Applied Mathematics Letters 179 (2026) 109944

Contents lists available at ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

Trend to equilibrium of the Wigner function describing a quantum particle subject to decoherence

Luigi Barletti [a], *, Francesco Salvarani [b,c]

[a] Dipartimento di Matematica e Informatica "U. Dini", Università di Firenze, Italy
[b] De Vinci Higher Education, De Vinci Research Center, Paris, France
[c] Dipartimento di Matematica "F. Casorati", Università di Pavia, Italy

ARTICLE INFO

Keywords:
Wigner equation
Decoherence
Long-time asymptotics

ABSTRACT

This paper studies the long-time behavior of a spatially homogeneous Wigner equation describing a quantum particle subject to decoherence. The model extends the classical Wigner–Fokker–Planck equation by including a general decoherence term derived from microscopic collision mechanisms. An explicit representation of the equilibrium state is obtained, and a necessary and sufficient condition ensuring that the equilibrium state belongs to L^2 is established, highlighting the balance between friction and decoherence. Under suitable assumptions on the initial data, exponential convergence in L^2 toward equilibrium is rigorously proved.

1. Introduction and formal analysis

The study of decoherence processes in quantum mechanics is of fundamental importance from both foundational and applied perspectives [1–3]. In particular, the mitigation of decoherence represents a major challenge in the development of quantum computing technologies [4]. Owing to its quasi-classical structure, the Wigner formalism [5] is a widely used framework for investigating the quantum-to-classical transition and, in particular, has proven to be an effective tool for describing decoherence phenomena, both for theoretical analysis and for numerical and applied purposes [1–3,6–11]. In this paper, we consider a spatially homogeneous Wigner equation supplemented with a general decoherence term encompassing several existing models, and we show that it admits a nontrivial asymptotic state. To the best of our knowledge, such a result was previously known only for the standard Wigner–Fokker–Planck model.

In Ref. [7], the following Wigner equation was proposed to describe a quantum particle subject to decoherence:

$$\frac{\partial w}{\partial t} + p\frac{\partial w}{\partial x} = \eta\frac{\partial (pw)}{\partial p} - \frac{1}{\tau}\gamma * w, \tag{1}$$

where $w = w(x, p, t)$ is the Wigner function of the particle (of unit mass) and $\gamma * w$ is the convolution, in the variable p, between w and the function

$$\gamma(p) = \hat{\Lambda}(p) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \mathrm{e}^{-ip\xi}\Lambda(\xi)\,d\xi. \tag{2}$$

The function $\Lambda(\xi)$ depends on the details of the decoherence interaction. In particular, (1)–(2) assumes that the particle undergoes collisions, at a rate $1/\tau$, with an environment represented by a gas of particles of asymptotically small mass. According to the

* Corresponding author.
E-mail addresses: luigi.barletti@unifi.it (L. Barletti), francesco.salvarani@unipv.it (F. Salvarani).

https://doi.org/10.1016/j.aml.2026.109944
Received 27 November 2025; Received in revised form 19 March 2026; Accepted 23 March 2026
Available online 23 March 2026
0893-9659/

microscopic theory developed in Ref. [12], if χ is the wave function of the environment particle and $r(k)$ is the reflection scattering coefficient, then

$$\Lambda(\xi) = \int_{-\infty}^{+\infty} (1 - e^{2ik\xi})|r(k)|^2 |\hat{\chi}(k)|^2 \, dk. \tag{3}$$

The first term on the right-hand side of Eq. (1) represents a form of "quantum friction", with a friction coefficient $\eta > 0$ [1]. Numerical simulations [2,6,7] indicate that this term is necessary to counterbalance the effects of the last term. Indeed, when $\eta = 0$, the Wigner function vanishes as $t \to \infty$, leading to the unphysical conclusion that the coherence length tends to zero [2].

Remark. In Ref. [7], it is shown that (1) includes, as particular cases, several decoherence models previously proposed in the literature. In particular, if Λ is approximated at second order, i.e. $\Lambda(\xi) \approx \Lambda_2 \xi^2$, then Eq. (1) reduces to the Wigner–Fokker–Planck equation

$$\frac{\partial w}{\partial t} + p \frac{\partial w}{\partial x} = \eta \frac{\partial (pw)}{\partial p} + \frac{1}{\tau} \frac{\partial^2 w}{\partial p^2}, \tag{4}$$

which is a standard model of quantum decoherence [1–3]. It is well known that, as $t \to \infty$, the solutions of the Wigner–Fokker–Planck Eq. (4) converge to Gaussian equilibrium states (or decay to zero if $\eta = 0$). The aim of this paper is to show that, at least in the spatially homogeneous setting, an analogous result holds for the solutions of Eq. (1), with the equilibrium state characterized by a fat-tailed Fourier transform.

