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RESEARCH

Semi-simple isometries of the manifold of symmetric positive definite real matrices with the trace metric

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Abstract

We study the semi-simple (or non-parabolic) isometries Φ of the manifold $\mathcal{P}_n$ of symmetric positive definite real matrices, endowed with the trace metric g and pay attention to the set $Min(\Phi)$ consisting of the points minimizing the displacement function $P \mapsto d(P, \Phi(P))$. In particular, for every isometry Φ of $(\mathcal{P}_n, g)$, we give the explicit expression and study the differential-geometric structure of the set $Min(\Phi)$.

Keywords Positive definite matrices · Trace metric · Hadamard manifolds · Symmetric totally geodesic Hadamard submanifolds · Semi-simple isometries · Axes · Orthogonal spaces

Mathematics Subject Classification 15B48 · 53C35

Contents

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Introduction

Symmetric positive definite matrices are widely present in several (but often interconnected) fields of pure and applied mathematics; as an example, we can mention the problem of the *geometric mean* of several positive definite matrices; among many other sources we can refer for instance to [5–7, 15].

Of course, this is just one of many examples: indeed positive definite matrices are involved in many other areas as Riemannian geometry, metric spaces of non-positive curvature, theory of diffusion tensor imaging, medical imaging, radar signal processing, statistical inference, matrix information geometry, mathematical optimization (for instance, among a lot of other sources, we refer to [1, 4, 18–20]).

Our point of view and the related problems are essentially differential-geometric; the results we prove could give some contribution in placing the studies coming from different areas in a geometrical setting. More precisely, we come back to study the isometries of the *symmetric Hadamard manifold* $(\mathcal{P}_n, g)$ of *symmetric positive definite* real matrices of order $n \geq 2$, endowed with the *trace metric*. This metric (called in literature in different ways as *Fisher-Rao*, *affine invariant*, δ_2,...) is one of the most natural on $\mathcal{P}_n$ and it has been intensively studied and also generalized.

We have already studied the trace metric in [10] and in [11]. In particular, in Theorem 4.2 of [10], we proved that the isometries of $(\mathcal{P}_n, g)$ are precisely the maps of one of the following forms: Γ_M, $\Gamma_M \circ \mathfrak{d}$, $\Gamma_M \circ \mathfrak{j}$, $\Gamma_M \circ \mathfrak{j} \circ \mathfrak{d}$, where M is an arbitrary non-singular real matrix, $\Gamma_M(X) := MXM^T$, $\mathfrak{j}(X) := X^{-1}$ and $\mathfrak{d}(X) := \det(X)^{-2/n} \cdot X$.

On the other hand, it is also known that it is possible to associate to each isometry Φ of $(\mathcal{P}_n, g)$ its *displacement function* $d_\Phi(P) := d(P, \Phi(P))$, its *translation length* $\mathfrak{L}(\Phi) := \inf\{d_\Phi(P) : P \in \mathcal{P}_n\}$ and its *minimum set* $Min(\Phi) := \{P \in \mathcal{P}_n : d_\Phi(P) = \mathfrak{L}(\Phi)\}$.

We say that Φ is *semi-simple* if $Min(\Phi) \neq \emptyset$. Since when $Min(\Phi) = \emptyset$ the isometry Φ is called *parabolic*, the semi-simple isometries are often called *non-parabolic*.

Semi-simple isometries are of two types: *elliptic* if $\mathfrak{L}(\Phi) = 0$ (i.e. if Φ has at least a fixed point) and *hyperbolic* (or *axial*) if $\mathfrak{L}(\Phi) > 0$. Elliptic isometries of $(\mathcal{P}_n, g)$ were the subject of our previous paper [11].

In the present paper we study all semi-simple isometries of $(\mathcal{P}_n, g)$ in a unified setting. In particular we explicit the geometric structure of the minimun set of an arbitrary semi-simple isometry Φ of $(\mathcal{P}_n, g)$. In elliptic case, the minimun set consists of all fixed points of Φ, while, in hyperbolic case, the minimun set consists of the disjoint union of all geodesics translated by Φ (these geodesics are called *axes* of Φ). In elliptic case we get again some results already obtained in [11].

Our analysis follows the different forms of the isometries and allows to describe the minimum sets and to discover their differential-geometric properties, which depend on the particular form of the isometry Φ. This is achieved in Sect. 4 for $\Phi = \Gamma_M$ (Theorem 4.6), in Sect. 5 for $\Phi = \Gamma_M \circ \mathfrak{d}$ (Theorem 5.7), in Sect. 6 for $\Phi = \Gamma_M \circ \mathfrak{j}$ (Theorem 6.5) and in Sect. 7 for $\Phi = \Gamma_M \circ \mathfrak{j} \circ \mathfrak{d}$ (Theorem 7.10).

In all cases, $Min(\Phi)$ is a symmetric totally geodesic Hadamard submanifold of $(\mathcal{P}_n, g)$ (Proposition 8.1) and its structure depends on the matrix M involved in the isometry Φ.

In the first three Sections we introduce the preliminary notions; in particular Sect. 2 is devoted to recall some basic facts about *Hadamard manifolds*. In Sect. 8 we study the hyperbolic isometries of $(\mathcal{P}_n, g)$. The minimum set of a hyperbolic isometry splits in a natural way into the Riemannian product of the euclidean line $\mathbb{R}$ and of an *orthogonal* space $\mathcal{Y}$, having the property that every axis of the isometry corresponds to a set of the form $\mathbb{R} \times \{P\}$, where P is a suitable point of $\mathcal{Y}$. We determine the explicit forms and the differential structures of the axes and of the orthogonal spaces of all hyperbolic isometries of $(\mathcal{P}_n, g)$ (see Theorems 8.3, 8.5). By means of these results on $(\mathcal{P}_n, g)$, it is also possible to determine the explicit forms and the differential properties of minimun sets, axes and orthogonal spaces of all semi-simple isometries of the irreducible Hadamard submanifold $\mathcal{P}_n(c) := \{Q \in \mathcal{P}_n \mid \det(Q) = c\}$, $c > 0$.

1 Preliminary notations

Notations 1.1
- $\mathbb{R}$, $\mathbb{C}$, $\mathbb{R}^+$: respectively, the field of real or complex numbers and the multiplicative group of positive real numbers; $\overline{z}$ is the *conjugate* of $z \in \mathbb{C}$.
- $\prod$: (depending on the context) both the usual product of numbers (for example as in $\prod_{j=1}^{p} \alpha_j := \alpha_1\alpha_2 \dots \alpha_p$, with $\alpha_1, \alpha_2, \dots, \alpha_p \in \mathbb{R}$) and the Cartesian product of sets (as in $\prod_{j=1}^{p} \mathcal{S}_j := \mathcal{S}_1 \times \mathcal{S}_2 \times \dots \times \mathcal{S}_p$, with $\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_p$ arbitrary sets).
- I_n: the identity matrix of order n.
- A^T, $\overline{A}$, A^{-1} and A^{-T}, where A is a square matrix: respectively, the transpose, the conjugate, the inverse and the transpose of the inverse A (when $\det(A)$, the determinant of A, is different from 0); $tr(A)$ is the trace of A.
- E_θ: the special orthogonal matrix $\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$ for every $\theta \in \mathbb{R}$; in particular $E_{\pi/2} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
- $diag(b_1, \dots, b_n)$: the diagonal matrix of order n, with diagonal entries $b_1, \dots, b_n$.
- If $B_1, \dots, B_m$ are square matrices (of possible distinct orders), $B_1 \oplus \dots \oplus B_m$ is the block diagonal square matrix with $B_1, \dots, B_m$ on its diagonal and, for every square matrix B, $B^{\oplus m}$ denotes $B \oplus \dots \oplus B$ (m times).
- If $\mathcal{S}_1, \dots, \mathcal{S}_m$ are sets of square matrices, then $\bigoplus_{j=1}^{m} \mathcal{S}_j := \mathcal{S}_1 \oplus \dots \oplus \mathcal{S}_m$ denotes the set of all matrices $\bigoplus_{j=1}^{m} B_j := B_1 \oplus \dots \oplus B_m$ with $B_j \in \mathcal{S}_j$ for every $j = 1 \dots, m$.
- M_n (and Sym_n): the vector space of the real square matrices of order n (which are symmetric).
- GL_n (and SL_n): the Lie group of non-singular real matrices of order n (and with determinant 1).
- $\mathcal{P}_n$ (and $\mathcal{P}_n(c)$): the manifold of *symmetric positive definite* matrices of order n (and with determinant $c > 0$).
- For every set $\mathcal{A} \subseteq \mathcal{P}_n$ and every $\lambda \in \mathbb{R}^+$ we denote $\lambda \cdot \mathcal{A} := \{\lambda \cdot P : P \in \mathcal{A}\} \subseteq \mathcal{P}_n$ and $\mathbb{R}^+ \cdot \mathcal{A} := \bigcup_{\lambda \in \mathbb{R}^+} (\lambda \cdot \mathcal{A}) \subseteq \mathcal{P}_n$. Note that $\mathcal{P}_n = \mathbb{R}^+ \cdot \mathcal{P}_n(c)$, for every $c > 0$.

- O_n (and SO_n): the Lie group of *orthogonal* real matrices of order n (with determinant 1).
- $SO_{(p,n-p)}$ ($p = 0, \dots, n-1$), the Lie group of *generalized orthogonal* real matrices of order n, signature $(p, n-p)$ and determinant equal to 1, while $SO^+_{(p,n-p)}$ is the connected component of $SO_{(p,n-p)}$ containing I_n.
- Sp_{2n}: the Lie group of *symplectic* real matrices of order $2n$, defined by $Sp_{2n} := \{X \in M_{2n} : X\, E_{\pi/2}^{\oplus n}\, X^T = E_{\pi/2}^{\oplus n}\}$.
- For each matrix $X \in GL_n$, we will denote by $\mathcal{C}(X)$ the closed subgroup of GL_n consisting of the non-singular real matrices commuting with X, i.e.

$$\mathcal{C}(X) := \{B \in GL_n : BX = XB\}.$$

- $M_n(\mathbb{C})$ (and $Herm_n$): the set of complex square matrices of order n (which are *hermitian*).
- $GL_n(\mathbb{C})$: the Lie group of non-singular complex matrices of order n.
- $\mathcal{H}_n$: the manifold of hermitian positive definite matrices of order n.
- U_n: the Lie group of *unitary* matrices of order n.
- $U_{(p,n-p)}$ ($p = 0, \dots, n-1$): the Lie group of *generalized unitary* matrices of order n and signature $(p, n-p)$.

We agree that in this paper the terms of the type $\sum_{i=1}^{h} (\cdots)$, $\bigoplus_{i=1}^{h} (\cdots)$, $\prod_{i=1}^{h} (\cdots)$ involved in some formulas, when $h = 0$, should be interpreted as missing.

Notations and Remarks 1.2 We set:

- Γ_M: $GL_n \to GL_n$ defined by $X \mapsto \Gamma_M(X) := MXM^T$, with M arbitrary matrix in GL_n (the *congruence* by M);
- $\mathfrak{d}$: $GL_n \to GL_n$ defined by $\mathfrak{d}(X) := |\det(X)|^{-2/n} X$;
- $\mathfrak{j}$: $GL_n \to GL_n$ defined by $X \mapsto \mathfrak{j}(X) := X^{-1}$ (the *inversion*).

We immediately obtain: $(\Gamma_M \circ \mathfrak{d})(X) = \frac{MXM^T}{|\det(X)|^{2/n}}$; $(\Gamma_M \circ \mathfrak{j})(X) = MX^{-1}M^T$; $(\Gamma_M \circ \mathfrak{j} \circ \mathfrak{d})(X) = |\det(X)|^{2/n}\, MX^{-1}M^T$. The restrictions to $\mathcal{P}_n$ of the maps Γ_M, $\Gamma_M \circ \mathfrak{d}$, $\Gamma_M \circ \mathfrak{j}$, $\Gamma_M \circ \mathfrak{j} \circ \mathfrak{d}$ (denoted by the same symbols) will play a fundamental role in this paper. The following equalities follow from standard computations; we collect them for convenience: $\mathfrak{j} \circ \mathfrak{d} = \mathfrak{d} \circ \mathfrak{j}$, $\mathfrak{d}^2 = \mathfrak{j}^2 = id_{GL_n}$ (the identity map of GL_n) and, for every $M, N \in GL_n$: $\Gamma_{(MN)} = \Gamma_M \circ \Gamma_N$, $\Gamma_M \circ \mathfrak{j} = \mathfrak{j} \circ \Gamma_{M^{-T}}$, $(\Gamma_M \circ \mathfrak{j})^2 = \Gamma_{(MM^{-T})}$, $\Gamma_M \circ \mathfrak{d} = \mathfrak{d} \circ \Gamma_{\mathfrak{d}(M)}$, $\left(\Gamma_M \circ \mathfrak{d}\right)^2 = \Gamma_{(M\,\mathfrak{d}(M))}$, $(\Gamma_M \circ \mathfrak{j} \circ \mathfrak{d})^2 = \Gamma_{(M\,[\mathfrak{d}(M)]^{-T})}$.

Recall 1.3 The *matrix exponential map*, $\exp : M_n(\mathbb{C}) \to GL_n(\mathbb{C})$, is defined by the formula $\exp(A) := I_n + \sum_{i=1}^{\infty} \dfrac{A^i}{i!}$ for every matrix $A \in M_n(\mathbb{C})$. It is known that, if $A = TCT^{-1}$ for some $T \in GL_n(\mathbb{C})$, then we have $\exp(A) = T \exp(C)\, T^{-1}$, and that, if $B_1, \dots, B_m$ are square matrices of arbitrary orders, then $\exp(\bigoplus_{j=1}^{m} B_j) = \bigoplus_{j=1}^{m} \exp(B_j)$, so that, if $\mathcal{S}_1, \dots, \mathcal{S}_m$ are sets of square matrices, then $\exp(\bigoplus_{j=1}^{m} \mathcal{S}_j) = \bigoplus_{j=1}^{m} \exp(\mathcal{S}_j)$. We still denote by exp the restriction of the matrix exponential map to Sym_n. It is known that $\exp : Sym_n \to \mathcal{P}_n$ is a diffeomorphism onto $\mathcal{P}_n$. We denote by $\log : \mathcal{P}_n \to Sym_n$ (the *matrix logarithm map*) its inverse diffeomorphism.

Similarly, the *squaring map*: $C \mapsto C^2$ is a diffeomorphism from $\mathcal{P}_n$ onto $\mathcal{P}_n$ and we denote the *square root map*: $B \mapsto \sqrt{B}$, the inverse diffeomorphism from $\mathcal{P}_n$ onto itself.

Moreover, for every $t \in \mathbb{R}$ and every $B \in \mathcal{P}_n$, it is possible to define the *t-th power* of B, as $B^t := \exp(t \cdot \log(B))$. Note that $B^{1/2} = \sqrt{B}$. It is also useful to note that, if $B \in \mathcal{P}_n$ and $\mu \in \mathbb{R}^+$, then $\mu \cdot B \in \mathcal{P}_n$ and $\log(\mu \cdot B) = \log(B) + \ln(\mu) \cdot I_n$; from this we obtain that $(\mu \cdot B)^t = \mu^t \cdot B^t$, for every $t \in \mathbb{R}$. Finally, if $B \in \mathcal{P}_n$ and H is any matrix of GL_n such that $B = H\, diag(b_1, \dots, b_n)\, H^{-1}$, then $\log(B) := H\, diag(\ln(b_1), \dots, \ln(b_n))\, H^{-1}$, $\sqrt{B} := H\, diag(\sqrt{b_1}, \dots, \sqrt{b_n})\, H^{-1}$, $B^t := H\, diag(b_1^t, \dots, b_n^t)\, H^{-1}$.

Remark 1.4 For any $h \geq 1$, we define the *decomplexification map* $\rho : M_h(\mathbb{C}) \to M_{2h}$, which maps the $h \times h$ complex matrix $Z = (z_{ij})$ to the $(2h) \times (2h)$ block real matrix $\rho(Z) := (Re(z_{ij})I_2 + Im(z_{ij})E_{\pi/2})$ having h^2 blocks of order 2×2. For simplicity, we will still denote by ρ the restriction of this map to any subset of $M_h(\mathbb{C})$.

It is known that $\rho : M_h(\mathbb{C}) \to M_{2h}$ is a monomorphism of $\mathbb{R}$-algebras, such that $\rho(\overline{Z}^T) = \rho(Z)^T$, for every $Z \in M_h(\mathbb{C})$. Moreover, if $\lambda_1, \dots, \lambda_h$ are the h eigenvalues of any matrix $Z \in M_h(\mathbb{C})$, then $\lambda_1, \overline{\lambda}_1, \dots, \lambda_h, \overline{\lambda}_h$ are the $2h$ eigenvalues of $\rho(Z) \in M_{2h}(\mathbb{R})$; so, in particular, we have $tr(\rho(Z)) = 2Re(tr(Z))$, $\det(\rho(Z)) = |\det(Z)|^2$.

Consequently, the following statements apply:

(i) $\rho : GL_h(\mathbb{C}) \to GL_{2h}^+ := \{X \in GL_{2h} : \det(X) > 0\}$ is a Lie group monomorphism;

(ii) $\rho : U_h \to \rho(U_h) = \rho(GL_h(\mathbb{C})) \cap SO_{2h}$ is a Lie group isomorphism;

(iii) $\rho : Herm_h \to \rho(Herm_h) = \rho(M_h(\mathbb{C})) \cap Sym_{2h}$ is a vector space isomorphism, which maps $\mathcal{H}_h$ onto $\rho(\mathcal{H}_h) = \rho(GL_h(\mathbb{C})) \cap \mathcal{P}_{2h}$.

Finally, note that the decomplexification map and the exponential map commute.

In the next Lemma we summarize a few simple and known results about the Lie group $\mathcal{C}(X)$ of non-singular matrices commuting with a given matrix $X \in GL_n$ (see for instance [22, Lemma 1.3] and [11, Lemma 3.3]).

Lemma 1.5 (a) *Let* $X = \bigoplus_{j=1}^t X_j$ *where any* X_j *is a semi-simple real square matrix of order* n_j *(with* $\sum_{j=1}^t n_j = n$*) and assume that* X_j *and* X_h *have no common eigenvalues as soon as* $j \neq h$*. Then we have* $\mathcal{C}(X) = \bigoplus_{j=1}^t \mathcal{C}(X_j)$.

(b) *Let* $\theta \in (0, \pi)$ *and* $m \geq 1$*. Then* $\mathcal{C}(E_\theta^{\oplus m}) = \rho\big(M_m(\mathbb{C})\big)$.

2 Some reminders about Hadamard manifolds

Remark-Definition 2.1 A *Hadamard manifold* (N, h) is a (smooth and without boundary) simply connected complete Riemannian manifold with non-positive sectional curvature. We denote $T_p(N)$ the tangent space to N at the point p and d_N the distance induced on N by the metric h.

If (N, h) is a Hadamard manifold, the well-known Cartan–Hadamard theorem states that the *Riemannian exponential map* $\exp_p : T_p(N) \to N$ is a diffeomorphism, for every $p \in N$. It is natural to denote the inverse diffeomorphism of $\exp_p$ by $\log_p : N \to T_p(N)$ (the *Riemannian logarithm map*). Therefore, given any two distinct points $p, p' \in N$, there exists a unique (minimizing) unit-speed geodesic $\alpha_{(p,p')} : \mathbb{R} \to (N, h)$ such that $\alpha_{(p,p')}(0) = p$, $\alpha_{(p,p')}(d_N(p, p')) = p'$, and clearly we can write this geodesic as follows: $\alpha_{(p,p')}(t) = \exp_p\left(t \cdot \frac{\log_p(p')}{d_N(p,p')}\right)$ for every $t \in \mathbb{R}$.

The Hadamard manifold (N, h) is said to be *symmetric*, if for every point $p \in M$ there exists an isometry σ_p of (N, h) such that $\sigma_p(\exp_p(v)) = \exp_p(-v)$, for every $v \in T_p(N)$.

We will also say that a submanifold N' of N is a *Hadamard submanifold* of (N, h) if it is a Hadamard manifold with respect to the metric inherited from (N, h). It is clear that, if (N, h) is a symmetric Hadamard manifold, then every totally geodesic Hadamard submanifold of (N, h) is also symmetric.

Proposition 2.2 *Let (N, h) be a Hadamard manifold.*

(a) *If N' is a connected complete totally geodesic submanifold of (N, h) and $p \in N'$, then $N' = \exp_p(T_p(N'))$ and, moreover, N' is a closed Hadamard submanifold of (N, h).*

(b) *If N' and N'' are totally geodesic Hadamard submanifolds of (N, h) and $N' \cap N'' \neq \emptyset$, then $N' \cap N''$ is a totally geodesic Hadamard submanifold of (N, h) too.*

Proof If N' is a totally geodesic Riemannian submanifold of the Hadamard manifold (N, h), then, by the well-known Gauss equation, all sectional curvatures of N' are non-positive and, moreover, the exponential map of N' at any point $p \in N'$ agrees with the restriction to $T_p(N')$ of the map $\exp_p : T_p(N) \to N$. So, if N' is connected and complete too, we obtain $N' = \exp_p(T_p(N'))$. Therefore, under all previous conditions, N' is simply connected and closed in N. This completes (a).

Now assume the hypotheses of (b) and fix $p \in N' \cap N''$.

Part (a) gives $N' \cap N'' = \left(\exp_p(T_p(N'))\right) \cap \left(\exp_p(T_p(N''))\right)$. Since $\exp_p : T_p(N) \to N$ is a bijection, we obtain $N' \cap N'' = \exp_p\Big(T_p(N') \cap T_p(N'')\Big)$. This implies that $N' \cap N''$ is a simply connected Riemannian submanifold of (N, h), with $T_p(N' \cap N'') = T_p(N') \cap T_p(N'')$.

Let $q \in N' \cap N''$ and let $v \in T_q(N' \cap N'') = T_q(N') \cap T_q(N'') \subseteq T_q(N)$; let $\alpha(t)$ (for $t \in \mathbb{R}$) be the geodesic of (N, h) such that $\alpha(0) = q$ and $\dfrac{d\alpha}{dt}(0) = v$. Since both N' and N'' are totally geodesic in (N, h), then $\alpha(t) \in N' \cap N''$ for every $t \in \mathbb{R}$. Thus $N' \cap N''$ is a complete totally geodesic submanifold of (N, h). Furthermore, from Gauss equation, all sectional curvatures of the Riemannian submanifold $N' \cap N''$ are non-positive. This completes the proof of part (b). $\square$

Proposition 2.3 (see [8, Lemma 3.2])
Let N' be a totally geodesic Hadamard submanifold of a Hadamard manifold (N, h). Then, for each $p \in N$, there exists a unique $p' \in N'$ such that $d(p, p') = \inf\{d(p, q) \mid q \in N'\}$; moreover (up to a reparametrization) the geodesic $\alpha_{(p,p')}$ joining p and p' is the unique geodesic of (N, h) passing through p and perpendicular to N'. The point p' is called the orthogonal projection *of p to N'.*

Definitions 2.4 Let (N, h) be a Hadamard manifold and let Φ be an isometry of (N, h).

