)+i\gamma_3(s)}, e^{\gamma_2(s)/2}\frac{z_2}{|z_2|}, e^{\gamma_3(s)/2}\frac{z_3}{|z_3|}\right) \in \mathcal{W}\,.$$

Finally, every point of the form $(1,e^{it_2},e^{it_3})$ can be joined to $(1,1,1)$ by means of the path

$$[0,1]\ni s\mapsto \left(1, e^{i(1-s)t_2}, e^{i(1-s)t_3}\right)\in\mathcal{W}\,.$$

The main body of $\mathcal{W}$ consists of those disks that have radius 1. Because of the choices made in Definition 2.1, the set $R^{-1}(1)$ of those $(z_2,z_3)\in(\mathbb{C}^*)^2$ such that $\eta(\log|z_2|^2,\log|z_3|^2)=0$ includes

$$A(0,e^{-\mu/2},e^{\mu/2})\times A(0,e^{-\mu/2},e^{\mu/2}).$$

Restricting to this main body for the sake of simplicity, we can study the fiber $\pi_1^{-1}(z_1)$ over each $z_1\in\triangle^*(0,2)$ as follows.

Proposition 2.4. *Fix any point $z_1=\rho_0e^{i\theta_0}$ with $0<\rho_0<2$ and $\theta_0\in\mathbb{R}$. For any $k\in\mathbb{Z}$, let*

$$r_k^{\pm} := \exp(\theta_0/2 \pm [\arccos(\rho_0/2)]/2 + \pi k)\,.$$

If $(z_2,z_3)\in R^{-1}(1)$, then (z_1,z_2,z_3) belongs to $\pi_1^{-1}(z_1)$ if and only if

$$z_2z_3 \in \bigcup_{k\in\mathbb{Z}} A(0,r_k^-,r_k^+)\,.$$

Proof. If $(z_2,z_3)\in R^{-1}(1)$, then $\eta(\log|z_2|^2,\log|z_3|^2)=0$. Also

$$\begin{aligned}\rho(z_1,z_2,z_3) &= |z_1|^2 - 2\,\mathrm{Re}(z_1e^{-i\log|z_2z_3|^2})\\ &= \rho_0^2 - 2\rho_0\,\mathrm{Re}(e^{i(\theta_0-\log|z_2z_3|^2)})\\ &= \rho_0(\rho_0 - 2\cos(\theta_0-\log|z_2z_3|^2))\,.\end{aligned}$$

This quantity is negative if and only if $\cos(\theta_0 - \log|z_2 z_3|^2) > \rho_0/2$. This is, in turn, equivalent to having

$$\theta_0 - \arccos(\rho_0/2) + 2\pi k < \log|z_2 z_3|^2 < \theta_0 + \arccos(\rho_0/2) + 2\pi k\,,$$

that is, $r_k^- < |z_2 z_3| < r_k^+$ for some $k \in \mathbb{Z}$. □

For each $z = (z_1, z_2, z_3) \in \mathbb{C} \times \mathbb{C}^* \times \mathbb{C}^*$, we adopt the following notations. We set $L := \log|z_2 z_3|^2$. Moreover, referring to the map $(t_2, t_3) \mapsto \eta(t_2, t_3)$, we set

$$\eta^\circ := \eta(\log|z_2|^2, \log|z_3|^2)$$

and, if η is smooth,

$$\eta'_j := \frac{\partial \eta}{\partial t_j}(\log|z_2|^2, \log|z_3|^2)$$

$$\eta''_{j,k} := \frac{\partial^2 \eta}{\partial t_j \partial t_k}(\log|z_2|^2, \log|z_3|^2)$$

for all $j, k \in \{2, 3\}$.

Remark 2.5. Fix $(z_1, z_2, z_3) \in \mathbb{C} \times \mathbb{C}^* \times \mathbb{C}^*$. By direct computation, we see that

$$\frac{\partial}{\partial z_1}\rho(z_1, z_2, z_3) = \overline{z}_1 - e^{-iL}\,.$$

Moreover, if η is smooth, then

$$\frac{\partial}{\partial z_j}\rho(z_1, z_2, z_3) = -2\,\mathrm{Im}(z_1 e^{-iL}) z_j^{-1} + \eta'_j z_j^{-1} \quad \text{for} \quad j \in \{2, 3\}\,.$$

Now, for appropriate choices of η, we can prove a first significant result about the geometry of $\mathcal{W}_\eta$.

Proposition 2.6. *Under the following additional assumptions:*

(I) *η is smooth in an open neighborhood of $\eta^{-1}([0,1])$;*

(II) *at each point of $\eta^{-1}(1)$ (if any), the gradient of η does not vanish;*

then $\mathcal{W}_\eta$ is a domain in $\mathbb{C}^3$ whose boundary is smooth except at the boundary points z with $z_2 z_3 = 0$ (if any). For each $z = (z_1, z_2, z_3) \in \partial\mathcal{W}_\eta$, if $z_1 \neq e^{iL}$ and $z_2 z_3 \neq 0$, then the complex tangent space to $\partial\mathcal{W}_\eta$ at z is spanned by the vectors

$$\mathbf{v}(z_1, z_2, z_3) := \begin{pmatrix} 2\,\mathrm{Im}(z_1 e^{-iL}) - \eta'_2 \\ z_2(\overline{z}_1 - e^{-iL}) \\ 0 \end{pmatrix}, \qquad \mathbf{w}(z_1, z_2, z_3) := \begin{pmatrix} 2\,\mathrm{Im}(z_1 e^{-iL}) - \eta'_3 \\ 0 \\ z_3(\overline{z}_1 - e^{-iL}) \end{pmatrix}.$$

In the (non generic) case when $z_1 = e^{iL}$ and $z_2 z_3 \neq 0$, the complex tangent space to $\partial\mathcal{W}_\eta$ at z is spanned by the vectors

$$\mathbf{v}(z_1, z_2, z_3) := \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{w}(z_1, z_2, z_3) := \begin{pmatrix} 0 \\ z_2 \eta'_3 \\ -z_3 \eta'_2 \end{pmatrix}.$$

Under the further additional assumption:

(III) *$\eta^{-1}([0,1])$ is a bounded subset of $\mathbb{R}^2$;*

then $\mathcal{W}_\eta$ is a smoothly bounded domain in $\mathbb{C}^3$.

Proof. We first work under assumptions (I) and (II) only. Suppose that $z = (z_1, z_2, z_3)$ is a boundary point of $\mathcal{W}_\eta$ with $z_2 z_3 \neq 0$, so that $\rho(z) = 0$. If $\partial\rho/\partial z_1(z) = 0$, then $z_1 = e^{iL}$. This in turn implies that

$$0 = \rho(z) = -1 + \eta(\log|z_2|^2, \log|z_3|^2)\,.$$

As a result, $(\log|z_2|^2, \log|z_3|^2) \in \eta^{-1}(1)$. Thus, $\eta'_j \neq 0$ for some $j \in \{2, 3\}$. The equality $z_1 = e^{iL}$ also implies that

$$\frac{\partial}{\partial z_j}\rho(z) = \eta'_j z_j^{-1}\,.$$

In conclusion, $\partial\rho/\partial z_j(z) \neq 0$. Since the gradient of ρ does not vanish at z, the boundary $\partial\mathcal{W}_\eta$ is smooth at z. The generators of the complex tangent space can be computed directly.

We now take all assumptions (I),(II),(III). In particular, there exists $\mu' > \mu > 0$ such that $\eta^{-1}([0,1])$ is included in a closed disk of radius μ' centered at the origin in $\mathbb{R}^2$. Now, if $z = (z_1, z_2, z_3) \in \mathcal{W}_\eta$, then $R(z_2, z_3) > 0$, whence $\eta(\log|z_2|^2, \log|z_3|^2) < 1$ and $e^{-\mu'/2} < |z_2|, |z_3| < e^{\mu'/2}$. Thus,

$$\mathcal{W}_\eta \subseteq \triangle^*(0,2) \times A(0, e^{-\mu'/2}, e^{\mu'/2}) \times A(0, e^{-\mu'/2}, e^{\mu'/2})$$

is a bounded domain and there exists no boundary point $w = (w_1, w_2, w_3)$ of $\mathcal{W}_\eta$ having $w_2 w_3 = 0$. □

Remark 2.7. *Under our additional hypotheses (I) and (II), we have that the complex tangent space to $\partial\mathcal{W}_\eta$ at each boundary point of the form $(0, w_2, w_3)$ equals*

$$\operatorname{span}_{\mathbb{C}}\left\{\begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}.$$

3. Study of pseudoconvexity

As it happened with the original worm domain of [DF77], we will now study the pseudoconvexity of $\mathcal{W}_\eta$. We begin with a useful remark.

Remark 3.1. *Fix $w = \big(w_1, w_2, w_3\big) \in \overline{\mathcal{W}}$ with $\varepsilon := \big|w_2\, w_3\big| > 0$. Then*

$$U_w := \Big\{(z_1, z_2, z_3) \in \mathbb{C}^3 \;:\; (z_2\, z_3)^2 \in \triangle\big((w_2\, w_3)^2, \varepsilon^2\big)\Big\}$$

is an open neighborhood of w in $\mathbb{C}^3$. Moreover, $\triangle\big((w_2\, w_3)^2, \varepsilon^2\big)$ is a simply connected open subset of $\mathbb{C}$ missing the origin, where a branch of logarithm log *(whence a branch* arg *of the argument function) is defined. In U_w, the product*

$$\begin{aligned}\widetilde{\rho}(z_1, z_2, z_3) :=& e^{\arg(z_2\, z_3)^2}\rho(z_1, z_2, z_3)\\ =& e^{\arg(z_2\, z_3)^2}|z_1|^2 - 2\operatorname{Re}(z_1 e^{-i\log(z_2\, z_3)^2}) + e^{\arg(z_2\, z_3)^2}\eta(\log|z_2|^2, \log|z_3|^2)\end{aligned}$$

is a local defining function for $\mathcal{W}_\eta$.

We are now ready for our first main theorem. In the proof, and throughout the paper, we will use the notation

$$h_M(\mathbf{u}_1, \mathbf{u}_2) := \mathbf{u}_1^t M \bar{\mathbf{u}}_2 \tag{1}$$

for every complex square matrix M of order 3 and any $\mathbf{u}_1, \mathbf{u}_2 \in \mathbb{C}^3$.

Theorem 3.2. *We take the extra assumptions (I),(II) on η, so that $\partial\mathcal{W}_\eta$ is smooth, except at the boundary points w where $w_2\, w_3 = 0$. The domain $\mathcal{W}_\eta$ is pseudoconvex if, and only if,*

$$\eta^\circ + \eta''_{22} \geq 0 \tag{2}$$

$$\eta''_{22}\eta''_{33} - (\eta''_{23})^2 + \eta^\circ(\eta''_{22} + \eta''_{33} - 2\eta''_{23}) - (\eta'_2 - \eta'_3)^2 \geq 0 \tag{3}$$

near $\partial\mathcal{W}_\eta$.

Proof. Fix $w = \big(w_1, w_2, w_3\big) \in \overline{\mathcal{W}_\eta}$ with $\big|w_2\, w_3\big| > 0$. Let us adopt the notations of Remark 3.1 and argue locally in U_w, studying whether the local defining function $\widetilde{\rho}$ is plurisubharmonic. The second addend in $\widetilde{\rho}(z)$, namely $-2\,\mathrm{Re}(z_1 e^{-i\log(z_2\, z_3)^2})$, is the real part of a holomorphic function, whence a real-valued pluriharmonic function. We are left with studying whether the sum of $m(z) := e^{\arg(z_2\, z_3)^2}|z_1|^2$ and $n(z) := e^{\arg(z_2\, z_3)^2}\eta(\log|z_2|^2, \log|z_3|^2)$ is plurisubharmonic. Let us compute the complex Hessian matrix of $m(z)$ at $z = (z_1, z_2, z_3) \in U_w$. By direct computation we see that

$$\begin{aligned}
\frac{\partial}{\partial z_1} m(z) &= e^{\arg(z_2\, z_3)^2}\overline{z}_1 \\
\frac{\partial}{\partial z_j} m(z) &= e^{\arg(z_2\, z_3)^2}\frac{|z_1|^2}{iz_j} \quad \text{for} \quad j \in \{2,3\}\,,
\end{aligned}$$

whence the complex Hessian matrix of $m(z)$ is $e^{\arg(z_2\, z_3)^2}M(z)$, where

$$M(z) := \begin{pmatrix} 1 & i\frac{\bar z_1}{\bar z_2} & i\frac{\bar z_1}{\bar z_3} \\ -i\frac{z_1}{z_2} & \frac{|z_1|^2}{|z_2|^2} & \frac{|z_1|^2}{z_2\bar z_3} \\ -i\frac{z_1}{z_3} & \frac{|z_1|^2}{\bar z_2 z_3} & \frac{|z_1|^2}{|z_3|^2} \end{pmatrix}.$$

Let us consider the vectors

$$\mathbf{p} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{n}_1 = \begin{pmatrix} 0 \\ z_2 \\ -z_3 \end{pmatrix}, \quad \mathbf{n}_2 = \begin{pmatrix} 2iz_1 \\ z_2 \\ z_3 \end{pmatrix}.$$

Since

$$M\bar{\mathbf{p}} = \begin{pmatrix} 1 \\ -i\frac{z_1}{z_2} \\ -i\frac{z_1}{z_3} \end{pmatrix}, \quad M\bar{\mathbf{n}}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad M\bar{\mathbf{n}}_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

the vectors $\mathbf{p}, \mathbf{n}_1, \mathbf{n}_2$ form an orthogonal basis with respect to h_M, the real number $h_M(\mathbf{p}, \mathbf{p}) = 1$ is positive and $\bar{\mathbf{n}}_1, \bar{\mathbf{n}}_2$ belong to the null space of M. In particular, M is positive semidefinite and m is plurisubharmonic.

Now assume η to fulfill the assumptions (I),(II) of Proposition 2.6. The equalities

$$\begin{aligned}
\frac{\partial}{\partial z_1} n(z) &= 0 \\
\frac{\partial}{\partial z_j} n(z) &= e^{\arg(z_2\, z_3)^2}\left(\frac{\eta^\circ}{iz_j} + \frac{\eta'_j}{z_j}\right) \quad \text{for} \quad j \in \{2,3\} \\
\frac{\partial^2}{\partial z_j \partial \bar z_k} n(z) &= \frac{e^{\arg(z_2\, z_3)^2}}{-i\bar z_k}\left(\frac{\eta^\circ}{iz_j} + \frac{\eta'_j}{z_j}\right) + e^{\arg(z_2\, z_3)^2}\left(\frac{\eta'_k}{iz_j\bar z_k} + \frac{\eta''_{j,k}}{z_j\bar z_k}\right) \quad \text{for} \quad j,k \in \{2,3\}
\end{aligned}$$

imply that the complex Hessian matrix of $n(z)$ is $e^{\arg(z_2\,z_3)^2}N(z)$, where

$$N(z) := \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{\eta^\circ+\eta''_{22}}{|z_2|^2} & \frac{\eta^\circ+\eta''_{23}+i(\eta'_2-\eta'_3)}{z_2\bar{z}_3} \\ 0 & \frac{\eta^\circ+\eta''_{23}+i(\eta'_3-\eta'_2)}{\bar{z}_2 z_3} & \frac{\eta^\circ+\eta''_{33}}{|z_3|^2} \end{pmatrix}.$$

Clearly, $\bar{\mathbf{p}}$ belongs to the null space of N. Thus, the sum $m+n$ (whence $\widetilde{\rho}$) is plurisuharmonic if, and only if, n is. We are therefore left with studying whether N is positive semidefinite, or, equivalently, whether the minor N_{11} is. The $(1,1)$ entry of N_{11} is non negative if, and only if, $\eta^\circ+\eta''_{22}\geq 0$. Moreover, $\det N_{11}\geq 0$ if, and only if,

$$\begin{aligned} |z_2 z_3|^2 \det N_{11} &= (\eta^\circ+\eta''_{22})(\eta^\circ+\eta''_{33}) - (\eta^\circ+\eta''_{23})^2 - (\eta'_2-\eta'_3)^2 \\ &= \eta''_{22}\eta''_{33} - (\eta''_{23})^2 + \eta^\circ(\eta''_{22}+\eta''_{33}-2\eta''_{23}) - (\eta'_2-\eta'_3)^2 \end{aligned}$$

is non negative. The thesis follows. □

Theorem 3.2 allows us to provide our first three examples of pseudoconvex domains in $\mathbb{C}^3$.

Corollary 3.3.

(i) If $\eta\equiv 0$, then $\mathcal{W}_\eta=\mathcal{W}_0$ is an unbounded pseudoconvex domain, whose boundary is smooth except at the points w with $w_2w_3=0$.

(ii) If η is the characteristic function of the complement of the square $(-\mu,\mu)\times(-\mu,\mu)$, then $\mathcal{W}_\eta$ is a bounded pseudoconvex domain, whose boundary is smooth except at the points w with $\log|w_2|^2=\pm\mu$ or $\log|w_3|^2=\pm\mu$.