In this paper, we investigate the spatially homogeneous case, which serves as a mathematical foundation for future work addressing the spatially inhomogeneous setting. Hence, we assume $w = w(p,t)$, and the Wigner equation reduces to

$$\frac{\partial w}{\partial t} = \eta \frac{\partial (pw)}{\partial p} - \frac{1}{\tau} \gamma * w. \tag{5}$$

Let $w = \hat{u}$, i.e. $u = u(x, \xi, t)$ is the inverse Fourier transform of w with respect to the variable p. The equation for u in the space-homogeneous case is

$$\frac{\partial u}{\partial t} = -\eta \xi \frac{\partial u}{\partial \xi} - \nu \Lambda u, \tag{6}$$

where we set $\nu = \sqrt{2\pi}/\tau$. Equation (6) can be explicitly solved by the method of characteristics. Denoting by $u_0(\xi) = u(\xi, 0)$ the initial datum, the equations of characteristics are

$$\begin{cases} \dot{\xi} = \eta \xi, \\ \xi(0) = s, \end{cases} \qquad \begin{cases} \dot{z} = -\nu \Lambda(\xi) z, \\ z(0) = u_0(s), \end{cases}$$

leading to

$$\xi(s,t) = e^{\eta t} s, \qquad z(s,t) = \exp\left(-\nu \int_0^t \Lambda(e^{\eta t'} s) \, dt' \right) u_0(s)$$

and then

$$u(\xi,t) = z(s(\xi,t),t) = z(\xi e^{-\eta t}, t) = \exp\left(-\nu \int_0^t \Lambda(\xi e^{\eta (t'-t)}) \, dt' \right) u_0(\xi e^{-\eta t}).$$

Putting $k = \xi e^{\eta(t'-t)}$ we arrive at

$$u(\xi,t) = \exp\left(-\frac{\nu}{\eta} \int_{\xi e^{-\eta t}}^{\xi} \frac{\Lambda(k)}{k} \, dk \right) u_0(\xi e^{-\eta t}). \tag{7}$$

For $t \to +\infty$ we see that, formally, $u(\xi,t)$ tends to the equilibrium function

$$u_{\mathrm{eq}}(\xi) := n_0 \exp\left(-\frac{\nu}{\eta} \int_0^{\xi} \frac{\Lambda(k)}{k} \, dk \right), \tag{8}$$

where

$$n_0 = u_0(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} w(p,0) \, dp$$

is the initial number of particles (which is a globally conserved quantity). Clearly, the equilibrium Wigner function $w_{\mathrm{eq}}(p)$ is, formally, the Fourier transform of u_{eq}. Note that, in the quadratic approximation $\Lambda(\xi) \approx \Lambda_2 \xi^2$, the function (8) is Gaussian:

$$u_{\mathrm{eq}}(\xi) = n_0 \exp\left(-\frac{\nu}{\eta} \Lambda_2 \int_0^{\xi} k \, dk \right) = n_0 \exp\left(-\frac{\nu}{\eta} \Lambda_2 \frac{\xi^2}{2} \right),$$

consistently with the fact that the dynamics is given by the (space-homogeneous version of the) Wigner–Fokker–Planck Eq. (4).

2. Estimations in L^2

In the previous section, we proceeded formally. We now want to prove the convergence to the equilibrium (8) in the space $L^2(\mathbb{R})$. The choice of such space is natural for the Wigner functions [13] and is also suitable for our approach, based on the Fourier transform. In particular, the L^2-convergence in time of $u(\cdot,t)$ to u_{eq} directly implies the convergence of $w(\cdot,t)$ to w_{eq}, with the same rate.