The *displacement function* $d_\Phi : N \to [0, +\infty)$ is defined by: $P \mapsto d_\Phi(P) := d_N(P, \Phi(P))$.

The *translation length* of Φ is the non-negative number $\mathfrak{L}(\Phi) := \inf\{d_\Phi(P) : P \in N\}$, while the *minimum set* of Φ is the set $Min(\Phi) := \{P \in N : d_\Phi(P) = \mathfrak{L}(\Phi)\}$.

The isometry Φ falls into one of the following three types:

- Φ is *elliptic*, if $\mathfrak{L}(\Phi) = 0$ and $Min(\Phi) \neq \emptyset$, i.e. if Φ has at least a fixed point;
- Φ is *hyperbolic* (or *axial*), if $\mathfrak{L}(\Phi) > 0$ and $Min(\Phi) \neq \emptyset$;
- Φ is *parabolic*, if $Min(\Phi) = \emptyset$.

Clearly, if the isometry Φ is elliptic, the set $Min(\Phi)$ is the set of all fixed points of Φ.

We say that the isometry Φ is *semi-simple*, if it is not parabolic (i.e. if $Min(\Phi) \neq \emptyset$).

The alternative name *axial isometry* is a consequence of the following result, whose proof is in [9, Thm. 6.8, pp. 231–232]).

Proposition 2.5 *Let Φ be an isometry of a Hadamard manifold (N, h).*

(a) *Φ is hyperbolic if and only if there exists an oriented unit-speed geodesic $c : \mathbb{R} \to N$ such that $\Phi(c(t)) = c(t + a)$, for some fixed $a > 0$ and for every $t \in \mathbb{R}$.*

The image $c(\mathbb{R})$ is said to be an axis *of Φ.*

(b) *If Φ is hyperbolic and $c, c' : \mathbb{R} \to N$ are distinct oriented unit-speed geodesics such that $\Phi(c(t)) = c(t + a)$ and $\Phi(c'(t)) = c'(t + a')$ for some $a, a' > 0$ and for every $t \in \mathbb{R}$, then we necessarily have: $a = a' = \mathfrak{L}(\Phi)$. Moreover, there exists $\chi > 0$ such that $d(c(t), c'(t)) \leq \chi$ for every $t \in \mathbb{R}$; we will shortly express this condition by saying that the axes $c(\mathbb{R})$ and $c'(\mathbb{R})$ are* parallel *or* asymptotic.

Remark 2.6 Let Φ be a hyperbolic isometry of a Hadamard manifold (N, h) and let $c(t)$ be a non-constant geodesic of (N, h), possibly not parametrized by arc length, i.e. with constant speed of norm not necessarily equal to one, and with arbitrary orientation. Then the image $c(\mathbb{R})$ is an axis of Φ if and only if there exists a real number $b \neq 0$ such that $\Phi(c(t)) = c(t + b)$ for every $t \in \mathbb{R}$. In general, $b \neq \mathfrak{L}(\Phi)$.

Proposition 2.7 *Let Φ be a semi-simple isometry of any Hadamard manifold (N, h).*

(a) *If Φ is elliptic, then the set $Min(\Phi) = \{P \in N : \phi(P) = P\}$ is a totally geodesic Hadamard submanifold of $(\mathcal{P}_n, g)$.*

(b) *If Φ is hyperbolic, then the set $Min(\Phi)$ is closed, convex and consists of the union of all axes of Φ.*

(c) *If Φ is hyperbolic and all sectional curvatures of (N, h) are strictly negative, then Φ has a unique axis, i.e. $Min(\Phi)$ consists of a unique geodesic of (N, h).*

Proof The proof can be found in the paper [8, Prop. 4.2] (see also [12, Prop. 1.9.2]). □

Proposition 2.8 *Let Φ be an isometry of a Hadamard manifold (N, h) and fix an integer $m \neq 0$. Then Φ and Φ^m are isometries of the same type. Furthermore, if Φ is semi-simple, then $Min(\Phi) \subseteq Min(\Phi^m)$ and $\mathfrak{L}(\Phi) = \frac{\mathfrak{L}(\Phi^m)}{|m|}$. In particular, when Φ is hyperbolic, every axis of Φ is also an axis of Φ^m.*

Proof From Proposition 2.5, it is easy to check that, if Φ is hyperbolic, then every axis of Φ is also an axis of Φ^m, $Min(\Phi) \subseteq Min(\Phi^m)$ and $\mathfrak{L}(\Phi) = \frac{\mathfrak{L}(\Phi^m)}{|m|}$. In particular, also Φ^m is hyperbolic. Clearly, if Φ is elliptic, then every fixed point of Φ is also a fixed point of Φ^m, i.e. $Min(\Phi) = \{P \in \mathcal{P}_n : \Phi(P) = P\} \subseteq \{P \in \mathcal{P}_n : \Phi^m(P) = P\} = Min(\Phi^m)$; so Φ^m is also elliptic and $0 = \mathfrak{L}(\Phi) = \frac{\mathfrak{L}(\Phi^m)}{|m|}$. Therefore, if Φ is semi-simple, then also Φ^m is semi-simple and of the same type of Φ. Conversely, by [9, Prop. 6.7, Thm. 6.8(2), pp. 231–232], if Φ^m is semi-simple then Φ is semi-simple and of the same type of Φ^m. Consequently, Φ is parabolic if and only if Φ^m is parabolic. □

Remark-Definition 2.9 Let Φ be a hyperbolic isometry of a Hadamard manifold (N, h) and fix an arbitrary point $Q \in Min(\Phi)$. We will denote by $\gamma_Q^\Phi : \mathbb{R} \to Min(\Phi) \subset N$ the unique oriented unit-speed geodesic of $(\mathcal{P}_n, g)$ that satisfies the following two conditions:

a) $\gamma_Q^\Phi(0) = Q$; b) $\Phi\big(\gamma_Q^\Phi(t)\big) = \gamma_Q^\Phi(t + \mathfrak{L}(\Phi))$ for every $t \in \mathbb{R}$.

Obviously the image of γ_Q^Φ is the axis of Φ passing through Q. The geodesic γ_Q^Φ is uniquely determined by the conditions $\gamma_Q^\Phi(0) = Q$, $\gamma_Q^\Phi(\mathfrak{L}(\Phi)) = \Phi(Q)$, with $\mathfrak{L}(\Phi) = d_N(Q, \Phi(Q))$. Therefore, remembering Remarks-Definitions 2.1, the geodesic γ_Q^Φ can be written as:

$$\gamma_Q^\Phi(t) = \exp_Q\left(t \cdot \frac{\log_Q(\Phi(Q))}{\mathfrak{L}(\Phi)}\right) \quad \text{for every } t \in \mathbb{R}.$$

We will say that the geodesic γ_Q^Φ is the *Q-axis* of the hyperbolic isometry Φ.

Proposition 2.10 *Let Φ and Ψ be isometries of a Hadamard manifold (N, h) and denote $\Phi' := \Psi \circ \Phi \circ \Psi^{-1}$. Then*

(i) $\mathfrak{L}(\Phi) = \mathfrak{L}(\Phi')$ *and* $Min(\Phi') = \Psi(Min(\Phi))$, *so, in particular, the conjugate isometries Φ and Φ' are of the same type;*

(ii) *in the hyperbolic case, for every $Q \in Min(\Phi)$ we have* $\Psi \circ \gamma_Q^\Phi = \gamma_{\Psi(Q)}^{\Phi'}$.

Proof The proofs easily follow by standard computations. □

Proposition 2.11 (see [9, Thm. 6.8, pp. 231–232, Thm. 2.14, p. 183])

Let Φ be a hyperbolic isometry of a Hadamard manifold (N, h). For every $Q \in Min(\Phi)$, there exists a uniquely determined closed convex subspace $\mathcal{Y}_Q(\Phi)$ of (N, h) contained in $Min(\Phi)$ and passing through Q, such that the map $F := F_{Q,\Phi} : \mathcal{Y}_Q(\Phi) \times \mathbb{R} \to Min(\Phi)$, defined by $F(P, t) = \gamma_P^\Phi(t)$ is an isometry from the product space $\mathcal{Y}_Q(\Phi) \times \mathbb{R}$ onto $Min(\Phi)$. Furthermore, the isometry $F^{-1} \circ \Phi \circ F : \mathcal{Y}_Q(\Phi) \times \mathbb{R} \to \mathcal{Y}_Q(\Phi) \times \mathbb{R}$ has the expression $(P, t) \mapsto (P, t + \mathfrak{L}(\Phi))$, for every $P \in \mathcal{Y}_Q(\Phi)$ and $t \in \mathbb{R}$.

Remark-Definition 2.12 Proposition 2.11 states the existence, for every $Q \in Min(\Phi)$, of a closed convex space $\mathcal{Y}_Q(\Phi)$ contained in $Min(\Phi)$ and passing through Q, such that $Min(\Phi) = \bigsqcup_{P \in \mathcal{Y}_Q(\Phi)} \gamma_P^\Phi(\mathbb{R})$, where each axis $\gamma_P^\Phi(\mathbb{R})$ is isometric to $\mathbb{R}$ and perpendicular to $\mathcal{Y}_Q(\Phi)$ at its initial point $\gamma_P^\Phi(0) = P \in \mathcal{Y}_Q(\Phi)$ (here the symbol $\bigsqcup$ denotes the disjoint union of the corresponding sets). Also note that $\mathcal{Y}_Q(\Phi)$ consists of the union of all geodesic arcs contained in $Min(\Phi)$, starting from Q and perpendicular at Q to the Q-axis γ_Q^Φ. For these reasons, we will say that $\mathcal{Y}_Q(\Phi)$ is the *Q-orthogonal space of* Φ.

Proposition 2.13 *Let Φ be a hyperbolic isometry of a Hadamard manifold (N, h) and let $Q \in Min(\Phi)$.*

(i) *We have* $\mathcal{Y}_Q(\Phi) = \{P \in Min(\Phi) :\ h_Q\big(\log_Q(P), \log_Q(\Phi(Q))\big) = 0\}$.

(ii) *If Ψ is another isometry of (N, h), then we have* $\mathcal{Y}_{\Psi(Q)}(\Psi \circ \Phi \circ \Psi^{-1}) = \Psi(\mathcal{Y}_Q(\Phi))$.

Proof Clearly $Q \in \mathcal{Y}_Q(\Phi) \cap \{P \in Min(\Phi) :\ h_Q\big(\log_Q(P), \log_Q(\Phi(Q))\big) = 0\}$. Now let $P \in \mathcal{Y}_Q(\Phi) \subset Min(\Phi)$, with $P \neq Q$, and set $d := d_N(Q, P) > 0$. Remembering Remarks-Definitions 2.1 and 2.9, the geodesic passing through the points Q and P is $\alpha_{(Q,P)}(t) = \exp_Q\big(t \cdot \frac{\log_Q(P)}{d}\big)$ (for $t \in \mathbb{R}$), while the Q-axis of Φ is the geodesic $\gamma_Q^\Phi(t) = \exp_Q\big(t \cdot \frac{\log_Q(\Phi(Q))}{\mathfrak{L}(\Phi)}\big)$ (for $t \in \mathbb{R}$). From Proposition 2.11 we obtain $\alpha_{(Q,P)}([0,d]) \subseteq \mathcal{Y}_Q(\Phi)$ and $h_Q\big(\frac{d\alpha_{(Q,P)}}{dt}(0), \frac{d\gamma_Q^\Phi}{dt}(0)\big) = 0$, from which we get $h_Q\big(\log_Q(P), \log_Q(\Phi(Q))\big) = 0$. Conversely let P be a point of $Min(\Phi)$, $P \neq Q$, such that $h_Q\big(\log_Q(P), \log_Q(\Phi(Q))\big) = 0$. Since the set $Min(\Phi)$ is convex, we have $\alpha_{(Q,P)}([0,d]) \subset Min(\Phi)$ (where, as before, $d := d_N(Q, P)$). The condition $h_Q\big(\log_Q(P), \log_Q(\Phi(Q))\big) = 0$ implies that the geodesics $\alpha_{(Q,P)}$ and $\gamma_Q^\Phi(t)$ are perpendicular at the point Q. Therefore, by Remark-Definition 2.12, we get: $\alpha_{(Q,P)}([0,d]) \subseteq \mathcal{Y}_Q(\Phi)$; in particular $P = \alpha_{(Q,P)}(d) \in \mathcal{Y}_Q(\Phi)$. So part (i) is fully proved. Finally, part (ii) easily follow from Propositions 2.10 and 2.11. $\square$

Proposition 2.14 *Let Φ be a semi-simple isometry of a real analytic Hadamard manifold (N, h). If $P \in N$, denote by $\exp_P : T_P(N) \to N$ the exponential map of (N, h) at P.*

(a) *The minimum set $Min(\Phi)$ is a totally geodesic real analytic Hadamard submanifold of (N, h).*

(b) *If the isometry Φ is hyperbolic and $P \in Min(\Phi)$, then the P-orthogonal space $\mathcal{Y}_P(\Phi)$ of Φ is a totally geodesic real analytic Hadamard submanifold of (N, h). Moreover, we have* $\mathcal{Y}_P(\Phi) = \exp_P(T_P(\mathcal{Y}_P(\Phi)))$.

Proof (a) Taking into account Proposition 2.2 (a), part (a) follows from Theorem 1.3.4 and Remark 1.3.6 of [21].

(b) By part (a), the set $Min(\Phi)$ is a totally geodesic real analytic Hadamard submanifold of (N, h). From Proposition 2.11, there exists a (real analytic) isometry F from $\mathcal{Y}_P(\Phi) \times \mathbb{R}$ onto $Min(\Phi)$ such that $F\big(\mathcal{Y}_P(\Phi) \times \{0\}\big) = \mathcal{Y}_P(\Phi)$. Hence $\mathcal{Y}_P(\Phi) \simeq \mathcal{Y}_P(\Phi) \times \{0\}$ is a totally geodesic real analytic Hadamard submanifold of $Min(\Phi)$ and therefore of (N, h) too. Finally, from Proposition 2.2 (a), we obtain $\mathcal{Y}_P(\Phi) = \exp_P(T_P(\mathcal{Y}_P(\Phi)))$. $\square$

For more information on Hadamard manifolds we refer to the classical books [2, 3, 9].

3 The trace metric

Recalls 3.1 For any integer $n \geq 2$, we call *trace metric* the Riemannian metric f on $\mathcal{H}_n$ defined by $f_A(V, W) = tr(A^{-1}VA^{-1}W)$, for any $A \in \mathcal{H}_n$ and $V, W \in T_A(\mathcal{H}_n) = Herm_n$.

For convenience, the restriction of f to $\mathcal{P}_n$ will be denoted by g (always called trace metric), so that $(\mathcal{P}_n, g)$ is a Riemannian submanifold of $(\mathcal{H}_n, f)$.

We will still denote by g the restriction of g to any submanifold of $\mathcal{P}_n$. We also set $||X|| := \sqrt{g_{I_n}(X, X)} = \sqrt{tr(X^2)}$, for every $X \in Sym_n = T_{I_n}(\mathcal{P}_n)$.

For the differential-geometric properties of $(\mathcal{P}_n, g)$ and of $(\mathcal{H}_n, f)$ we refer for instance to [5, 6, 9, 10, 20]. In particular, the following facts are well known:

(a) $t \mapsto P^{1/2}\exp(t \cdot P^{-1/2}VP^{-1/2})P^{1/2}$ $(t \in \mathbb{R})$ is the geodesic passing through $P \in \mathcal{P}_n$ whose tangent vector at P is $V \in T_P(\mathcal{P}_n) = Sym_n$;

(b) the map $c : [0, 1] \to \mathcal{P}_n$ defined by $c(t) = A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2}$ is the unique geodesic arc of $(\mathcal{P}_n, g)$ such that $c(0) = A$ and $c(1) = B$, for every $A, B \in \mathcal{P}_n$;

(c) if $A, B \in \mathcal{P}_n$ and $\lambda_1, \ldots, \lambda_n$ are the n eigenvalues of the matrix $A^{-1/2}BA^{-1/2} \in \mathcal{P}_n$, then the distance $d(A, B)$ induced by the metric g is given by:

$$d(A, B) = \sqrt{\sum_{i=1}^{n} (\ln(\lambda_i))^2} = || \log(A^{-1/2}BA^{-1/2})||.$$

In particular, note that $d(A, B) = d(t \cdot A,\ t \cdot B)$, for every $t > 0$ and $A, B \in \mathcal{P}_n$; in other words, the map $P \mapsto t \cdot P$ is an isometry of $(\mathcal{P}_n, g)$, for every $t > 0$.

For more information, see [5, § 2], [6, Ch. 6 § 1] and also [10, Prop. 3.4].

Lemma 3.2 *For every integer $h \geq 1$, the decomplexification map*
$\rho : \big(\mathcal{H}_h, 2f\big) \to \big(\rho(GL_h(\mathbb{C})) \cap \mathcal{P}_{2h}, g\big)$ *is an isometry of Riemannian manifolds.*

Proof Denoting by ρ_* the differential map of ρ, it suffices to note that, if $A \in \mathcal{H}_h$ and $Z, W \in Herm_h$, we have:

$$g_{\rho(A)}\big(\rho_*(Z), \rho_*(W)\big) = tr\big(\rho(A)^{-1}\rho(Z)\,\rho(A)^{-1}\rho(W)\big) = tr\big(\rho(A^{-1}ZA^{-1}W)\big) =$$
$$= 2Re\big(tr(A^{-1}ZA^{-1}W)\big) = 2Re\big(f_A(Z, W)\big) = 2f_A(Z, W).$$ □

Proposition 3.3

(a) *The real analytic Riemannian manifold* $(\mathcal{P}_n, g)$ *is a symmetric Hadamard manifold diffeomorphic to the homogeneous space* $\frac{GL_n}{O_n}$.

(b) *A mapping $\Phi : (\mathcal{P}_n, g) \to (\mathcal{P}_n, g)$ is an isometry if and only if there exists a matrix $M \in GL_n$ such that one of the following cases occurs:*

$$\Phi(X) = \Gamma_M(X) = MXM^T;$$
$$\Phi(X) = (\Gamma_M \circ \mathfrak{d})(X) = \frac{MXM^T}{\big(\det(X)\big)^{2/n}};$$
$$\Phi(X) = (\Gamma_M \circ \mathrm{j})(X) = MX^{-1}M^T;$$
$$\Phi(X) = (\Gamma_M \circ \mathrm{j} \circ \mathfrak{d})(X) = \big(\det(X)\big)^{2/n}\, MX^{-1}M^T.$$

Proof Part (a) is well known (see for instance [14, Ch. XII]), while for part (b) see [10, Thm. 4.2]. □

Proposition 3.4 *Fix $M \in GL_n$ and consider the isometry $\Phi : (\mathcal{P}_n, g) \to (\mathcal{P}_n, g)$, where $\Phi := \Gamma_M$ or $\Phi := \Gamma_M \circ \mathrm{j} \circ \mathfrak{d}$.*

(a) $\Phi : (\mathcal{P}_n, g) \to (\mathcal{P}_n, g)$ *is a* positively homogeneous map, *i.e.* $\Phi(t \cdot P) = t \cdot \Phi(P)$, *for any* $t > 0$ *and* $P \in \mathcal{P}_n$.

(b) *The set* $Min(\Phi)$ *is a* positive cone *of* $\mathcal{P}_n$, *i.e. for every* $P \in Min(\Phi)$ *and for every* $t > 0$, *we necessarily have* $t \cdot P \in Min(\Phi)$.

Proof Part (a) can be verified by means of a simple calculation, while part (b) easily follows from (a), taking into account that $d(A, B) = d(t \cdot A,\ t \cdot B)$, for every $t > 0$ and $A, B \in \mathcal{P}_n$. □

Lemma 3.5 *Let* $Q \in \mathcal{P}_n$. *Denote by* $\exp_Q : T_Q(\mathcal{P}_n) \to \mathcal{P}_n$ *and* $\log_Q : \mathcal{P}_n \to T_Q(\mathcal{P}_n)$, *respectively, the Riemannian exponential map and the Riemannian logarithm map both at the point* Q *with respect to the metric* g. *Denote also by* $\exp : Sym_n \to \mathcal{P}_n$ *and* $\log : \mathcal{P}_n \to Sym_n$, *respectively, the matrix exponential map and the matrix logarithm map. Then* $\exp_Q = \Gamma_{Q^{1/2}} \circ \exp \circ \Gamma_{Q^{-1/2}}$ *and* $\log_Q = \Gamma_{Q^{1/2}} \circ \log \circ \Gamma_{Q^{-1/2}}$.

Proof It is easy to note that we have $\exp_{I_n} = \exp$. Since $\Gamma_{Q^{1/2}}$ is a linear isometry of $(\mathcal{P}_n, g)$ which maps I_n to Q, we have: $\exp_Q \circ \Gamma_{Q^{1/2}} = \Gamma_{Q^{1/2}} \circ \exp_{I_n} = \Gamma_{Q^{1/2}} \circ \exp$. The first statement follows from this. The second one follows easily from the first. □

Definitions 3.6 Let G be a closed subgroup of GL_n. We say that G is $\mathcal{P}_n$-compatible if, whenever Q belongs to $G \cap \mathcal{P}_n$, every real power $Q^t = \exp(t \cdot \log(Q))$ of Q also belongs to G (as well as, of course, to $\mathcal{P}_n$). The group G is said to be *reductive* if $A^T \in G$ as soon as $A \in G$. Finally, we say that G is *algebraic* if there exists a finite system of polynomials in the entries of the matrices of M_n such that G is the intersection of GL_n with the set of common zeros of this system.

Proposition 3.7 *Let* G *be a* $\mathcal{P}_n$*-compatible reductive closed subgroup of* GL_n. *Then* $\mathcal{P}_n \cap G$ *is a symmetric totally geodesic Hadamard submanifold of* $(\mathcal{P}_n, g)$ *diffeomorphic to the homogeneous space* $\dfrac{G}{G \cap O_n}$.

Proof Let $\mathfrak{g} = T_{I_n}(G) \subseteq M_n$ be the Lie algebra of G. We have $\exp(Sym_n) = \mathcal{P}_n$ and $\exp(\mathfrak{g}) \subseteq G$, so $\exp(Sym_n \cap \mathfrak{g}) \subseteq \mathcal{P}_n \cap G$. Since G is $\mathcal{P}_n$-compatible, for any $Q \in \mathcal{P}_n \cap G$ we have $\alpha(t) := Q^t = \exp(t \cdot \log(Q)) \in G$, for every $t \in \mathbb{R}$. Therefore, $\frac{d\alpha}{dt}(0) = \log(Q)$ belongs to $\mathfrak{g} = T_{I_n}(G)$. Since $\exp(\log(Q)) = Q$ and $\log(Q) \in Sym_n$, we conclude that $\exp(Sym_n \cap \mathfrak{g}) = \mathcal{P}_n \cap G$.