(iii) $\mathcal{W}_\eta$ is an unbounded pseudoconvex domain with smooth boundary if we pick

$$\eta(t_2,t_3)=\phi(t_2+t_3), \quad \text{i.e.,} \quad \eta^\circ=\phi(\log|z_2z_3|^2)\,,$$

for a smooth, strictly convex and even function ϕ having $\phi^{-1}(0)=[-\mu,\mu]$ and $\phi^{-1}([0,1])=[-\mu',\mu']$ for some real number $\mu'>\mu$.

Proof. To prove the first and third statements, we will apply Theorem 3.2. The second statement will follow from the first statement.

(i) If $\eta\equiv 0$, then the additional assumptions (I) and (II) are fulfilled and, for all $z\in\mathbb{C}\times\mathbb{C}^*\times\mathbb{C}^*$, equality obviously holds in the inequalities (2), (3) of Theorem 3.2.

(ii) If η is the characteristic function of the complement of the square $(-\mu,\mu)\times(-\mu,\mu)$, then $\mathcal{W}_\eta$ is the intersection between the unbounded pseudoconvex domain $\mathcal{W}_0$ of $\mathbb{C}\times\mathbb{C}^*\times\mathbb{C}^*$ and the pseudoconvex domain $\mathbb{C}\times A(0,e^{-\mu/2},e^{\mu/2})\times A(0,e^{-\mu/2},e^{\mu/2})$ of $\mathbb{C}^3$. It follows that $\mathcal{W}_\eta$ is a pseudoconvex domain in $\mathbb{C}^3$. Every boundary point $w=\big(w_1,w_2,w_3\big)$ either is a boundary point of $\mathcal{W}_0$ with $w_2\,w_3\neq 0$ or has $|w_j|=e^{\pm\mu/2}$ for some $j\in\{2,3\}$. The thesis follows.

(iii) If $\eta(t_2,t_3)=\phi(t_2+t_3)$, then the additional assumptions (I) and (II) are fulfilled. Moreover, for all $z\in\mathbb{C}\times\mathbb{C}^*\times\mathbb{C}^*$ we find that $\eta'_2=\eta'_3=\phi'(\log|z_2z_3|^2)$ and that $\eta''_{22}=\eta''_{33}=\eta''_{23}=\phi''(\log|z_2z_3|^2)$. Thus, inequality (2) is fulfilled and an equality holds in (3). Finally, each $w=\big(w_1,w_2,w_3\big)$ in the finite boundary $\partial\mathcal{W}_\eta$ has $\log|w_2\,w_3|^2\in[-\mu',\mu']$, whence $w_2\,w_3\neq 0$. □

For the unbounded non smooth domain defined in case (i) and for the bounded non smooth domain defined in case (ii) of Corollary 3.3, it is convenient to set the following notations.

Definition 3.4. We set the notations $C_\mu := \mathbb{R}^2 \setminus (-\mu,\mu)\times(-\mu,\mu)$ and $\mathcal{W}'_\mu := \mathcal{W}_{\chi_{C_\mu}}$, as well as $\mathcal{W}'_\infty := \mathcal{W}_0$.

Remark 3.5. The equality $\mathcal{W}'_\infty = \bigcup_{\nu\in(0,+\infty)} \mathcal{W}'_\nu$ holds true.

If we fix $z_3 = 1$ in the bounded non smooth domain $\mathcal{W}'_\mu$ constructed in case (ii), we recover the truncated worm domains studied in [Kis91] and in subsequent literature. If we fix $z_3 = 1$ in the unbounded smooth domain constructed in case (iii) of Corollary 3.3, we recover the original Diederich-Fornaess worm domains of $\mathbb{C}^2$. Our next goal is showing that the following class of smoothly bounded pseudoconvex domains is not empty.

Definition 3.6. We let $\mathscr{C}_\mu$ denote the class of all $\mathcal{W}_\eta$ with η such that the extra assumptions (I),(II),(III) of Proposition 2.6 and the inequalities (2), (3) are all fulfilled.

Our first attempt to prove that $\mathscr{C}_\mu$ is not empty leads to a negative result, which is however instructive.

Corollary 3.7. *If we pick*

$$\eta(t_2,t_3) = \phi(t_2) + \psi(t_3)$$

for smooth, convex and even functions ϕ,ψ having $\phi^{-1}(0) = \psi^{-1}(0) = [-\mu,\mu]$ and $\phi^{-1}([0,1]) = [-\mu',\mu'], \psi^{-1}([0,1]) = [-\xi,\xi]$ (for some $\mu',\xi > \mu$), then $\mathcal{W}_\eta$ is a smoothly bounded domain in $\mathbb{C}^3$ that is **not** *pseudoconvex.*

Proof. Under such hypotheses, we find $\eta'_2 = \phi'(\log|z_2|^2), \eta'_3 = \psi'(\log|z_3|^2), \eta''_{22} = \phi''(\log|z_2|^2), \eta''_{33} = \psi''(\log|z_3|^2), \eta''_{23} \equiv 0$. For $z = (z_1,z_2,z_3) \in \mathbb{C}\times\mathbb{C}^*\times A(0,e^{-\mu/2},e^{\mu/2})$, we get that $\eta^\circ = \phi(\log|z_2|^2)$, whence $\eta'_3 = \eta''_{33} = \eta''_{23} = 0$. Thus, inequality (3) becomes

$$\eta^\circ\eta''_{22} - (\eta'_2)^2 \geq 0\,.$$

Thus,

$$\phi\phi'' - (\phi')^2 \geq 0$$

in $(\mu,\mu+\varepsilon)$ for some $\varepsilon > 0$. This means that ϕ coincides away from $[-\mu,\mu]$ with e^f, for some f with $f'' \geq 0$ in $(\mu,\mu+\varepsilon)$. In other words, ϕ is logarithmically convex in $(\mu,\mu+\varepsilon)$. But this contradicts the hypothesis $\lim_{t\to\mu^+}\phi(t) = 0$. Indeed, if $\lim_{t\to\mu^+} f(t) = -\infty$, then

$$\lim_{t\to\mu^+}\int_t^{\mu'} f'(\tau)d\tau = \lim_{t\to\mu^+}[f(\mu') - f(t)] = +\infty\,.$$

As a consequence, there exists a sequence $\{s_n\}_{n\in\mathbb{N}} \subset (\mu,\mu')$ with $\lim_{n\to+\infty} s_n = \mu$ such that $\lim_{n\to+\infty} f'(s_n) = +\infty$. Thus,

$$\lim_{n\to+\infty}\int_{s_n}^{\mu'} f''(\tau)d\tau = \lim_{n\to+\infty}[f'(\mu') - f'(s_n)] = -\infty$$

This implies that $\liminf_{\tau\to\mu^+} f''(\tau) = -\infty$. □

The computations made in the last proof motivate the next remark and the subsequent proposition.

Remark 3.8. Convexity of η suffices to guarantee inequality (2) but not inequality (3).

Proposition 3.9. *Assume $\eta = \chi_U e^f$ for some open subset U of $\mathbb{R}^2$ and some smooth function $f : U \to \mathbb{R}$. Set $f^\circ := f(\log|z_2|^2, \log|z_3|^2)$, $f'_j := \frac{\partial f}{\partial t_j}(\log|z_2|^2, \log|z_3|^2)$ and $f''_{j,k} := \frac{\partial^2 f}{\partial t_j \partial t_k}(\log|z_2|^2, \log|z_3|^2)$. Then the equalities*

$$\begin{aligned}\eta^\circ &= e^{f^\circ}\,,\\ \eta'_j &= \eta^\circ f'_j\,,\\ \eta''_{j,k} &= \eta^\circ(f'_j f'_k + f''_{j,k})\,,\end{aligned}$$

hold true in U for all $j,k \in \{2,3\}$. If we set $H_f := \begin{pmatrix} f''_{22} & f''_{23} \\ f''_{23} & f''_{33}\end{pmatrix}$, $v := \begin{pmatrix} f'_3 \\ -f'_2 \end{pmatrix}$, $u := \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ then, for the terms appearing in inequality (3), we have

$$\begin{aligned}&\eta''_{22}\eta''_{33} - (\eta''_{23})^2 = \det H_\eta = (\eta^\circ)^2[h_{H_f}(v,v) + \det H_f]\\ &\eta^\circ(\eta''_{22} + \eta''_{33} - 2\eta''_{23}) - (\eta'_2 - \eta'_3)^2 = \eta^\circ h_{H_\eta}(u,u) - (\eta'_2 - \eta'_3)^2 = (\eta^\circ)^2 h_{H_f}(u,u)\,.\end{aligned}$$

Therefore, inequalities (2), (3) are equivalent, respectively, to

$$(f'_2)^2 + f''_{22} + 1 \geq 0\,, \tag{4}$$

$$h_{H_f}(v,v) + \det H_f + h_{H_f}(u,u) \geq 0\,. \tag{5}$$

Proof. We compute

$$\begin{aligned}\eta''_{22}\eta''_{33} - (\eta''_{23})^2 &= (\eta^\circ)^2\{[(f'_2)^2 + f''_{22}][(f'_3)^2 + f''_{33}] - (f'_2 f'_3 + f''_{23})^2\}\\ &= (\eta^\circ)^2[(f'_2)^2 f''_{33} + (f'_3)^2 f''_{22} - 2f'_2 f'_3 f''_{23} + f''_{22}f''_{33} - (f''_{23})^2]\\ &= (\eta^\circ)^2[v^t H_f v + \det H_f]\\ \eta^\circ(\eta''_{22} + \eta''_{33} - 2\eta''_{23}) - (\eta'_2 - \eta'_3)^2 &= (\eta^\circ)^2[(f'_2)^2 + f''_{22} + (f'_3)^2 + f''_{33} - 2f'_2 f'_3 - 2f''_{23} - (f'_2 - f'_3)^2]\\ &= (\eta^\circ)^2[f''_{22} + f''_{33} - 2f''_{23}]\\ &= (\eta^\circ)^2 u^t H_f u\,.\end{aligned}$$

□

The last result is useful to prove that the class $\mathscr{C}_\mu$ is not empty, thus finding examples of smoothly-bounded pseudoconvex domains $\mathcal{W}_\eta$. The next proposition will also be useful in the construction.

Proposition 3.10. *Let $U_0, \dots, U_l$ be open subsets of $\mathbb{R}^2$, take smooth functions $\eta_0, \dots, \eta_l : \mathbb{R}^2 \to \mathbb{R}$ and set*

$$\eta := \eta_0 + \ldots + \eta_l\,.$$

If each η_k fulfils inequality (2) separately, then η itself fulfils inequality (2). This is automatically true if $\eta_0, \dots, \eta_l$ are convex. If we assume $l = 1$, i.e., $\eta := \eta_0 + \eta_1$, we can compute the quantity

$$\mathcal{P}(\eta) = \det H_\eta + \eta^\circ h_{H_\eta}(u,u) - (\eta'_2 - \eta'_3)^2$$

appearing on the left-hand side of inequality (3) as follows:

$$\begin{aligned}\mathcal{P}(\eta) &= \mathcal{P}(\eta_0) + \mathcal{P}(\eta_1) + \eta_0^\circ \det H_{\eta_1} + \eta_1^\circ \det H_{\eta_0}\\ &\quad + (\eta_0)''_{22}(\eta_1)''_{33} + (\eta_1)''_{22}(\eta_0)''_{33} - 2(\eta_0)''_{23}(\eta_1)''_{23} - 2\left[(\eta_0)'_2 - (\eta_0)'_3\right]\left[(\eta_1)'_2 - (\eta_1)'_3\right]\,.\end{aligned}$$

Proof. The first statement follows from the equality

$$\eta^\circ + \eta''_{22} = \sum_{k=1}^{l} ((\eta_k)^\circ + (\eta_k)''_{22}) .$$

For the second statement, we compute

$$\begin{aligned}
\mathcal{P}(\eta) &= \det H_\eta + \eta^\circ h_{H_\eta}(u,u) - (\eta'_2 - \eta'_3)^2 \\
&= \left[(\eta_0)''_{22} + (\eta_1)''_{22}\right] \left[(\eta_0)''_{33} + (\eta_1)''_{33}\right] - \left[(\eta_0)''_{23} + (\eta_1)''_{23}\right]^2 \\
&\quad + (\eta_0^\circ + \eta_1^\circ) \left[h_{H_{\eta_0}}(u,u) + h_{H_{\eta_1}}(u,u)\right] \\
&\quad - ((\eta_0)'_2 + (\eta_1)'_2 - (\eta_0)'_3 - (\eta_1)'_3)^2 \\
&= \mathcal{P}(\eta_0) + \mathcal{P}(\eta_1) + (\eta_0)''_{22}(\eta_1)''_{33} + (\eta_1)''_{22}(\eta_0)''_{33} - 2(\eta_0)''_{23}(\eta_1)''_{23} \\
&\quad + \eta_0^\circ \det H_{\eta_1} + \eta_1^\circ \det H_{\eta_0} - 2((\eta_0)'_2 - (\eta_0)'_3)((\eta_1)'_2 - (\eta_1)'_3) .
\end{aligned}$$

□

We are now ready to prove that the class $\mathscr{C}_\mu$ is not empty, constructing a class of examples of η's such that the corresponding $\mathcal{W}_\eta$ are smoothly-bounded pseudoconvex domains.

Theorem 3.11. *Pick $A_\pm > B_\pm \geq \sqrt{2e^\mu}$ and let $c_\pm > 0$. Let us define $\eta := \eta_+ + \eta_-$, where*

$$\begin{aligned}
\eta_\pm(t_2,t_3) &:= \chi_{(B_\pm^2,+\infty)}(e^{\pm t_2} + e^{\pm t_3})\, e^{f_\pm(t_2,t_3)} , \\
f_\pm(t_2,t_3) &:= \frac{c_\pm}{A_\pm^2 - B_\pm^2} - \frac{c_\pm}{e^{\pm t_2} + e^{\pm t_3} - B_\pm^2} .
\end{aligned}$$

In other words,

$$\begin{aligned}
\eta^\circ = \chi_{(B_+^2,+\infty)}&(|z_2|^2 + |z_3|^2) \exp\left(\frac{c_+}{A_+^2 - B_+^2} \frac{|z_2|^2 + |z_3|^2 - A_+^2}{|z_2|^2 + |z_3|^2 - B_+^2} \right) \\
&+ \chi_{(B_-^2,+\infty)}(|z_2|^{-2} + |z_3|^{-2}) \exp\left(\frac{c_-}{A_-^2 - B_-^2} \frac{|z_2|^{-2} + |z_3|^{-2} - A_-^2}{|z_2|^{-2} + |z_3|^{-2} - B_-^2} \right) .
\end{aligned}$$

Then $\mathcal{W}_\eta$ is a smoothly bounded domain in $\mathbb{C}^3$ that contains $\mathcal{W}'_\mu$ (and is contained in $\mathcal{W}'_{\mu'}$ for $\mu' \geq 2\log A_+, 2\log A_-$). Moreover, there exists $C_+ \geq A_+^2$ such that, for every $c_+ > C_+$, there exist $C_- = C_-(c_+) \geq A_-^2$ such that, for every $c_- > C_-$ the domain $\mathcal{W}_\eta$ is pseudoconvex.

Proof. The functions η, η_+, η_- are continuous because $f_\pm(t_2,t_3) \to -\infty$ as $e^{\pm t_2} + e^{\pm t_3} \to B_\pm^+$. Moreover, η, η_+, η_- are C^2 because

$$\begin{aligned}
\frac{\partial \eta_\pm}{\partial t_j}(t_2,t_3) &= \chi_{(B_\pm^2,+\infty)}(e^{\pm t_2} + e^{\pm t_3}) e^{f_\pm(t_2,t_3)} \frac{\partial f_\pm}{\partial t_j}(t_2,t_3) , \\
\frac{\partial^2 \eta_\pm}{\partial t_j \partial t_k}(t_2,t_3) &= \chi_{(B_\pm^2,+\infty)}(e^{\pm t_2} + e^{\pm t_3}) e^{f_\pm(t_2,t_3)} \left(\frac{\partial f_\pm}{\partial t_j} \frac{\partial f_\pm}{\partial t_k} + \frac{\partial^2 f_\pm}{\partial t_j \partial t_k} \right)(t_2,t_3) ,
\end{aligned}$$

where

$$\begin{aligned}
\frac{\partial f_\pm}{\partial t_j}(t_2,t_3) &= \frac{\pm c_\pm e^{\pm t_j}}{(e^{\pm t_2}+e^{\pm t_3}-B_\pm^2)^2}\\
\frac{\partial^2 f_\pm}{\partial t_2^2}(t_2,t_3) &= \frac{c_\pm e^{\pm t_2}(e^{\pm t_2}+e^{\pm t_3}-B_\pm^2)^2 - c_\pm e^{\pm t_2} 2(e^{\pm t_2}+e^{\pm t_3}-B_\pm^2)e^{\pm t_2}}{(e^{\pm t_2}+e^{\pm t_3}-B_\pm^2)^4}\\
&= c_\pm e^{\pm t_2}\frac{-e^{\pm t_2}+e^{\pm t_3}-B_\pm^2}{(e^{\pm t_2}+e^{\pm t_3}-B_\pm^2)^3}\\
\frac{\partial^2 f_\pm}{\partial t_3^2}(t_2,t_3) &= c_\pm e^{\pm t_3}\frac{e^{\pm t_2}-e^{\pm t_3}-B_\pm^2}{(e^{\pm t_2}+e^{\pm t_3}-B_\pm^2)^3}\\
\frac{\partial^2 f_\pm}{\partial t_2 \partial t_3}(t_2,t_3) &= \frac{-2c_\pm e^{\pm t_2}e^{\pm t_3}}{(e^{\pm t_2}+e^{\pm t_3}-B_\pm^2)^3}\,.
\end{aligned}$$

Similar reasonings apply to successive derivatives and prove that η, η_+, η_- are smooth. Clearly, the gradient of $\eta_\pm$ never vanishes where $\eta_\pm \neq 0$.