In order to shorten the notation, from now on we put

$$\phi(k) = |r(k)|^2 |\hat{\chi}(k)|^2. \tag{9}$$

Lemma 1. *Assume that $\phi(k)$ is an even function such that*

$$\int_{-\infty}^{\infty} \big(1 + |\log(|k|)|\big)\phi(k)\, dk < \infty.$$

Then, there exist three constants, $A \geq 0$, B and $B' \geq 0$, depending on ϕ, such that

$$\max\{0, A\log|\xi| + B\} \leq \int_0^\xi \frac{\Lambda(k)}{k}\, dk \leq \max\{B', A\log|\xi| + B'\}. \tag{10}$$

Proof. By assumption, $\phi(k)$ is even and non-negative. Then, from (3),

$$\int_0^\xi \frac{\Lambda(k)}{k}\, dk = \int_0^\xi \int_{-\infty}^{\infty} \frac{1 - e^{2ikk'}}{k}\, \phi(k')\, dk' dk = \int_{-\infty}^{\infty} \int_0^\xi \frac{2\sin^2(kk')}{k}\, \phi(k')\; dk\, dk'.$$

Hence, with the change of variable $s = kk'$ (and eventually renaming k' as k to shorten the notation), we arrive at

$$\int_0^\xi \frac{\Lambda(k)}{k}\, dk = \int_{-\infty}^{\infty} I(\xi k)\, \phi(k)\, dk, \tag{11}$$

where

$$I(x) = \int_0^x \frac{2\sin^2(s)}{s}\, ds$$

is an even and positive function of $x \in \mathbb{R}$. First of all we note that two constants, $0 < a < 1$ and $b_1 > 1$, exist such that

$$2ax \leq \frac{2\sin^2(s)}{s} \leq b_1, \qquad \text{for } 0 \leq x \leq 1.$$

Let Ci be the "cosine integral" function, denoted as follows for all $x \in \mathbb{R}_+$:

$$\mathrm{Ci}(x) = -\int_x^{+\infty} \frac{\cos(t)}{t}\, dt.$$

Then, for $x > 1$, we can write

$$\int_1^x \frac{2\sin^2(s)}{s}\, ds = \int_1^x \frac{1 - \cos(2s)}{s}\, ds = \log(x) + \mathrm{Ci}(2) - \mathrm{Ci}(2x).$$

Since $\mathrm{Ci}(2) - \mathrm{Ci}(2x)$ is positive and bounded for $x > 1$, a second constant $b_2 > 0$ exists such that

$$\log(x) \leq \int_1^x \frac{2\sin^2(s)}{s}\, ds \leq \log(x) + b_2 \qquad \text{for } x > 1.$$

Taking also into account that $I(-x) = I(x)$, we conclude that

$$\begin{cases} ax^2, & \text{for } |x| \leq 1, \\ \log(|x|) + a, & \text{for } |x| > 1. \end{cases} \leq I(x) \leq \begin{cases} b|x|, & \text{for } |x| \leq 1, \\ \log(|x|) + b, & \text{for } |x| > 1, \end{cases} \tag{12}$$

where $b = \max\{b_1, b_2\}$. For $x = \xi k$, starting with the estimate from below, we can simply write

$$I(\xi k) \geq \begin{cases} 0, & \text{for } |\xi k| \leq 1, \\ \log(|\xi k|), & \text{for } |\xi k| > 1. \end{cases} \geq \log(|\xi k|), \quad \text{for } \xi, k \in \mathbb{R}.$$

Hence, from (11),

$$\int_0^\xi \frac{\Lambda(k)}{k}\, dk \geq \int_{-\infty}^{+\infty} \log(|\xi k|)\, \phi(k)\, dk \tag{13}$$

so that, taking also into account that the integral is non-negative, the first inequality in (10) holds with

$$A = \int_{-\infty}^{+\infty} \phi(k)\, dk, \qquad B = \int_{-\infty}^{+\infty} \log(|k|)\, \phi(k)\, dk. \tag{14}$$

Passing to the estimate from above, we can write

$$I(\xi k) \leq \begin{cases} b|\xi k|, & \text{for } |\xi k| \leq 1, \\ b + \log(|\xi k|), & \text{for } |\xi k| > 1. \end{cases} \leq \begin{cases} b, & \text{for } |\xi|, |k| \leq 1, \\ b + \log(|k|), & \text{for } |\xi| \leq 1,\ |k| > 1, \\ b + \log(|\xi|), & \text{for } |\xi| > 1,\ |k| \leq 1, \\ b + \log(|\xi k|), & \text{for } |\xi|, |k| > 1, \end{cases}$$

and therefore, from (11),

$$\int_0^\xi \frac{\Lambda(k)}{k}\, dk \leq \int_{|k|<1} b\, \phi(k)\, dk + \int_{|k|>1} (b + \log(|k|))\, \phi(k)\, dk,$$

for $|\xi| \leq 1$, and

$$\int_0^\xi \frac{\Lambda(k)}{k}\, dk \leq \int_{|k|<1} (b + \log(|\xi|))\, \phi(k)\, dk + \int_{|k|>1} (b + \log(|\xi k|))\, \phi(k)\, dk,$$

for $|\xi| > 1$. Consequently, the second inequality in (10) holds with

$$B' = A + b \int_{|k|>1} \log(|k|)\, \phi(k)\, dk \tag{15}$$

(note that $B' > B$). □

As an immediate corollary of the preceding Lemma we obtain a necessary and sufficient condition for the equilibrium Wigner function to belong to $L^2(\mathbb{R})$.