Keeping in mind that the exponential map is a diffeomorphism from Sym_n onto $\mathcal{P}_n$, we deduce that $\mathcal{P}_n \cap G = \exp(Sym_n \cap \mathfrak{g})$ is a simply connected Riemannian submanifold of $(\mathcal{P}_n, g)$ whose tangent space at the point I_n is $Sym_n \cap \mathfrak{g}$. Taking into account that G is also reductive, the map; $(Q, P) \in (G, \mathcal{P}_n \cap G) \mapsto \Gamma_Q(P) = QPQ^T \in \mathcal{P}_n \cap G$ defines a left action of the Lie group G on the manifold $\mathcal{P}_n \cap G$. Since G is $\mathcal{P}_n$-compatible, if $Q \in \mathcal{P}_n \cap G$ then also $Q^{1/2}$ belongs to G; therefore, since every $Q \in \mathcal{P}_n \cap G$ can be written as $Q = \Gamma_{Q^{1/2}}(I_n)$ (with $Q^{1/2} \in G$), we conclude that the action of G on $\mathcal{P}_n \cap G$ is transitive. Obviously, the stabilizer of I_n with respect to this action is $G \cap O_n$; this implies that the manifold $\mathcal{P}_n \cap G$ is diffeomorphic to the homogeneous space $\frac{G}{G \cap O_n}$. Also note that the equality $Q = \Gamma_{Q^{1/2}}(I_n)$ implies that $T_Q(\mathcal{P}_n \cap G) = \Gamma_{Q^{1/2}}(Sym_n \cap \mathfrak{g})$, for every $Q \in \mathcal{P}_n \cap G$.

Now consider the geodesic $\beta(t)$ of $(\mathcal{P}_n, g)$ passing through an arbitrary point $Q \in \mathcal{P}_n \cap G$ and having, as tangent vector at the point Q, an arbitrary vector $Q^{1/2} V\, Q^{1/2}$ of $T_Q(\mathcal{P}_n \cap G)$, where $V \in Sym_n \cap \mathfrak{g}$. Clearly we have $\beta(t) = Q^{1/2} \exp(t \cdot V)\, Q^{1/2}$ ($t \in \mathbb{R}$). Remembering that $\exp(Sym_n \cap \mathfrak{g}) = \mathcal{P}_n \cap G$ and $Q^{1/2} \in G$, we can conclude that $\beta(t) \in \mathcal{P}_n \cap G$, for every $t \in \mathbb{R}$, i.e. $\mathcal{P}_n \cap G$ is a complete totally geodesic submanifold of $(\mathcal{P}_n, g)$. From this, it easily follows that $(\mathcal{P}_n \cap G, g)$ is symmetric and has non-positive sectional curvature. This completes the proof. $\square$

Proposition 3.7 is essentially stated in [9, Thm. 10.58, pp. 328–329]. For convenience of the reader, we have decided to simplify the statement for our purposes and provide a simple proof.

Proposition 3.8 *Let G be an algebraic subgroup of GL_n. Let G^0 denote the connected component of G containing the identity. Then*

(a) *G is $\mathcal{P}_n$-compatible;*

(b) *if G is also reductive, then $\mathcal{P}_n \cap G^0 = \mathcal{P}_n \cap G$ and G^0 is a $\mathcal{P}_n$-compatible reductive closed subgroup of GL_n.*

Proof (a) The proof of part (a) is in [9, Lemma 10.59, pp. 329–330].

(b) From part (a) and Proposition 3.7, $\mathcal{P}_n \cap G$ is a Hadamard submanifold of $(\mathcal{P}_n, g)$. In particular $\mathcal{P}_n \cap G$ is connected, so $\mathcal{P}_n \cap G \subseteq G^0$ and from this: $\mathcal{P}_n \cap G^0 = \mathcal{P}_n \cap G$. The remainder of statement (b) follows easily from (a) and this equality. $\square$

Proposition 3.9 *Fix $c > 0$ and denote by ϵ the Euclidean metric of $\mathbb{R}$. Then*

(1) *$(\mathcal{P}_n(c), g)$ is a symmetric totally geodesic Hadamard submanifold of $(\mathcal{P}_n, g)$ diffeomorphic to the homogeneous space $\dfrac{SL_n}{SO_n}$;*

(2) *the map $F : (\mathbb{R}, \epsilon) \times (\mathcal{P}_n(c), g) \to (\mathcal{P}_n, g)$ defined by $F(x, A) := e^{x/\sqrt{n}} \cdot A$ is an isometry and the symmetric Hadamard manifold $(\mathcal{P}_n(c), g)$ is irreducible (i.e. it is not isometric to a product of positive-dimensional symmetric Hadamard manifolds);*

(3) *for every $P \in \mathcal{P}_n$, the sets $\mathbb{R}^+ \cdot \{P\}$ and $\mathcal{P}_n(\det(P))$ are symmetric totally geodesic Hadamard submanifolds of $(\mathcal{P}_n, g)$ which intersect orthogonally at the point P.*

Proof If $A \in GL_n$ and $\det(A) = \sqrt{c}$, the isometry Γ_A of $(\mathcal{P}_n, g)$ maps $\mathcal{P}_n(1)$ onto $\mathcal{P}_n(c)$; so it suffices to prove (1) for $c = 1$. We have $\mathcal{P}_n(1) = SL_n \cap \mathcal{P}_n$, and SL_n is a $\mathcal{P}_n$-compatible reductive closed subgroup of GL_n. Therefore, part (1) follows from Proposition 3.7.

For part (2), see [9, Prop. 10.53, p. 325]); part (3) directly follows from (1) and (2). $\square$

Remark 3.10 Since $(\mathcal{P}_n(c), g)$ is isometric to $(\mathcal{P}_n(1), g)$ for every $c > 0$, by means of the list of the isometries of $(\mathcal{P}_n(1), g)$ (see [10, Thm. 4.1]), it is easy to prove that the isometries of $(\mathcal{P}_n(c), g)$ are precisely the maps of the form Γ_M or $\Gamma_M \circ \mathfrak{j} \circ \mathfrak{d}$, with $\det(M) = \pm 1$, i.e. they are the restrictions to $\mathcal{P}_n(c)$ of the isometries of $(\mathcal{P}_n, g)$ mapping $\mathcal{P}_n(c)$ onto itself.

Proposition 3.11 *Let Φ be a hyperbolic isometry of $(\mathcal{P}_n, g)$ and let $Q \in Min(\Phi)$. Then*

(a) *the Q-axis of Φ has the following expression:*

$\gamma_Q^\Phi(t) = Q^{1/2} \exp(t \cdot \frac{\log(Q^{-1/2}\,\Phi(Q)\,Q^{-1/2})}{\mathfrak{L}(\Phi)})\, Q^{1/2} = Q^{1/2}\big(Q^{-1/2}\,\Phi(Q)\,Q^{-1/2}\big)^{\frac{t}{\mathfrak{L}(\Phi)}}\, Q^{1/2}$ *for every $t \in \mathbb{R}$, where $\mathfrak{L}(\Phi) = ||\log(Q^{-1/2}\,\Phi(Q)\,Q^{-1/2})||$;*

(b) *if $M \in GL_n$ and we set $P := {\Gamma_M}^{-1}(Q)$, $\Phi' := {\Gamma_M}^{-1} \circ \Phi \circ \Gamma_M$, we have $\gamma_Q^\Phi = \Gamma_M \circ \gamma_P^{\Phi'}$.*

Proof Since $\mathfrak{L}(\Phi) = d(Q, \Phi(Q)) = ||\log(Q^{-1/2}\Phi(Q)\,Q^{-1/2})||$, part (a) easily follows from Remark-Definition 2.9 and Lemma 3.5, while Proposition 2.10 (ii) implies part (b). $\square$

Proposition 3.12 *Let Φ be a hyperbolic isometry of $(\mathcal{P}_n, g)$ with translation length $\mathfrak{L}$. Fix $Q \in Min(\Phi)$, set $L := \dfrac{\log(Q^{-1/2}\,\Phi(Q)\,Q^{-1/2})}{\mathfrak{L}}$ and consider the Q-axis of Φ: $\gamma_Q^\Phi(t) = Q^{1/2}\big(\exp(t \cdot L)\big)Q^{1/2}$ $(t \in \mathbb{R})$. The following occurs:*

(a) *if $\Phi = \Gamma_M \circ \mathfrak{d}$ or $\Phi = \Gamma_M \circ \mathfrak{j}$ (for some $M \in GL_n$), then $tr(L) = 0$; for these isometries we necessarily have $\det(\gamma_Q^\Phi(t)) = |\det(M)|$ for every $t \in \mathbb{R}$, so that $Min(\Phi) \subseteq \mathcal{P}_n(|\det(M)|)$;*

(b) *if $\Phi = \Gamma_M$ or $\Phi = \Gamma_M \circ \mathfrak{j} \circ \mathfrak{d}$ (for some $M \in GL_n$), then $tr(L) = \dfrac{2\ln(|\det(M)|)}{\mathfrak{L}}$; when $\det(M) = \pm 1$ for these isometries we get $\det(\gamma_Q^\Phi(t)) = \det(Q)$ for every $t \in \mathbb{R}$.*

Proof Remember that we have $\det(\exp(X)) = e^{tr(X)}$ for every $X \in M_n$.

First assume $\Phi = \Gamma_M \circ \mathfrak{d}$; hence $M \dfrac{\gamma_Q^\Phi(t)}{\big(\det(\gamma_Q^\Phi(t))\big)^{2/n}} M^T = \gamma_Q^\Phi(t + \mathfrak{L})$ for every $t \in \mathbb{R}$.

So we get $\big(\det(M)\big)^2 = \det(\gamma_Q^\Phi(t)) \cdot \det\big(\gamma_Q^\Phi(t + \mathfrak{L})\big)$ for every $t \in \mathbb{R}$, or, equivalently,

$e^{(2t+\mathfrak{L})\cdot tr(L)} = \dfrac{\big(\det(M)\big)^2}{\big(\det(Q)\big)^2}$ for every $t \in \mathbb{R}$. Differentiating with respect to t, we obtain $tr(L) = 0$, from which it follows that $\det(\gamma_Q^\Phi(t)) = \det(Q) = |\det(M)|$ for every $t \in \mathbb{R}$. The case $\Phi = \Gamma_M \circ \mathfrak{j}$ can be treated with similar arguments. This concludes part (a).

Now assume $\Phi = \Gamma_M$; from $M\gamma_Q^\Phi(t)M^T = \gamma_Q^\Phi(t + \mathfrak{L})$, by calculating their respective determinants, we get $\big(\det(M)\big)^2 = e^{\mathfrak{L}\cdot tr(L)}$, and so $tr(L) = \frac{2\ln(|\det(M)|)}{\mathfrak{L}}$. Then, if $\det(M) = \pm 1$, we have $tr(L) = 0$, and this implies: $\det(\gamma_Q^\Phi(t)) = \det(Q)$ for every $t \in \mathbb{R}$. The case $\Phi = \Gamma_M \circ \mathfrak{j} \circ \mathfrak{d}$ can be treated similarly, so part (b) is also proved. $\square$

Proposition 3.13 *Let $M \in GL_n$ be a matrix with $\det(M) = \pm 1$ and let F be the isometry of $(\mathcal{P}_n, g)$ defined as $F := \Gamma_M$ or as $F := \Gamma_M \circ \mathfrak{j} \circ \mathfrak{d}$, so that $F(\mathcal{P}_n(c)) = \mathcal{P}_n(c)$ and the restriction $f := F|_{\mathcal{P}_n(c)}$ is an arbitrary isometry of $(\mathcal{P}_n(c), g)$.*

(i) *We have: $Min(F) = \mathbb{R}^+ \cdot Min(f)$, $Min(f) = Min(F) \cap \mathcal{P}_n(c)$ and $\mathcal{L}(f) = \mathcal{L}(F)$.*

(ii) *The isometries f and F are of the same type.*

Furthermore, assume that f is hyperbolic (so F is hyperbolic too) and fix $Q \in Min(f)$.

(iii) *The Q-axis of f is precisely the Q-axis of F, i.e. $\gamma^f_Q(t) = \gamma^F_Q(t)$, for every $t \in \mathbb{R}$.*

(iv) *For the Q-orthogonal spaces of F and f, we have:*
$\mathcal{Y}_Q(F) = \mathbb{R}^+ \cdot \mathcal{Y}_Q(f)$ *and* $\mathcal{Y}_Q(f) = \mathcal{Y}_Q(F) \cap \mathcal{P}_n(c)$.

Proof Part (i) follows from the fact that $Min(F)$ is a positive cone of $\mathcal{P}_n$ (see Proposition 3.4 (b)), remembering that the map $P \mapsto t \cdot P$ is an isometry of $(\mathcal{P}_n, g)$, for every $t > 0$. Part (ii) directly follows from part (i). Part (iii) follows from Proposition 3.12 (b). Finally, part (iv) directly follows from parts (i) and (iii). □

The next Sections are devoted to determine the minimum sets and the relative orthogonal spaces of all isometries of $(\mathcal{P}_n, g)$. As stated in Proposition 2.14, these are always totally geodesic real analytic Hadamard submanifolds of $(\mathcal{P}_n, g)$. There are many studies regarding totally geodesic submanifolds of $(\mathcal{P}_n, g)$; among them, we refer to the classical contribution of Mostow [17] and to the recent paper of Tumpach and Larotonda [23].

4 Semi-simple isometries of the form Γ_M

Remark-Definition 4.1 For convenience, let us fix on $\mathbb{C}$ any strict total ordering (denoted by $<$) that extends the usual ordering of $\mathbb{R}$, with the property that $z_1 < z_2$ implies $\mu z_1 < \mu z_2$ for all $z_1, z_2 \in \mathbb{C}$, $\mu \in \mathbb{R}^+$, and such that $\lambda < z$ for all $\lambda \in \mathbb{R}$ and all $z \in \mathbb{C}\backslash\mathbb{R}$.

Let $M \in GL_n$ be a matrix whose distinct real eigenvalues are: a_1 of multiplicity r_1, a_2 of multiplicity r_2, up to a_p of multiplicity r_p, while its distinct pairs of non-real eigenvalues are: $\rho_1 e^{\pm i\theta_1}$ of multiplicity m_1, $\rho_2 e^{\pm i\theta_2}$ of multiplicity m_2, up to $\rho_q e^{\pm i\theta_q}$ of multiplicity m_q $\big(\theta_1, \ldots, \theta_q \in (0, \pi)\big)$, and assume $a_1 < a_2 < \cdots < a_r < \rho_1 e^{i\theta_1} < \rho_2 e^{i\theta_2} < \cdots < \rho_q e^{i\theta_q}$. So, it is uniquely determined the real matrix $\Delta_M := \Big(\bigoplus_{j=1}^p (a_j \cdot I_{r_j})\Big) \oplus \Big(\bigoplus_{h=1}^q (\rho_h \cdot E_{\theta_h})^{\oplus m_h}\Big)$.

Note that Δ_M is a real Jordan form of the semi-simple part of the Jordan-Chevalley decomposition of M; so Δ_M is a real Jordan form of M if and only if M is semi-simple. From Lemma 1.5, we deduce that $\mathcal{C}(\Delta_M) = \Big(\bigoplus_{j=1}^p GL_{r_j}\Big) \oplus \Big(\bigoplus_{j=1}^q \rho\big(GL_{m_j}(\mathbb{C})\big)\Big)$, and therefore we have $\mathcal{C}(\Delta_M) \cap \mathcal{P}_n = \big(\bigoplus_{j=1}^p \mathcal{P}_{r_j}\big) \oplus \big(\bigoplus_{j=1}^q \rho(\mathcal{H}_{m_j})\big)$.

By the polar decomposition Theorem (see e.g. [13, Thm. 7.3.1 p. 449]), there exist a unique $\mathcal{S}_M \in \mathcal{P}_n$ and a unique $\mathcal{R}_M \in O_n$ such that $\Delta_M = \mathcal{S}_M \mathcal{R}_M$. As is known, we have $\mathcal{S}_M = \sqrt{\Delta_M \Delta_M^T} = \Big(\bigoplus_{j=1}^p (|a_j| \cdot I_{r_j})\Big) \oplus \Big(\bigoplus_{h=1}^q (\rho_h \cdot I_{2m_h})\Big)$; therefore the matrix $\mathcal{S}_M$ belongs to the center of the group $\mathcal{C}(\Delta_M)$; furthermore, if we denote the n eigenvalues of M by $\eta_1, \eta_2, \ldots, \eta_n$, then the n eigenvalues of $\mathcal{S}_M$ are $|\eta_1|, |\eta_2|, \ldots, |\eta_n|$.

Lemma 4.2 *Fix $M \in GL_n$, $P_0 \in \mathcal{P}_n$, and let $\Delta_M \in GL_n$ be as in Remark-Definition 4.1. The following three conditions are equivalent:*
(i) $M^2 P_0 (M^T)^2 = M P_0 M^T P_0^{-1} M P_0 M^T$;
(ii) *the matrix* $P_0^{-1/2} M P_0^{1/2}$ *is normal;*
(iii) *there exists* $A_0 \in GL_n$ *such that* $M = A_0 \Delta_M A_0^{-1}$ *and* $P_0 = A_0 A_0^T$.

Proof The equivalence (i) $\Leftrightarrow$ (ii) is just a simple calculation.

Now, if statement (ii) holds, by real spectral Theorem (see for instance [13, p. 136]), the matrix M is semi-simple (so Δ_M is a real Jordan form of M) and there exists $U_0 \in O_n$ such that $P_0^{-1/2} M P_0^{1/2} = U_0 \Delta_M U_0^T$; hence $M = (P_0^{1/2} U_0) \Delta_M (P_0^{1/2} U_0)^{-1}$ and therefore we get (iii) with $A_0 = P_0^{1/2} U_0$. The converse, (iii) $\Rightarrow$(ii), follows from the polar decomposition of A_0, after noting that Δ_M is normal. This concludes the proof. ◻

Proposition 4.3 *Fix $M \in GL_n$ and let Δ_M be as in Remark-Definition 4.1.*
(a) *The isometry Γ_M of $(\mathcal{P}_n, g)$ is elliptic if and only if the matrix M is semi-simple and its real Jordan form Δ_M is orthogonal.*
(b) *The isometry Γ_M of $(\mathcal{P}_n, g)$ is hyperbolic if and only if the matrix M is semi-simple and its real Jordan form Δ_M is not orthogonal.*
(c) *The isometry Γ_M of $(\mathcal{P}_n, g)$ is semi-simple if and only if the matrix M is semi-simple.*
(d) *If the isometry Γ_M is semi-simple, the following equality holds:*
$Min(\Gamma_M) = \{AA^T \,:\, A \in GL_n \text{ and } M = A\,\Delta_M A^{-1}\}$.

Proof Clearly, part (c) will be a trivial consequence of (a) and (b). Part (a) has been already proved in [11, Prop. 3.1], while, in elliptic case, part (d) has been proved in [11, Prop. 3.2].

Now, it remains to prove that the isometry Γ_M is hyperbolic if and only if the matrix M is semi-simple and Δ_M is not orthogonal, and that, in this case, we have
$Min(\Gamma_M) = \{AA^T \,:\, A \in GL_n \text{ and } M = A\,\Delta_M A^{-1}\}$.

Suppose that Γ_M is hyperbolic and let $P_0 \in Min(\Gamma_M)$; so there is an axis of Γ_M passing through P_0. Keeping in mind Remark 2.6, there exist a geodesic $c(t) = P_0^{1/2} \exp(t \cdot V) P_0^{1/2}$ (with $V \in Sym_n \backslash \{0\}$) and a real number $b \neq 0$ such that, for every $t \in \mathbb{R}$, we have
(*) $M P_0^{1/2} \exp(t \cdot V) P_0^{1/2} M^T = P_0^{1/2} \exp(t \cdot V) \exp(b \cdot V) P_0^{1/2}$.
In particular, setting $t = 0$ and $t = b$ in (*), we obtain, respectively:
(**) $\exp(b \cdot V) = P_0^{-1/2} M P_0 M^T P_0^{-1/2}$;
(***) $M P_0^{1/2} \exp(b \cdot V) P_0^{1/2} M^T = P_0^{1/2} \big(\exp(b \cdot V)\big)^2 P_0^{1/2}$.
Now, from (**) and (***), we easily get: $M^2 P_0 (M^T)^2 = M P_0 M^T P_0^{-1} M P_0 M^T$. So, from Lemma 4.2, we obtain that there exists $A_0 \in GL_n$ such that $M = A_0 \Delta_M A_0^{-1}$ (i.e. M is semi-simple) and $P_0 = A_0 A_0^T$ (i.e. $\{AA^T \,:\, A \in GL_n \text{ and } M = A\,\Delta_M A^{-1}\} \supseteq Min(\Gamma_M)$). Moreover, by part (a) the matrix Δ_M cannot be orthogonal.

For the converse, assume that M is semi-simple and its real Jordan form Δ_M is not orthogonal. Let $A_0 \in GL_n$ be such that $M = A_0 \Delta_M A_0^{-1}$ and denote $P_0 := A_0 A_0^T \in \mathcal{P}_n$. Since Δ_M is not orthogonal, part (a) implies that P_0 is not a fixed point of Γ_M, i.e. $P_0^{-1/2} M P_0 M^T P_0^{-1/2} \in \mathcal{P}_n \backslash \{I_n\}$; therefore $V := \log(P_0^{-1/2} M P_0 M^T P_0^{-1/2}) \in$

$Sym_n \backslash \{0\}$ and the geodesic $c_0(t) = P_0^{1/2} \exp(t \cdot V) P_0^{1/2}$ is non-constant. Furthermore, by Lemma 4.2, we have $M^2 P_0 (M^T)^2 = M P_0 M^T P_0^{-1} M P_0 M^T$. Now, consider the two geodesics $\alpha(t) := \Gamma_M(c_0(t))$ and $\beta(t) := c_0(t+1)$. We have: $\alpha(0) = M P_0 M^T = P_0^{1/2} \exp(V) P_0^{1/2} = c_0(1) = \beta(0)$, and $\alpha(1) = M c_0(1) M^T = M P_0^{1/2} \exp(V) P_0^{1/2} M^T = M^2 P_0 (M^T)^2 = M P_0 M^T P_0^{-1} M P_0 M^T = P_0^{1/2} \big(\exp(V)\big)^2 P_0^{1/2} = P_0^{1/2} \exp(2 \cdot V) P_0^{1/2} = c_0(2) = \beta(1)$.

Since $(\mathcal{P}_n, g)$ is a Hadamard manifold, we necessarily have $\alpha(t) = \beta(t)$ for every $t \in \mathbb{R}$, i.e. $\Gamma_M(c_0(t)) = c_0(t+1)$ for every $t \in \mathbb{R}$. By Remark 2.6, we conclude that the image $c_0(\mathbb{R})$ is an axis of Γ_M passing through P_0, so Γ_M is hyperbolic and $P_0 = A_0 A_0^T$ belongs to $Min(\Gamma_M)$. This concludes the proof. □

Remark 4.4 Let M be a semi-simple matrix of GL_n and let $\lambda \in \mathbb{R}^+$. Then the isometries Γ_M and $\Gamma_{(\lambda M)}$ are both semi-simple and $Min\big(\Gamma_M\big) = Min\big(\Gamma_{(\lambda M)}\big)$.

This follows directly from Proposition 4.3 (c), (d), keeping in mind that $\Delta_{(\lambda M)} = \lambda\, \Delta_M$.