The function η vanishes identically in the (compact) intersection of the sets

$$\begin{aligned}
(\eta_+)^{-1}(0) &= \{(t_2,t_3)\in\mathbb{R}^2 : e^{t_2}+e^{t_3}\le B_+^2\}\,,\\
(\eta_-)^{-1}(0) &= \{(t_2,t_3)\in\mathbb{R}^2 : e^{-t_2}+e^{-t_3}\le B_-^2\}\,,
\end{aligned}$$

which includes the square $(-\mu,\mu)\times(-\mu,\mu)$ because $e^{-\mu} < e^{t_2}, e^{t_3} < e^{\mu}$ implies $e^{t_2}+e^{t_3} < 2e^{\mu} \le B_+^2$ and $e^{-t_2}+e^{-t_3} < 2e^{\mu}\le B_-^2$. In particular, $\mathcal{W}'_\mu \subset \mathcal{W}_\eta$. The function η takes positive values wherever η_+ or η_- is strictly positive. Clearly, each positive level set of η_+ is a straight line $e^{t_2}+e^{t_3} = l_+ > B_+^2 > 2$ in the (e^{t_2}, e^{t_3})-quadrant, while each positive level set of η_- is a hyperbola $e^{-t_2}+e^{-t_3} = l_- > B_-^2 > 2$ (or equivalently $e^{t_2}e^{t_3} - l_-^{-1}(e^{t_2}+e^{t_3}) = 0$) in the (e^{t_2}, e^{t_3})-quadrant. This straight line and this hyperbola intersect at exactly two points, namely

$$\left(e^{t_2}, e^{t_3}\right) = \left(\frac{l_+\pm\sqrt{\Delta}}{2}, \frac{l_+\mp\sqrt{\Delta}}{2}\right), \qquad \Delta := l_+^2 - 4l_+l_-^{-1} = l_+l_-^{-1}(l_+l_- - 4) > 0\,.$$

We point out, for future reference, that $\eta_\pm(t_2,t_3) > 0$ with $e^{t_2} = e^{t_3}$ implies $\eta_\mp(t_2,t_3) = 0$.

The set

$$(\eta_+)^{-1}((0,1]) = \{(t_2,t_3)\in\mathbb{R}^2 : B_+^2 < e^{t_2}+e^{t_3}\le A_+^2\}\,,$$

corresponds to the set between the dotted straight lines in Figure 1, while the set

$$(\eta_-)^{-1}((0,1]) = \{(t_2,t_3)\in\mathbb{R}^2 : B_-^2 < e^{-t_2}+e^{-t_3}\le A_-^2\}\,,$$

corresponds to the set between the two dotted hyperbola branches in Figure 1 (bold curve included, finer curve excluded). The intersection $(\eta_+)^{-1}((0,1])\cap(\eta_-)^{-1}((0,1])$ has two connected components U_2, U_3, with U_2 included in the half-plane $t_3 < t_2$ and U_3 included in the half-plane $t_2 < t_3$. The compact set $\eta^{-1}([0,1])$ is properly included in the (compact) intersection between the sets

$$\begin{aligned}
(\eta_+)^{-1}([0,1]) &= \{(t_2,t_3)\in\mathbb{R}^2 : e^{t_2}+e^{t_3}\le A_+^2\}\,,\\
(\eta_-)^{-1}([0,1]) &= \{(t_2,t_3)\in\mathbb{R}^2 : e^{-t_2}+e^{-t_3}\le A_-^2\}\,.
\end{aligned}$$

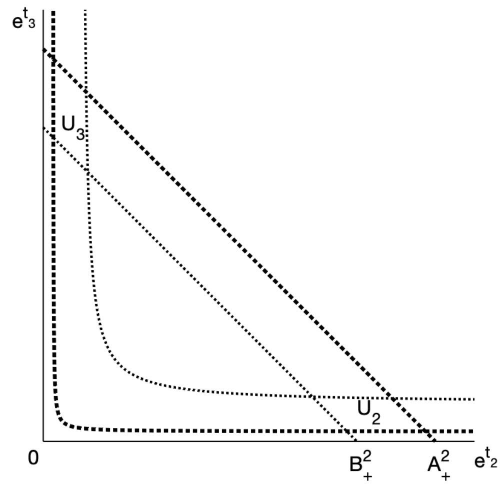

FIGURE 1. The curves $e^{t_2}+e^{t_3}=A_+^2, e^{-t_2}+e^{-t_3}=A_-^2$ (in bold dots) and $e^{t_2}+e^{t_3}=B_+^2, e^{-t_2}+e^{-t_3}=B_-^2$ (in finer dots) within the (e^{t_2},e^{t_3})-quadrant.

In particular, if $\eta(t_2,t_3)\in[0,1]$, then $A_-^{-2}\le e^{t_2},e^{t_3}\le A_+^2$, whence $-2\log A_-<t_2,t_3<2\log A_+$. Thus, $\mathcal{W}_\eta\subset\mathcal{W}'_{\mu'}$ for $\mu'\ge 2\log A_+, 2\log A_-$.

More precisely, the simple closed curve $\eta^{-1}(1)$ bounding the compact set $\eta^{-1}([0,1])$ is the union of four arcs $\gamma_+,\gamma_-,\gamma_2,\gamma_3$, where

$$\gamma_+:\begin{cases} e^{t_2}+e^{t_3}=A_+^2\\ e^{-t_2}+e^{-t_3}\le B_-^2\end{cases}$$

is part of the bold straight line in Figure 1,

$$\gamma_-:\begin{cases} e^{-t_2}+e^{-t_3}=A_-^2\\ e^{t_2}+e^{t_3}\le B_+^2\end{cases}$$

is part of the bold hyperbola in Figure 1, while the arcs

$$\gamma_j:\begin{cases} e^{f^-(t_2,t_3)}+e^{f^+(t_2,t_3)}=1\\ (t_2,t_3)\in U_j\end{cases}$$

(with $j\in\{2,3\}$) are not drawn in Figure 1. Clearly, the gradient of η never vanishes in γ_+, where it equals the gradient of η_+, nor in γ_-, where it equals the gradient of η_-. Moreover, the gradient of η never vanishes in γ_2,γ_3 because it never vanishes in $U_2\cup U_3$: if it did, then

$$e^{f_+(t_2,t_3)}\frac{c_+e^{t_2}}{(e^{t_2}+e^{t_3}-B_+^2)^2}-e^{f_-(t_2,t_3)}\frac{c_-e^{-t_2}}{(e^{-t_2}+e^{-t_3}-B_-^2)^2}=0\,,$$
$$e^{f_+(t_2,t_3)}\frac{c_+e^{t_3}}{(e^{t_2}+e^{t_3}-B_+^2)^2}-e^{f_-(t_2,t_3)}\frac{c_-e^{-t_3}}{(e^{-t_2}+e^{-t_3}-B_-^2)^2}=0\,,$$

whence

$$e^{2t_2}=\frac{c_-}{c_+}\frac{e^{f_-(t_2,t_3)}}{e^{f_+(t_2,t_3)}}\frac{(e^{t_2}+e^{t_3}-B_+^2)^2}{(e^{-t_2}+e^{-t_3}-B_-^2)^2}=e^{2t_3}$$

and $t_2 = t_3$. We would thus find a contradiction with the properties of U_2, U_3. We have therefore proven that the gradient of η never vanishes in $\eta^{-1}(1)$. Overall, all assumptions (I),(II),(III) are fulfilled. By Theorem 3.2, the set $\mathcal{W}_\eta$ is a smoothly bounded domain in $\mathbb{C}^3$, which is pseudoconvex if, and only if, inequalities (2), (3) are fulfilled near $\partial\mathcal{W}_\eta$. We will now prove that, for sufficiently large c_+, the function η_+ is convex in a neighborhood of $(\eta_+)^{-1}([0,1])$. Then we will prove an analogous statement for η_-. This will allow us to easily address inequality (2), thanks to Proposition 3.10. Later, we will study inequality (3) for η.

Omitting the sign $+$ in f_+, c_+, A_+, B_+ for the sake of readability, we have

$$\begin{aligned} f'_j &= \frac{c|z_j|^2}{(|z_2|^2+|z_3|^2-B^2)^2} \\ f''_{22} &= c|z_2|^2 \frac{-|z_2|^2+|z_3|^2-B^2}{(|z_2|^2+|z_3|^2-B^2)^3} \\ f''_{33} &= c|z_3|^2 \frac{|z_2|^2-|z_3|^2-B^2}{(|z_2|^2+|z_3|^2-B^2)^3} \\ f''_{23} &= c\frac{-2|z_2|^2|z_3|^2}{(|z_2|^2+|z_3|^2-B^2)^3}\,. \end{aligned}$$

The inequality $(\eta_+)''_{22} \geq 0$ is trivial where η_+° vanishes and equivalent to $(f'_2)^2 + f''_{22} \geq 0$ elsewhere by Proposition 3.9. In order to study this last inequality, we compute

$$\begin{aligned} c^{-1}|z_2|^{-4}(|z_2|^2+|z_3|^2-B^2)^4\left[(f'_2)^2+f''_{22}\right] &= c+|z_2|^{-2}(-|z_2|^2+|z_3|^2-B^2)(|z_2|^2+|z_3|^2-B^2) \\ &= c-|z_2|^2+|z_2|^{-2}(|z_3|^2-B^2)^2\,. \end{aligned}$$

The inequality $(f'_2)^2 + f''_{22} \geq 0$ is certainly true when $c \geq |z_2|^2$. Choosing $c = c_+ > A_+^2$ guarantees that $(\eta_+)''_{22} \geq 0$ in a neighborhood of $(\eta_+^\circ)^{-1}([0,1])$. We now study the inequality $\det H_{\eta_+} \geq 0$, which is trivial where η_+° vanishes and equivalent to the inequality $h_{H_f}(v,v) + \det H_f \geq 0$ elsewhere by Proposition 3.9. Here,

$$H_f = \frac{c}{(|z_2|^2+|z_3|^2-B^2)^3}\begin{pmatrix} |z_2|^2(-|z_2|^2+|z_3|^2-B^2) & -2|z_2|^2|z_3|^2 \\ -2|z_2|^2|z_3|^2 & |z_3|^2(|z_2|^2-|z_3|^2-B^2) \end{pmatrix}$$

and

$$v = v_+ := \begin{pmatrix} f'_3 \\ -f'_2 \end{pmatrix} = \frac{c}{(|z_2|^2+|z_3|^2-B^2)^2}\begin{pmatrix} |z_3|^2 \\ -|z_2|^2 \end{pmatrix}.$$

Recalling the notation set in formula (1), we compute

$$\begin{aligned} &c^{-3}|z_2|^{-2}|z_3|^{-2}(|z_2|^2+|z_3|^2-B^2)^7 h_{H_f}(v,v) \\ &= |z_2|^{-2}|z_3|^{-2}\left(c^{-1}(|z_2|^2+|z_3|^2-B^2)^2 v^t\right)\left(c^{-1}(|z_2|^2+|z_3|^2-B^2)^3 H_f\right) \\ &\quad \cdot \left(c^{-1}(|z_2|^2+|z_3|^2-B^2)^2 \bar{v}\right) \\ &= |z_2|^{-2}|z_3|^{-2}(|z_3|^2, -|z_2|^2)\begin{pmatrix} |z_2|^2|z_3|^2(-|z_2|^2+|z_3|^2-B^2)+2|z_2|^4|z_3|^2 \\ -2|z_2|^2|z_3|^4-|z_2|^2|z_3|^2(|z_2|^2-|z_3|^2-B^2) \end{pmatrix} \\ &= |z_3|^2(-|z_2|^2+|z_3|^2-B^2)+2|z_2|^2|z_3|^2+2|z_2|^2|z_3|^2+|z_2|^2(|z_2|^2-|z_3|^2-B^2) \\ &= |z_3|^2(-|z_2|^2+|z_3|^2-B^2+2|z_2|^2)+|z_2|^2(|z_2|^2-|z_3|^2-B^2+2|z_3|^2) \\ &= (|z_2|^2+|z_3|^2)(|z_2|^2+|z_3|^2-B^2)\,, \end{aligned}$$

whence

$$h_{H_f}(v,v) = c^3|z_2|^2|z_3|^2\frac{|z_2|^2+|z_3|^2}{(|z_2|^2+|z_3|^2-B^2)^6} = c^3|z_2|^2|z_3|^2\frac{t^2}{(t^2-B^2)^6}\,,$$

where $t = t_+ := \sqrt{|z_2|^2+|z_3|^2}$. Moreover,

$$\begin{aligned}
&c^{-2}|z_2|^{-2}|z_3|^{-2}(|z_2|^2+|z_3|^2-B^2)^6\det H_f\\
&= |z_2|^{-2}|z_3|^{-2}\det\left(c^{-1}(|z_2|^2+|z_3|^2-B^2)^3H_f\right)\\
&= |z_2|^{-2}|z_3|^{-2}\det\begin{pmatrix}|z_2|^2(-|z_2|^2+|z_3|^2-B^2) & -2|z_2|^2|z_3|^2\\ -2|z_2|^2|z_3|^2 & |z_3|^2(|z_2|^2-|z_3|^2-B^2)\end{pmatrix}\\
&= (-|z_2|^2+|z_3|^2-B^2)(|z_2|^2-|z_3|^2-B^2)-4|z_2|^2|z_3|^2\\
&= B^4-|z_2|^4-|z_3|^4-2|z_2|^2|z_3|^2\\
&= B^4-(|z_2|^2+|z_3|^2)^2\\
&= -(|z_2|^2+|z_3|^2+B^2)(|z_2|^2+|z_3|^2-B^2)\,,
\end{aligned}$$

whence

$$\det H_f = -c^2|z_2|^2|z_3|^2\frac{|z_2|^2+|z_3|^2+B^2}{(|z_2|^2+|z_3|^2-B^2)^5} = -c^2|z_2|^2|z_3|^2\frac{t^2+B^2}{(t^2-B^2)^5}\,.$$

We conclude that

$$\begin{aligned}
c^{-2}|z_2|^{-2}|z_3|^{-2}t^{-2}(t^2-B^2)^6\left[h_{H_f}(v,v)+\det H_f\right] &= c - t^{-2}(t^2+B^2)(t^2-B^2)\\
&= c - t^{-2}(t^4-B^4)\\
&= c - t^2 + \frac{B^4}{t^2}\,,
\end{aligned}$$

is nonnegative when $c \geq t^2 = |z_2|^2+|z_3|^2$. Therefore, choosing $c = c_+ > A_+^2$ guarantees not only $(\eta_+)''_{22} \geq 0$, but also $\det H_{\eta_+} \geq 0$ in a neighborhood of $(\eta_+^\circ)^{-1}([0,1])$, whence the convexity of η_+ in a neighborhood of $\eta_+^{-1}([0,1])$. For future reference, we now compute the quantity $\mathcal{P}(\eta_+) = \det H_{\eta_+} + \eta^\circ h_{H_{\eta_+}}(u,u) - ((\eta_+)'_2-(\eta_+)'_3)^2$, defined in Proposition 3.10. Here, $u := (1,1)^t$. By Proposition 3.9, $\mathcal{P}(\eta_+) = (\eta_+^\circ)^2[h_{H_f}(v,v)+\det H_f + h_{H_f}(u,u)]$ wherever η_+° does not vanish. We compute

$$\begin{aligned}
&c^{-1}(|z_2|^2+|z_3|^2-B^2)^3h_{H_f}(u,u)\\
&= u^t\left(c^{-1}(|z_2|^2+|z_3|^2-B^2)^3H_f\right)\bar{u}\\
&= (1,-1)\begin{pmatrix}|z_2|^2(-|z_2|^2+|z_3|^2-B^2)+2|z_2|^2|z_3|^2\\ -2|z_2|^2|z_3|^2-|z_3|^2(|z_2|^2-|z_3|^2-B^2)\end{pmatrix}\\
&= |z_2|^2(-|z_2|^2+|z_3|^2-B^2)+|z_3|^2(|z_2|^2-|z_3|^2-B^2)+4|z_2|^2|z_3|^2\\
&= -|z_2|^4-|z_3|^4+6|z_2|^2|z_3|^2-(|z_2|^2+|z_3|^2)B^2\\
&= 8|z_2|^2|z_3|^2-(|z_2|^2+|z_3|^2)(|z_2|^2+|z_3|^2+B^2)\,,
\end{aligned}$$

whence

$$h_{H_f}(u,u) = c\,\frac{8|z_2|^2|z_3|^2-(|z_2|^2+|z_3|^2)(|z_2|^2+|z_3|^2+B^2)}{(|z_2|^2+|z_3|^2-B^2)^3} = c\,\frac{8|z_2|^2|z_3|^2-t^2(t^2+B^2)}{(t^2-B^2)^3}\,.$$