Theorem 1. *In the assumptions of Lemma 1, $u_{\rm eq}$ (and therefore $w_{\rm eq}$) belongs to $L^2(\mathbb{R})$ if and only if*

$$\eta < 2\nu A, \tag{16}$$

where A is given by (14).

Proof. Recalling (8), we have that $u_{\rm eq}(\xi)$ is even and positive. From Lemma 1 we have that $\xi_0 > 0$ exists such that $u_{\rm eq}$ is bounded for $|\xi| \leq \xi_0$ and

$$n_0\, \mathrm{e}^{-\frac{\nu B'}{\eta}}\, |\xi|^{-\frac{\nu A}{\eta}} \leq u_{\rm eq}(\xi) \leq n_0\, \mathrm{e}^{-\frac{\nu B}{\eta}}\, |\xi|^{-\frac{\nu A}{\eta}}, \qquad \text{for } |\xi| > \xi_0.$$

Hence, $u_{\rm eq} \in L^2(\mathbb{R})$ if and only condition (16) holds. □

It is worth pointing out the behavior of the system in the limiting regimes $\eta \to 0$ and $\nu \to 0$. In the former case, one obtains $u_{\rm eq}(\xi) = 0$, corresponding to the aforementioned unphysical situation in which the coherence length tends to zero [2,7]. In the latter case, one finds $u_{\rm eq}(\xi) = n_0$ ($w_{\rm eq}(p) = \sqrt{2\pi} n_0 \delta(p)$), which is clearly non-integrable and corresponds to a situation in which friction slows down the particle so that the momentum concentrates at 0. The two limiting regimes show that inequality (16) provides a natural setting for our problem, in which friction is necessary but it is not the dominant phenomenon.

3. Trend to equilibrium

Let us now turn to the analysis of the trend to equilibrium. We shall prove that, for a suitable class of initial data, the solution $u(\xi, t)$ to Eq. (6) converges in $L^2(\mathbb{R})$ to $u_{\rm eq}$, with exponential rate.

The explicit expression (7) defines the evolution semigroup of Eq. (6):

$$\left(T(t)u_0\right)(\xi) = \exp\left(-\frac{\nu}{\eta} \int_{\xi \mathrm{e}^{-\eta t}}^{\xi} \frac{\Lambda(k)}{k}\, dk\right) u_0(\xi \mathrm{e}^{-\eta t}), \qquad t \geq 0.$$

It is easily seen that the semigroup acts on $L^2(\mathbb{R})$, with

$$\|T(t)u_0\|_2^2 \leq \mathrm{e}^{\eta t} \|u_0\|_2^2$$

for all $u_0 \in L^2(\mathbb{R})$, where $\|\cdot\|_2$ denotes the L^2-norm. The latter estimate must now be refined. Recalling Eq. (11), we write

$$\|T(t)u_0\|_2^2 = \int_{-\infty}^{+\infty} \exp\left(-\frac{2\nu}{\eta} \int_{-\infty}^{+\infty} \left[I(\xi k) - I(\xi k \mathrm{e}^{-\eta t})\right] \phi(k)\, dk\right) |u_0(\xi \mathrm{e}^{-\eta t})|^2 d\xi$$

$$= \mathrm{e}^{\eta t} \int_{-\infty}^{+\infty} \exp\left(-\frac{2\nu}{\eta} \int_{-\infty}^{+\infty} \left[I(\xi k \mathrm{e}^{\eta t}) - I(\xi k)\right] \phi(k)\, dk\right) |u_0(\xi)|^2 d\xi.$$

From (12), we have:

$$I(x) - I(y) \geq \begin{cases} ax^2 - b|y| \geq ax^2 - b, & \text{for } |y| < |x| < 1, \\ \log(|x|) + a - b|y| \geq \log(|x|) - b, & \text{for } |y| < 1 < |x|, \\ \log(|x|/|y|) + a - b \geq \log(|x|/|y|) - b, & \text{for } 1 < |y| < |x|. \end{cases}$$