Clearly, parts (a), (b), (c) of Proposition 4.3 can be reformulated as follows:

Proposition 4.5 *Let $M \in GL_n$. The isometry Γ_M of $(\mathcal{P}_n, g)$ is*

(a) *elliptic if and only if M is semi-simple and all of its eigenvalues have absolute value* 1*;*

(b) *hyperbolic if and only if M is semi-simple and at least one of its eigenvalues has an absolute value other than* 1*;*

(c) *parabolic if and only if M is not semi-simple.*

Theorem 4.6 *Let $M \in GL_n$ be such that Γ_M is a semi-simple isometry of $(\mathcal{P}_n, g)$. Consider the matrices $\Delta_M, \mathcal{S}_M \in GL_n$ and the Lie group $\mathcal{C}(\Delta_M)$ defined as in Remark-Definition* 4.1*. Let $A \in GL_n$ be such that $M = A\, \Delta_M A^{-1}$ (see Proposition* 4.3 (c)*).*

(a) *The translation length of Γ_M is $\mathfrak{L}(\Gamma_M) = 2\, || \log(\mathcal{S}_M) || = 2 \sqrt{\sum_{j=1}^n \big(\ln(|\eta_j|)\big)^2}$ where $\eta_1, \eta_2, \ldots, \eta_n$ are the n eigenvalues of M.*

(b) *Γ_{Δ_M} is also a semi-simple isometry of $(\mathcal{P}_n, g)$ and $Min(\Gamma_M) = \Gamma_A\big(Min(\Gamma_{\Delta_M})\big)$. Moreover, we have $Min(\Gamma_{\Delta_M}) = \mathcal{C}(\Delta_M) \cap \mathcal{P}_n = \big(\bigoplus_{j=1}^p \mathcal{P}_{r_j}\big) \oplus \big(\bigoplus_{j=1}^q \rho(\mathcal{H}_{m_j})\big) = \{P \in \mathcal{P}_n :\ \Gamma_{\Delta_M}(P) = P^{1/2} \mathcal{S}_M^2 P^{1/2}\}$.*

c) $(Min(\Gamma_M), g)$ is a symmetric totally geodesic Hadamard submanifold of $(\mathcal{P}_n, g)$ isometric to the Riemannian product $\Big(\prod_{j=1}^p (\mathcal{P}_{r_j}, g)\Big) \times \Big(\prod_{h=1}^q (\mathcal{H}_{m_h}, 2f)\Big)$.

Proof From the equality $M = A\, \Delta_M A^{-1}$ we obtain $\Gamma_M = \Gamma_A \circ \Gamma_{\Delta_M} \circ \big(\Gamma_A\big)^{-1}$. Therefore, by Proposition 2.10 (i) we deduce that the isometry Γ_{Δ_M} is also semi-simple, and that furthermore we have $Min(\Gamma_M) = \Gamma_A\big(Min(\Gamma_{\Delta_M})\big)$ and $\mathfrak{L}(\Gamma_{\Delta_M}) = \mathfrak{L}(\Gamma_M)$.

From Proposition 4.3 (d) (replacing M with Δ_M), we obtain:

$Min(\Gamma_{\Delta_M}) = \big\{B B^T :\ B \in GL_n \text{ and } \Delta_M = B\, \Delta_M B^{-1}\big\}$, or equivalently, $Min(\Gamma_{\Delta_M}) = \mathcal{C}(\Delta_M) \cap \mathcal{P}_n = \Big(\bigoplus_{j=1}^p \mathcal{P}_{r_j}\Big) \oplus \Big(\bigoplus_{h=1}^q \rho(\mathcal{H}_{m_h})\Big)$ (see Remark-Definition 4.1). So, if $P \in Min(\Gamma_{\Delta_M})$ then $P^{1/2} \in Min(\Gamma_{\Delta_M})$ too. Since $\mathcal{S}_M^2$

belongs to the center of the group $\mathcal{C}(\Delta_M)$, we have $\Gamma_{\Delta_M}(P) = \Delta_M P \Delta_M^T = P \Delta_M \Delta_M^T = P\mathcal{S}_M^2 = P^{1/2}\mathcal{S}_M^2 P^{1/2}$; this proves the inclusion $Min(\Gamma_{\Delta_M}) \subseteq \{P \in \mathcal{P}_n : \Gamma_{\Delta_M}(P) = P^{1/2}\mathcal{S}_M^2 P^{1/2}\}$. From Recalls 3.1 (c), we get $\mathfrak{L}(\Gamma_M) = \mathfrak{L}(\Gamma_{\Delta_M}) = d(P, \Gamma_{\Delta_M}(P)) = \|\log(\mathcal{S}_M^2)\| = 2\|\log(\mathcal{S}_M)\| = 2\sqrt{\sum_{j=1}^n \left(\ln(|\eta_j|)\right)^2}$. This proves part (a).

Let us now prove the reverse inclusion $\{P \in \mathcal{P}_n : \Gamma_{\Delta_M}(P) = P^{1/2}\mathcal{S}_M^2 P^{1/2}\} \subseteq Min(\Gamma_{\Delta_M})$. If P is any point of $\mathcal{P}_n$ such that $\Gamma_{\Delta_M}(P) = P^{1/2}\mathcal{S}_M^2 P^{1/2}$, then we have $d(P, \Gamma_{\Delta_M}(P)) = \|\log(\mathcal{S}_M^2)\| = \mathfrak{L}(\Gamma_{\Delta_M})$, and so $P \in Min(\Gamma_{\Delta_M})$. Therefore, part (b) is fully proved.

Note that $Min(\Gamma_{\Delta_M}) = G \cap \mathcal{P}_n$, where $G := \left(\bigoplus_{j=1}^p GL_{r_j}\right) \oplus \left(\bigoplus_{j=1}^q \rho\left(GL_{m_j}(\mathbb{C})\right)\right)$ is a $\mathcal{P}_n$-compatible reductive closed subgroup of GL_n. So, we get part (c) remembering that Γ_A is an isometry of $(\mathcal{P}_n, g)$ and keeping in mind Lemma 3.2 and Proposition 3.7. □

5 Semi-simple isometries of the form $\Gamma_M \circ \mathfrak{d}$

Remark-Definition 5.1 Given any matrix $M \in GL_n$, we set $\pi(M) := \dfrac{M}{|\det(M)|^{1/n}}$.

Of course, $\det(\pi(M)) = \pm 1$ and, if $\eta_1, \eta_2, \ldots, \eta_n$ are the n eigenvalues of M then the n eigenvalues of $\pi(M)$ are $\frac{\eta_1}{|\det(M)|^{1/n}}, \frac{\eta_2}{|\det(M)|^{1/n}}, \ldots, \frac{\eta_n}{|\det(M)|^{1/n}}$. Moreover, M is semi-simple if and only if $\pi(M)$ is semi-simple, and (with the notations of Remark-Definition 4.1) we have $\Delta_{\pi(M)} = \frac{\Delta_M}{|\det(M)|^{1/n}}$, $\mathcal{S}_{\pi(M)} = \frac{\mathcal{S}_M}{|\det(M)|^{1/n}}$.

Lemma 5.2 *Given any matrix $M \in GL_n$, the maps $\Gamma_M \circ \mathfrak{d}$ and $\Gamma_{\pi(M)}$ are isometries of $(\mathcal{P}_n, g)$ of the same type and $\mathfrak{L}(\Gamma_M \circ \mathfrak{d}) = \mathfrak{L}(\Gamma_{\pi(M)})$.*

Proof We have $\Gamma_M \circ \mathfrak{d} = \mathfrak{d} \circ \Gamma_{\mathfrak{d}(M)}$, so $\left(\Gamma_M \circ \mathfrak{d}\right)^2 = \Gamma_{(M\,\mathfrak{d}(M))} = \Gamma_{(\pi(M))^2} = \left(\Gamma_{\pi(M)}\right)^2$; therefore we conclude by Proposition 2.8. □

Proposition 5.3 *Let $M \in GL_n$. The isometry $\Gamma_M \circ \mathfrak{d}$ of $(\mathcal{P}_n, g)$ is*

(a) *elliptic if and only if M is semi-simple and its eigenvalues have the same absolute value;*

(b) *hyperbolic if and only if M is semi-simple and its eigenvalues do not have the same absolute value;*

(c) *parabolic if and only if M is not semi-simple.*

Proof It is an easy consequence of Proposition 4.5 and Lemma 5.2, taking into account Remark-Definition 5.1. □

Proposition 5.4 *Let $M \in GL_n$ be such that $\Gamma_M \circ \mathfrak{d}$ is a semi-simple isometry of $(\mathcal{P}_n, g)$ and denote by $\eta_1, \ldots, \eta_n$ the n eigenvalues of M. Then the translation length of $\Gamma_M \circ \mathfrak{d}$ is $\mathfrak{L}(\Gamma_M \circ \mathfrak{d}) = 2\sqrt{\sum_{j=1}^n \left(\ln(\frac{|\eta_j|}{|\det(M)|^{1/n}})\right)^2} = 2\left\|\log\left(\frac{\mathcal{S}_M}{|\det(M)|^{1/n}}\right)\right\|$.*

Proof By Lemma 5.2, the isometry $\Gamma_{\pi(M)}$ is semi-simple and $\mathfrak{L}(\Gamma_M \circ \mathfrak{d}) = \mathfrak{L}(\Gamma_{\pi(M)})$.

We conclude the proof by Theorem 4.6 (a), taking into account Remark-Definition 5.1. □

Proposition 5.5 *Consider any semi-simple matrix $M \in GL_n$ and let $A \in GL_n$ be any matrix such that $M = A\,\Delta_M A^{-1}$ (remember the notations of Remark-Definition 4.1).*

(a) *The isometries $\Gamma_M \circ \mathfrak{d}$, $\Gamma_{\pi(M)}$ and Γ_M are all semi-simple, and we have $Min(\Gamma_M \circ \mathfrak{d}) = Min(\Gamma_{\pi(M)}) \cap \mathcal{P}_n\big(|\det(M)|\big) = Min(\Gamma_M) \cap \mathcal{P}_n\big(|\det(M)|\big)$.*

(b) *If the isometry $\Psi := \Gamma_M \circ \mathfrak{d}$ (and so also $\Psi' := \Gamma_{\pi(M)}$) is hyperbolic, then for every point $Q \in Min(\Gamma_M \circ \mathfrak{d})$ we have $\gamma_Q^{\Psi}(t) = \gamma_Q^{\Psi'}(t)$ $(t \in \mathbb{R})$ (so every axis of $\Gamma_M \circ \mathfrak{d}$ is also an axis of $\Gamma_{\pi(M)}$).*

Proof By Propositions 4.3 (c) and 5.3, the isometries $\Gamma_M \circ \mathfrak{d}$, $\Gamma_{\pi(M)}$ and Γ_M are all semi-simple. We set $\mathfrak{L} := \mathfrak{L}(\Psi) = \mathfrak{L}(\Psi')$ (where $\Psi := \Gamma_M \circ \mathfrak{d}$, $\Psi' := \Gamma_{\pi(M)}$). From Remark 4.4, we deduce that $Min(\Gamma_{\pi(M)}) \cap \mathcal{P}_n\big(|\det(M)|\big) = Min(\Gamma_M) \cap \mathcal{P}_n\big(|\det(M)|\big)$.

Let us now prove the equality $Min(\Gamma_M \circ \mathfrak{d}) = Min(\Gamma_{\pi(M)}) \cap \mathcal{P}_n\big(|\det(M)|\big)$ and that, if Ψ (and therefore Ψ') is hyperbolic, then $\gamma_Q^{\Psi}(t) = \gamma_Q^{\Psi'}(t)$ for every $Q \in Min(\Gamma_M \circ \mathfrak{d}), t \in \mathbb{R}$.

In the elliptic case the equality is proved in [11, Prop. 4.1 (2)]. Now, assume that Ψ (and so Ψ') is hyperbolic. For any $Q \in Min(\Psi)$, consider the Q-axis $\gamma_Q^{\Psi}(t)$ of Ψ . So, we have $\dfrac{M\gamma_Q^{\Psi}(t)M^T}{\big(\det(\gamma_Q^{\Psi}(t))\big)^{2/n}} = \gamma_Q^{\Psi}(t+\mathfrak{L})$, while from Proposition 3.12 (a) we deduce that $\det(\gamma_Q^{\Psi}(t)) = \det(Q) = |\det(M)|$ for every $t \in \mathbb{R}$. From these equalities we easily obtain $\pi(M)\,\gamma_Q^{\Psi}(t)\,\pi(M)^T = \gamma_Q^{\Psi}(t+\mathfrak{L})$ for all $t \in \mathbb{R}$. Therefore $\gamma_Q^{\Psi}(t) = \gamma_Q^{\Psi'}(t)$ for every $t \in \mathbb{R}$, $Q \in Min(\Psi')$, and $\gamma_Q^{\Psi}(\mathbb{R})$ is also an axis of the isometry $\Psi' = \Gamma_{\pi(M)}$.

For the converse, assume that $\det(Q) = |\det(M)|$, $Q \in Min(\Psi')$, so that we have $\pi(M)\,\gamma_Q^{\Psi'}(t)\,\pi(M)^T = \dfrac{M\gamma_Q^{\Psi'}(t)M^T}{|\det(M)|^{2/n}} = \gamma_Q^{\Psi'}(t+\mathfrak{L})$ for every $t \in \mathbb{R}$. Since $\det(\pi(M)) = \pm 1$, from Proposition 3.12 (b) we obtain $\det(\gamma_Q^{\Psi'}(t)) = \det(Q)$ for every $t \in \mathbb{R}$; so, we get $(\Gamma_M \circ \mathfrak{d})(\gamma_Q^{\Psi'}(t)) = \dfrac{M\gamma_Q^{\Psi'}(t)M^T}{\big(\det(\gamma_Q^{\Psi'}(t))\big)^{2/n}} = \dfrac{M\gamma_Q^{\Psi'}(t)M^T}{\big(\det(Q)\big)^{2/n}} = \dfrac{M\gamma_Q^{\Psi'}(t)M^T}{|\det(M)|^{2/n}} = \gamma_Q^{\Psi'}(t+\mathfrak{L})$ for every $t \in \mathbb{R}$. Therefore $Q \in Min(\Gamma_M \circ \mathfrak{d})$, and $\gamma_Q^{\Psi}(t) = \gamma_Q^{\Psi'}(t)$ $(t \in \mathbb{R})$. $\square$

Notation 5.6 We denote by $\mathbf{S}\Big(\big(\prod_{j=1}^{p}(\mathcal{P}_{r_j}, g)\big) \times \big(\prod_{h=1}^{q}(\mathcal{H}_{m_h}, 2f)\big)\Big)$ the Riemannian submanifold of the Riemannian product $\big(\prod_{j=1}^{p}(\mathcal{P}_{r_j}, g)\big) \times \big(\prod_{h=1}^{q}(\mathcal{H}_{m_h}, 2f)\big)$, consisting of the elements $(A_1, \ldots, A_p, B_1, \ldots, B_q) \in \big(\prod_{j=1}^{p}\mathcal{P}_{r_j}\big) \times \big(\prod_{h=1}^{q}\mathcal{H}_{m_h}\big)$ such that $\prod_{j=1}^{p}\det(A_j)\cdot\prod_{h=1}^{q}(\det(B_h))^2 = 1$.

Theorem 5.7 *Let $M \in GL_n$ be such that $\Gamma_M \circ \mathfrak{d}$ is a semi-simple isometry of $(\mathcal{P}_n, g)$ and assume that the eigenvalues of M and Δ_M are as in Remark-Definition 4.1. Choose $A_0 \in GL_n$ such that $M = A_0\Delta_M A_0^{-1}$ with $|\det(A_0)| = \sqrt{|\det(M)|}$. Then the minimum set of $\Gamma_M \circ \mathfrak{d}$ is $Min(\Gamma_M \circ \mathfrak{d}) = \Gamma_{A_0}\Big(\big(\big(\bigoplus_{j=1}^{p}\mathcal{P}_{r_j}\big) \oplus \big(\bigoplus_{h=1}^{q}\rho(\mathcal{H}_{m_h})\big)\big) \cap$*

$\Big(\mathcal{P}_n(1)\Big)\Big)$ *and it is a symmetric totally geodesic Hadamard submanifold of* $(\mathcal{P}_n, g)$ *isometric to* $\mathbf{S}\Big(\Big(\prod_{j=1}^{p}(\mathcal{P}_{r_j}, g)\Big)\times\Big(\prod_{h=1}^{q}(\mathcal{H}_{m_h}, 2f)\Big)\Big)$.

Proof From Proposition 5.5 (a), we have $Min(\Gamma_M \circ \mathfrak{d}) = Min(\Gamma_{\pi(M)}) \cap \mathcal{P}_n\big(|\det(M)|\big)$. So, by Proposition 2.2 (b), $Min(\Gamma_M \circ \mathfrak{d})$ is a symmetric totally geodesic Hadamard submanifold of $(\mathcal{P}_n, g)$, because $Min(\Gamma_{\pi(M)})$ and $\mathcal{P}_n\big(|\det(M)|\big)$ are both totally geodesic Hadamard submanifolds of $(\mathcal{P}_n, g)$. We also have $\pi(M) = A_0\, \Delta_{\pi(M)} A_0^{-1}$; so, from Theorem 4.6 (b), we obtain $Min(\Gamma_{\pi(M)}) = \Gamma_{A_0}\Big(\big(\bigoplus_{j=1}^{p}\mathcal{P}_{r_j}\big)\oplus\big(\bigoplus_{h=1}^{q}\rho(\mathcal{H}_{m_h})\big)\Big)$. Since $\det(A_0) = \pm\sqrt{|\det(M)|}$, we also have $\mathcal{P}_n\big(|\det(M)|\big) = \Gamma_{A_0}\big(\mathcal{P}_n(1)\big)$; consequently:

$Min(\Gamma_M \circ \mathfrak{d}) = \Gamma_{A_0}\Big(\Big(\big(\bigoplus_{j=1}^{p}\mathcal{P}_{r_j}\big)\oplus\big(\bigoplus_{h=1}^{q}\rho(\mathcal{H}_{m_h})\big)\Big)\cap\Big(\mathcal{P}_n(1)\Big)\Big)$. We get the statement, since $\Big(\Big(\big(\bigoplus_{j=1}^{p}\mathcal{P}_{r_j}\big)\oplus\big(\bigoplus_{h=1}^{q}\rho(\mathcal{H}_{m_h})\big)\Big)\cap\Big(\mathcal{P}_n(1)\Big), g\Big)$ and $\mathbf{S}\Big(\Big(\prod_{j=1}^{p}(\mathcal{P}_{r_j}, g)\Big)\times\Big(\prod_{h=1}^{q}(\mathcal{H}_{m_h}, 2f)\Big)\Big)$ are clearly isometric Riemannian manifolds and Γ_{A_0} is an isometry of $(\mathcal{P}_n, g)$. □

6 Semi-simple isometries of the form $\Gamma_M \circ \mathfrak{j}$

Proposition 6.1 *Let* $M \in GL_n$. *The isometry* $\Gamma_M \circ \mathfrak{j}$ *of* $(\mathcal{P}_n, g)$ *is*

(a) *elliptic if and only if* MM^{-T} *is semi-simple and all of its eigenvalues have absolute value equal to* 1*;*

(b) *hyperbolic if and only if* MM^{-T} *is semi-simple and at least one of its eigenvalues has absolute value different from* 1*;*

(c) *parabolic if and only if* MM^{-T} *is not semi-simple.*

Proof Since $(\Gamma_M \circ \mathfrak{j})^2 = \Gamma_{(MM^{-T})}$, the statements follow from Propositions 2.8, 4.5. □

Notations 6.2 (a) Let $M \in GL_n$ be such that $\Gamma_M \circ \mathfrak{j}$ is a semi-simple isometry of $(\mathcal{P}_n, g)$; then, as seen in Proposition 6.1, the matrix MM^{-T} is semi-simple. Remembering [13, pp. 294–295], we list and denote all distinct eigenvalues of MM^{-T}, with their multiplicity, as follows:

- 1, with multiplicity $p \geq 0$;
- λ_j and λ_j^{-1}, both with multiplicity $p_j \geq 1$, for $j = 1, \dots, l_1$, where every $\lambda_j \in \mathbb{R}$ and $1 < \lambda_1 < \lambda_2 < \cdots < \lambda_{l_1}$ (when $l_1 \geq 1$);
- -1, with multiplicity $2q \geq 0$;
- $-\mu_j$ and $-\mu_j^{-1}$, both with multiplicity $q_j \geq 1$, for $j = 1, \dots, l_2$, where every $\mu_j \in \mathbb{R}$ and $1 < \mu_1 < \mu_2 < \cdots < \mu_{l_2}$ (when $l_2 \geq 1$);
- $e^{\pm\psi_j \mathbf{i}}$, both with multiplicity $t_j \geq 0$,
  $r_{j1}e^{\pm\psi_j \mathbf{i}}$ and $r_{j1}^{-1}e^{\pm\psi_j \mathbf{i}}$, both with multiplicity $s_{j1} \geq 1$,

$r_{j2}e^{\pm\psi_j \mathbf{i}}$ and $r_{j2}^{-1}e^{\pm\psi_j \mathbf{i}}$, both with multiplicity $s_{j2} \geq 1$, $\cdots$ up to
$r_{jk_j}e^{\pm\psi_j \mathbf{i}}$ and $r_{jk_j}^{-1}e^{\pm\psi_j \mathbf{i}}$, both with multiplicity $s_{jk_j} \geq 1$,
where every $r_{jh} \in \mathbb{R}$ and $1 < r_{j1} < r_{j2} < \cdots < r_{jk_j}$ (when $k_j \geq 1$),
with the condition $t_j + k_j \geq 1$, for $j = 1, \ldots, l_3$, and finally,
where every $\psi_j \in \mathbb{R}$ and $0 < \psi_1 < \psi_2 < \cdots < \psi_{l_3} < \pi$ (when $l_3 \geq 1$).

Clearly: $p + 2\sum_{j=1}^{l_1} p_j + 2\big(q + \sum_{j=1}^{l_2} q_j\big) + 2\sum_{j=1}^{l_3}\big(t_j + \sum_{h=1}^{k_j} 2s_{jh}\big) = n$.
So a real Jordan form $\widehat{\Delta}_M$ of MM^{-T} is as follows:

$$\widehat{\Delta}_M := \Big[I_p\Big] \oplus \Big[\bigoplus_{j=1}^{l_1}\big[(\lambda_j I_{p_j}) \oplus (\lambda_j^{-1} I_{p_j})\big]\Big] \oplus \Big[-I_{2q}\Big] \oplus \Big[\bigoplus_{j=1}^{l_2}\big[(-\mu_j I_{q_j}) \oplus (-\mu_j^{-1} I_{q_j})\big]\Big] \oplus \Big[\bigoplus_{j=1}^{l_3}\Big[\Big(E_{\psi_j}^{\oplus t_j}\Big) \oplus \Big(\bigoplus_{h=1}^{k_j}\big[(r_{jh}E_{\psi_j}^{\oplus s_{jh}}) \oplus (r_{jh}^{-1}E_{\psi_j}^{\oplus s_{jh}})\big]\Big)\Big]\Big].$$

Since MM^{-T} is semi-simple, we can fix $Z_M \in GL_n$ such that $MM^{-T} = Z_M \widehat{\Delta}_M Z_M^{-1}$.