We conclude that

$$\begin{aligned}
&c^{-1}|z_2|^{-2}|z_3|^{-2}t^{-2}(t^2-B^2)^6\left[h_{H_f}(v,v)+\det H_f+h_{H_f}(u,u)\right]\\
&=c^2+c\left(-t^2+\frac{B^4}{t^2}\right)+[8t^{-2}-|z_2|^{-2}|z_3|^{-2}(t^2+B^2)](t^2-B^2)^3\\
&=c^2+c\left(-t^2+\frac{B^4}{t^2}\right)-|z_2|^{-2}|z_3|^{-2}(t^4-B^4)(t^2-B^2)^2+8t^{-2}(t^2-B^2)^3\\
&\geq c^2+c\left(-t^2+\frac{B^4}{t^2}\right)-|z_2|^{-2}|z_3|^{-2}(t^4-B^4)(t^2-B^2)^2\,,
\end{aligned}$$

where the last expression is nonegative for

$$c=c_+\geq\left(t_+^2-\frac{B_+^4}{t_+^2}\right)\left(\frac{1}{2}+\frac{1}{2}\sqrt{1+4|z_2|^{-2}|z_3|^{-2}t_+^4(t_+^2-B_+^2)(t_+^2+B_+^2)^{-1}}\right).$$

Similarly, we can prove that choosing $c_->A_-^2$ guarantees the convexity of η_- in a neighborhood of $\eta_-^{-1}([0,1])$. Indeed, omitting the sign $-$ in f_-,c_-,A_-,B_- for the sake of readability, we have

$$\begin{aligned}
f'_j&=\frac{-c|z_j|^{-2}}{(|z_2|^{-2}+|z_3|^{-2}-B^2)^2}\\
f''_{22}&=c|z_2|^{-2}\frac{-|z_2|^{-2}+|z_3|^{-2}-B^2}{(|z_2|^{-2}+|z_3|^{-2}-B^2)^3}\\
f''_{33}&=c|z_3|^{-2}\frac{|z_2|^{-2}-|z_3|^{-2}-B^2}{(|z_2|^{-2}+|z_3|^{-2}-B^2)^3}\\
f''_{23}&=c\frac{-2|z_2|^{-2}|z_3|^{-2}}{(|z_2|^{-2}+|z_3|^{-2}-B^2)^3}\,.
\end{aligned}$$

The inequality $(\eta_-)''_{22}\geq 0$ holds where η_-° does not vanish if, and only if, $(f'_2)^2+f''_{22}\geq 0$, where

$$\begin{aligned}
&c^{-1}|z_2|^4(|z_2|^{-2}+|z_3|^{-2}-B^2)^4\left[(f'_2)^2+f''_{22}\right]\\
&=c+|z_2|^2(-|z_2|^{-2}+|z_3|^{-2}-B^2)(|z_2|^{-2}+|z_3|^{-2}-B^2)\\
&=c-|z_2|^{-2}+|z_2|^2(|z_3|^{-2}-B^2)^2\,.
\end{aligned}$$

The last inequality is certainly true when $c\geq|z_2|^{-2}$. On the other hand, $\det H_{\eta_-}\geq 0$ where η_-° does not vanish if, and only if, (after setting $t=t_-:=\sqrt{|z_2|^{-2}+|z_3|^{-2}}$)

$$0\leq c^{-2}|z_2|^2|z_3|^2t^{-2}(t^2-B^2)^6\left[h_{H_f}(v,v)+\det H_f\right]=c-t^2+\frac{B^4}{t^2}\,.$$

The last inequality is certainly true when $c\geq t^2=|z_2|^{-2}+|z_3|^{-2}$. A suitable choice is therefore $c=c_->A_-^2$, as announced. For future reference, we now compute the quantity $\mathcal{P}(\eta_-)=$

$\det H_{\eta_-} + \eta^\circ h_{H_{\eta_-}}(u,u) - ((\eta_-)'_2 - (\eta_-)'_3)^2 = (\eta^\circ_-)^2[h_{H_f}(v,v) + \det H_f + h_{H_f}(u,u)]$. We have

$$\begin{aligned} &c^{-1}|z_2|^2|z_3|^2 t^{-2}(t^2-B^2)^6 \left[h_{H_f}(v,v) + \det H_f + h_{H_f}(u,u)\right] \\ &= c^2 + c\left(-t^2 + \frac{B^4}{t^2}\right) - |z_2|^2|z_3|^2(t^4-B^4)(t^2-B^2)^2 + 8t^{-2}(t^2-B^2)^3 \\ &\geq c^2 + c\left(-t^2 + \frac{B^4}{t^2}\right) - |z_2|^2|z_3|^2(t^4-B^4)(t^2-B^2)^2 \,, \end{aligned}$$

where the last expression is nonegative for

$$c = c_- \geq \left(t_-^2 - \frac{B_-^4}{t_-^2}\right)\left(\frac{1}{2} + \frac{1}{2}\sqrt{1 + 4|z_2|^2|z_3|^2 t_-^4 (t_-^2 - B_-^2)(t_-^2 + B_-^2)^{-1}}\right) .$$

Consider now the compact set

$$\begin{aligned} K &:= (\eta^\circ)^{-1}([0,1]) \subseteq (\eta^\circ_+)^{-1}([0,1]) \cap (\eta^\circ_-)^{-1}([0,1]) \\ &= \{(z_2,z_3) \in \mathbb{C}^2 : t_+ \leq A_+^2, t_- \leq A_-^2\} . \end{aligned}$$

For future reference, we point out that $(z_2,z_3) \in K$ implies $A_-^{-1} \leq |z_2|, |z_3| \leq A_+$ and that $\frac{\sqrt{2}}{B_-} \leq |z_2|, |z_3| \leq \frac{B_+}{\sqrt{2}}$ implies $\eta^\circ(z) = 0$. We established that η_+, η_- are both convex in a neighborhood of K when $c_+ > A_+^2$ and $c_- > A_-^2$. Proposition 3.10 guarantees that, when $c_+ > A_+^2$ and $c_- > A_-^2$, then inequality (2) is fulfilled in a neighborhood of K for $\eta = \eta_+ + \eta_-$. Our final aim is studying inequality (3) for $\eta = \eta_+ + \eta_-$. According to Proposition 3.10, inequality (3) is the same as

$$\begin{aligned} 0 \leq \mathcal{P}(\eta) &= \mathcal{P}(\eta_+) + \mathcal{P}(\eta_-) + \eta^\circ_+ \det H_{\eta_-} + \eta^\circ_- \det H_{\eta_+} \\ &+ (\eta_+)''_{22}(\eta_-)''_{33} + (\eta_-)''_{22}(\eta_+)''_{33} - 2(\eta_+)''_{23}(\eta_-)''_{23} - 2((\eta_+)'_2 - (\eta_+)'_3)((\eta_-)'_2 - (\eta_-)'_3) . \end{aligned}$$

We already established that $\det H_{\eta_+} \geq 0$ in a neighborhood of $(\eta^\circ_+)^{-1}([0,1])$ when $c_+ > A_+^2$ and that $\det H_{\eta_-} \geq 0$ in a neighborhood of $(\eta^\circ_-)^{-1}([0,1])$ when $c_- > A_-^2$. We also know that $\mathcal{P}(\eta_+), \mathcal{P}(\eta_-) \geq 0$ in a neighborhood of K if we choose $c_+ > \mathcal{M}_+$ and $c_- > \mathcal{M}_-$, where

$$\begin{aligned} \mathcal{M}_+ &:= \max_{(z_2,z_3)\in K} \left(t_+^2 - \frac{B_+^4}{t_+^2}\right)\left(\frac{1}{2} + \frac{1}{2}\sqrt{1 + 4|z_2|^{-2}|z_3|^{-2} t_+^4 (t_+^2 - B_+^2)(t_+^2 + B_+^2)^{-1}}\right) , \\ \mathcal{M}_- &:= \max_{(z_2,z_3)\in K} \left(t_-^2 - \frac{B_-^4}{t_-^2}\right)\left(\frac{1}{2} + \frac{1}{2}\sqrt{1 + 4|z_2|^2|z_3|^2 t_-^4 (t_-^2 - B_-^2)(t_-^2 + B_-^2)^{-1}}\right) . \end{aligned}$$

We are left with studying the quantity

$$\mathcal{Q}(\eta) := (\eta_+)''_{22}(\eta_-)''_{33} + (\eta_-)''_{22}(\eta_+)''_{33} - 2(\eta_+)''_{23}(\eta_-)''_{23} - 2\left[(\eta_+)'_2 - (\eta_+)'_3\right]\left[(\eta_-)'_2 - (\eta_-)'_3\right] .$$

We have $\mathcal{Q}(\eta) = 0$ wherever $\eta^\circ_+ = 0$ or $\eta^\circ_- = 0$. Moreover,

$$\begin{aligned} &(\eta^\circ)^{-2}\mathcal{Q}(\eta) \\ &= \left[((f_+)'_2)^2 + (f_+)''_{22}\right]\left[((f_-)'_3)^2 + (f_-)''_{33}\right] + \left[((f_-)'_2)^2 + (f_-)''_{22}\right]\left[((f_+)'_3)^2 + (f_+)''_{33}\right] \\ &\quad - 2\left[(f_+)'_2(f_+)'_3 + (f_+)''_{23}\right]\left[(f_-)'_2(f_-)'_3 + (f_-)''_{23}\right] - 2\left[(f_+)'_2 - (f_+)'_3\right]\left[(f_-)'_2 - (f_-)'_3\right] \end{aligned}$$

at each point (z_2, z_3) where neither η_+° nor η_-° vanishes (which implies $|z_2| \neq |z_3|$), i.e., at each point (z_2, z_3) in the disjoint union $V_2 \cup V_3$ with

$$V_2 : \begin{cases} t_+ = |z_2|^2 + |z_3|^3 > B_+ \\ t_- = |z_2|^{-2} + |z_3|^{-2} > B_- \\ |z_3| < |z_2| \end{cases}, \qquad V_3 : \begin{cases} t_+ = |z_2|^2 + |z_3|^3 > B_+ \\ t_- = |z_2|^{-2} + |z_3|^{-2} > B_- \\ |z_2| < |z_3| \end{cases}.$$

We already know that

$$\begin{aligned} c_+^{-1}(t_+^2 - B_+^2)^2 (f_+)'_2 &= |z_2|^2 \,, \\ c_-^{-1}(t_-^2 - B_-^2)^2 (f_-)'_2 &= -|z_2|^{-2} \,, \\ c_+^{-1}(t_+^2 - B_+^2)^4 \left[((f_+)'_2)^2 + (f_+)''_{22}\right] &\geq |z_2|^4 (c_+ - |z_2|^2) \,, \\ c_-^{-1}(t_-^2 - B_-^2)^4 \left[((f_-)'_2)^2 + (f_-)''_{22}\right] &\geq |z_2|^{-4} (c_- - |z_2|^{-2}) \,. \end{aligned}$$

Similarly,

$$\begin{aligned} c_+^{-1}(t_+^2 - B_+^2)^2 (f_+)'_3 &= |z_3|^2 \,, \\ c_-^{-1}(t_-^2 - B_-^2)^2 (f_-)'_3 &= -|z_3|^{-2} \,, \\ c_+^{-1}(t_+^2 - B_+^2)^4 \left[((f_+)'_3)^2 + (f_+)''_{33}\right] &\geq |z_3|^4 (c_+ - |z_3|^2) \,, \\ c_-^{-1}(t_-^2 - B_-^2)^4 \left[((f_-)'_3)^2 + (f_-)''_{33}\right] &\geq |z_3|^{-4} (c_- - |z_3|^{-2}) \,. \\ c_+^{-1}(t_+^2 - B_+^2)^4 \left[(f_+)'_2 (f_+)'_3 + (f_+)''_{23}\right] &= |z_2|^2 |z_3|^2 \left[c_+ - 2(t_+^2 - B_+^2)\right] \,, \\ c_-^{-1}(t_-^2 - B_-^2)^4 \left[(f_-)'_2 (f_-)'_3 + (f_-)''_{23}\right] &= |z_2|^{-2} |z_3|^{-2} \left[c_- - 2(t_-^2 - B_-^2)\right] \,. \end{aligned}$$

Let us set $x := |z_2|^2 |z_3|^{-2}$. If $(z_2, z_3) \in V_2$, then $|z_2|^2 > \frac{B_+^2}{2}$ and $|z_3|^{-2} > \frac{B_-^2}{2}$, whence $x > \frac{B_+^2 B_-^2}{4} > 1$; if $(z_2, z_3) \in V_3$, then $|z_3|^2 > \frac{B_+^2}{2}$ and $|z_2|^{-2} > \frac{B_-^2}{2}$, whence $x < \frac{4}{B_+^2 B_-^2} < 1$. We also know that $(z_2, z_3) \in K$ implies $\frac{1}{A_+^2 A_-^2} \leq x \leq A_+^2 A_-^2$. For $(z_2, z_3) \in V_2 \cup V_3$, we have

$$\begin{aligned} & c_+^{-1} c_-^{-1} (t_+^2 - B_+^2)^4 (t_-^2 - B_-^2)^4 (\eta^\circ)^{-2} \mathcal{Q}(\eta) \\ & \geq |z_2|^4 |z_3|^{-4} (c_+ - |z_2|^2)(c_- - |z_3|^{-2}) + |z_2|^{-4} |z_3|^4 (c_- - |z_2|^{-2})(c_+ - |z_3|^2) \\ & \quad - 2\left[c_+ - 2(t_+^2 - B_+^2)\right]\left[c_- - 2(t_-^2 - B_-^2)\right] \\ & \quad - 2(t_+^2 - B_+^2)^2 (t_-^2 - B_-^2)^2 \left(|z_2|^2 - |z_3|^2\right)\left(-|z_2|^{-2} + |z_3|^{-2}\right) \\ & = c_+ c_- (x^2 + x^{-2} - 2) + c_+ \left[-x^2 |z_3|^{-2} - x^{-2} |z_2|^{-2} + 4(t_-^2 - B_-^2)\right] + \\ & \quad + c_- \left[-x^2 |z_2|^2 - x^{-2} |z_3|^2 + 4(t_+^2 - B_+^2)\right] + x^3 + x^{-3} - 8(t_+^2 - B_+^2)(t_-^2 - B_-^2) \\ & \quad - 2(t_+^2 - B_+^2)(t_-^2 - B_-^2)(x + x^{-1} - 2) \\ & \geq \alpha_{11} c_+ c_- + \alpha_{10} c_+ + \alpha_{01} c_- + \alpha_{00} \,, \end{aligned}$$

with

$$\begin{aligned} \alpha_{11} &:= (x - x^{-1})^2 > 0 \\ \alpha_{10} &:= -x^2 |z_3|^{-2} - x^{-2} |z_2|^{-2} \\ \alpha_{01} &:= -x^2 |z_2|^2 - x^{-2} |z_3|^2 \\ \alpha_{00} &:= -2(t_+^2 - B_+^2)(t_-^2 - B_-^2) \left[(x^{1/2} - x^{-1/2})^2 + 4(t_+^2 - B_+^2)(t_-^2 - B_-^2)\right] \,. \end{aligned}$$

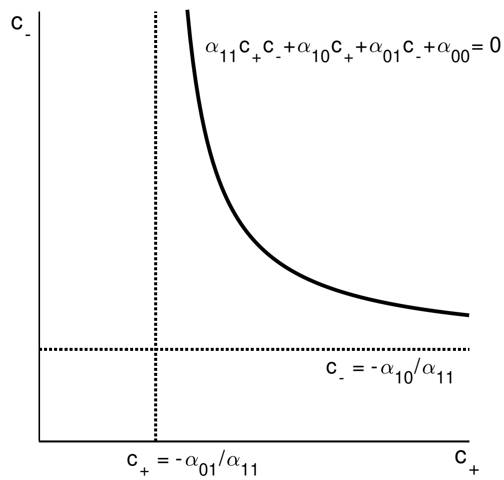

Figure 2. The right branch of the rectangular hyperbola $\alpha_{11}c_+c_- + \alpha_{10}c_+ + \alpha_{01}c_- + \alpha_{00} = 0$.