Hence, using $x = \xi k e^{\eta t}$ and $y = \xi k$, we can write

$$\int_{-\infty}^{+\infty} \left[I(\xi k e^{\eta t}) - I(\xi k) \right] \phi(k) dk \geq \int_{|\xi k| \leq e^{-\eta t}} (a\xi^2 k^2 e^{2\eta t} - b)\phi(k) dk$$

$$+ \int_{e^{-\eta t} < |\xi k| < 1} (\log(|\xi k|) - b)\phi(k) dk \quad + \int_{|\xi k| \geq e^{-\eta t}} (\eta t - b)\phi(k) dk$$

$$= -bA + a e^{2\eta t} \xi^2 \int_{|\xi k| \leq e^{-\eta t}} k^2 \phi(k) dk + \int_{e^{-\eta t} < |\xi k| < 1} \log(|\xi k|)\, \phi(k) dk + \eta t \int_{|\xi k| \geq e^{-\eta t}} \phi(k) dk$$

(where A is defined in (14)). The third term can be estimated independently on t as follows:

$$\int_{e^{-\eta t} < |\xi k| < 1} \log(|\xi k|)\, \phi(k)\, dk \geq \int_{|\xi k| < 1} \log(|\xi k|)\, \phi(k)\, dk \geq A \min\{\log(|\xi|), 0\} + \int_{|k| \leq 1} \log(|k|)\, \phi(k)\, dk.$$

Consequently, we obtain

$$\|T(t)u_0\|_2^2 \leq C \int_{-\infty}^{+\infty} \exp\left(\eta t - \frac{2\nu}{\eta} F(\xi, t) \right) \max\{|\xi|^{-\frac{2\nu A}{\eta}}, 1\}\, |u_0(\xi)|^2\, d\xi, \tag{17}$$

where

$$F(\xi, t) = a\, e^{2\eta t}\, \xi^2 \int_{|\xi k| \leq e^{-\eta t}} k^2 \phi(k)\, dk + \eta t \int_{|\xi k| \geq e^{-\eta t}} \phi(k)\, dk \tag{18}$$

and

$$C = \exp\left(\frac{2\nu b A}{\eta} - \frac{2\nu}{\eta} \int_{|k| < 1} \log(|k|)\, \phi(k)\, dk \right).$$

Note that all integrals in the previous computations are finite because of the assumptions of Lemma 1. We have furthermore that

$$F(0, t) = 0, \qquad \lim_{|\xi| \to +\infty} F(\xi, t) = \eta A t$$

and (by direct inspection) $\partial F / \partial |\xi| > 0$ for $t > 1/\eta$. This means that, for t large enough, the exponential contained in the integral in Eq. (17) decreases from $e^{\eta t}$ to $e^{(\eta - 2\nu A)t}$ as $|\xi|$ goes from 0 to $+\infty$, where (from Theorem 1) $\eta - 2\nu A < 0$ if u_{eq} is to be in L^2. Hence, inequality (17) clearly shows that the long-term behavior of the dynamics strongly depends on the initial datum $u_0(\xi)$, which weights exponentially growing and decaying "modes".

We are now in position to state and prove the main result of this paper.

Theorem 2. *Let us assume that $\phi(k)$ satisfies the assumptions of Lemma 1 and that $\eta < 2\nu A$. Moreover, let us assume that, in a neighborhood of $\xi = 0$, the initial datum $u_0 \in L^2(\mathbb{R})$ is continuous and there exists $\epsilon > 0$ such that*

$$\xi^{-(\frac{\nu A}{\eta} + \epsilon)} \left(u_0(\xi) - u_{eq}(\xi) \right) < \infty. \tag{19}$$

Then, a time $t_0 > 0$ and a constant A', with $0 \leq A' \leq A$, exist such that, if $\eta < 2\nu A'$, then

$$\|T(t)u_0 - u_{eq}\|_2 \leq M\, e^{-\alpha t} \tag{20}$$

for all $t > t_0$, with $\alpha > 0$ and $M > 0$.