(b) As usual, we now denote by $\mathcal{C}(\widehat{\Delta}_M)$ the closed subgroup of GL_n consisting of all matrices of GL_n that commute with $\widehat{\Delta}_M$. Remembering Lemma 1.5, we obtain:

$$\mathcal{C}(\widehat{\Delta}_M) = \Big[GL_p\Big] \oplus \Big[\bigoplus_{j=1}^{l_1}\big(GL_{p_j} \oplus GL_{p_j}\big)\Big] \oplus \Big[GL_{2q}\Big] \oplus \Big[\bigoplus_{j=1}^{l_2}\big(GL_{q_j} \oplus GL_{q_j}\big)\Big]$$
$$\oplus \Big[\bigoplus_{j=1}^{l_3}\Big[\Big(\rho\big(GL_{t_j}(\mathbb{C})\big)\Big) \oplus \Big(\bigoplus_{h=1}^{k_j}\big[\rho\big(GL_{s_{jh}}(\mathbb{C})\big) \oplus \rho\big(GL_{s_{jh}}(\mathbb{C})\big)\big]\Big)\Big]\Big],$$

where ρ is the decomplexification map. Therefore, it is clear that $\mathcal{C}(\widehat{\Delta}_M)$ is a $\mathcal{P}_n$-compatible reductive closed subgroup of GL_n and that we have:

$$\mathcal{C}(\widehat{\Delta}_M) \cap \mathcal{P}_n = \Big[\mathcal{P}_p\Big] \oplus \Big[\bigoplus_{j=1}^{l_1}\big(\mathcal{P}_{p_j} \oplus \mathcal{P}_{p_j}\big)\Big] \oplus \Big[\mathcal{P}_{2q}\Big] \oplus \Big[\bigoplus_{j=1}^{l_2}\big(\mathcal{P}_{q_j} \oplus \mathcal{P}_{q_j}\big)\Big]$$
$$\oplus \Big[\bigoplus_{j=1}^{l_3}\Big[\Big(\rho\big(\mathcal{H}_{t_j}\big)\Big) \oplus \Big(\bigoplus_{h=1}^{k_j}\big[\rho(\mathcal{H}_{s_{jh}}) \oplus \rho(\mathcal{H}_{s_{jh}})\big]\Big)\Big]\Big].$$

By the polar decomposition Theorem, there exists a unique $\mathcal{D}_M \in \mathcal{C}(\widehat{\Delta}_M) \cap \mathcal{P}_n$ and a unique $\mathcal{O}_M \in \mathcal{C}(\widehat{\Delta}_M) \cap O_n$ such that $\widehat{\Delta}_M = \mathcal{D}_M \mathcal{O}_M$. Clearly, we have:

$$\mathcal{D}_M = \Big[I_p\Big] \oplus \Big[\bigoplus_{j=1}^{l_1}\big[(\lambda_j I_{p_j}) \oplus (\lambda_j^{-1} I_{p_j})\big]\Big] \oplus \Big[I_{2q}\Big] \oplus \Big[\bigoplus_{j=1}^{l_2}\big[(\mu_j I_{q_j}) \oplus (\mu_j^{-1} I_{q_j})\big]\Big]$$
$$\oplus \Big[\bigoplus_{j=1}^{l_3}\Big[\Big(I_{2t_j}\Big) \oplus \Big(\bigoplus_{h=1}^{k_j}\big[(r_{jh} I_{2s_{jh}}) \oplus (r_{jh}^{-1} I_{2s_{jh}})\big]\Big)\Big]\Big],$$

$$\mathcal{O}_M = \Big[I_{p'}\Big] \oplus \Big[-I_{2q'}\Big] \oplus \Big[\bigoplus_{j=1}^{l_3}\big(E_{\psi_j}^{\oplus s'_j}\big)\Big], \qquad \text{where we set:}$$

$p' := p + 2\sum_{j=1}^{l_1} p_j$ (i.e. p' is the total number of positive eigenvalues of MM^{-T}, counted with their multiplicities),

$q' := q + \sum_{j=1}^{l_2} q_j$ (i.e. $2q'$ is the total number of negative eigenvalues of MM^{-T}, counted with their multiplicities), and

$s'_j := t_j + \sum_{h=1}^{k_j} 2s_{jh}$ (i.e. s'_j is the total number of non-real eigenvalues of MM^{-T} having argument $\psi_j \in (0, \pi)$, counted with their multiplicities), for every $j = 1, \ldots, l_3$.

Denoting the n eigenvalues of MM^{-T} by $\sigma_1, \sigma_2, \ldots, \sigma_n$, the n eigenvalues of $\mathcal{D}_M$ are $|\sigma_1|, |\sigma_2|, \ldots, |\sigma_n|$. Looking at their expressions, it is clear that all matrices $\widehat{\Delta}_M$, $\mathcal{D}_M$, $\sqrt{\mathcal{D}_M}$, $\mathcal{D}_M^{-1/2}$, $\mathcal{O}_M$ belong to the center of the group $\mathcal{C}(\widehat{\Delta}_M)$. We also note that $\mathcal{C}(\mathcal{O}_M) = \{A \in GL_n : A\mathcal{O}_M = \mathcal{O}_M A\} = GL_{p'} \oplus GL_{2q'} \oplus \Big(\bigoplus_{j=1}^{l_3} \rho\big(GL_{s'_j}(\mathbb{C})\big)\Big)$; therefore: $\mathcal{C}(\mathcal{O}_M) \cap \mathcal{P}_n = \mathcal{P}_{p'} \oplus \mathcal{P}_{2q'} \oplus \Big(\bigoplus_{j=1}^{l_3} \rho\big(\mathcal{H}_{s'_j}\big)\Big)$.

Proposition 6.3 *Let $M \in GL_n$ be such that $\Gamma_M \circ \mathrm{j}$ is a semi-simple isometry of $(\mathcal{P}_n, g)$; let $\mathcal{D}_M \in \mathcal{P}_n$ be as in Notations 6.2* (b) *and let us denote with $\sigma_1, \sigma_2, \ldots, \sigma_n$ the n eigenvalues of MM^{-T}. Then:* $\mathfrak{L}(\Gamma_M \circ \mathrm{j}) = ||\log(\mathcal{D}_M)|| = \sqrt{\sum_{j=1}^n \big(\ln(|\sigma_j|)\big)^2}$.

Proof The equality $(\Gamma_M \circ \mathrm{j})^2 = \Gamma_{(MM^{-T})}$, Proposition 2.8 and Theorem 4.6 (a) imply the statement. □

Proposition 6.4 *Let $M \in GL_n$ be such that $\Gamma_M \circ \mathrm{j}$ is a semi-simple isometry of $(\mathcal{P}_n, g)$; consider the matrices $\widehat{\Delta}_M, \mathcal{D}_M, \mathcal{O}_M$ and the Lie group $\mathcal{C}(\widehat{\Delta}_M)$ defined in Notations 6.2. Let $Z_M \in GL_n$ be such that $MM^{-T} = Z_M \widehat{\Delta}_M Z_M^{-1}$ and set $\widehat{M} := Z_M^{-1} M Z_M^{-T}$. Then*

(a) *$\Gamma_{\widehat{M}} \circ \mathrm{j}$ is a semi-simple isometry of $(\mathcal{P}_n, g)$ and $Min(\Gamma_M \circ \mathrm{j}) = \Gamma_{Z_M}\big(Min(\Gamma_{\widehat{M}} \circ \mathrm{j})\big)$;*

(b) *$Min(\Gamma_{\widehat{M}} \circ \mathrm{j}) = \{P \in \mathcal{P}_n : (\Gamma_{\widehat{M}} \circ \mathrm{j})(P) = P^{1/2}\mathcal{D}_M P^{1/2}\} \subseteq \mathcal{C}(\widehat{\Delta}_M) \cap \mathcal{P}_n$;*

(c) *fix $P_0 \in Min(\Gamma_{\widehat{M}} \circ \mathrm{j})$; then there exists $Q_0 \in O_n$ such that $\widehat{M} = \sqrt{\mathcal{D}_M}\, N_0 J_0 N_0^T$, where $N_0 = P_0^{1/2} Q_0$ and J_0 is an orthogonal square root of $\mathcal{O}_M$ of the following form:* $J_0 := \Big[I_{\widehat{p}} \Big] \oplus \Big[-I_{(p'-\widehat{p})} \Big] \oplus \Big[E_{\pi/2}^{\oplus q'} \Big] \oplus \Big[\bigoplus_{j=1}^{l_3} \Big(\big(E_{\psi_j/2}^{\oplus \widehat{s}_j}\big) \oplus \big(-E_{\psi_j/2}^{\oplus (s'_j - \widehat{s}_j)}\big) \Big) \Big]$,

for some $\widehat{p} \in \{0, 1, \ldots, p'\}$, $\widehat{s}_j \in \{0, 1, \ldots, s'_j\}$ (for every $j = 1, \ldots, l_3$);

(d) *if N_0 is the matrix defined in* (c)*, then we can write $N_0 = S \oplus T \oplus \big(\bigoplus_{j=1}^{l_3} \rho(Y_j)\big)$, for some $S \in GL_{p'}$, $T \in GL_{2q'}$, $Y_j \in GL_{s'_j}(\mathbb{C})$ (for every $j = 1, \ldots, l_3$).*

Proof (a) Since $\Gamma_{Z_M^{-T}} \circ \mathrm{j} = \mathrm{j} \circ \Gamma_{Z_M}$, we have $\Gamma_{\widehat{M}} \circ \mathrm{j} = \Gamma_{Z_M^{-1}} \circ \Gamma_M \circ \mathrm{j} \circ \Gamma_{Z_M}$. So, from Proposition 2.10 (i), we get that $\Gamma_{\widehat{M}} \circ \mathrm{j}$ is semi-simple, $Min(\Gamma_M \circ \mathrm{j}) = \Gamma_{Z_M}\big(Min(\Gamma_{\widehat{M}} \circ \mathrm{j})\big)$, and $\mathfrak{L}(\Gamma_M \circ \mathrm{j}) = \mathfrak{L}(\Gamma_{\widehat{M}} \circ \mathrm{j})$. Therefore, part (a) is proved.

(b) First of all, note that we have $\widehat{M}\,\widehat{M}^{-T} = Z_M^{-1} MM^{-T} Z_M = \widehat{\Delta}_M = \mathcal{D}_M \mathcal{O}_M = \mathcal{O}_M \mathcal{D}_M$.

Note also that: $Min(\Gamma_{\widehat{M}} \circ \mathrm{j}) \subseteq Min(\Gamma_{\widehat{\Delta}_M}) \subseteq \mathcal{C}(\widehat{\Delta}_M) \cap \mathcal{P}_n$. Indeed, the first inclusion follows from Proposition 2.8, taking into account that $(\Gamma_{\widehat{M}} \circ \mathrm{j})^2 = \Gamma_{\widehat{M}\widehat{M}^{-T}} = \Gamma_{\widehat{\Delta}_M}$; while, for the second inclusion, we note that, for every $P \in Min(\Gamma_{\widehat{\Delta}_M})$, by Proposition 4.3 (d) there exists $W \in GL_n$ such that $P = WW^T$ and $\widehat{\Delta}_M = W\widehat{\Delta}_M W^{-1}$, i.e. $W\widehat{\Delta}_M = \widehat{\Delta}_M W$; so $W \in \mathcal{C}(\widehat{\Delta}_M)$, and consequently $P = WW^T$, $P^{1/2}$, $P^{-1/2} \in \mathcal{C}(\widehat{\Delta}_M) \cap \mathcal{P}_n$.

If the isometry $\Gamma_{\widehat{M}} \circ \mathrm{j}$ is elliptic, then $(\Gamma_{\widehat{M}} \circ \mathrm{j})^2 = \Gamma_{\widehat{\Delta}_M}$ is also elliptic and so $\widehat{\Delta}_M$ is orthogonal; therefore we have $\mathcal{D}_M = I_n$ and part (b) is trivial.

Now, assume that the isometry $\Gamma_{\widehat{M}} \circ \mathrm{j}$ is hyperbolic and let $P \in Min(\Gamma_{\widehat{M}} \circ \mathrm{j})$. As seen above, $P,\ P^{1/2},\ P^{-1/2} \in \mathcal{C}(\widehat{\Delta}_M) \cap \mathcal{P}_n$, so $P,\ P^{1/2},\ P^{-1/2}$ also commute with $\mathcal{D}_M,\ \sqrt{\mathcal{D}_M},\ \mathcal{D}_M^{-1/2},\ \mathcal{O}_M$. Denote by $c(t) = P^{1/2}\exp(t\cdot L)P^{1/2}$ (with $L \in Sym_n \setminus \{0\}$) the non-constant geodesic of $(\mathcal{P}_n, g)$ passing through P whose image $c(\mathbb{R})$ is an axis of $\Gamma_{\widehat{M}} \circ \mathrm{j}$. So there exists a real number $b \neq 0$ such that

(*) $(\Gamma_{\widehat{M}} \circ \mathrm{j})\,(c(t)) = c(t+b)$, for every $t \in \mathbb{R}$, i.e.

$\widehat{M}P^{-1/2}\exp(-t\cdot L)P^{-1/2}\widehat{M}^T = P^{1/2}\exp(t\cdot L)\exp(b\cdot L)P^{1/2}$, for every $t \in \mathbb{R}$. In particular, for $t=0$ we get $b\cdot L = \log(P^{-1/2}\widehat{M}P^{-1}\widehat{M}^T P^{-1/2})$. If we set $\mathcal{A} := P^{-1/2}\widehat{M}P^{-1/2}$, we have $b\cdot L = \log(\mathcal{A}\mathcal{A}^T)$. Denoting with $\mathcal{A} = \mathcal{D}\mathcal{O}$ (with $\mathcal{D} \in \mathcal{P}_n$ and $\mathcal{O} \in O_n$) the polar decomposition of $\mathcal{A}$, we can write $\mathcal{A}\mathcal{A}^T = \mathcal{D}^2$, so that $b\cdot L = \log(\mathcal{D}^2) = 2\cdot\log(\mathcal{D})$. Since $(\Gamma_{\widehat{M}} \circ \mathrm{j})^2 = \Gamma_{\widehat{\Delta}_M}$, from (*) we obtain $\Gamma_{\widehat{\Delta}_M}(c(t)) = c(t+2b)$, for every $t \in \mathbb{R}$. In particular, for $t=0$ we get $\mathcal{D}_M\mathcal{O}_M P\mathcal{O}_M^T\mathcal{D}_M = \widehat{\Delta}_M P\big(\widehat{\Delta}_M\big)^T = P^{1/2}\exp(2b\cdot L)P^{1/2} = P^{1/2}\exp\big(4\cdot\log(\mathcal{D})\big)P^{1/2} = P^{1/2}\mathcal{D}^4P^{1/2}$. Since $P^{1/2}$ commutes with both $\mathcal{D}_M$ and $\mathcal{O}_M$, from the previous equality we conclude that $\mathcal{D}_M^2 = \mathcal{D}^4$, i.e. $\mathcal{D}^2 = \mathcal{D}_M$, i.e. $\mathcal{D} = \sqrt{\mathcal{D}_M}$. So, by setting $t=0$ in (*), we get $(\Gamma_{\widehat{M}} \circ \mathrm{j})\,(P) = P^{1/2}\mathcal{D}^2P^{1/2} = P^{1/2}\mathcal{D}_M P^{1/2}$. We conclude that $Min(\Gamma_{\widehat{M}} \circ \mathrm{j}) \subseteq \{P \in \mathcal{P}_n :\ (\Gamma_{\widehat{M}} \circ \mathrm{j})(P) = P^{1/2}\mathcal{D}_M P^{1/2}\}$.

Conversely, if $(\Gamma_{\widehat{M}} \circ \mathrm{j})\,(P) = P^{1/2}\mathcal{D}_M P^{1/2}$, by Recalls 3.1 (c) and Proposition 6.3, we have $d((\Gamma_{\widehat{M}} \circ \mathrm{j})\,(P), P) = ||\log(\mathcal{D}_M)|| = \mathfrak{L}(\Gamma_{\widehat{M}} \circ \mathrm{j})$. Consequently $P \in Min(\Gamma_{\widehat{M}} \circ \mathrm{j})$ and this concludes the proof of part (b).

c) Fix $P_0 = W_0W_0^T \in Min(\Gamma_{\widehat{M}} \circ \mathrm{j})$, with $W_0 \in \mathcal{C}(\widehat{\Delta}_M)$, and set $\mathcal{A}_0 := P_0^{-1/2}\widehat{M}P_0^{-1/2}$. Let $\mathcal{A}_0 = \mathcal{D}_0\mathcal{O}_0 = \mathcal{O}_0\widehat{\mathcal{D}}_0$ be the polar decompositions of the matrix $\mathcal{A}_0$ (where we have $\mathcal{D}_0 = \sqrt{\mathcal{A}_0\mathcal{A}_0^T},\ \ \widehat{\mathcal{D}}_0 = \sqrt{\mathcal{A}_0^T\mathcal{A}_0} \in \mathcal{P}_n$ and $\mathcal{O}_0 \in O_n$). In part (b) we proved that, if $\Gamma_{\widehat{M}} \circ \mathrm{j}$ is hyperbolic, then $\mathcal{D}_0 = \sqrt{\mathcal{D}_M}$. This is trivially true even if $\Gamma_{\widehat{M}} \circ \mathrm{j}$ is elliptic, since in this case the matrices $\mathcal{A}_0\mathcal{A}_0^T,\ \mathcal{D}_0,\ \mathcal{D}_M$ are necessarily all equal to I_n.

<u>*Claim*</u>. We have $\mathcal{O}_0^2 = \mathcal{O}_M$.

First assume that $\Gamma_{\widehat{M}} \circ \mathrm{j}$ is elliptic. As seen before, in this case: $\mathcal{D}_0 = \mathcal{D}_M = I_n$, and so: $\widehat{M} = P_0^{1/2}\mathcal{O}_0P_0^{1/2}$. Since $P_0^{1/2}$ and $\mathcal{O}_M$ commute, then $P_0^{1/2}\mathcal{O}_M P_0^{-1/2} = \mathcal{O}_M = \widehat{\Delta}_M = \widehat{M}\widehat{M}^{-T} = P_0^{1/2}\mathcal{O}_0P_0^{1/2}P_0^{-1/2}\mathcal{O}_0^{-T}P_0^{-1/2} = P_0^{1/2}\mathcal{O}_0^2P_0^{-1/2}$. Clearly, this implies the Claim.

Now suppose that $\Gamma_{\widehat{M}} \circ \mathrm{j}$ is hyperbolic. From the equality $\mathcal{D}_0 = \sqrt{\mathcal{D}_M}$ it follows that $\mathcal{O}_0 = \mathcal{D}_M^{-1/2}\mathcal{A}_0 = \mathcal{D}_M^{-1/2}P_0^{-1/2}\widehat{M}P_0^{-1/2} = P_0^{-1/2}\mathcal{D}_M^{-1/2}\widehat{M}\,P_0^{-1/2}$, because $\mathcal{D}_M^{-1/2}$ and $P_0^{-1/2}$ commute. In the proof of part (b) we saw that the axis of $(\Gamma_{\widehat{M}} \circ \mathrm{j})$ passing through P_0 is the image of the geodesic $c_0(t) := P_0^{1/2}\exp\big(\frac{t}{b}\cdot\log(\mathcal{A}_0\mathcal{A}_0^T)\big)P_0^{1/2}$ (with $t \in \mathbb{R}$), for some $b \in \mathbb{R}$. Therefore, the equality (*) for $t = b$ (with c_0 in place of c) becomes $\widehat{M}P_0^{-1/2}\exp\big(-\log(\mathcal{A}_0\mathcal{A}_0^T)\big)P_0^{-1/2}\widehat{M}^T = P_0^{1/2}\big(\mathcal{A}_0\mathcal{A}_0^T\big)^2P_0^{1/2}$, or equivalently $\mathcal{A}_0\big(\mathcal{A}_0\mathcal{A}_0^T\big)^{-1}\mathcal{A}_0^T = \mathcal{A}_0\mathcal{A}_0^T\mathcal{A}_0\mathcal{A}_0^T$; so $\mathcal{D}_M^{-1} = (\mathcal{A}_0\mathcal{A}_0^T)^{-1} = \mathcal{A}_0^T\mathcal{A}_0 = \widehat{\mathcal{D}}_0^2$. We conclude that $\widehat{\mathcal{D}}_0 = \mathcal{D}_M^{-1/2}$, so we can write $\mathcal{A}_0 = \mathcal{O}_0\mathcal{D}_M^{-1/2}$ (as well as $\mathcal{A}_0 = \mathcal{D}_M^{1/2}\mathcal{O}_0$).

Now let $W_0 = P_0^{1/2}H_0$ (with $H_0 \in O_n$) be the polar decomposition of W_0. It is clear that $H_0,\ H_0^T \in \mathcal{C}(\widehat{\Delta}_M) \cap O_n$, so, in particular, H_0^T commutes with $\mathcal{D}_M$ and $\mathcal{O}_M$.

Therefore, we have $\mathcal{D}_M\mathcal{O}_M = \widehat{\Delta}_M = W_0^{-1}\widehat{\Delta}_M W_0 = H_0^T P_0^{-1/2}\widehat{M}\,\widehat{M}^{-T}P_0^{1/2}H_0 = H_0^T\big(P_0^{-1/2}\widehat{M}P_0^{-1/2}\big)\big(P_0^{1/2}\widehat{M}^{-T}P_0^{1/2}\big)H_0 = H_0^T\mathcal{A}_0\mathcal{A}_0^{-T}H_0 = H_0^T\mathcal{D}_M^{1/2}\mathcal{O}_0\big(\mathcal{D}_M^{1/2}\mathcal{O}_0\big)^{-T}H_0 = H_0^T\mathcal{D}_M^{1/2}(\mathcal{O}_0\mathcal{D}_M^{-1/2})\,\mathcal{O}_0H_0 = H_0^T\mathcal{D}_M^{1/2}(\mathcal{D}_M^{1/2}\mathcal{O}_0)\,\mathcal{O}_0H_0 = H_0^T\mathcal{D}_M\mathcal{O}_0^2H_0 = \mathcal{D}_M H_0^T\mathcal{O}_0^2H_0$.

This implies: $\mathcal{O}_M = H_0^T\mathcal{O}_0^2H_0$, i.e. $\mathcal{O}_0^2 = H_0\mathcal{O}_M H_0^T = \mathcal{O}_M$, because $H_0\mathcal{O}_M = \mathcal{O}_M H_0$. This concludes the proof of the Claim. The proof proceeds as follows.