Therefore, $\mathcal{Q}(\eta) \geq 0$ in $V_2 \cup V_3$ if (c_+, c_-) lies to the right of the right branch of the rectangular hyperbola $\alpha_{11}c_+c_- + \alpha_{10}c_+ + \alpha_{01}c_- + \alpha_{00} = 0$ depicted in Figure 2. This happens exactly when the two following inequalities are fulfilled:

$$\begin{aligned} c_+ &\geq -\frac{\alpha_{01}}{\alpha_{11}} = \frac{x^2|z_2|^2 + x^{-2}|z_3|^2}{(x - x^{-1})^2} = \frac{x^4|z_2|^2 + |z_3|^2}{(x^2-1)^2}\,, \\ c_- &\geq \frac{-\alpha_{10}c_+ - \alpha_{00}}{\alpha_{11}c_+ + \alpha_{01}}\,. \end{aligned}$$

We can set

$$\mathcal{N}_+ := \max_{(z_2,z_2)\in K} \left(\frac{x^4|z_2|^2 + |z_3|^2}{(x^2-1)^2} \right)\,,$$

pick a c_+ larger than $A_+^2, \mathcal{M}_+$ and $\mathcal{N}_+$, set

$$\mathcal{N}_- = \mathcal{N}_-(c_+) := \max_{(z_2,z_2)\in K} \frac{-\alpha_{10}c_+ - \alpha_{00}}{\alpha_{11}c_+ + \alpha_{01}}\,,$$

and pick a c_- larger than $A_-^2, \mathcal{M}_-$ and $\mathcal{N}_-$ to conclude that $\mathcal{W}_\eta$ is a smoothly bounded pseudoconvex domain. □

By inspection in the last proof, the following property could be added to the statement: there exists $C_- \geq A_-^2$ such that, for every $c_- > C_-$, there exist $C_+ = C_+(c_-) \geq A_+^2$ such that, for every $c_+ > C_+$ the domain $\mathcal{W}_\eta$ is pseudoconvex.

4. Study of the smoothly-bounded pseudoconvex worm domains in $\mathscr{C}_\mu$

This section is devoted to the study of the class $\mathscr{C}_\mu$ of smoothly-bounded pseudoconvex domains constructed in Definition 3.6. We make the following remark, which involves the bounded non smooth domain $\mathcal{W}'_\mu$ and the unbounded domain $\mathcal{W}'_\infty$ defined in Definition 3.4.

Remark 4.1. Let $\Omega \in \mathscr{C}_\mu$. Then

$$\mathcal{W}'_\mu \subset \Omega \subset \mathcal{W}'_{\mu'} \subset \mathcal{W}'_\infty \tag{6}$$

for sufficiently large $\mu' > \mu$. Namely, if $\Omega = \mathcal{W}_\eta$, it suffices for the open disk of radius μ' centered at the origin in $\mathbb{R}^2$ to include the set $\eta^{-1}([0,1))$.

It is useful to define the following functions on $\mathcal{W}'_\infty$.

Definition 4.2. For all $z = (z_1, z_2, z_3) \in \mathcal{W}'_\infty$ and all $\kappa \in \mathbb{C}$, we define

$$\begin{aligned} \ell(z) &:= \log\left(z_1 e^{-i\log|z_2z_3|^2}\right) + i\log|z_2z_3|^2 \,, \\ E_\kappa(z) &:= \exp\left(\kappa\ell(z)\right)\,, \end{aligned}$$

where log denotes the principal branch of logarithm on $\mathbb{C}\setminus(-\infty, 0]$.

We can make the following remark and prove the subsequent proposition concerning the Bergman space $A^2(\Omega)$ of any $\Omega \in \mathscr{C}_\mu$.

Remark 4.3. Let $z \in \mathcal{W}'_\infty$. If $\kappa \in \mathbb{Z}$, then $E_\kappa(z) = z_1^\kappa$. If $\kappa \in \mathbb{R}$, then $|E_\kappa(z)| = |z_1|^\kappa$. If $\kappa \in \mathbb{C}$ and $z \in \mathcal{W}'_{\frac{\pi}{2}}$, then $E_\kappa(z) = z_1^\kappa := \exp(\kappa\log(z_1))$.

Proposition 4.4. *The functions $\ell, E_\kappa : \mathcal{W}'_\infty \to \mathbb{C}$ are holomorphic and locally constant in z_2 and in z_3. Moreover, $\frac{\partial \ell}{\partial z_1}(z) = \frac{1}{z_1}$ and $\frac{\partial E_\kappa}{\partial z_1} = \kappa\, E_{\kappa-1}$. Finally, the following are equivalent:*

(i) $\mathrm{Re}(\kappa) > -1$*;*
(ii) $E_\kappa(z)$ *belongs to* $A^2(\mathcal{W}'_\mu)$ *for some* $\mu > 0$*;*
(iii) $E_\kappa(z)z_2^jz_3^k$ *belongs to* $A^2(\mathcal{W}'_\mu)$ *for all* $\mu > 0$ *and all* $j,k \in \mathbb{Z}$*;*
(iv) $E_\kappa(z)z_2^jz_3^k$ *belongs to* $A^2(\Omega)$ *for all* $\mu > 0$*, for all* $\Omega \in \mathscr{C}_\mu$ *and for all* $j,k \in \mathbb{Z}$*.*

Proof. The function ℓ (whence all functions E_κ with $\kappa \in \mathbb{C}$) are holomorphic because

$$\begin{aligned} \frac{\partial \ell}{\partial \bar z_1}(z) &= \frac{0}{z_1e^{-i\log|z_2z_3|^2}} + 0 \equiv 0\,, \\ \frac{\partial \ell}{\partial \bar z_2}(z) &= \frac{z_1e^{-i\log|z_2z_3|^2}(-i)}{z_1e^{-i\log|z_2z_3|^2}}\frac{\partial\log|z_2z_3|^2}{\partial \bar z_2} + i\frac{\partial\log|z_2z_3|^2}{\partial \bar z_2} \equiv 0\,, \\ \frac{\partial \ell}{\partial \bar z_3}(z) &= \frac{z_1e^{-i\log|z_2z_3|^2}(-i)}{z_1e^{-i\log|z_2z_3|^2}}\frac{\partial\log|z_2z_3|^2}{\partial \bar z_3} + i\frac{\partial\log|z_2z_3|^2}{\partial \bar z_3} \equiv 0\,. \end{aligned}$$

Analogous computations prove that $\frac{\partial \ell}{\partial z_2} \equiv 0 \equiv \frac{\partial \ell}{\partial z_3}$, whence ℓ and E_κ with $\kappa \in \mathbb{C}$ are locally constant in z_2 and in z_3. Moreover,

$$\frac{\partial \ell}{\partial z_1}(z) = \frac{e^{-i\log|z_2z_3|^2}}{z_1e^{-i\log|z_2z_3|^2}} + 0 = \frac{1}{z_1}\,.$$

It easily follows that $E_\kappa = e^{\kappa\ell}$ has $\frac{\partial E_\kappa}{\partial z_1} = \kappa\, E_{\kappa-1}$.

Now let us study L^2-integrability. For any $a, b \in \mathbb{R}, j, k \in \mathbb{Z}, \mu > 0$, we remark that

$$\|E_{a+ib} z_2^j z_3^k\|^2_{A^2(\mathcal{W}'_\mu)} = \int_{\mathcal{W}'_\mu} \left|E_{a+ib}(z) z_2^j z_3^k\right|^2 dV(z)$$

$$= \int_{-\mu<\log|z_2|^2<\mu} \int_{-\mu<\log|z_3|^2<\mu} \int_{\triangle(e^{i\log|z_2z_3|^2},1)} |z_1|^{2a}|z_2|^{2j}|z_3|^{2k}$$

$$\cdot \exp\left\{-2b[\arg(z_1 e^{-i\log|z_2z_3|^2}) + \log|z_2z_3|^2]\right\} dV(z_1)dV(z_2)dV(z_3)$$

$$= \int_{-\mu<\log|z_2|^2<\mu} \int_{-\mu<\log|z_3|^2<\mu} \int_{\triangle(1,1)} |\zeta|^{2a}|z_2|^{2j}|z_3|^{2k}$$

$$\cdot \exp\left\{-2b(\arg(\zeta) + \log|z_2z_3|^2)\right\} dV(\zeta)\, dV(z_2)\, dV(z_3)$$

$$= 4\pi^2 \int_{e^{-\mu/2}}^{e^{\mu/2}} \int_{e^{-\mu/2}}^{e^{\mu/2}} \int_0^{\frac{\pi}{2}} \int_0^{2\cos\theta} r^{2a+1} \rho_2^{2j+1} \rho_3^{2k+1} e^{-2b(\theta+\log(\rho_2\rho_3)^2)}\, dr\, d\theta\, d\rho_2\, d\rho_3$$

$$= \pi^2 \int_0^{\frac{\pi}{2}} \int_{\frac{\theta}{2}-\mu}^{\frac{\theta}{2}+\mu} \int_{\frac{\theta}{2}-\mu}^{\frac{\theta}{2}+\mu} \int_{-\infty}^{\log(2\cos\theta)} e^{2(a+1)s} e^{(x_2-\frac{\theta}{2})(j+1)} e^{(x_3-\frac{\theta}{2})(k+1)} e^{-2b(x_2+x_3)}\, ds\, dx_2\, dx_3\, d\theta$$

$$= \pi^2 \int_0^{\frac{\pi}{2}} \int_{\frac{\theta}{2}-\mu}^{\frac{\theta}{2}+\mu} \int_{\frac{\theta}{2}-\mu}^{\frac{\theta}{2}+\mu} \int_{-\infty}^{\log(2\cos\theta)} e^{2(a+1)s} ds\, e^{x_2(j+1-2b)}\, dx_2 e^{x_3(k+1-2b)}\, dx_3\, e^{-\frac{\theta}{2}(j+k+2)}\, d\theta$$

is finite if, and only if, $a > -1$. It follows at once that properties (i),(ii),(iii) are mutually equivalent. But the chain of inclusions (6) guarantees that (iii) and (iv) are mutually equivalent. The proof is therefore complete. □

We are now ready to prove that, for sufficiently large μ and for $\Omega \in \mathscr{C}_\mu$, the Nebenhülle of Ω is nontrivial. We actually prove a bit more about the space $A^2(\overline{\Omega})$, defined as the closure within the Bergman space $A^2(\Omega)$ of the subspace of those elements of $A^2(\Omega)$ that extend to holomorphic functions on neighborhoods of $\overline{\Omega}$.

Theorem 4.5. *Let $\mu > 4\pi$. If $f \in A^2(\overline{\mathcal{W}'_\mu})$, then f admits a holomorphic extension to the domain*

$$\widehat{\mathcal{W}'_\mu} := \mathcal{W}'_\mu \cup \bigcup_{-\frac{\mu}{2}<a<\frac{\mu}{2}-2\pi} \left\{(z_1, z_2, z_3) : |z_1 - e^{2ia}| < 1, a < \log|z_2|^2, \log|z_3|^2 < a + 2\pi\right\}. \quad (7)$$

As a consequence, $A^2(\overline{\mathcal{W}'_\mu}) \subsetneq A^2(\mathcal{W}'_\mu)$.

The same conclusions hold true with any $\Omega \in \mathscr{C}_\mu$ in place of $\mathcal{W}'_\mu$.

Proof. First suppose f to be a holomorphic function on a domain $Y \supset \overline{\mathcal{W}'_\mu}$. Take $a \in \left[-\frac{\mu}{2}, \frac{\mu}{2} - 2\pi\right]$ and consider the annulus

$$\mathscr{A}_a := A\left(0, e^{\frac{a}{2}}, e^{\frac{a}{2}+\pi}\right) = \left\{\zeta \in \mathbb{C} : a < \log|\zeta|^2 < a + 2\pi\right\},$$

whose oriented boundary $\partial\mathscr{A}_a$ consists of the circles $\log|\zeta|^2 = a$ and $\log|\zeta|^2 = a + 2\pi$. Consider also the unit disk centered at e^{2ia}, namely $\triangle_a = \triangle(e^{2ia}, 1)$. Since $2a \in [-\mu, \mu - 4\pi]$, both of the following compact subsets of $\mathbb{C}^3$ are contained in $\overline{\mathcal{W}'_\mu}$:

$$\{0\} \times \overline{\mathscr{A}}_a \times \overline{\mathscr{A}}_a = \left\{(0, z_2, z_3) : a \le \log|z_2|^2 \le a + 2\pi, a \le \log|z_3|^2 \le a + 2\pi\right\}$$

and

$$\overline{\triangle}_a \times \partial\mathscr{A}_a \times \partial\mathscr{A}_a = \left\{(z_1,z_2,z_3) : \, |z_1 - e^{2ia}| \le 1, \log|z_2|^2, \log|z_3|^2 \in \{a, a+2\pi\}\right\}.$$

Set

$$F_a(z_1,z_2,z_3) := \frac{1}{(2\pi i)^2}\int_{\partial\mathscr{A}_a}\int_{\partial\mathscr{A}_a} \frac{f(z_1,\zeta_2,\zeta_3)}{(\zeta_2-z_2)(\zeta_3-z_3)}\, d\zeta_2 d\zeta_3\,. \tag{8}$$

This defines a function F_a that is holomorphic in a neighborhood of the set

$$\overline{\triangle}_a \times \mathscr{A}_a \times \mathscr{A}_a = \left\{(z_1,z_2,z_3) : \, |z_1 - e^{2ia}| \le 1, a < \log|z_2|^2, \log|z_3|^2 < a+2\pi\right\}$$

and coincides with f in a neighborhood of $\{0\} \times \mathscr{A}_a \times \mathscr{A}_a$ by Cauchy's integral formula. Thus, F_a coincides with f in $\overline{\mathcal{W}'_\mu} \cap (\overline{\triangle}_a \times \mathscr{A}_a \times \mathscr{A}_a)$. We have thus constructed a holomorphic extension $\widehat{f}$ of f to the domain $\widehat{Y} := Y \cup \bigcup_{-\frac{\mu}{2}<a<\frac{\mu}{2}-2\pi} \triangle_a \times \mathscr{A}_a \times \mathscr{A}_a$.

Now consider a sequence $\{f_n\}_{n\in\mathbb{N}} \subset A^2(\mathcal{W}'_\mu)$ of functions that admit holomorphic extensions to domains $Y_n \supset \overline{\mathcal{W}'_\mu}$, whence holomorphic extensions $\widehat{f_n}$ to the domains $\widehat{Y}_n := Y_n \cup \bigcup_{-\frac{\mu}{2}<a<\frac{\mu}{2}-2\pi} \triangle_a \times \mathscr{A}_a \times \mathscr{A}_a$. Let us assume $f_n \to f$ in $A^2(\mathcal{W}'_\mu)$, whence $f_n \to f$ uniformly in every compact subset of $\mathcal{W}'_\mu$. Taking into account, for all $a \in \left(-\frac{\mu}{2}, \frac{\mu}{2} - 2\pi\right)$ and for all $(z_1,z_2,z_3) \in \triangle_a \times \mathscr{A}_a \times \mathscr{A}_a$, the Cauchy integral formula

$$\begin{aligned}\widehat{f_n}(z_1,z_2,z_3) &= \frac{1}{(2\pi i)^2}\int_{\partial\mathscr{A}_a}\int_{\partial\mathscr{A}_a} \frac{\widehat{f_n}(z_1,\zeta_2,\zeta_3)}{(\zeta_2-z_2)(\zeta_3-z_3)}\, d\zeta_2 d\zeta_3 \\ &= \frac{1}{(2\pi i)^2}\int_{\partial\mathscr{A}_a}\int_{\partial\mathscr{A}_a} \frac{f_n(z_1,\zeta_2,\zeta_3)}{(\zeta_2-z_2)(\zeta_3-z_3)}\, d\zeta_2 d\zeta_3\,,\end{aligned}$$

we conclude that the sequence $\{\widehat{f_n}\}_{n\in\mathbb{N}}$, too, converges uniformly in every compact subset of $\widehat{\mathcal{W}'_\mu} := \mathcal{W}'_\mu \cup \bigcup_{-\frac{\mu}{2}<a<\frac{\mu}{2}-2\pi} \triangle_a \times \mathscr{A}_a \times \mathscr{A}_a$. These uniform limits define a holomorphic function $\widehat{f} : \widehat{\mathcal{W}'_\mu} \to \mathbb{C}$, which coincides with f in $\mathcal{W}'_\mu$ by construction.

Now take any $\Omega \in \mathscr{C}_\mu$. By (6), every $f \in A^2(\overline{\Omega})$ also belongs to $A^2(\overline{\mathcal{W}'_\mu})$. Thus, f extends holomorphically to $\widehat{\Omega} := \Omega \cup \bigcup_{-\frac{\mu}{2}<a<\frac{\mu}{2}-2\pi} \triangle_a \times \mathscr{A}_a \times \mathscr{A}_a$.

For any $\kappa \in \mathbb{C} \setminus \mathbb{Z}$, the function E_κ (belonging to both $A^2(\mathcal{W}'_\mu)$ and $A^2(\Omega)$) cannot be holomorphically extended to $\triangle_a \times \mathscr{A}_a \times \mathscr{A}_a$ for any $a \in \left(-\frac{\mu}{2}, \frac{\mu}{2} - 2\pi\right)$. This proves the proper inclusions $A^2(\overline{\mathcal{W}'_\mu}) \subsetneq A^2(\mathcal{W}'_\mu)$ and $A^2(\overline{\Omega}) \subsetneq A^2(\Omega)$. □

Our final result concerns the irregularity of the Bergman projection of every $\Omega \in \mathscr{C}_\mu$. Just as in the case of the Diederich-Fornaess worm domains, see [Bar92], the Sobolev space $W^{r,2}(\mathcal{W})$ is not preserved by the Bergman projection for sufficiently large r.