Proof. Note that assuming $\eta \leq 2\nu A'$, with $A' \leq A$, implies that $u_{eq} \in L^2(\mathbb{R})$, by Theorem 1. Let $h(\xi) = u_0(\xi) - u_{eq}(\xi)$. Since $T(t)u_{eq} = u_{eq}$, from inequality (17) we have

$$\|T(t)u_0 - u_{eq}\|_2^2 = \|T(t)h\|_2^2 \leq C \int_{-\infty}^{+\infty} e^{\eta t - \frac{2\nu}{\eta} F(\xi,t)} \max\{|\xi|^{-\frac{2\nu A}{\eta}}, 1\}\, |h(\xi)|^2\, d\xi.$$

As already observed, $F(\xi, t)$ is an increasing function of $|\xi|$ for t sufficiently large. Hence, for $|\xi| > e^{-\eta t}$ we can write

$$F(\xi, t) \geq F(e^{-\eta t}, t) = a \int_{|k| \leq 1} k^2 \phi(k)\, dk + \eta t \int_{|k| > 1} \phi(k)\, dk$$

and, therefore,

$$C \int_{|\xi| > e^{-\eta t}} e^{\eta t - \frac{2\nu}{\eta} F(\xi,t)} \max\{|\xi|^{-\frac{2\nu A}{\eta}}, 1\}\, |h(\xi)|^2\, d\xi \leq$$

$$C' e^{(\eta - 2\nu A')t} \left(\int_{e^{-\eta t} < |\xi| < 1} \left|\xi^{-\frac{\nu A}{\eta}} h(\xi)\right|^2 d\xi + \int_{|\xi| > 1} |h(\xi)|^2\, d\xi \right)$$

where

$$A' = \int_{|k| > 1} \phi(k)\, dk \leq A, \qquad C' = C \exp\left(-\frac{2\nu a}{\eta} \int_{|k| \leq 1} k^2 \phi(k)\, dk \right).$$

Because of our assumptions on u_0, $\xi^{-\frac{\nu A}{\eta}} h(\xi)$ is a bounded function between 0 and 1, and $h \in L^2(\mathbb{R})$. Then, the two integrals in brackets are bounded. Note also that $\eta - 2\nu A' < 0$, by assumption.
Turning to $|\xi| \le \mathrm{e}^{-\eta t}$, from the mean value theorem, a ξ^* with $0 < |\xi_*| < \mathrm{e}^{-\eta t}$ exists such that

$$\int_{|\xi|\le \mathrm{e}^{-\eta t}} \mathrm{e}^{\eta t - \frac{2\nu}{\eta} F(\xi,t)} \, |\xi^{-\frac{2\nu A}{\eta}} h(\xi)|^2 \, d\xi = 2\mathrm{e}^{-\frac{2\nu}{\eta} F(\xi_*,t)} |\xi_*^{-\frac{\nu A}{\eta}} h(\xi_*)|^2.$$

Since $\xi^{-(\frac{\nu A}{\eta}+\epsilon)} h(\xi)$ is bounded and continuous in a neighborhood of $\xi = 0$, then, for t large enough,

$$|\xi_*^{-\frac{\nu A}{\eta}} h(\xi_*)|^2 \le c\, |\xi_*|^{2\epsilon} \le C''\, \mathrm{e}^{-2\epsilon\eta t}$$

for some constant $C'' > 0$ depending on u_0.

Then, we can conclude that, for t larger than a suitable t_0, inequality (20) holds, with

$$\alpha^2 = \min\{2\nu A' - \eta, 2\epsilon\eta\} \tag{21}$$

and $M = \max\{C', C''\}$. □

Remark. Condition (19) imposes further restrictions on ν, η and $\phi(k)$. In fact, from (8) we have $u_0(0) = u_{\mathrm{eq}}(0) = n_0$ and $u'_{\mathrm{eq}}(0) = 0$. Hence, $u_0(\xi) - u_{\mathrm{eq}}(\xi) = u'_0(0)\xi + o(\xi)$ and (19) require $\eta > \nu A$, which is compatible with the condition $\eta < 2\nu A'$ only if $A' > A/2$. If we assume $u'_0(0) = 0$ (which means that the first-order moment of the initial Wigner function vanishes), then the previous conditions relax to $\eta > \nu A/2$ and $A' > A/4$. More in general, if we assume that $u_0^{(j)}(0) = u_{\mathrm{eq}}^{(j)}(0)$ for $j = 1, 2, \ldots, m$ (which means that w_0 and w_{eq} have the same moments up to order m), then we have to impose the conditions

$$\eta > \frac{\nu A}{m+1}, \qquad A' > \frac{A}{2(m+1)}.$$

Acknowledgments

This paper has been partially supported by INdAM, GNFM group. This work originated from discussions that took place during L.B.'s visit to the De Vinci Research Center, whose support is gratefully acknowledged. The authors thank the referees for their useful comments and suggestions.

Data availability

No data was used for the research described in the article.

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