From the equality $\mathcal{O}_0^2 = \mathcal{O}_M$ we deduce that a real Jordan form J_0 of $\mathcal{O}_0$ is of the form:

$$J_0 := \Big[\,I_{\widehat{p}}\,\Big]\oplus\Big[-I_{(p'-\widehat{p})}\Big]\oplus\Big[E_{\pi/2}^{\oplus q'}\Big]\oplus\Big[\bigoplus_{j=1}^{l_3}\Big(\big(E_{\psi_j/2}^{\oplus\widehat{s}_j}\big)\oplus\big(-E_{\psi_j/2}^{\oplus(s'_j-\widehat{s}_j)}\big)\Big)\Big],$$

for some $\widehat{p}\in\{0,1,\ldots,p'\}$, $\widehat{s}_j\in\{0,1,\ldots,s'_j\}$ (for every $j=1,\ldots,l_3$), where J_0 is a suitable orthogonal square root of $\mathcal{O}_M$. So there exists $Q_0\in O_n$ such that $\mathcal{O}_0 = Q_0J_0Q_0^T$; then $\mathcal{D}_M^{-1/2}\widehat{M} = \mathcal{D}_M^{-1/2}P_0^{1/2}\mathcal{A}_0P_0^{1/2} = P_0^{1/2}\mathcal{D}_M^{-1/2}\mathcal{D}_0\mathcal{O}_0P_0^{1/2} = P_0^{1/2}\mathcal{O}_0P_0^{1/2} = N_0J_0N_0^T$, where $N_0 := P_0^{1/2}Q_0$. Therefore $\widehat{M} = \sqrt{\mathcal{D}_M}\,N_0J_0N_0^T$ and this concludes part (c).

(d) In part (c) we saw that $\mathcal{O}_M = \mathcal{O}_0^2 = (Q_0J_0Q_0^T)^2 = Q_0J_0^2\,Q_0^T = Q_0\mathcal{O}_M\,Q_0^T$. So Q_0 (as well as $P_0^{1/2}$) commutes with $\mathcal{O}_M$. We conclude that $N_0 = P_0^{1/2}Q_0$ also commutes with $\mathcal{O}_M$. Therefore N_0 can be written as in the statement of part (d). □

Theorem 6.5 *Let $M\in GL_n$ be such that $\Gamma_M\circ \mathrm{j}$ is a semi-simple isometry of $(\mathcal{P}_n, g)$; denote the eigenvalues of MM^{-T} (with their multiplicities) as in Notations 6.2 and let $Z_M\in GL_n$, $N_0 = S\oplus T\oplus\big(\bigoplus_{j=1}^{l_3}\rho(Y_j)\big)\in GL_{p'}\oplus GL_{2q'}\oplus\Big(\bigoplus_{j=1}^{l_3}\rho\big(GL_{s'_j}(\mathbb{C})\big)\Big)$ be as defined by Proposition 6.4.*

(1) *We have* $Min(\Gamma_M\circ \mathrm{j}) = \Gamma_{Z_M}(\mathfrak{S}_M)$, *where*

$$\mathfrak{S}_M := \Bigg[\Bigg((\mathcal{P}_p)\oplus\Bigg(\bigoplus_{j=1}^{l_1}\mathcal{P}_{p_j}\Bigg)\oplus\Bigg(\bigoplus_{j=1}^{l_1}\mathcal{P}_{p_j}\Bigg)\Bigg)\cap\Big(\Gamma_S\big((SO^+_{(\widehat{p},\,p'-\widehat{p})})\cap(\mathcal{P}_{p'})\big)\Big)\Bigg]$$
$$\oplus\Bigg[\Bigg((\mathcal{P}_{2q})\oplus\Bigg(\bigoplus_{j=1}^{l_2}\mathcal{P}_{q_j}\Bigg)\oplus\Bigg(\bigoplus_{j=1}^{l_2}\mathcal{P}_{q_j}\Bigg)\Bigg)\cap\Big(\Gamma_T\big((Sp_{2q'})\cap(\mathcal{P}_{2q'})\big)\Big)\Bigg]$$
$$\oplus\Bigg[\bigoplus_{j=1}^{l_3}\Bigg(\Big[\big(\mathcal{P}_{2t_j}\big)\oplus\Bigg(\bigoplus_{h=1}^{k_j}\big(\mathcal{P}_{2s_{jh}}\oplus\mathcal{P}_{2s_{jh}}\big)\Bigg)\Big]\cap\Big[\Gamma_{\rho(Y_j)}\Big(\rho\big((U_{(\widehat{s}_j,\,s'_j-\widehat{s}_j)})\cap(\mathcal{H}_{s'_j})\big)\Big)\Big]\Bigg)\Bigg],$$

for some $\widehat{p}\in\{0,1,\ldots,p'\}$, $\widehat{s}_j\in\{0,1,\ldots,s'_j\}$ (for every $j=1,\ldots,l_3$).

(2) *The set $Min(\Gamma_M\circ \mathrm{j})$ is a symmetric totally geodesic Hadamard submanifold of $(\mathcal{P}_n, g)$ contained in $\mathcal{P}_n(|\det(M)|)$.*

Proof In this proof, we will use the same notations as in Proposition 6.4.

If $P\in Min(\Gamma_{\widehat{M}}\circ \mathrm{j})$, then $\sqrt{\mathcal{D}_M}$ and $P^{1/2}$ commute; so Proposition 6.4 (a), (b) implies: $Min(\Gamma_M\circ \mathrm{j}) = \Gamma_{Z_M}\big(\{P\in\mathcal{C}(\widehat{\Delta}_M)\cap\mathcal{P}_n:\ (\Gamma_{\widehat{M}}\circ \mathrm{j})(P) = \sqrt{\mathcal{D}_M}P\sqrt{\mathcal{D}_M}\}\big)$. From Proposition 6.4 (c) we have: $\widehat{M} = \sqrt{\mathcal{D}_M}N_0J_0N_0^T$, so the condition $(\Gamma_{\widehat{M}}\circ \mathrm{j})(P) = \sqrt{\mathcal{D}_M}P\sqrt{\mathcal{D}_M}$ is equivalent to the condition $(N_0J_0N_0^T)P^{-1}(N_0J_0^TN_0^T) = P$, or, equivalently, to the condition $J_0 = \big(N_0^{-1}PN_0^{-T}\big)J_0\big(N_0^{-1}PN_0^{-T}\big) = \Gamma_{N_0^{-1}}(P)J_0\,\Gamma_{N_0^{-1}}(P) = QJ_0Q$, where $Q := \Gamma_{N_0^{-1}}(P)$. Note that the matrices N_0, N_0^{-1}, N_0^{-T} commute with $\mathcal{O}_M$ and that $\mathcal{C}(\widehat{\Delta}_M)\subseteq\mathcal{C}(\mathcal{O}_M)$; therefore, if $P\in\mathcal{C}(\widehat{\Delta}_M)\cap\mathcal{P}_n$ we have $Q = \Gamma_{N_0^{-1}}(P)\in\mathcal{C}(\mathcal{O}_M)\cap\mathcal{P}_n$. Therefore, we can conclude that $\left\{P \in \mathcal{C}(\widehat{\Delta}_M) \cap \mathcal{P}_n \,:\, (\Gamma_{\widehat{M}} \circ \mathrm{j})(P) = \sqrt{\mathcal{D}_M}\, P \sqrt{\mathcal{D}_M}\right\} = \left\{P \in \mathcal{C}(\widehat{\Delta}_M) \cap \mathcal{P}_n : P = \Gamma_{N_0}(Q),\ \text{with } QJ_0Q = J_0 \text{ and } Q \in \mathcal{C}(\mathcal{O}_M) \cap \mathcal{P}_n\right\}$.

By Notations 6.2 (b), if $Q \in \mathcal{C}(\mathcal{O}_M) \cap \mathcal{P}_n$ there exist $Q_1 \in \mathcal{P}_{p'}$, $Q_2 \in \mathcal{P}_{2q'}$, $K_j \in \mathcal{H}_{s'_j}$ (for $j = 1, \dots, l_3$) such that $Q = Q_1 \oplus Q_2 \oplus \left(\bigoplus_{j=1}^{l_3} \rho(K_j)\right)$.

<u>*Claim*</u>. Let $Q = Q_1 \oplus Q_2 \oplus \left(\bigoplus_{j=1}^{l_3} \rho(K_j)\right)$ (with $Q_1 \in \mathcal{P}_{p'}$, $Q_2 \in \mathcal{P}_{2q'}$, $K_j \in \mathcal{H}_{s'_j}$ for every $j = 1, \dots, l_3$). Recalling the expression of J_0 (see Proposition 6.4 (c)), the following conditions (a) and (b) are equivalent:

a) $QJ_0Q = J_0$ b) $\begin{cases} b_1)\ Q_1 \in \left(SO^+_{(\widehat{p},\, p'-\widehat{p})}\right) \cap \left(\mathcal{P}_{p'}\right); \\ b_2)\ Q_2 \in \left(Sp_{2q'}\right) \cap \left(\mathcal{P}_{2q'}\right); \\ b_3)\ K_j \in \left(U_{(\widehat{s}_j,\, s'_j-\widehat{s}_j)}\right) \cap \left(\mathcal{H}_{s'_j}\right) \ \text{for } j = 1, \dots, l_3. \end{cases}$

Clearly, condition (a) is equivalent to the following three equalities:

$a_1)$ $Q_1\big((I_{\widehat{p}}) \oplus (-I_{(p'-\widehat{p})})\big)Q_1 = (I_{\widehat{p}}) \oplus (-I_{(p'-\widehat{p})})$;

$a_2)$ $Q_2\big(E_{\pi/2}^{\oplus q'}\big)Q_2 = E_{\pi/2}^{\oplus q'}$;

$a_3)$ $\rho(K_j)\left(\big(E_{\psi_j/2}^{\oplus\widehat{s}_j}\big) \oplus \big(-E_{\psi_j/2}^{\oplus(s'_j-\widehat{s}_j)}\big)\right)\rho(K_j) = \big(E_{\psi_j/2}^{\oplus\widehat{s}_j}\big) \oplus \big(-E_{\psi_j/2}^{\oplus(s'_j-\widehat{s}_j)}\big)$ $j = 1, \dots, l_3$.

Note that $SO_{(\widehat{p},\, p'-\widehat{p})}$ is a reductive algebraic subgroup of $GL_{p'}$, so Proposition 3.8 (b) implies that $\left(SO_{(\widehat{p},\, p'-\widehat{p})}\right) \cap \left(\mathcal{P}_{p'}\right) = \left(SO^+_{(\widehat{p},\, p'-\widehat{p})}\right) \cap \left(\mathcal{P}_{p'}\right)$. Therefore, taking into account that the matrices Q_1 and Q_2 are symmetric and positive definite, we obtain that (a_1) is equivalent to (b_1) and (a_2) is equivalent to (b_2). It remains to be seen that (a_3) is equivalent to (b_3). Fix any $j \in \{1, \dots, l_3\}$. Since $E_{\psi_j/2} = \rho(e^{\frac{\psi_j}{2}\mathbf{i}})$, we have $\big(E_{\psi_j/2}^{\oplus\widehat{s}_j}\big) \oplus \big(-E_{\psi_j/2}^{\oplus(s'_j-\widehat{s}_j)}\big) = \rho\left(e^{\frac{\psi_j}{2}\mathbf{i}} \cdot [(I_{\widehat{s}_j}) \oplus (-I_{(s'_j-\widehat{s}_j)})]\right)$. Furthermore, the decomplexificatiom map ρ is a monomorphism of $\mathbb{R}$-algebras, so the condition

$\rho(K_j)\left(\big(E_{\psi_j/2}^{\oplus\widehat{s}_j}\big) \oplus \big(-E_{\psi_j/2}^{\oplus(s'_j-\widehat{s}_j)}\big)\right)\rho(K_j) = \big(E_{\psi_j/2}^{\oplus\widehat{s}_j}\big) \oplus \big(-E_{\psi_j/2}^{\oplus(s'_j-\widehat{s}_j)}\big)$ is equivalent to the condition $K_j\left((I_{\widehat{s}_j}) \oplus (-I_{(s'_j-\widehat{s}_j)})\right)K_j = (I_{\widehat{s}_j}) \oplus (-I_{(s'_j-\widehat{s}_j)})$ (with $K_j = \overline{K}_j^T$), i.e. to the condition $K_j \in \left(U_{(\widehat{s}_j,\, s'_j-\widehat{s}_j)}\right) \cap \left(\mathcal{H}_{s'_j}\right)$.

Hence (a_3) is equivalent to (b_3) and this concludes the proof of the Claim.

When $Q \in \mathcal{C}(\mathcal{O}_M) \cap \mathcal{P}_n$ we have $P := \Gamma_{N_0}(Q) = \Gamma_S(Q_1) \oplus \Gamma_T(Q_2) \oplus \left(\bigoplus_{j=1}^{l_3} \Gamma_{\rho(Y_j)}(\rho(K_j))\right)$. Therefore, from the Claim we have: $QJ_0Q = J_0$ if and only if P belongs to the set

$$\left[\Gamma_S\big((SO^+_{(\widehat{p},\, p'-\widehat{p})}) \cap (\mathcal{P}_{p'})\big)\right] \oplus \left[\Gamma_T\big((Sp_{2q'}) \cap (\mathcal{P}_{2q'})\big)\right] \oplus \left[\bigoplus_{j=1}^{l_3} \left[\Gamma_{\rho(Y_j)}\big(\rho((U_{(\widehat{s}_j,\, s'_j-\widehat{s}_j)}) \cap (\mathcal{H}_{s'_j}))\big)\right]\right].$$

So, recalling Notations 6.2 (b), we have $Min(\Gamma_M \circ \mathrm{j}) = \Gamma_{Z_M}(\mathfrak{S}_M)$, where

$$\mathfrak{S}_M := \left[\left(\left(\mathcal{P}_p\right) \oplus \left(\bigoplus_{j=1}^{l_1} \mathcal{P}_{p_j}\right) \oplus \left(\bigoplus_{j=1}^{l_1} \mathcal{P}_{p_j}\right)\right) \cap \left(\Gamma_S\big((SO^+_{(\widehat{p},\, p'-\widehat{p})}) \cap (\mathcal{P}_{p'})\big)\right)\right]$$
$$\oplus\left[\left(\left(\mathcal{P}_{2q}\right) \oplus \left(\bigoplus_{j=1}^{l_2} \mathcal{P}_{q_j}\right) \oplus \left(\bigoplus_{j=1}^{l_2} \mathcal{P}_{q_j}\right)\right) \cap \left(\Gamma_T\big((Sp_{2q'}) \cap (\mathcal{P}_{2q'})\big)\right)\right]$$

$$\oplus\Big[\bigoplus_{j=1}^{l_3}\Big(\Big[\big(\rho(\mathcal{H}_{t_j})\big)\oplus\Big(\bigoplus_{h=1}^{k_j}\big[\rho\big(\mathcal{H}_{s_{jh}}\big)\oplus\rho\big(\mathcal{H}_{s_{jh}}\big)\big]\Big)\Big]\cap\Big[\Gamma_{\rho(Y_j)}\big(\rho((U_{(\widehat{s}_j,\,s'_j-\widehat{s}_j)})\cap(\mathcal{H}_{s'_j}))\big)\Big]\Big)\Big].$$

Since $\rho(\mathcal{H}_h)=\rho(GL_h(\mathbb{C}))\cap\mathcal{P}_{2h}$ (for $h\geq 1$), we obtain the statement of part (1).

Let $G_1:=(GL_p)\oplus(\bigoplus_{j=1}^{l_1}GL_{p_j})\oplus(\bigoplus_{j=1}^{l_1}GL_{p_j})\oplus(GL_{2q})\oplus(\bigoplus_{j=1}^{l_2}GL_{q_j})\oplus(\bigoplus_{j=1}^{l_2}GL_{q_j})\oplus\Big(\bigoplus_{j=1}^{l_3}\Big[\big(GL_{2t_j}\big)\oplus\Big(\bigoplus_{h=1}^{k_j}\big(GL_{2s_{jh}}\oplus GL_{2s_{jh}}\big)\Big)\Big]\Big)$ and $G_2:=\Big(SO^+_{(\widehat{p},\,p'-\widehat{p})}\Big)\oplus\Big(Sp_{2q'}\Big)\oplus\Big(\bigoplus_{j=1}^{l_3}\rho(U_{(\widehat{s}_j,\,s'_j-\widehat{s}_j)})\Big)$.

Keeping in mind Proposition 3.8, we deduce that G_1 and G_2 are $\mathcal{P}_n$-compatible reductive closed subgroups of GL_n; hence, by Proposition 3.7, both $G_1\cap\mathcal{P}_n$ and $G_2\cap\mathcal{P}_n$ are totally geodesic Hadamard submanifolds of $(\mathcal{P}_n,g)$. Since the congruences are isometries of $(\mathcal{P}_n,g)$, also $\Gamma_{Z_M}(G_1\cap\mathcal{P}_n)$ and $\Gamma_{Z_MN_0}(G_2\cap\mathcal{P}_n)$ are both totally geodesic Hadamard submanifolds of $(\mathcal{P}_n,g)$. By part (1), we have $\Big(\Gamma_{Z_M}(G_1\cap\mathcal{P}_n)\Big)\cap\Big(\Gamma_{Z_MN_0}(G_2\cap\mathcal{P}_n)\Big)=Min(\Gamma_M\circ\mathfrak{j})\neq\emptyset$. Therefore, by Proposition 2.2 (b), the set $Min(\Gamma_M\circ\mathfrak{j})$ is a symmetric totally geodesic Hadamard submanifold of $(\mathcal{P}_n,g)$. Finally, remembering Proposition 3.12 (a), we obtain the inclusion $Min(\Gamma_M\circ\mathfrak{j})\subseteq\mathcal{P}_n(|\det(M)|)$. So the proof is complete. □

7 Semi-simple isometries of the form $\Gamma_M\circ\mathfrak{j}\circ\mathfrak{d}$

Remark 7.1 A key point of this Section is to prove that, for every $M\in GL_n$, we have: $Min\big(\Gamma_M\circ\mathfrak{j}\circ\mathfrak{d}\big)=\mathbb{R}^+\cdot Min\big(\Gamma_M\circ\mathfrak{j}\big)$ (see Proposition 7.9 (c)). We have already proved this result when $\Gamma_M\circ\mathfrak{j}\circ\mathfrak{d}$ is elliptic ([11, Prop, 6.2]): in that case the isometry $\Gamma_M\circ\mathfrak{j}$ is elliptic too. Instead, when $\Gamma_M\circ\mathfrak{j}\circ\mathfrak{d}$ is hyperbolic, the isometry $\Gamma_M\circ\mathfrak{j}$ can be either elliptic or hyperbolic (see Remark 7.3).

Proposition 7.2 *Let $M\in GL_n$. The isometry $\Gamma_M\circ\mathfrak{j}\circ\mathfrak{d}$ of $(\mathcal{P}_n,g)$ is*

(a) *elliptic if and only if MM^{-T} is semi-simple and all its eigenvalues have absolute value equal to $|\det(M)|^{-2/n}$;*

(b) *hyperbolic if and only if MM^{-T} is semi-simple and at least one of its eigenvalues has absolute value different from $|\det(M)|^{-2/n}$;*

(c) *parabolic if and only if MM^{-T} is not semi-simple.*

Proof Since $(\Gamma_M\circ\mathfrak{j}\circ\mathfrak{d})^2=\Gamma_{(M\,[\mathfrak{d}(M)]^{-T})}$, by Proposition 2.8 the isometries $\Gamma_M\circ\mathfrak{d}\circ\mathfrak{j}$ and $\Gamma_{(M\,[\mathfrak{d}(M)]^{-T})}$ are of the same type. We have $M\big(\mathfrak{d}(M)\big)^{-T}=|\det(M)|^{2/n}MM^{-T}$; so the matrix $M\big(\mathfrak{d}(M)\big)^{-T}$ is semi-simple if and only if MM^{-T} is semi-simple; moreover, if the n eigenvalues of MM^{-T} are $\sigma_1,\sigma_2,\ldots,\sigma_n$, then the n eigenvalues of $M\left(\mathfrak{d}(M)\right)^{-T}$ are $|\det(M)|^{2/n}\sigma_1,|\det(M)|^{2/n}\sigma_2,\ldots,|\det(M)|^{2/n}\sigma_n$. We conclude by Proposition 4.5. □

Remark 7.3 Keeping in mind Propositions 6.1 and 7.2, we easily deduce that, for every $M\in GL_n$, the isometry $\Gamma_M\circ\mathfrak{j}\circ\mathfrak{d}$ is semi-simple if and only if the isometry

$\Gamma_M \circ \mathrm{j}$ is also semi-simple. However, if $\Gamma_M \circ \mathrm{j} \circ \mathfrak{d}$ is hyperbolic, $\Gamma_M \circ \mathrm{j}$ is not necessarily hyperbolic. Indeed, consider for example the matrices $M_1 = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$ and $M_2 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. From Proposition 7.2, we easily obtain that $\Gamma_{M_1} \circ \mathrm{j} \circ \mathfrak{d}$ and $\Gamma_{M_2} \circ \mathrm{j} \circ \mathfrak{d}$ are both hyperbolic isometries of $\mathcal{P}_2$, while, from Proposition 6.1 we deduce that the isometry $\Gamma_{M_1} \circ \mathrm{j}$ is hyperbolic and the isometry $\Gamma_{M_2} \circ \mathrm{j}$ is elliptic.

Notations 7.4 Let $M \in GL_n$. In this Section we will use exactly the same notations, relating to the matrix MM^{-T}, as in Notations 6.2.

Proposition 7.5 *Let $M \in GL_n$. Then*

(a) $\Gamma_M \circ \mathrm{j} \circ \mathfrak{d}$ *is a semi-simple isometry of $(\mathcal{P}_n, g)$ if and only if the matrix MM^{-T} is semi-simple;*

(b) *if the isometry $\Gamma_M \circ \mathrm{j} \circ \mathfrak{d}$ is semi-simple and $\sigma_1, \ldots, \sigma_n$ are the n eigenvalues of MM^{-T}, then $\mathfrak{L}\big(\Gamma_M \circ \mathrm{j} \circ \mathfrak{d}\big) = \sqrt{\sum_{j=1}^n \Big(\ln\big(|\det(M)|^{2/n} \cdot |\sigma_j|\big)\Big)^2} = \|\log\big(|\det(M)|^{2/n} \cdot \mathcal{D}_M\big)\|$.*

Proof Part (a) follows from Proposition 7.2.

The statement of part (b) easily follows from Theorem 4.6 (a) and from the equalities
$$\mathfrak{L}\big(\Gamma_M \circ \mathrm{j} \circ \mathfrak{d}\big) = \frac{\mathfrak{L}\big((\Gamma_M \circ \mathrm{j} \circ \mathfrak{d})^2\big)}{2} = \frac{\mathfrak{L}\big(\Gamma_{(M\,[\mathfrak{d}(M)]^{-T})}\big)}{2} = \frac{\mathfrak{L}\big(\Gamma_{(|\det(M)|^{2/n} MM^{-T})}\big)}{2}. \square$$

Remark 7.6 For every $\lambda \neq 0$ and every $M \in GL_n$ the isometry $\Gamma_M \circ \mathrm{j} \circ \mathfrak{d}$ is semi-simple if and only if the isometry $\Gamma_{(\lambda M)} \circ \mathrm{j} \circ \mathfrak{d}$ is semi-simple too.

This follows from Proposition 7.2 by remarking that $(\lambda M)(\lambda M)^{-T} = MM^{-T}$.