Theorem 4.6. *Let $\mu > \frac{\pi}{2}$ and set $\nu = \frac{\pi}{2\mu}$. For all $\Omega \in \mathscr{C}_\mu$, the Bergman projection associated to Ω does not map $W^{r,2}(\Omega)$ into itself when $r \ge \nu$.*

Our proof follows the lines of [Bar92] and is postponed to the next section.

5. Proof of Theorem 4.6

Although the proof of Theorem 4.6 closely follows the proof given in [Bar92] for the classical worm domain in $\mathbb{C}^2$, we prefer to include it for the sake of completeness and for the reader's convenience. We will need several tools, starting with the next remark.

Remark 5.1. Let Ω be a domain in $\mathbb{C}^n$, invariant with respect to the rotation

$$w \mapsto (w_1, \dots, w_{m-1}, e^{i\theta} w_m, w_{m+1}, \dots, w_n)$$

for all $\theta \in \mathbb{R}$. The Bergman space $A^2(\Omega)$ decomposes as $\bigoplus_{k\in\mathbb{Z}} \mathcal{H}^{(0,\dots,k,\dots,0)}(\Omega)$ where

$$\begin{aligned}\mathcal{H}^{(0,\dots,k,\dots,0)}(\Omega) &= \Big\{F \in A^2(\Omega) : \ F(w_1, \dots, e^{i\theta} w_m, \dots, w_n) = e^{ik\theta} F(w) \ \forall\, \theta \in \mathbb{R}\Big\} \\ &= \Big\{F \in A^2(\Omega) : F(w) w_m^{-k} \text{ is locally constant in } w_m\Big\} .\end{aligned}$$

The projection $Q_{(0,\dots,k,\dots,0)} : A^2(\Omega) \to \mathcal{H}^{(0,\dots,k,\dots,0)}(\Omega)$ is given by

$$Q_{(0,\dots,k,\dots,0)} F(w) = \frac{1}{2\pi} \int_0^{2\pi} F(w_1, \dots, e^{i\theta} w_m, \dots, w_n) e^{-ik\theta}\, d\theta .$$

If Ω is also invariant with respect to rotations in the variable w_l, we set

$$\mathcal{H}^{(0,\dots,j,\dots,k,\dots,0)} := \mathcal{H}^{(0,\dots,j,\dots,0,\dots,0)} \cap \mathcal{H}^{(0,\dots,0,\dots,k,\dots,0)} .$$

The projection $Q_{(0,\dots,j,\dots,k,\dots,0)} : A^2(\Omega) \to \mathcal{H}^{(0,\dots,j,\dots,k,\dots,0)}(\Omega)$ is given by

$$\begin{aligned}&Q_{(0,\dots,j,\dots,k,\dots,0)} F(w) = Q_{(0,\dots,j,\dots,0,\dots,0)} Q_{(0,\dots,0,\dots,k,\dots,0)} F \\ &= \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} F(w_1, \dots, e^{i\theta_l} w_l, \dots, e^{i\theta_m} w_m, \dots, w_n)\, e^{-ik\theta_m}\, d\theta_m\, e^{-ij\theta_l}\, d\theta_l .\end{aligned}$$

Just as in [Bar92], rather than working directly with the domains in the class $\mathscr{C}_\mu$, we construct model domains with the next definition. Here, and in the sequel, we adopt the notations

$$\begin{aligned} I_\alpha &= (-\alpha, \alpha) , \\ S_\beta &= \{\zeta \in \mathbb{C} : \operatorname{Im} \zeta \in I_\beta\} \end{aligned}$$

for all $\alpha, \beta > 0$.

Definition 5.2. Fix a real number $\mu > 0$. We set

$$\begin{aligned} D_\mu &= \Big\{(z_1, z_2, z_3) \in \mathbb{C}^3 : \operatorname{Re}\Big(z_1 e^{-i \log |z_2 z_3|^2}\Big) > 0, \log |z_2|^2, \log |z_3|^2 \in I_\mu\Big\} , \\ D'_\mu &= \Big\{(w_1, w_2, w_3) \in \mathbb{C}^3 : \operatorname{Im}(w_1) - \log |w_2 w_3|^2 \in I_{\frac{\pi}{2}}, \log |w_2|^2, \log |w_3|^2 \in I_\mu\Big\} . \end{aligned}$$

Clearly, $\mathcal{W}'_\mu \subset D_\mu$ for all $\mu > 0$.

Remark 5.3. *If $\Omega \in \mathscr{C}_\mu$, if $\Omega = D_\mu$ or if $\Omega = D'_\mu$, then Ω is invariant with respect to rotations in the second and third variables. Therefore,*

$$A^2(\Omega) = \bigoplus_{j,k\in\mathbb{Z}} \mathcal{H}^{(0,j,k)}(\Omega) .$$

For the sake of simplicity, we will denote $\mathcal{H}^{(0,j,k)}(\Omega)$ as $\mathcal{H}^{j,k}(\Omega)$.

More useful tools are provided by the next remarks and lemmas.

Remark 5.4. The domain D_μ is biholomorphic to D'_μ via the "unwinding" map $z=(z_1,z_2,z_3)\mapsto(\ell(z),z_2,z_3)$, whose inverse is $(w_1,w_2,w_3)\mapsto(e^{w_1},w_2,w_3)$. Thus, we have an isometric isomorphism $T:A^2(D_\mu)\to A^2(D'_\mu)$ with $(TF)(w_1,w_2,w_3)=e^{w_1}F(e^{w_1},w_2,w_3)$ and $(T^{-1}G)(z_1,z_2,z_3)=z_1^{-1}G(\ell(z),z_2,z_3)$. Clearly, T maps $\mathcal{H}^{j,k}(D_\mu)$ isomorphically and isometrically onto $\mathcal{H}^{j,k}(D'_\mu)$ for all $j,k\in\mathbb{Z}$.

The advantage of D'_μ over D_μ is that its fiber over each couple (w_1,w_2) and its fiber over each couple (w_1,w_3) are both connected, a property that allows to prove the next lemma. Before its statement and proof, we recall a few useful properties.

Remark 5.5. *Let $\beta>0$ and consider, on the strip S_β, a weight $\omega:S_\beta\to\mathbb{R}$ of the form $\omega(x+iy)=\alpha(y)$ for some integrable $\alpha:\mathbb{R}\to[0,+\infty)$ with $\operatorname{supp}(\alpha)\subseteq I_\beta$. Set $\widetilde{\alpha}(\xi)=\widehat{\alpha}(-2i\xi)=\int_{I_\beta}\alpha(y)\,e^{-2y\xi}\,dy$. The Fourier transform associating to any $\phi\in L^2(\mathbb{R},\widetilde{\alpha})$ the complex function*

$$S_\beta\to\mathbb{C}\quad \zeta\mapsto\frac{1}{2\pi}\int_{\mathbb{R}}\phi(\xi)\,e^{i\zeta\xi}\,d\xi\,,$$

is an isometric isomorphism from $L^2(\mathbb{R},\widetilde{\alpha})$ to $A^2(S_\beta,\omega)$ (see [Bar92, Proof of Lemma 2]*). In particular, according to* [Bar92, §2]*, the reproducing kernel K_ω of $A^2(S_\beta,\omega)$ has the form*

$$K_\omega(\zeta,\zeta')=\frac{1}{2\pi}\int_{\mathbb{R}}\frac{e^{i(\zeta-\bar\zeta')\xi}}{\widehat{\alpha}(-2i\xi)}\,d\xi\,.$$

Lemma 5.6. *Let $j,k\in\mathbb{Z},\mu>0$, set $\beta=2\mu+\frac{\pi}{2}$ and define $\omega_{j,k}=\omega^{\mu}_{j,k}:S_\beta\to\mathbb{R}$ as*

$$\omega_{j,k}(x+iy)=\pi^2(e^{(j+1)(\cdot)}\chi_{I_\mu})*(e^{(k+1)(\cdot)}\chi_{I_\mu})*\chi_{I_{\frac{\pi}{2}}}(y)\,.$$

Then

$$L_{j,k}:\mathcal{H}^{j,k}(D'_\mu)\to A^2(S_\beta,\omega_{j,k})\qquad (L_{j,k}G)(\zeta)=G(\zeta,w_2,w_3)w_2^{-j}w_3^{-k}$$

is an isometric isomorphism. Thus: if $K_{\omega_{j,k}}$ denotes the reproducing kernel of $A^2(S_\beta,\omega_{j,k})$, then the reproducing kernel of $\mathcal{H}^{j,k}(D'_\mu)$ is $(w,w')\mapsto K_{\omega_{j,k}}(w_1,w'_1)\,(w_2\,\overline{w}'_2)^j\,(w_3\,\overline{w}'_3)^k$. Moreover,

$$K_{\omega_{-1,-1}}(\zeta,\zeta')=\int_{\mathbb{R}}\frac{\xi^3\,e^{i(\zeta-\bar\zeta')\xi}}{2\pi^3\sinh^2(2\mu\xi)\sinh(\pi\xi)}\,d\xi\,.$$

For $\mu>\frac{\pi}{2}$, we derive

$$\begin{aligned}K_{\omega_{-1,-1}}(\zeta,\zeta')&=e^{-(\zeta-\bar\zeta')\nu}(C_\nu(\zeta-\bar\zeta')+C'_\nu)+O\left(e^{-\operatorname{Re}(\zeta-\bar\zeta')\nu'}\right)&&\text{if }\operatorname{Re}(\zeta-\bar\zeta')>0\\&=e^{(\zeta-\bar\zeta')\nu}(C_\nu(\zeta-\bar\zeta')-C'_\nu)+O\left(e^{\operatorname{Re}(\zeta-\bar\zeta')\nu'}\right)&&\text{if }\operatorname{Re}(\zeta-\bar\zeta')<0\,,\end{aligned}$$

where $\nu=\frac{\pi}{2\mu}<1,\nu'=\min\{2\nu,1\}>\nu,C_\nu=-\frac{i\nu^5}{2\pi^5\sin(\nu\pi)}$ and $C'_\nu=\frac{i\nu^4}{2\pi^5}\frac{3-\nu\pi\cot(\nu\pi)}{\sin(\nu\pi)}$.

Proof. Let $\pi_1:D'_\mu\to\mathbb{C}$ denote the projection map onto the first variable: we have

$$\begin{aligned}\pi_1(D'_\mu)&=\left\{w_1\in\mathbb{C}:\exists\ t_2,t_3\in I_\mu\text{ s.t. }\operatorname{Im}(w_1)-t_2-t_3\in I_{\frac{\pi}{2}}\right\}\\&=S_\beta\quad\text{with }\beta=2\mu+\frac{\pi}{2}\,.\end{aligned}$$

Over each point $x+iy\in S_\beta$, we have the fiber

$$\pi_1^{-1}(x+iy)=\left\{(x+iy,w_2,w_3):\log|w_2|^2,\log|w_3|^2\in I_\mu,y-\log|w_2|^2-\log|w_3|^2\in I_{\frac{\pi}{2}}\right\}\,.$$

We now construct weights $\omega_{j,k} : S_\beta \to \mathbb{R}$ such that

$$M_{j,k} : A^2(S_\beta, \omega_{j,k}) \to \mathcal{H}^{j,k}(D'_\mu) \qquad (M_{j,k}f)(w_1, w_2, w_3) = f(w_1)w_2^j w_3^k$$

is an injective linear map and an isometry. For any holomorphic $f, g : S_\beta \to \mathbb{C}$, we have

$$\begin{aligned}\langle M_{j,k}f, M_{j,k}g\rangle_{\mathcal{H}^{j,k}(D'_\mu)} &= \int_{S_\beta} f(w_1)\overline{g(w_1)} \int_{\pi_1^{-1}(w_1)} |w_2|^{2j}|w_3|^{2k}\, dV(w_3)\, dV(w_2)\, dV(w_1)\\ &= \int_{S_\beta} f(w_1)\overline{g(w_1)}\, \omega_{j,k}(w_1)\, dV(w_1) = \langle f, g\rangle_{A^2(S_\beta,\omega_{j,k})}\end{aligned}$$

if we set, for all $x + iy \in S_\beta$,

$$\begin{aligned}\omega_{j,k}(x+iy) &= \int_{\pi_1^{-1}(x+iy)} |w_2|^{2j}|w_3|^{2k}\, dV(w_3)\, dV(w_2)\\ &= 4\pi^2 \iint_{\mathbb{R}^2} \chi_{I_\mu}\left(\log r_2^2\right)\, \chi_{I_\mu}\left(\log r_3^2\right)\, \chi_{I_{\frac{\pi}{2}}}\left(y - \log r_2^2 - \log r_3^2\right)\, r_2^{2j+1} r_3^{2k+1}\, dr_3\, dr_2\\ &= \pi^2 \int_{\mathbb{R}} e^{(j+1)s_2}\chi_{I_\mu}(s_2) \int_{\mathbb{R}} e^{(k+1)s_3}\chi_{I_\mu}(s_3)\chi_{I_{\frac{\pi}{2}}}(y - s_2 - s_3)\, ds_3\, ds_2\\ &= \pi^2 \int_{\mathbb{R}} e^{(j+1)s_2}\chi_{I_\mu}(s_2) \left(e^{(k+1)(\cdot)}\chi_{I_\mu} * \chi_{I_{\frac{\pi}{2}}}\right)(y - s_2)\, ds_2\\ &= \pi^2\, (e^{(j+1)(\cdot)}\chi_{I_\mu}) * (e^{(k+1)(\cdot)}\chi_{I_\mu}) * \chi_{I_{\frac{\pi}{2}}}(y)\,.\end{aligned}$$

We now fix any $G \in \mathcal{H}^{j,k}(D'_\mu) \subset \mathcal{H}^k(D'_\mu)$, recalling that $G(w_1, w_2, w_3)w_3^{-k}$ is locally constant in w_3 and $G(w_1, w_2, w_3)w_2^{-j}$ is locally constant in w_2. We are going to show that the function $G(w_1, w_2, w_3)w_2^{-j}w_3^{-k}$ is constant in (w_2, w_3): the resulting function of w_1 will automatically be an element of $A^2(S_\beta, \omega_{j,k})$ (which proves that $M_{j,k}$ is surjective and an isomorphism, as desired). Let $\pi_{1,2} : D'_\mu \to \mathbb{C}^2$ denote the projection map onto the first two variables: then

$$\begin{aligned}\pi_{1,2}(D'_\mu) &= \left\{(w_1, w_2) \in \mathbb{C}^2 : \log|w_2|^2 \in I_\mu,\ \exists\ t_3 \in I_\mu \text{ s.t. } \operatorname{Im}(w_1) - \log|w_2|^2 - t_3 \in I_{\frac{\pi}{2}}\right\}\\ &= \left\{(w_1, w_2) \in \mathbb{C}^2 : \log|w_2|^2 \in I_\mu,\ \operatorname{Im}(w_1) - \log|w_2|^2 \in I_{\mu+\frac{\pi}{2}}\right\}\,.\end{aligned}$$

Over each pair $(w_1, w_2) \in \pi_{1,2}(D'_\mu)$, we have the fiber

$$\begin{aligned}&\pi_{1,2}^{-1}(w_1, w_2)\\ &= \left\{(w_1, w_2, w_3) : \log|w_3|^2 \in I_\mu \cap \left(\operatorname{Im}(w_1) - \log|w_2|^2 - \frac{\pi}{2}, \operatorname{Im}(w_1) - \log|w_2|^2 + \frac{\pi}{2}\right)\right\}\,,\end{aligned}$$

which is a nonempty annulus because $\operatorname{Im}(w_1) - \log|w_2|^2 - \frac{\pi}{2} < \mu$ and $\operatorname{Im}(w_1) - \log|w_2|^2 + \frac{\pi}{2} > -\mu$. Since every fiber $\pi_{1,2}^{-1}(w_1, w_2)$ is connected, the function $G(w_1, w_2, w_3)w_3^{-k}$ is not only locally constant but constant in w_3. An analogous argument proves that the function $G(w_1, w_2, w_3)w_2^{-j}$ is not only locally constant but constant in w_2. The proof of the first statement is complete.

The fact that $M_{j,k} : A^2(S_\beta, \omega_{j,k}) \to \mathcal{H}^{j,k}(D'_\mu)$, which maps f into $(w_1, w_2, w_3) \mapsto f(w_1)w_2^j w_3^k$, is an isometry yields the second statement.

Our next aim is computing and estimating the reproducing kernel $K_{\omega_{-1,-1}}$ of $A^2(S_\beta, \omega_{-1,-1})$. We have

$$\omega_{-1,-1}(x + iy) = \pi^2\, \chi_{I_\mu} * \chi_{I_\mu} * \chi_{I_{\frac{\pi}{2}}}(y)$$

and the Fourier-Laplace transform of the latter function, computed at $-2i\xi$, is

$$\pi^2\ (\widehat{\chi}_{I_\mu})^2(-2i\xi)\ \widehat{\chi}_{I_{\frac{\pi}{2}}}(-2i\xi) = \pi^2\xi^{-3}\sinh^2(2\mu\xi)\sinh(\pi\xi)\,.$$

By Remark 5.5,

$$K_{\omega_{-1,-1}}(\zeta,\zeta') = \int_{\mathbb{R}} \mathcal{I}(\xi)\,d\xi\,, \qquad \mathcal{I}(\xi) = \frac{\xi^3\,e^{i(\zeta-\bar{\zeta}')\xi}}{2\pi^3\sinh^2(2\mu\xi)\sinh(\pi\xi)}\,.$$

When viewed as a function of a complex variable $\xi = x+iy$, the integrand function $\mathcal{I}(\xi)$ has squared modulus

$$|\mathcal{I}(x+iy)|^2 = \frac{(x^2+y^2)^3\,e^{-2\operatorname{Im}(\zeta-\bar{\zeta}')x}\,e^{-2\operatorname{Re}(\zeta-\bar{\zeta}')y}}{2\pi^3(\sinh^2(2\mu x)+\sin^2(2\mu y))^2(\sinh^2(\pi x)+\sin^2(\pi y))}\,.$$

Moreover, $\mathcal{I}$ has: a simple pole at each point ik with $k\in\mathbb{Z}^*$, because

$$\sinh(\pi\xi) = (-1)^k\sum_{n\in\mathbb{N}}\frac{(\pi\xi-ik\pi)^{2n+1}}{(2n+1)!} = (-1)^k\sum_{n\in\mathbb{N}}\frac{\pi^{2n+1}}{(2n+1)!}(\xi-ik)^{2n+1}\,;$$

a double pole at each point $ik\nu$ with $k\in\mathbb{Z}^*$, where $\nu=\frac{\pi}{2\mu}$, because

$$\sinh(2\mu\xi) = (-1)^k\sum_{n\in\mathbb{N}}\frac{(2\mu)^{2n+1}}{(2n+1)!}(\xi-ik\nu)^{2n+1} = (-1)^k\left(2\mu(\xi-ik\nu)+\frac{4}{3}\mu^3(\xi-ik\nu)^3+\dots\right)\,.$$

Our hypothesis $\mu>\frac{\pi}{2}$ yields $\nu<1$. Thus, the poles nearest to $\mathbb{R}$ are $\pm i\nu$, followed by $\pm i\nu'$ with $\nu'=\min\{2\nu,1\}>\nu$. If $\operatorname{Re}(\zeta-\bar{\zeta}')>0$, let $y_0\in(\nu,\nu')$ and $p_0=i\nu$; if $\operatorname{Re}(\zeta-\bar{\zeta}')<0$, let $y_0\in(-\nu',-\nu)$ and $p_0=-i\nu$. Along the line $y=y_0$, the modulus $|\mathcal{I}(x+iy)|$ decays well as $x\to\pm\infty$. A standard contour integration argument yields that

$$K_{\omega_{-1,-1}}(\zeta,\zeta') = \int_{\mathbb{R}}\mathcal{I}(\xi)\,d\xi = \int_{\mathbb{R}+iy_0}\mathcal{I}(\xi)\,d\xi + \operatorname{Res}(\mathcal{I},p_0)\,. \tag{9}$$

Clearly,

$$\int_{\mathbb{R}+iy_0}\mathcal{I}(\xi)\,d\xi = O\left(e^{-\operatorname{Re}(\zeta-\bar{\zeta}')y_0}\right)\,.$$

To compute $\operatorname{Res}(\mathcal{I},p_0)$ at $p_0=\pm i\nu$, we first define the functions $\mathcal{G}_1^{\pm}(\xi)=(\xi\mp i\nu)\sinh^{-1}(2\mu\xi)$, which have $\mathcal{G}_1^{\pm}(\pm i\nu) = -(2\mu)^{-1} = -\frac{\nu}{\pi}$ and $(\mathcal{G}_1^{\pm})'(\pm i\nu)=0$, as well as the function $\mathcal{G}_2(\xi) = (2\pi^3)^{-1}\xi^3\sinh^{-1}(\pi\xi)e^{i(\zeta-\bar{\zeta}')\xi}$. We then compute

$$\begin{aligned}
\operatorname{Res}(\mathcal{I},\pm i\nu) &= \lim_{\xi\to\pm i\nu}\frac{d}{d\xi}\left((\xi\mp i\nu)^2\mathcal{I}(\xi)\right) = \lim_{\xi\to\pm i\nu}\frac{d}{d\xi}\left(\mathcal{G}_1^{\pm}(\xi)^2\mathcal{G}_2(\xi)\right)\\
&= 2\mathcal{G}_1^{\pm}(\pm i\nu)\,(\mathcal{G}_1^{\pm})'(\pm i\nu)\,\mathcal{G}_2(\pm i\nu)+\mathcal{G}_1^{\pm}(\pm i\nu)^2\,\mathcal{G}_2'(\pm i\nu)\\
&= 0+\frac{\nu^2}{\pi^2}\mathcal{G}_2'(\pm i\nu)\\
&= \frac{\nu^2}{\pi^2}\left(\frac{\xi^2 e^{i(\zeta-\bar{\zeta}')\xi}}{2\pi^3\sinh(\pi\xi)}\left(3-\pi\xi\coth(\pi\xi)+i(\zeta-\bar{\zeta}')\xi\right)\right)_{|\xi=\pm i\nu}\\
&= e^{\mp(\zeta-\bar{\zeta}')\nu}(C_\nu(\zeta-\bar{\zeta}')\pm C_\nu')\,,
\end{aligned}$$

where

$$C_\nu = \frac{\nu^2}{2\pi^5}\frac{i\xi^3}{\sinh(\pi\xi)}_{|\xi=\pm i\nu} = \frac{\nu^5}{2\pi^5}\frac{1}{\sinh(i\nu\pi)} = \frac{\nu^5}{2\pi^5}\frac{1}{i\sin(\nu\pi)} = -\frac{i\nu^5}{2\pi^5\sin(\nu\pi)}$$

$$C'_\nu = -\frac{\nu^4}{2\pi^5}\frac{3-i\nu\pi\coth(i\nu\pi)}{\sinh(i\nu\pi)} = \frac{\nu^4}{2\pi^5}\frac{3-\nu\pi\cot(\nu\pi)}{-i\sin(\nu\pi)} = \frac{i\nu^4}{2\pi^5}\frac{3-\nu\pi\cot(\nu\pi)}{\sin(\nu\pi)}\,.$$

If $\mathrm{Re}(\zeta-\bar\zeta')>0$, by letting $y_0\to(\nu')^-$ in formula (9), we find that

$$K_{\omega_{-1,-1}}(\zeta,\zeta') = e^{-(\zeta-\bar\zeta')\nu}(C_\nu(\zeta-\bar\zeta')+C'_\nu) + O\left(e^{-\mathrm{Re}(\zeta-\bar\zeta')\nu'}\right)\,.$$

Similarly, if $\mathrm{Re}(\zeta-\bar\zeta')<0$, by letting $y_0\to(-\nu')^+$ in formula (9), we find that

$$K_{\omega_{-1,-1}}(\zeta,\zeta') = e^{(\zeta-\bar\zeta')\nu}(C_\nu(\zeta-\bar\zeta')-C'_\nu) + O\left(e^{\mathrm{Re}(\zeta-\bar\zeta')\nu'}\right)\,,$$

as desired. □

For future use, we prove the next lemma.

Lemma 5.7. *Let $0<\mu<\mu'$. Associating to each $F\in A^2(D_{\mu'})$ its restriction $F_{|D_\mu}$ defines a linear map*

$$A^2(D_{\mu'})\to A^2(D_\mu)\,,$$

whose image is dense in $A^2(D_\mu)$.

Proof. Thanks to Remark 5.4 and to Lemma 5.6, it suffices to prove that, after setting $\beta=2\mu+\frac{\pi}{2}$ and $\beta'=2\mu'+\frac{\pi}{2}$, for all $j,k\in\mathbb{Z}$ the restriction to S_β defines a linear map

$$A^2(S_{\beta'},\omega^{\mu'}_{j,k})\to A^2(S_\beta,\omega^{\mu}_{j,k})\,,$$

whose image is dense in $A^2(S_\beta,\omega^\mu_{j,k})$.

Since

$$\omega^\mu_{j,k}(x+iy) = \pi^2(e^{(j+1)(\cdot)}\chi_{I_\mu})*(e^{(k+1)(\cdot)}\chi_{I_\mu})*\chi_{I_{\frac{\pi}{2}}}(y)\,,$$
$$\omega^{\mu'}_{j,k}(x+iy) = \pi^2(e^{(j+1)(\cdot)}\chi_{I_{\mu'}})*(e^{(k+1)(\cdot)}\chi_{I_{\mu'}})*\chi_{I_{\frac{\pi}{2}}}(y)\,,$$

our Remark 5.5 guarantees that the spaces $A^2(S_\beta,\omega^\mu_{j,k})$, $A^2(S_{\beta'},\omega^{\mu'}_{j,k})$ are the image through the inverse Fourier transform of the spaces $L^2(\mathbb{R},\phi)$, $L^2(\mathbb{R},\psi)$, where

$$\phi(\xi) = \pi^2\,\frac{\sinh\left(2\mu\left(\xi-\frac{j+1}{2}\right)\right)}{\xi-\frac{j+1}{2}}\,\frac{\sinh\left(2\mu\left(\xi-\frac{k+1}{2}\right)\right)}{\xi-\frac{k+1}{2}}\,\frac{\sinh(\pi\xi)}{\xi}\,,$$

$$\psi(\xi) = \pi^2\,\frac{\sinh\left(2\mu'\left(\xi-\frac{j+1}{2}\right)\right)}{\xi-\frac{j+1}{2}}\,\frac{\sinh\left(2\mu'\left(\xi-\frac{k+1}{2}\right)\right)}{\xi-\frac{k+1}{2}}\,\frac{\sinh(\pi\xi)}{\xi}\,.$$

It follows at once that the restriction to S_β maps $A^2(S_{\beta'},\omega^{\mu'}_{j,k})$ into $A^2(S_\beta,\omega^\mu_{j,k})$. Since $C^\infty_0(\mathbb{R})$ is dense in both $L^2(\mathbb{R},\phi)$ and $L^2(\mathbb{R},\psi)$, we immediately conclude that the image of the restriction map $A^2(S_{\beta'},\omega^{\mu'}_{j,k})\to A^2(S_\beta,\omega^\mu_{j,k})$ is dense in $A^2(S_\beta,\omega^\mu_{j,k})$. □

We now turn back to D_μ.

Lemma 5.8. *The reproducing kernel of $\mathcal{H}^{j,k}(D_\mu)$ is*

$$K_{j,k}(z,z') = K_{\omega_{j,k}}(\ell(z),\ell(z'))(z_1\,\overline{z}_1')^{-1}(z_2\,\overline{z}_2')^j\,(z_3\,\overline{z}_3')^k\,.$$

In particular, recalling that (for $\kappa\in\mathbb{C}$) $E_\kappa(z)$ is the holomorphic extension of z_1^κ constructed in Definition 4.2, we have

$$z_2\,\overline{z}_2'\,z_3\,\overline{z}_3'\,K_{-1,-1}(z,z') = \int_{\mathbb{R}} \frac{\xi^3\,E_{i\xi-1}(z)\,\overline{E_{i\xi-1}(z')}}{2\pi^3\sinh^2(2\mu\xi)\sinh(\pi\xi)}\,d\xi\,. \tag{10}$$

For $\mu>\frac{\pi}{2}, \nu=\frac{\pi}{2\mu}, \nu'=\min\{2\nu,1\}, C_\nu=-\frac{i\nu^5}{2\pi^5\sin(\nu\pi)}$ and $C_\nu'=\frac{i\nu^4}{2\pi^5}\frac{3-\nu\pi\cot(\nu\pi)}{\sin(\nu\pi)}$, we have

$$\begin{aligned} &z_2\,\overline{z}_2'\,z_3\,\overline{z}_3'\,K_{-1,-1}(z,z')\\ &= E_{-\nu-1}(z)\,\overline{E_{\nu-1}(z')}\left(C_\nu\left(\ell(z)-\overline{\ell(z')}\right)+C_\nu'\right)+O\left(\frac{|z_1'|^{\nu'-1}}{|z_1|^{\nu'+1}}\right) \quad \text{if } |z_1|>|z_1'|\\ &= E_{\nu-1}(z)\,\overline{E_{-\nu-1}(z')}\left(C_\nu\left(\ell(z)-\overline{\ell(z')}\right)-C_\nu'\right)+O\left(\frac{|z_1|^{\nu'-1}}{|z_1'|^{\nu'+1}}\right) \quad \text{if } |z_1|<|z_1'|. \end{aligned}$$

As a consequence: for each $\mu>\frac{\pi}{2}, z'\in D_\mu, m\in\mathbb{N}, s\in[0,1)$, the function

$$z\mapsto\left(\operatorname{Re}\left(z_1e^{-i\log|z_2z_3|^2}\right)\right)^s\,\frac{\partial^m}{\partial z_1^m}K_{-1,-1}(z,z')$$

does not belong to $L^2(D_\mu)$ if $m-s\geq\nu$.

Proof. The first statement follows from Lemma 5.6 through the isometry $T^{-1}:\mathcal{H}^{j,k}(D_\mu')\to\mathcal{H}^{j,k}(D_\mu)$, which maps each F into $T^{-1}F(z)=F(\ell(z),z_2,z_3)z_1^{-1}$.

To prove formula (10), we compute

$$\begin{aligned} z_2\,\overline{z}_2'\,z_3\,\overline{z}_3'\,K_{-1,-1}(z,z') &= (z_1\,\overline{z}_1')^{-1}K_{\omega_{-1,-1}}(\ell(z),\ell(z'))\\ &= (z_1\,\overline{z}_1')^{-1}\int_{\mathbb{R}}\frac{\xi^3\,e^{i(\ell(z)-\overline{\ell(z')})\xi}}{2\pi^3\sinh^2(2\mu\xi)\sinh(\pi\xi)}\,d\xi\\ &= \int_{\mathbb{R}}\frac{\xi^3\,E_{i\xi-1}(z)\,\overline{E_{i\xi-1}(z')}}{2\pi^3\sinh^2(2\mu\xi)\sinh(\pi\xi)}\,d\xi\,, \end{aligned}$$

where we took into account the equalities $E_{i\xi-1}(z)=e^{(i\xi-1)\ell(z)}=e^{i\ell(z)\xi}z_1^{-1}$ and the equalities $\overline{E_{i\xi-1}(z')}=\overline{e^{i\ell(z')\xi}(z_1')^{-1}}=e^{-i\overline{\ell(z')}\xi}(\overline{z}_1')^{-1}$.

We assume henceforth $\mu>\frac{\pi}{2}$ and apply the estimates for $K_{\omega_{-1,-1}}$ obtained in Lemma 5.6. For $|z_1|>|z_1'|$, which is the same as $\operatorname{Re}\left(\ell(z)-\overline{\ell(z')}\right)>0$, we find that

$$\begin{aligned} z_1\,\overline{z}_1'\,z_2\,\overline{z}_2'\,z_3\,\overline{z}_3'\,K_{-1,-1}(z,z') &= e^{-(\ell(z)-\overline{\ell(z')})\nu}\left(C_\nu\left(\ell(z)-\overline{\ell(z')}\right)+C_\nu'\right)+O\left(e^{-\operatorname{Re}\left(\ell(z)-\overline{\ell(z')}\right)\nu'}\right)\\ &= E_{-\nu}(z)\,\overline{E_\nu(z')}\left(C_\nu\left(\ell(z)-\overline{\ell(z')}\right)+C_\nu'\right)+O\left(\left|z_1'/z_1\right|^{\nu'}\right)\,,\\ z_2\,\overline{z}_2'\,z_3\,\overline{z}_3'\,K_{-1,-1}(z,z') &= E_{-\nu-1}(z)\,\overline{E_{\nu-1}(z')}\left(C_\nu\left(\ell(z)-\overline{\ell(z')}\right)+C_\nu'\right)+O\left(\frac{|z_1'|^{\nu'-1}}{|z_1|^{\nu'+1}}\right)\,, \end{aligned}$$

as stated. For $|z_1| < |z'_1|$, we find that

$$\begin{aligned} z_1\,\overline{z}'_1\,z_2\,\overline{z}'_2\,z_3\,\overline{z}'_3\,K_{-1,-1}(z,z') &= e^{(\ell(z)-\overline{\ell(z')})\nu}\left(C_\nu\left(\ell(z)-\overline{\ell(z')}\right)-C'_\nu\right)+O\left(e^{\mathrm{Re}\left(\ell(z)-\overline{\ell(z')}\right)\nu'}\right)\\ &= E_\nu(z)\,\overline{E_{-\nu}(z')}\left(C_\nu\left(\ell(z)-\overline{\ell(z')}\right)-C'_\nu\right)+O\left(\left|z_1/z'_1\right|^{\nu'}\right),\\ z_2\,\overline{z}'_2\,z_3\,\overline{z}'_3\,K_{-1,-1}(z,z') &= E_{\nu-1}(z)\,\overline{E_{-\nu-1}(z')}\left(C_\nu\left(\ell(z)-\overline{\ell(z')}\right)-C'_\nu\right)+O\left(\frac{|z_1|^{\nu'-1}}{|z'_1|^{\nu'+1}}\right), \end{aligned}$$

as desired.