Proposition 7.7 *Let $M \in GL_n$ be such that the isometry $\Gamma_M \circ \mathrm{j} \circ \mathfrak{d}$ is semi-simple. Then $Min(\Gamma_M \circ \mathrm{j}) = Min(\Gamma_{\pi(M)} \circ \mathrm{j} \circ \mathfrak{d}) \cap \mathcal{P}_n(|\det(M)|)$, where $\pi(M) := \dfrac{M}{|\det(M)|^{1/n}}$.*

Proof The isometries $\Gamma_M \circ \mathrm{j} \circ \mathfrak{d}$, $\Gamma_{\pi(M)} \circ \mathrm{j} \circ \mathfrak{d}$ and $\Gamma_M \circ \mathrm{j}$ are all semi-simple (remember Remarks 7.3 and 7.6). We have the inclusion: $Min(\Gamma_M \circ \mathrm{j}) \subseteq \mathcal{P}_n(|\det(M)|)$. Indeed, if $\Gamma_M \circ \mathrm{j}$ is hyperbolic, it follows from Proposition 3.12 (a); while, if $\Gamma_M \circ \mathrm{j}$ is elliptic, the set $Min(\Gamma_M \circ \mathrm{j})$ consists of the points $P \in \mathcal{P}_n$ such that $MP^{-1}M^T = P$, and clearly these points have determinant equal to $|\det(M)|$.

Now we prove the inclusion : $Min(\Gamma_M \circ \mathrm{j}) \subseteq Min(\Gamma_{\pi(M)} \circ \mathrm{j} \circ \mathfrak{d})$. It is easy to check that the maps $\Gamma_M \circ \mathrm{j}$ and $\Gamma_{\pi(M)} \circ \mathrm{j} \circ \mathfrak{d}$ overlap on the submanifold $\mathcal{P}_n(|\det(M)|)$. Moreover, we have $\mathfrak{L}\big(\Gamma_M \circ \mathrm{j}\big) = \mathfrak{L}\big(\Gamma_{\pi(M)} \circ \mathrm{j} \circ \mathfrak{d}\big)$, by Propositions 6.3, 7.5. If $P \in Min(\Gamma_M \circ \mathrm{j})$, then $d\big(P, (\Gamma_M \circ \mathrm{j})(P)\big) = \mathfrak{L}\big(\Gamma_M \circ \mathrm{j}\big)$ and, as seen above, $P \in \mathcal{P}_n(|\det(M)|)$. Therefore, $d\big(P, (\Gamma_{\pi(M)} \circ \mathrm{j} \circ \mathfrak{d})(P)\big) = d\big(P, (\Gamma_M \circ \mathrm{j})(P)\big) = \mathfrak{L}\big(\Gamma_M \circ \mathrm{j}\big) = \mathfrak{L}\big(\Gamma_{\pi(M)} \circ \mathrm{j} \circ \mathfrak{d}\big)$, so P belongs to $Min(\Gamma_{\pi(M)} \circ \mathrm{j} \circ \mathfrak{d})$ and from this we get the requested inclusion. With similar arguments we get the reverse inclusion : $Min(\Gamma_{\pi(M)} \circ \mathrm{j} \circ \mathfrak{d}) \cap \mathcal{P}_n(|\det(M)|) \subseteq Min(\Gamma_M \circ \mathrm{j})$. $\square$

Proposition 7.8 *Let $M \in GL_n$ be such that $\Gamma_M \circ \mathrm{j} \circ \mathfrak{d}$ is a semi-simple isometry of $(\mathcal{P}_n, g)$; consider the matrices $\widehat{\Delta}_M$, $\mathcal{D}_M$ and the Lie group $\mathcal{C}(\widehat{\Delta}_M)$ defined in Notations*

6.2. *Fix* $Z_M \in GL_n$ *such that* $MM^{-T} = Z_M \widehat{\Delta}_M Z_M^{-1}$ *and set* $\widetilde{M} := |\det(Z_M)|^{2/n} \cdot Z_M^{-1} M Z_M^{-T}$. *Then the following statements hold:*

(a) *the isometry* $\Gamma_{\widetilde{M}} \circ \mathrm{j} \circ \mathfrak{d}$ *is semi-simple and* $Min\big(\Gamma_M \circ \mathrm{j} \circ \mathfrak{d}\big) = \Gamma_{Z_M}\big(Min\big(\Gamma_{\widetilde{M}} \circ \mathrm{j} \circ \mathfrak{d}\big)\big)$;

(b) $Min\big(\Gamma_{\widetilde{M}} \circ \mathrm{j} \circ \mathfrak{d}\big) = \{P \in \mathcal{P}_n : \ (\Gamma_{\widetilde{M}} \circ \mathrm{j} \circ \mathfrak{d})(P) = |\det(M)|^{2/n}\, P^{1/2} \mathcal{D}_M P^{1/2}\}$ *and* $Min\big(\Gamma_{\widetilde{M}} \circ \mathrm{j} \circ \mathfrak{d}\big) \subseteq \mathcal{C}(\widehat{\Delta}_M) \cap \mathcal{P}_n$.

Proof By standard calculations we easily get that $\Gamma_{\widetilde{M}} \circ \mathrm{j} \circ \mathfrak{d} = \Gamma_{Z_M^{-1}} \circ \Gamma_M \circ \mathrm{j} \circ \mathfrak{d} \circ \Gamma_{Z_M}$, where $\widetilde{M} := |\det(Z_M)|^{2/n} \cdot Z_M^{-1} M Z_M^{-T}$. Consequently, Proposition 2.10 (i) implies that $Min\big(\Gamma_M \circ \mathrm{j} \circ \mathfrak{d}\big) = \Gamma_{Z_M}\big(Min\big(\Gamma_{Z_M^{-1}} \circ \Gamma_M \circ \mathrm{j} \circ \mathfrak{d} \circ \Gamma_{Z_M}\big)\big) = \Gamma_{Z_M}\big(Min\big(\Gamma_{\widetilde{M}} \circ \mathrm{j} \circ \mathfrak{d}\big)\big)$, $\mathfrak{L}(\Gamma_M \circ \mathrm{j} \circ \mathfrak{d}) = \mathfrak{L}(\Gamma_{\widetilde{M}} \circ \mathrm{j} \circ \mathfrak{d})$ and $\Gamma_{\widetilde{M}} \circ \mathrm{j} \circ \mathfrak{d}$ is semi-simple. Therefore, part (a) is proved.

For part (b), first of all note that $\big(\Gamma_{\widetilde{M}} \circ \mathrm{j} \circ \mathfrak{d}\big)^2 = \Gamma_{\widetilde{M}\,[\mathfrak{d}(\widetilde{M})]^{-T}} = \Gamma_{(|\det(M)|^{2/n} \cdot \widehat{\Delta}_M)}$, since we have $\det(\widetilde{M}) = \det(M)$ and $\widetilde{M}\widetilde{M}^{-T} = \widehat{\Delta}_M$. Proposition 4.3 (d) and inclusion $Min\big(\Gamma_{\widetilde{M}} \circ \mathrm{j} \circ \mathfrak{d}\big) \subseteq Min\big(\big(\Gamma_{\widetilde{M}} \circ \mathrm{j} \circ \mathfrak{d}\big)^2\big) = Min\big(\Gamma_{(|\det(M)|^{2/n} \cdot \widehat{\Delta}_M)}\big)$ imply that, if $P \in Min\big(\Gamma_{\widetilde{M}} \circ \mathrm{j} \circ \mathfrak{d}\big)$, there exists a suitable matrix $V \in GL_n$ such that $P = VV^T$ and $|\det(M)|^{2/n} \cdot \widehat{\Delta}_M = V(|\det(M)|^{2/n} \cdot \widehat{\Delta}_M)V^{-1}$, i.e. $V\widehat{\Delta}_M = \widehat{\Delta}_M V$. Therefore, $V \in \mathcal{C}(\widehat{\Delta}_M)$ and $P = VV^T,\ P^{1/2},\ P^{-1/2} \in \mathcal{C}(\widehat{\Delta}_M) \cap \mathcal{P}_n$.

If $\Gamma_M \circ \mathrm{j} \circ \mathfrak{d}$ is elliptic, then $\Gamma_{\widetilde{M}} \circ \mathrm{j} \circ \mathfrak{d}$ is also elliptic; moreover, by Proposition 7.2 (a) we have $\mathcal{D}_M = |\det(M)|^{-2/n} \cdot I_n$; so part (b) is trivially true.

Now assume that $\Gamma_M \circ \mathrm{j} \circ \mathfrak{d}$ is hyperbolic; therefore, $\Gamma_{\widetilde{M}} \circ \mathrm{j} \circ \mathfrak{d}$ is also hyperbolic. Let $P \in Min\big(\Gamma_{\widetilde{M}} \circ \mathrm{j} \circ \mathfrak{d}\big)$. Arguing as in the proof of Proposition 6.4 (b), let $\beta(t) = P^{1/2}\exp(t \cdot \widetilde{L})P^{1/2}$ (with $\widetilde{L} \in Sym_n \backslash \{0\}$) be a geodesic of $(\mathcal{P}_n, g)$ such that there exists $b_0 \in \mathbb{R}\backslash\{0\}$ such that we have

($\triangledown$) $\big(\Gamma_{\widetilde{M}} \circ \mathrm{j} \circ \mathfrak{d}\big)(\beta(t)) = \beta(t + b_0)$, for every $t \in \mathbb{R}$, i.e.

$(\det(P))^{2/n}\, e^{2t \cdot tr(\widetilde{L})/n} \cdot \widetilde{M} P^{-1/2} \exp(-t \cdot \widetilde{L}) P^{-1/2} \widetilde{M}^T = P^{1/2}\exp(t \cdot \widetilde{L})\exp(b_0 \cdot \widetilde{L})P^{1/2}$ $(t \in \mathbb{R})$.

When $t = 0$ we get $b_0 \cdot \widetilde{L} = \log\big((\det(P))^{2/n} \cdot P^{-1/2}\widetilde{M}P^{-1}\widetilde{M}^T P^{-1/2}\big)$; so, after setting $\widetilde{\mathcal{A}} := (\det(P))^{1/n} \cdot P^{-1/2}\widetilde{M}P^{-1/2}$, we have $b_0 \cdot \widetilde{L} = \log(\widetilde{\mathcal{A}}\widetilde{\mathcal{A}}^T)$. If $\widetilde{\mathcal{A}} = \widetilde{\mathcal{D}}\widetilde{\mathcal{O}}$ (with $\widetilde{\mathcal{D}} \in \mathcal{P}_n$ and $\widetilde{\mathcal{O}} \in O_n$) is the polar decomposition of $\widetilde{\mathcal{A}}$, from $\widetilde{\mathcal{A}}\widetilde{\mathcal{A}}^T = \widetilde{\mathcal{D}}^2$ we obtain $b_0 \cdot \widetilde{L} = \log(\widetilde{\mathcal{D}}^2)$. Since $\big(\Gamma_{\widetilde{M}} \circ \mathrm{j} \circ \mathfrak{d}\big)^2 = \Gamma_{(|\det(M)|^{2/n} \cdot \widehat{\Delta}_M)}$, from ($\triangledown$) we get $\Gamma_{(|\det(M)|^{2/n} \cdot \widehat{\Delta}_M)}(P) = \beta(2b_0)$, i.e. $|\det(M)|^{4/n} \cdot \widehat{\Delta}_M P \widehat{\Delta}_M^T = P^{1/2}\exp(2b_0 \cdot \widetilde{L})P^{1/2} = P^{1/2}\widetilde{\mathcal{D}}^4 P^{1/2}$. Since $P^{1/2} \in \mathcal{C}(\widehat{\Delta}_M)$ and $\widehat{\Delta}_M\widehat{\Delta}_M^T = \mathcal{D}_M^2$, we get $|\det(M)|^{4/n} \cdot \mathcal{D}_M^2 = \widetilde{\mathcal{D}}^4$, i.e. $\widetilde{\mathcal{D}}^2 = |\det(M)|^{2/n} \cdot \mathcal{D}_M$. So: $(\Gamma_{\widetilde{M}} \circ \mathrm{j} \circ \mathfrak{d})(P) = \beta(b_0) = |\det(M)|^{2/n} \cdot P^{1/2}\mathcal{D}_M P^{1/2}$.

Conversely, if $(\Gamma_{\widetilde{M}} \circ \mathrm{j} \circ \mathfrak{d})(P) = |\det(M)|^{2/n} \cdot P^{1/2}\mathcal{D}_M P^{1/2}$, from Recalls 3.1 (c) and Proposition 7.5 (b), we obtain that $d\big(P, (\Gamma_{\widetilde{M}} \circ \mathrm{j} \circ \mathfrak{d})(P)\big) = ||\log\big(|\det(M)|^{2/n} \cdot \mathcal{D}_M\big)|| = \mathfrak{L}(\Gamma_{\widetilde{M}} \circ \mathrm{j} \circ \mathfrak{d})$; therefore $P \in Min(\Gamma_{\widetilde{M}} \circ \mathrm{j})$, and part (b) is fully proved. □

Proposition 7.9 *Let* $M \in GL_n$ *be such that the isometry* $\Gamma_M \circ \mathrm{j} \circ \mathfrak{d}$ *is semi-simple.*

(a) $Min\big(\Gamma_M \circ \mathrm{j} \circ \mathfrak{d}\big) = Min\big(\Gamma_{(\lambda M)} \circ \mathrm{j} \circ \mathfrak{d}\big)$, *for every* $\lambda \in \mathbb{R}\backslash\{0\}$.

(b) $Min(\Gamma_M \circ \mathrm{j}) = Min(\Gamma_M \circ \mathrm{j} \circ \mathfrak{d}) \cap \mathcal{P}_n(|\det(M)|)$.

(c) $Min\big(\Gamma_M \circ \mathrm{j} \circ \mathfrak{d}\big) = \mathbb{R}^+ \cdot Min\big(\Gamma_M \circ \mathrm{j}\big)$.

Proof Fix $\lambda \in \mathbb{R}\backslash\{0\}$. Clearly $(\lambda M)(\lambda M)^{-T} = MM^{-T}$; so, with the same notations as in Proposition 7.8, we get: $Z_{(\lambda M)} = Z_M$, $\widetilde{(\lambda M)} = \lambda \widetilde{M}$, $\mathcal{D}_{(\lambda M)} = \mathcal{D}_M$. Hence, by the same Proposition, we easily obtain:

($\star$) $Min\big(\Gamma_{(\lambda M)} \circ \mathrm{j} \circ \mathfrak{d}\big) = \Gamma_{Z_M}\big(Min\big(\Gamma_{(\lambda \widetilde{M})} \circ \mathrm{j} \circ \mathfrak{d}\big)\big)$ and

($\star\star$) $Min\big(\Gamma_{(\lambda \widetilde{M})} \circ \mathrm{j} \circ \mathfrak{d}\big) = \{P \in \mathcal{P}_n : (\Gamma_{(\lambda \widetilde{M})} \circ \mathrm{j} \circ \mathfrak{d})(P) = \lambda^2 |\det(M)|^{2/n} P^{1/2} \mathcal{D}_M P^{1/2}\}$.

Since $\Gamma_{(\lambda \widetilde{M})} = \lambda^2 \cdot \Gamma_{\widetilde{M}}$, again from Proposition 7.8, the equality ($\star\star$) is equivalent to:

($\star\star\star$) $Min\big(\Gamma_{(\lambda \widetilde{M})} \circ \mathrm{j} \circ \mathfrak{d}\big) = Min\big(\Gamma_{\widetilde{M}} \circ \mathrm{j} \circ \mathfrak{d}\big) = \Gamma_{Z_M^{-1}}\big(Min\big(\Gamma_M \circ \mathrm{j} \circ \mathfrak{d}\big)\big)$.

Part (a) follows from ($\star$) and ($\star\star\star$), while part (b) follows from (a) and Proposition 7.7.

Since $Min(\Gamma_M \circ \mathrm{j} \circ \mathfrak{d})$ is a positive cone of $\mathcal{P}_n$, for any $P \in Min\big(\Gamma_M \circ \mathrm{j} \circ \mathfrak{d}\big)$ the point $P' := \left(\frac{|\det(M)|}{\det(P)}\right)^{1/n} \cdot P$ belongs to $Min(\Gamma_M \circ \mathrm{j} \circ \mathfrak{d}) \cap \mathcal{P}_n(|\det(M)|)$; therefore, by part (b), $P' \in Min(\Gamma_M \circ \mathrm{j})$ and, consequently, $P = \left(\frac{\det(P)}{|\det(M)|}\right)^{1/n} \cdot P \in \mathbb{R}^+ \cdot Min\big(\Gamma_M \circ \mathrm{j}\big)$.

Let now $P := t \cdot Q \in \mathbb{R}^+ \cdot Min\big(\Gamma_M \circ \mathrm{j}\big)$ (where $t > 0$ and $Q \in Min\big(\Gamma_M \circ \mathrm{j}\big)$). Then, by part (b), $Q \in Min(\Gamma_M \circ \mathrm{j} \circ \mathfrak{d})$; so, from Proposition 3.4 (b), we conclude that $P = t \cdot Q$ belongs to $Min(\Gamma_M \circ \mathrm{j} \circ \mathfrak{d})$. Therefore, part (c) is fully proved. □

Theorem 7.10 *Let $M \in GL_n$ be such that $\Gamma_M \circ \mathrm{j} \circ \mathfrak{d}$ is a semi-simple isometry of $(\mathcal{P}_n, g)$; let us denote the eigenvalues of MM^{-T} (with their multiplicities) and $Z_M \in GL_n$ as in Notations 6.2. Then*

(1) *we have: $Min(\Gamma_M \circ \mathrm{j} \circ \mathfrak{d}) = \Gamma_{Z_M}(\mathbb{R}^+ \cdot \mathfrak{S}_M) = \mathbb{R}^+ \cdot \Gamma_{Z_M}(\mathfrak{S}_M)$, where $\mathfrak{S}_M$ is the set defined in Theorem 6.5* (1)*;*

(2) *the set $Min(\Gamma_M \circ \mathrm{j} \circ \mathfrak{d})$ is a symmetric totally geodesic Hadamard submanifold of $(\mathcal{P}_n, g)$, isometric to the Riemannian product $\big(\mathbb{R}, \epsilon\big) \times \big(Min(\Gamma_M \circ \mathrm{j}), g\big)$, where ϵ is the Euclidean metric of $\mathbb{R}$.*

Proof Part (1) follows directly form Proposition 7.9 (c) and from Theorem 6.5 (1), remembering that the isometry $\Gamma_M \circ \mathrm{j}$ is semi-simple too.

By Theorem 6.5 (2), $\big(Min(\Gamma_M \circ \mathrm{j}), g\big)$ is a symmetric totally geodesic Hadamard submanifold of $(\mathcal{P}_n(|\det(M)|), g)$. Then we obtain part (2), by noting that the isometry $F : \big(\mathbb{R}, \epsilon\big) \times \big(\mathcal{P}_n(|\det(M)|), g\big) \to \big(\mathcal{P}_n, g\big)$ defined by $F(x, A) = e^{x/\sqrt{n}} \cdot A$ maps the symmetric totally geodesic Hadamard submanifold $\mathbb{R} \times \big(Min(\Gamma_M \circ \mathrm{j})\big)$ onto the symmetric totally geodesic Hadamard submanifold $\mathbb{R}^+ \cdot \big(Min(\Gamma_M \circ \mathrm{j})\big) = Min(\Gamma_M \circ \mathrm{j} \circ \mathfrak{d})$. □

8 Minimum set, axes, orthogonal spaces of a hyperbolic isometry of $(\mathcal{P}_n, g)$

The following Proposition is a particular case of a more general result contained in [21] (see Proposition 2.14 (a)); note, however, that it can be independently obtained as a consequence of Proposition 3.3 (b) and Theorems 4.6 (c), 5.7, 6.5 (2), 7.10 (2).

Proposition 8.1 *For every semi-simple isometry Φ of $(\mathcal{P}_n, g)$, the set $Min(\Phi)$ is a symmetric totally geodesic Hadamard submanifold of $(\mathcal{P}_n, g)$.*

Remark 8.2 Note that, if $c > 0$ and φ is a semi-simple isometry of $(\mathcal{P}_n(c), g)$, then also the minimum set $Min(\varphi)$ is a symmetric totally geodesic Hadamard submanifold of both $(\mathcal{P}_n(c), g)$ and $(\mathcal{P}_n, g)$.

As above, this result can be directly deduced from Proposition 2.14 (a); however, it can be independently obtained from Propositions 8.1, 3.13 (i), 2.2 (b) and Remark 3.10.

Theorem 8.3 *Let Φ be a hyperbolic isometry of $(\mathcal{P}_n, g)$ with translation length $\mathfrak{L}(\Phi)$ (so $\mathfrak{L}(\Phi) > 0$) and let $Q \in Min(\Phi)$.*

(1) *Assume that $\Phi := \Gamma_M$ or $\Phi := \Gamma_M \circ \mathfrak{d}$ (so $M \in GL_n$ is semi-simple); let $\Delta_M, \mathcal{S}_M$ be as in Remark-Definition 4.1; let $A_0 \in GL_n$ be such that $M = A_0 \Delta_M A_0^{-1}$ with $|\det(A_0)| = \sqrt{|\det(M)|}$ and denote $P := \Gamma_{A_0^{-1}}(Q)$. The Q-axis of Φ is*

(a) $$\gamma_Q^{\Phi}(t) = A_0 P^{1/2} \exp\left(2t \cdot \frac{\log(\mathcal{S}_M)}{\mathcal{L}(\Phi)}\right) P^{1/2} A_0^T = A_0 P^{1/2} \left(\mathcal{S}_M\right)^{\frac{2t}{\mathcal{L}(\Phi)}} P^{1/2} A_0^T$$
$(t \in \mathbb{R})$,
when $\Phi := \Gamma_M$;

(b) $$\gamma_Q^{\Phi}(t) = A_0 P^{1/2} \exp\left(\frac{2t}{\mathcal{L}(\Phi)} \cdot \left[\log(\mathcal{S}_M) - \ln(|\det(M)|^{1/n}) \cdot I_n\right]\right) P^{1/2} A_0^T =$$
$$= |\det(M)|^{-\frac{2t}{n\mathcal{L}(\Phi)}} \cdot A_0 P^{1/2} \left(\mathcal{S}_M\right)^{\frac{2t}{\mathcal{L}(\Phi)}} P^{1/2} A_0^T \quad (t \in \mathbb{R})$$
when $\Phi := \Gamma_M \circ \mathfrak{d}$.

(2) *Assume that $\Phi := \Gamma_M \circ \mathfrak{j}$ or $\Phi := \Gamma_M \circ \mathfrak{j} \circ \mathfrak{d}$ (so $MM^{-T} \in GL_n$ is semi-simple); let $\widehat{\Delta}_M, \mathcal{D}_M$ be the matrices defined in Notations 6.2; let $Z_M \in GL_n$ be such that $MM^{-T} = Z_M \widehat{\Delta}_M Z_M^{-1}$ and denote $P := \Gamma_{Z_M^{-1}}(Q)$. The Q-axis of Φ is*

(c) $$\gamma_Q^{\Phi}(t) = Z_M P^{1/2} \exp\left(t \cdot \frac{\log(\mathcal{D}_M)}{\mathcal{L}(\Phi)}\right) P^{1/2} Z_M^T = Z_M P^{1/2} \left(\mathcal{D}_M\right)^{\frac{t}{\mathcal{L}(\Phi)}} P^{1/2} Z_M^T$$
$(t \in \mathbb{R})$,
when $\Phi := \Gamma_M \circ \mathfrak{j}$;

(d) $$\gamma_Q^{\Phi}(t) = Z_M P^{1/2} \exp\left(t \cdot \frac{\log(|\det(M)|^{2/n} \cdot \mathcal{D}_M)}{\mathcal{L}(\Phi)}\right) P^{1/2} Z_M^T =$$
$$|\det(M)|^{\frac{2t}{n\mathcal{L}(\Phi)}} \cdot Z_M P^{1/2} \left(\mathcal{D}_M\right)^{\frac{t}{\mathcal{L}(\Phi)}} P^{1/2} Z_M^T \qquad (t \in \mathbb{R}),$$
when $\Phi := \Gamma_M \circ \mathfrak{j} \circ \mathfrak{d}$.