We now turn to the last statement, still under the assumption $\mu > \frac{\pi}{2}$. Since $\frac{\partial E_\kappa}{\partial z_1} = \kappa E_{\kappa-1}$ and $\frac{\partial}{\partial z_1}\left(E_\kappa(z)\,\ell(z)\right) = \kappa E_{\kappa-1}(z)\,\ell(z) - E_\kappa(z) z_1^{-1} = E_{\kappa-1}\left(\kappa\,\ell(z)-1\right)$, for any $m\in\mathbb{N}$ and any $z'\in D_\mu$ there exist $\phi_\nu^m = \phi_\nu^m(z'), \psi_\nu^m = \psi_\nu^m(z') \in \mathbb{C}$ such that

$$z_2\,z_3\,\frac{\partial^m}{\partial z_1^m}K_{-1,-1}(z,z') = E_{\nu-m-1}(z)\left(\phi_\nu^m\ell(z)+\psi_\nu^m\right)+O\left(|z_1|^{\nu'-m-1}\right)$$

in the region $D_\mu^{z'} := \{z\in D_\mu : |z_1| < |z'_1|\}$. If the function $\frac{\partial^m K_{-1,-1}}{\partial z_1^m}(\cdot,z')$ belongs to $L^2(D_\mu)$, then $E_{\nu-m-1}(z)\left(\phi_\nu^m\ell(z)+\psi_\nu^m\right)(z_2\,z_3)^{-1}$ is square-integrable in $D_\mu^{z'}$. Using the notation $H^{z'}$ for the half-disk $\{\zeta\in\mathbb{C} : \mathrm{Re}(\zeta)>0, |\zeta| < |z'_1|\}$, we compute

$$\begin{aligned} &\int_{D_\mu^{z'}} |E_{\nu-m-1}(z)|^2\,|\phi_\nu^m\ell(z)+\psi_\nu^m|^2\,|z_2\,z_3|^{-2}\,dV(z)\\ &= 4\pi^2\int_{\log r_3^2\in I_\mu}\int_{\log r_2^2\in I_\mu}\int_{z_1 e^{-i\log(r_2r_3)^2}\in H^{z'}} |z_1|^{2(\nu-m-1)}\,|\phi_\nu^m\,I+\psi_\nu^m|^2\,dV(z_1)\frac{dr_2}{r_2}\frac{dr_3}{r_3}\\ &= \pi^2\int_{s_3\in I_\mu}\int_{s_2\in I_\mu}\int_{z_1e^{-i(s_2+s_3)}\in H^{z'}} |z_1|^{2(\nu-m-1)}\,|\phi_\nu^m\,II+\psi_\nu^m|^2\,dV(z_1)\,ds_2\,ds_3\\ &= \pi^2\int_{s_3\in I_\mu}\int_{s_2\in I_\mu}\int_{s_2+s_3-\frac{\pi}{2}}^{s_2+s_3+\frac{\pi}{2}}\int_0^{|z'_1|} r_1^{2(\nu-m)-1}\,|\phi_\nu^m\,III+\psi_\nu^m|^2\;dr_1\,d\theta_1\,ds_2\,ds_3\,, \end{aligned}$$

where

$$\begin{aligned} I &= \log\left(z_1e^{-i\log(r_2r_3)^2}\right)+i\log(r_2r_3)^2\\ II &= \log\left(z_1e^{-i(s_2+s_3)}\right)+i(s_2+s_3)\\ III &= \log\left(r_1e^{i(\theta_1-s_2-s_3)}\right)+i(s_2+s_3) = \log r_1 + i\theta_1\,. \end{aligned}$$

If the last integral converges, then $\nu-m>0$. Similarly, if $s\in[0,1), m\in\mathbb{N}, z'\in D_\mu$ and if the function

$$z\mapsto\left(\mathrm{Re}\left(z_1e^{-i\log|z_2z_3|^2}\right)\right)^s\,\frac{\partial^m}{\partial z_1^m}K_{-1,-1}(z,z')$$

belongs to $L^2(D_\mu)$, then $\nu+s-m>0$, i.e., $m-s<\nu$. □

We are now in a position to prove our Theorem 4.6 about the irregularity of the Bergman projection of every $\Omega\in\mathscr{C}_\mu$. For the sake of simplicity, we will write $W^r(\Omega)$ instead of $W^{r,2}(\Omega)$.

Proof of Theorem 4.6. Fix $\Omega = \mathcal{W}_\eta \in \mathscr{C}_\mu$. For $\lambda \geq 1$, we set $\tau_\lambda(z) = (\lambda z_1, z_2, z_3)$ for all $z \in \Omega$ and $\Omega^\lambda = \tau_\lambda(\Omega)$. Setting $T_\lambda f = f \circ \tau_\lambda$ defines an isomorphism $T_\lambda : W^r(\Omega^\lambda) \to W^r(\Omega)$ for all $r \geq 0$, with inverse $T_\lambda^{-1} g = g \circ \tau_\lambda^{-1}$. Although T_λ is not an isometry in general, we have

$$\|T_\lambda f\|_{W^r(\Omega)} \leq \lambda^{r-1} \|f\|_{W^r(\Omega^\lambda)}$$

for all $r \in \mathbb{N}$ and, by interpolation, for all $r \geq 0$. Consider the defining function

$$\rho(z) = |z_1|^2 - 2\,\mathrm{Re}(z_1 e^{-i \log |z_2 z_3|^2}) + \eta(\log |z_2|^2, \log |z_3|^2)$$

of $\Omega = \mathcal{W}_\eta$ and notice that

$$\rho_\lambda(z) = \lambda \rho \circ \tau_\lambda^{-1}(z) = \lambda^{-1}|z_1|^2 - 2\,\mathrm{Re}(z_1 e^{-i \log |z_2 z_3|^2}) + \lambda \eta(\log |z_2|^2, \log |z_3|^2)$$

is a defining function of Ω^λ with

$$\rho_\lambda(z)\, \chi_{I_\mu}(\log |z_2|^2)\, \chi_{I_\mu}(\log |z_3|^2) = \left(\lambda^{-1}|z_1|^2 - 2\,\mathrm{Re}(z_1 e^{-i \log |z_2 z_3|^2})\right) \chi_{I_\mu}(\log |z_2|^2)\, \chi_{I_\mu}(\log |z_3|^2)$$

converging as $\lambda \to +\infty$ to the defining function of D_μ

$$\rho_\infty(z)\, \chi_{I_\mu}(\log |z_2|^2)\, \chi_{I_\mu}(\log |z_3|^2)\,, \qquad \rho_\infty(z) = -2\,\mathrm{Re}(z_1 e^{-i \log |z_2 z_3|^2})\,.$$

We note, for future reference, that $(\Omega^\lambda \cap D_\mu) \nearrow D_\mu$ as $\lambda \nearrow +\infty$, whence every compact subset of D_μ is contained in Ω^λ for sufficiently large λ; while every compact subset of $\mathbb{C}^3 \setminus \overline{D}_\mu$ does not intersect Ω^λ for sufficiently large λ. Let P, P_λ denote the Bergman projections of Ω, Ω^λ, respectively, related by the equality $P_\lambda = T_\lambda^{-1} P T_\lambda$. Then

$$\begin{aligned}
\left\| |\rho_\lambda|^s \frac{\partial^m}{\partial z_1^m} P_\lambda f \right\|_{L^2(\Omega^\lambda)} &= \left\| |\rho_\lambda|^s \frac{\partial^m}{\partial z_1^m} (T_\lambda^{-1} P T_\lambda f) \right\|_{L^2(\Omega^\lambda)} \\
&= \lambda^{s-m} \left\| |\rho \circ \tau_\lambda^{-1}|^s \left(\frac{\partial^m}{\partial z_1^m} (P T_\lambda f) \right) \circ \tau_\lambda^{-1} \right\|_{L^2(\Omega^\lambda)} \\
&= \lambda^{1-m+s} \left\| |\rho|^s \frac{\partial^m}{\partial z_1^m} (P T_\lambda f) \right\|_{L^2(\Omega)} \\
&\leq C_1\, \lambda^{1-m+s}\, \|P T_\lambda f\|_{W^{m-s}(\Omega)}
\end{aligned}$$

for some constant $C_1 > 0$, according to [Lig87]. Assume that $P(W^{m-s}(\Omega)) \subseteq W^{m-s}(\Omega)$, whence there exists a constant $C > 0$ such that $\|Pg\|_{W^{m-s}(\Omega)} \leq C\, \|g\|_{W^{m-s}(\Omega)}$. If we set $C_2 := C_1\, C$, then the previous chain of inequalities yields

$$\begin{aligned}
\left\| |\rho_\lambda|^s \frac{\partial^m}{\partial z_1^m} P_\lambda f \right\|_{L^2(\Omega^\lambda)} &\leq C_2\, \lambda^{1-m+s}\, \|T_\lambda f\|_{W^{m-s}(\Omega)} \\
&\leq C_2\, \|f\|_{W^{m-s}(\Omega^\lambda)}\,.
\end{aligned} \tag{11}$$

We claim that inequality (11) implies that

$$\left\| |\rho_\infty|^s \frac{\partial^m}{\partial z_1^m} P_\infty f \right\|_{L^2(D_\mu)} \leq C_2\, \|f\|_{W^{m-s}(\mathbb{C}^3)} \tag{12}$$

for all $f \in W^{m-s}(\mathbb{C}^3)$ compactly supported in $\overline{D}_\mu$. Assuming this claim, let $K(\cdot, \cdot)$ denote the Bergman kernel of D_μ. For any fixed $z' \in D_\mu$ and for any open ball $B = B(z', R) \subset D_\mu$, we can choose a function $f \in C_0^\infty(\mathbb{C}^3)$, supported in B and radial in B, such that $K(\cdot, z') = P_\infty f$: it suffices, see [BL80], to set $f(z' + su) = \phi(s)$ when $s > 0, u \in \mathbb{C}^3, |u| = 1$, for some $\phi : [0, +\infty) \to \mathbb{C}$

supported in the interval $[0,R]$ and such that $\int_{\mathbb{C}^3} f(w)dV(w) = 1$. From the last inequality, we conclude that

$$\left\| |\rho_\infty|^s \frac{\partial^m}{\partial z_1^m} K(\cdot, z') \right\|_{L^2(D_\mu)} < +\infty\,,$$

whence $m - s < \nu$ because of Lemma 5.8.

Our claim can be proven as follows. For sufficiently large λ, the function $\chi_{\Omega^\lambda} P_\lambda f$ is well defined and

$$\|\chi_{\Omega^\lambda} P_\lambda f\|_{L^2(\mathbb{C}^3)} \leq \|f\|_{L^2(\mathbb{C}^3)}\,.$$

Therefore, there exists a sequence $\{\lambda_n\}_{n\in\mathbb{N}} \subset (1,+\infty)$ with $\lambda_n \to +\infty$ as $n \to +\infty$ such that the sequence $\{\chi_{\Omega^{\lambda_n}} P_{\lambda_n} f\}_{n\in\mathbb{N}}$ (is well defined and) weakly converges to a function h in $L^2(\mathbb{C}^3)$. Since $\{\Omega^{\lambda_n} \cap D_\mu\}_{n\in\mathbb{N}}$ is an increasing sequence of domains converging to D_μ, the limit function h must be holomorphic in D_μ. Moreover, $h \equiv 0$ in $\mathbb{C}^3 \setminus \overline{D}_\mu$. Our next aim is showing that $h = \chi_{D_\mu} P_\infty f$. If we choose $\mu' > \mu$ such that the inclusions (6) hold true, then for every $n \in \mathbb{N}$ the inclusion $\Omega^{\lambda_n} \subset D_{\mu'}$ holds true. Thus, the property $f - P_{\lambda_n} f \perp A^2(\Omega^{\lambda_n})$ implies the property $f - \chi_{\Omega^{\lambda_n}} P_{\lambda_n} f \perp A^2(D_{\mu'})$. By taking the weak limit as $n \to +\infty$, we conclude that $f - h \perp A^2(D_{\mu'})$. Lemma 5.7 now yields $f - h \perp A^2(D_\mu)$. Therefore, $h = \chi_{D_\mu} P_\infty f$, as desired. Finally, we remark that $\chi_{D_\mu} |\rho_\infty|^s \frac{\partial^m}{\partial z_1^m} P_\infty f$ is the weak limit in $L^2(\mathbb{C}^3)$ of a subsequence of

$$\left\{ \chi_{\Omega^{\lambda_n}} |\rho_\lambda|^s \frac{\partial^m}{\partial z_1^m} P_{\lambda_n} f \right\}_{n\in\mathbb{N}}.$$

Taking into account inequality (11), we immediately derive inequality (12). This completes the proof of our claim. □

6. Closing remarks

For more than forty-five years the worm domain has provided analytic and geometric insight into important complex analytic phenomena of several variables. The worm has been a decisive counterexample for many longstanding problems.

In this paper we have constructed some new, geometrically natural, 3-dimensional variants of the classical two-dimensional Diederich-Fornæss worm domain. We show that they are smoothly bounded, pseudoconvex, and have nontrivial Nebenhülle. We also show that their Bergman projections are not bounded in the Sobolev topology for sufficiently large Sobolev indices.

We plan to further this study, in the spirit of [ADF23, §3], to fully understand which geometric features of our newly constructed class of smoothly bounded pseudoconvex domains play an essential role. We expect from this further study a significant generalization step in the already rich realm of worm domains.

References

[ADF23] Leandro Arosio, Gian Maria Dall'Ara, and Matteo Fiacchi. Worm domains are not Gromov hyperbolic. *J. Geom. Anal.*, 33(8):Paper No. 257, 16, 2023.

[Bar92] David Barrett. Behavior of the Bergman projection on the Diederich-Fornæss worm. *Acta Math.*, 168(1-2):1–10, 1992.

[Bar98] David E. Barrett. The Bergman projection on sectorial domains. In *Operator theory for complex and hypercomplex analysis (Mexico City, 1994)*, volume 212 of *Contemp. Math.*, pages 1–24. Amer. Math. Soc., Providence, RI, 1998.

[BL80] Steve Bell and Ewa Ligocka. A simplification and extension of Fefferman's theorem on biholomorphic mappings. *Invent. Math.*, 57(3):283–289, 1980.

[BŞ12] David Barrett and Sönmez Şahutoğlu. Irregularity of the Bergman projection on worm domains in $\mathbb{C}^n$. *Michigan Math. J.*, 61(1):187–198, 2012.
[Chr96] Michael Christ. Global C^∞ irregularity of the $\overline{\partial}$-Neumann problem for worm domains. *J. Amer. Math. Soc.*, 9(4):1171–1185, 1996.
[CS01] So-Chin Chen and Mei-Chi Shaw. *Partial Differential Equations in Several Complex Variables*, volume 19. American Mathematical Society, Providence, RI, 2001.
[ČŞ18] Željko Čučković and Sönmez Şahutoğlu. Essential norm estimates for the $\overline{\partial}$-Neumann operator on convex domains and worm domains. *Indiana Univ. Math. J.*, 67(1):267–292, 2018.
[DF77] Klas Diederich and John Erik Fornaess. Pseudoconvex domains: an example with nontrivial Nebenhülle. *Math. Ann.*, 225(3):275–292, 1977.
[DM23] Gian Maria Dall'Ara and Samuele Mongodi. The core of the Levi distribution. *J. Éc. polytech. Math.*, 10:1047–1095, 2023.
[Kis91] Christer O. Kiselman. A study of the Bergman projection in certain Hartogs domains. In *Several complex variables and complex geometry, Part 3 (Santa Cruz, CA, 1989)*, volume 52 of *Proc. Sympos. Pure Math.*, pages 219–231. Amer. Math. Soc., Providence, RI, 1991.
[KMPS23] Steven G. Krantz, Alessandro Monguzzi, Marco M. Peloso, and Caterina Stoppato. Irregularity of the Bergman projection on smooth unbounded worm domains. *Mediterr. J. Math.*, 20(3):Paper No. 128, 19, 2023.
[KPS16] Steven G. Krantz, Marco M. Peloso, and Caterina Stoppato. Bergman kernel and projection on the unbounded Diederich-Fornæss worm domain. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 16(4):1153–1183, 2016.
[KPS19] Steven G. Krantz, Marco M. Peloso, and Caterina Stoppato. Completeness on the worm domain and the Müntz-Szász problem for the Bergman space. *Math. Res. Lett.*, 26(1):231–251, 2019.
[Lig87] Ewa Ligocka. Estimates in Sobolev norms $\|\cdot\|_p^s$ for harmonic and holomorphic functions and interpolation between Sobolev and Hölder spaces of harmonic functions. *Studia Math.*, 86(3):255–271, 1987.
[Liu19] Bingyuan Liu. The Diederich-Fornæss index I: For domains of non-trivial index. *Adv. Math.*, 353:776–801, 2019.
[Str10] Emil J. Straube. *Lectures on the $\mathscr{L}^2$-Sobolev theory of the $\overline{\partial}$-Neumann problem*. ESI Lectures in Mathematics and Physics. European Mathematical Society (EMS), Zürich, 2010.

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