Proof We will consider each of the four cases of the statement separately.

Case 1 (a). Assume $\Phi = \Gamma_M$. From Theorem 4.6 (b), we get that $P \in Min(\Gamma_{\Delta_M})$ and $\Gamma_{\Delta_M}(P) = P^{1/2}\mathcal{S}_M^2 P^{1/2}$. On the other hand, from Proposition 3.11, we obtain that $\gamma_Q^\Phi(t) = A_0 P^{1/2} \exp\left(t \cdot \dfrac{\log(\mathcal{S}_M^2)}{\mathcal{L}(\Gamma_{\Delta_M})}\right) P^{1/2} A_0^T$. Since we have $\mathcal{L}(\Phi) = \mathcal{L}(\Gamma_{\Delta_M})$ and $\log(\mathcal{S}_M^2) = 2 \cdot \log(\mathcal{S}_M)$, we get the requested expressions for $\gamma_Q^\Phi(t)$.

Case 1 (b). Assume $\Phi = \Gamma_M \circ \mathfrak{d}$. From Proposition 5.5 (b), we have $\gamma_Q^\Phi = \gamma_Q^\Theta$, where $\Theta := \Gamma_{\pi(M)}$. Since $\Delta_{\pi(M)} = \dfrac{\Delta_M}{|\det(M)|^{1/n}}$, we obtain that $\mathcal{S}_{\pi(M)} = \dfrac{\mathcal{S}_M}{|\det(M)|^{1/n}}$. So we have $\log(\mathcal{S}_{\pi(M)}) = \log(\mathcal{S}_M) - \ln(|\det(M)|^{1/n}) \cdot I_n$ and $\left(\mathcal{S}_{\pi(M)}\right)^u = |\det(M)|^{-u/n} \cdot \left(\mathcal{S}_M\right)^u$, for every $u \in \mathbb{R}$. By Lemma 5.2, we have $\mathcal{L}(\Phi) = \mathcal{L}(\Theta)$; hence, from (1) (a) we obtain:

$$\gamma_Q^\Phi(t) = \gamma_Q^\Theta(t) = A_0 P^{1/2} \exp\left(2t \cdot \frac{\log(\mathcal{S}_{\pi(M)})}{\mathcal{L}(\Phi)}\right) P^{1/2} A_0^T$$

$$= A_0 P^{1/2} \left(\mathcal{S}_{\pi(M)}\right)^{\frac{2t}{\mathcal{L}(\Phi)}} P^{1/2} A_0^T$$

$$= A_0 P^{1/2} \exp\left(\frac{2t}{\mathcal{L}(\Phi)} \cdot \left[\log(\mathcal{S}_M) - \ln(|\det(M)|^{1/n}) \cdot I_n\right]\right) P^{1/2} A_0^T$$

$$= |\det(M)|^{-\frac{2t}{n\mathcal{L}(\Phi)}} \cdot A_0 P^{1/2} \left(\mathcal{S}_M\right)^{\frac{2t}{\mathcal{L}(\Phi)}} P^{1/2} A_0^T, \quad \text{for every } t \in \mathbb{R}.$$

Case 2 (c). Assume $\Phi := \Gamma_M \circ \mathfrak{j}$ and set $\widehat{M} := Z_M^{-1} M Z_M^{-T}$. By Proposition 6.4, we get $P \in Min\left(\Gamma_{\widehat{M}} \circ \mathfrak{j}\right)$, $(\Gamma_{\widehat{M}} \circ \mathfrak{j})(P) = P^{1/2}\mathcal{D}_M P^{1/2}$ and $\mathfrak{L}(\Phi) = \mathfrak{L}(\Gamma_{\widehat{M}} \circ \mathfrak{j})$. Consequently, from Proposition 3.11 we get the requested formula.

Case 2 (d). Assume $\Phi := \Gamma_M \circ \mathfrak{j} \circ \mathfrak{d}$. Set $\widetilde{M} := |\det(Z_M)|^{2/n} \cdot Z_M^{-1} M Z_M^{-T}$, $\Lambda := \Gamma_{\widetilde{M}} \circ \mathfrak{j} \circ \mathfrak{d}$. By Proposition 7.8, we obtain that Λ is hyperbolic, $P \in Min(\Lambda)$, $\mathfrak{L}(\Phi) = \mathfrak{L}(\Lambda)$, and $\Lambda(P) = |\det(M)|^{2/n} P^{1/2}\mathcal{D}_M P^{1/2}$. So, from Proposition 3.11 we get the statement. □

Proposition 8.4 *Let Φ be a hyperbolic isometry of $(\mathcal{P}_n, g)$ and let $P \in Min(\Phi)$. Then*

(a) *the P-orthogonal space $\mathcal{Y}_P(\Phi)$ of Φ is a symmetric totally geodesic Hadamard submanifold of $(\mathcal{P}_n, g)$ and furthermore, if we set $L_P := \log(P^{-1/2}\,\Phi(P)\,P^{-1/2})$, we have: $\mathcal{Y}_P(\Phi) =$*
$\Gamma_{P^{1/2}}\left(\{X : \log(X) \in \Gamma_{P^{-1/2}}\left(T_P(Min(\Phi))\right) \text{ and } \det\left(\exp(\log(X)L_P)\right) = 1\}\right)$;

(b) *if $M \in GL_n$ and we set $Q := \Gamma_M(P)$, then we have*
$\mathcal{Y}_Q(\Phi) = \Gamma_M\left(\mathcal{Y}_P(\Gamma_{M^{-1}} \circ \Phi \circ \Gamma_M)\right)$.

Proof By Proposition 2.14 (b), the set $\mathcal{Y}_P(\Phi)$ is a symmetric totally geodesic Hadamard submanifold of $(\mathcal{P}_n, g)$ and we have $\mathcal{Y}_P(\Phi) = \exp_P(T_P(\mathcal{Y}_P(\Phi)))$.

Set $L_P := \log(P^{-1/2}\,\Phi(P)\,P^{-1/2})$. From Proposition 3.11 (a), we obtain that a non-zero tangent vector to the P-axis of Φ at P is $P^{1/2}L_P P^{1/2}$; hence, using Proposition 2.13 (i), we deduce that $T_P\left(\mathcal{Y}_P(\Phi)\right) = \{V \in T_P(Min(\Phi)) : g_P(V,\ P^{1/2}L_P P^{1/2}) = 0\}$. Since $g_P(V, P^{1/2}L_P P^{1/2}) = tr(P^{-1}VP^{-1/2}L_P P^{1/2}) = tr(P^{-1/2}VP^{-1/2}L_P)$, we get:

$\mathcal{Y}_P(\Phi) = \exp_P(T_P(\mathcal{Y}_P(\Phi))) = \exp_P\left(\left\{V \in T_P(Min(\Phi)) \;:\; tr(\Gamma_{P^{-1/2}}(V)\, L_P) = 0\right\}\right) = \exp_P\left(\left\{V \in T_P(Min(\Phi)) \;:\; \det\left(\exp\left(\Gamma_{P^{-1/2}}(V)\, L_P\right)\right) = 1\right\}\right) = \exp_P\left(\Gamma_{P^{1/2}}\left(\left\{W \in \Gamma_{P^{-1/2}}\left(T_P(Min(\Phi))\right) \;:\; \det\left(\exp(W L_P)\right) = 1\right\}\right)\right)$.

From Lemma 3.5 we have $\exp_P = \Gamma_{P^{1/2}} \circ \exp \circ \Gamma_{P^{-1/2}}$; therefore, we can write: $\mathcal{Y}_P(\Phi) = \Gamma_{P^{1/2}}\left(\left\{\exp(W) \;:\; W \in \Gamma_{P^{-1/2}}\left(T_P(Min(\Phi))\right) \text{ and } \det\left(\exp(W L_P)\right) = 1\right\}\right) = \Gamma_{P^{1/2}}\left(\left\{X \;:\; \log(X) \in \Gamma_{P^{-1/2}}\left(T_P(Min(\Phi))\right) \text{ and } \det\left(\exp\left(\log(X)\, L_P\right)\right) = 1\right\}\right)$. This proves part (a), while part (b) follows directly from Proposition 2.13 (ii). □

Theorem 8.5 *Let Φ be a hyperbolic isometry of $(\mathcal{P}_n, g)$ with translation length $\mathfrak{L}(\Phi)$ (so $\mathfrak{L}(\Phi) > 0$) and let $Q \in Min(\Phi)$.*

(1) *Assume that $\Phi := \Gamma_M$ or $\Phi := \Gamma_M \circ \mathfrak{d}$ (so $M \in GL_n$ is semi-simple). Consider the matrix $\Delta_M \in GL_n$ and denote the eigenvalues of M (with their multiplicities) as in Remark-*Definition 4.1. *Let $A_0 \in GL_n$ be such that $M = A_0 \Delta_M A_0^{-1}$ with $|\det(A_0)| = \sqrt{|\det(M)|}$ and denote $P := \Gamma_{A_0^{-1}}(Q)$. The Q-orthogonal space of Φ is as follows:*

(a) *when $\Phi := \Gamma_M$, we have* $\mathcal{Y}_Q(\Phi) = \Gamma_{(A_0 P^{1/2})}(\mathcal{B})$,

where $\mathcal{B}$ is the set of matrices of the form $\left(\bigoplus_{j=1}^{p} R_j\right) \oplus \left(\bigoplus_{l=1}^{q} \rho(H_l)\right) \in \left(\bigoplus_{j=1}^{p} \mathcal{P}_{r_j}\right) \oplus \left(\bigoplus_{l=1}^{q} \rho(\mathcal{H}_{m_l})\right)$ *such that* $\prod_{j=1}^{p} \left(\det(R_j)\right)^{\ln(|a_j|)} \cdot \prod_{l=1}^{q} \left(\det(H_l)\right)^{2\ln(\rho_l)} = 1$;

(b) *when $\Phi := \Gamma_M \circ \mathfrak{d}$, we have* $\mathcal{Y}_Q(\Phi) = \Gamma_{(A_0 P^{1/2})}(\widehat{\mathcal{B}})$,

where $\widehat{\mathcal{B}}$ is the set of matrices of the form $\left(\bigoplus_{j=1}^{p} R_j\right) \oplus \left(\bigoplus_{l=1}^{q} \rho(H_l)\right) \in \left(\bigoplus_{j=1}^{p} \mathcal{P}_{r_j}\right) \oplus \left(\bigoplus_{l=1}^{q} \rho(\mathcal{H}_{m_l})\right)$ *such that* $\prod_{j=1}^{p} \left(\det(R_j)\right)^{\ln(|a_j|)} \cdot \prod_{l=1}^{q} \left(\det(H_l)\right)^{2\ln(\rho_l)} = \prod_{j=1}^{p} \left(\det(R_j)\right) \cdot \prod_{l=1}^{q} \left(\det(H_l)\right)^{2} = 1$.

(2) *Assume that $\Phi := \Gamma_M \circ \mathfrak{j}$ or $\Phi := \Gamma_M \circ \mathfrak{j} \circ \mathfrak{d}$ (so $MM^{-T} \in GL_n$ is semi-simple). Consider the matrix $\widehat{\Delta}_M \in GL_n$ and denote the eigenvalues of MM^{-T} (with their multiplicities) as in Notations* 6.2. *Let $\mathfrak{S}_M$ be the set defined in Theorem* 6.5 *and $Z_M \in GL_n$ be such that $MM^{-T} = Z_M \widehat{\Delta}_M Z_M^{-1}$. Set $P := \Gamma_{Z_M^{-1}}(Q)$. The Q-orthogonal space of Φ is as follows:*

(c) *when $\Phi := \Gamma_M \circ \mathfrak{j}$, we have $\mathcal{Y}_Q(\Phi) = \Gamma_{(Z_M P^{1/2})}(\mathcal{E})$, where $\mathcal{E}$ is the set consisting of matrices* $E := E_1 \oplus \left(\bigoplus_{j=1}^{l_1} \left(E_{2;j} \oplus E_{3;j}\right)\right) \oplus E_4 \oplus \left(\bigoplus_{j=1}^{l_2} \left(E_{5;j} \oplus E_{6;j}\right)\right) \oplus \left(\bigoplus_{j=1}^{l_3} \left[\left(\rho\left(E_{7;j}\right)\right) \oplus \left(\bigoplus_{h=1}^{k_j} \left[\rho\left(E_{8;j,h}\right) \oplus \rho\left(E_{9;j,h}\right)\right]\right)\right]\right) \in \exp\left(\Gamma_{P^{-1/2}}\left(T_P(\mathfrak{S}_M)\right)\right)$ *such that*

$$\left(\prod_{j=1}^{l_1} \left[\left(\det(E_{2;j})\right)^{\ln(\lambda_j)} \cdot \left(\det(E_{3;j})\right)^{-\ln(\lambda_j)}\right]\right) \cdot \left(\prod_{j=1}^{l_2} \left[\left(\det(E_{5;j})\right)^{\ln(\mu_j)} \left(\det(E_{6;j})\right)^{-\ln(\mu_j)}\right]\right) \cdot \left(\prod_{j=1}^{l_3} \prod_{h=1}^{k_j} \left[\left(\det(E_{8;j,h})\right)^{2\ln(r_{jh})} \cdot \left(\det(E_{9;j,h})\right)^{-2\ln(r_{jh})}\right]\right) = 1;$$

(d) *when* $\Phi := \Gamma_M \circ \mathfrak{j} \circ \mathfrak{d}$, *we have* $\mathcal{Y}_Q(\Phi) = \Gamma_{(Z_M P^{1/2})}(\widehat{\mathcal{E}})$, *where* $\widehat{\mathcal{E}}$ *is the set of matrices* $E := E_1 \oplus \left(\bigoplus_{j=1}^{l_1} \left(E_{2;j} \oplus E_{3;j}\right)\right) \oplus E_4 \oplus \left(\bigoplus_{j=1}^{l_2} \left(E_{5;j} \oplus E_{6;j}\right)\right)$
$\oplus \left(\bigoplus_{j=1}^{l_3} \left[\left(\rho\left(E_{7;j}\right)\right) \oplus \left(\bigoplus_{h=1}^{k_j} \left[\rho\left(E_{8;j,h}\right) \oplus \rho\left(E_{9;j,h}\right)\right]\right)\right]\right) \in \exp\left(\Gamma_{P^{-1/2}}\left(T_P(\mathbb{R}^+ \cdot \mathfrak{S}_M)\right)\right)$
such that

$$\left(\prod_{j=1}^{l_1} \left[\left(\det(E_{2;j})\right)^{\ln(\lambda_j)} \cdot \left(\det(E_{3;j})\right)^{-\ln(\lambda_j)}\right]\right)$$
$$\left(\prod_{j=1}^{l_2} \left[\left(\det(E_{5;j})\right)^{\ln(\mu_j)} \left(\det(E_{6;j})\right)^{-\ln(\mu_j)}\right]\right)$$
$$\cdot\left(\prod_{j=1}^{l_3} \prod_{h=1}^{k_j} \left[\left(\det(E_{8;j,h})\right)^{2\ln(r_{jh})} \cdot \left(\det(E_{9;j,h})\right)^{-2\ln(r_{jh})}\right]\right) = \left(\det(E)\right)^{-\frac{2\cdot\ln(|\det(M)|)}{n}}.$$

Proof Also in this proof we will consider each of the four cases of the statement separately.

Case 1 (a). Assume $\Phi := \Gamma_M$. We have $\Phi = \Gamma_{A_0} \circ \Gamma_{\Delta_M} \circ \Gamma_{A_0^{-1}}$. Proposition 8.4 and Theorem 4.6 imply: $P \in Min(\Gamma_{\Delta_M}) = \left(\bigoplus_{j=1}^p \mathcal{P}_{r_j}\right) \oplus \left(\bigoplus_{l=1}^q \rho(\mathcal{H}_{m_l})\right)$ (so $T_P\left(Min(\Gamma_{\Delta_M})\right) = \left(\bigoplus_{j=1}^p Sym_{r_j}\right) \oplus \left(\bigoplus_{j=1}^q \rho(Herm_{m_j})\right)$), and $\mathcal{Y}_Q(\Phi) = \Gamma_{A_0}\left(\mathcal{Y}_P(\Gamma_{\Delta_M})\right) = \Gamma_{(A_0 P^{1/2})}(\mathcal{B})$, where $\mathcal{B} := \Big\{X \in \left(\bigoplus_{j=1}^p \mathcal{P}_{r_j}\right) \oplus \left(\bigoplus_{l=1}^q \rho(\mathcal{H}_{m_l})\right) : \ \det\left(\exp(\log(X)\ \log(\mathcal{S}_M^2))\right) = 1\Big\} =$
$\Big\{X \in \left(\bigoplus_{j=1}^p \mathcal{P}_{r_j}\right) \oplus \left(\bigoplus_{l=1}^q \rho(\mathcal{H}_{m_l})\right) : \ \det\left(\exp(\log(X)\ \log(\mathcal{S}_M))\right) = 1\Big\}$.

If $X := \left(\bigoplus_{j=1}^p R_j\right) \oplus \left(\bigoplus_{l=1}^q \rho(H_l)\right) \in \left(\bigoplus_{j=1}^p \mathcal{P}_{r_j}\right) \oplus \left(\bigoplus_{l=1}^q \rho(\mathcal{H}_{m_l})\right)$, it is clear that $\log(X) = \left(\bigoplus_{j=1}^p \log(R_j)\right) \oplus \left(\bigoplus_{l=1}^q \log(\rho(H_l))\right)$ and $\log(\mathcal{S}_M) = \left(\bigoplus_{j=1}^p (\ln(|a_j|) \cdot I_{r_j})\right) \oplus \left(\bigoplus_{l=1}^q (\ln(\rho_l) \cdot I_{2m_l})\right)$. Therefore we get $\log(X) \cdot \log(\mathcal{S}_M) = \left(\bigoplus_{j=1}^p (\ln(|a_j|) \cdot \log(R_j))\right) \oplus \left(\bigoplus_{l=1}^q (\ln(\rho_l) \cdot \log(\rho(H_l)))\right)$, and so $\exp\left(\log(X) \cdot \log(\mathcal{S}_M)\right) = \bigoplus_{j=1}^p (R_j)^{\ln(|a_j|)} \bigoplus_{l=1}^q (\rho(H_l))^{\ln(\rho_l)}$. From the known identities $\det(\rho(B)) = |\det(B)|^2$, $\det(B^t) = (\det(B))^t$ $(t \in \mathbb{R})$, $\det(B_1 \oplus B_2) = \det(B_1)\det(B_2)$, we get: $\det\left(\exp\left(\log(X)\log(\mathcal{S}_M)\right)\right) = \prod_{j=1}^p \left(\det(R_j)\right)^{\ln(|a_j|)} \cdot \prod_{l=1}^q \left(\det(H_l)\right)^{2\ln(\rho_l)}$, and this implies statement (1) (a).

Case 1 (b). Assume $\Phi := \Gamma_M \circ \mathfrak{d}$. From Proposition 5.5 we deduce that $\mathcal{Y}_Q\left(\Gamma_M \circ \mathfrak{d}\right) = \mathcal{Y}_Q(\Gamma_{\pi(M)}) \cap \mathcal{P}_n(|\det(M)|)$. Since $(\det(A_0))^2 \det(P) = \det(Q) = |\det(M)|$, we conclude that $\Gamma_{(A_0 P^{1/2})}\left(\mathcal{P}_n(1)\right) = \mathcal{P}_n(|\det(M)|)$. So, from part (1) (a) we obtain $\mathcal{Y}_Q\left(\Gamma_M \circ \mathfrak{d}\right) = \mathcal{Y}_Q(\Gamma_{\pi(M)}) \cap \mathcal{P}_n(|\det(M)|) = \Gamma_{(A_0 P^{1/2})}(\mathcal{B}_0) \cap \mathcal{P}_n(|\det(M)|) = \Gamma_{(A_0 P^{1/2})}\left(\mathcal{B}_0 \cap \mathcal{P}_n(1)\right)$, where $\mathcal{B}_0 \cap \mathcal{P}_n(1)$ is the set of matrices, with determinant equal to 1, of the form:

$$B := \left(\bigoplus_{j=1}^p R_j\right) \oplus \left(\bigoplus_{l=1}^q \rho(H_l)\right) \in \left(\bigoplus_{j=1}^p \mathcal{P}_{r_j}\right) \oplus \left(\bigoplus_{l=1}^q \rho(\mathcal{H}_{m_l})\right) \text{ such that}$$
$$\prod_{j=1}^p \left(\det(R_j)\right)^{\ln\left(\frac{|a_j|}{|\det(M)|^{1/n}}\right)} \cdot \prod_{l=1}^q \left(\det(H_l)\right)^{2\ln\left(\frac{\rho_l}{|\det(M)|^{1/n}}\right)} = 1, \text{ or equivalently,}$$

$$\prod_{j=1}^{p}\left(\det(R_j)\right)^{\ln(|a_j|)}\cdot\prod_{l=1}^{q}\left(\det(H_l)\right)^{2\ln(\rho_l)}$$
$$=\left(\prod_{j=1}^{p}\left(\det(R_j)\right)\cdot\prod_{l=1}^{q}\left(\det(\rho(H_l))\right)\right)^{\ln(|\det(M)|^{1/n})}$$
$=\left(\det(B)\right)^{\ln(|\det(M)|^{1/n})} = 1$. So $\mathcal{B}_0 \cap \mathcal{P}_n(1) = \widehat{\mathcal{B}}$, where $\widehat{\mathcal{B}}$ is the set of the statement. Therefore part (1) (b) is proved.

Cases 2 (c), 2 (d). In these cases, the proofs proceed with arguments analogous to those of case 1 (a), keeping in mind the Proposition 8.4 and the results of Sects. 6 and 7. □

Remark 8.6 (a) It is clear that, for every $c > 0$ and any hyperbolic isometry φ of $(\mathcal{P}_n(c), g)$, explicit expressions of all Q-axes of φ and all Q-orthogonal spaces to φ can be easily obtained from Theorems 8.3 (a), (d), 8.5 (a), (d) and Proposition 3.13 (iii), (iv).

(b) We believe that, with similar methods and predictably without major complications, it is possible to obtain results analogous to those of the present paper also for all the semi-simple isometries of $(\mathcal{H}_n, f)$. In fact, there is a result, due to Molnãr (see [16, Thm. 3]) and concerning the group of isometries of $(\mathcal{H}_n, f)$, which is very similar to our Proposition 3.3 (b): the isometries of $(\mathcal{P}_n, g)$ are of four types, while the isometries of $(\mathcal{H}_n, f)$ are of eight types, which however generalise in a very simple and intuitive way the four types of isometries of $(\mathcal{P}_n, g)$.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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