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ORIGINAL RESEARCH

Remark on a Regularity Criterion in Terms of Pressure for the 3*D* Inviscid Boussinesq–Voigt Equations

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Abstract
We consider the three-dimensional inviscid Boussinesq–Voigt system which is a regularization model for the inviscid Boussinesq equations. We prove a regularity criterion for the weak solutions (in particular, for the time-derivative of the velocity), in L^p-spaces, involving first derivatives of the pressure.

Keywords Regularity criterion · Boussinesq equations · Navier–Stokes equations · Turbulent flows · Large Eddy Simulation (LES)

Mathematics Subject Classification 35Q35 · 35Q30 · 35B65 · 76F65 · 76D03 · 76F20

Introduction

Consider the following three-dimensional inviscid Boussinesq–Voigt system (see, e.g., [25]), on the torus $\mathbb{T}^3$, i.e.

$$\begin{aligned}
&\partial_t(I-\alpha^2\Delta)u + (u\cdot\nabla)u + \nabla\pi = \theta e_3,\\
&\partial_t\theta + (u\cdot\nabla)\theta - \kappa\Delta\theta = 0,\\
&\nabla\cdot u = 0,\\
&u(x,0)=u_0(x), \quad \theta(x,0)=\theta_0(x),
\end{aligned} \tag{1}$$

where $u = u(x,t)$, $\theta = \theta(x,t)$ and $\pi = \pi(x,t)$ are respectively the velocity, the temperature and the pressure of the fluid, and they depend on spatial position $x \in \mathbb{T}^3$ and on time $t > 0$. Also, $\kappa > 0$ is the diffusion coefficient, and $e_3 = (0,0,1)^T$. The above system is supplemented by the initial conditions (u_0, θ_0), with $\nabla\cdot u_0 = 0$.

For it, after introducing a suitable class of weak solutions for (1), we provide a control for the time-derivative of the velocity $\partial_t u$ in terms of the partial derivative of the pressure $\partial_3\pi$. Our analysis moves from results presented in [28, 31, 33].

System (1) consists of the non-dimensional Euler-Boussinesq equations (see [9, 24, 25]) with the additional term $-\alpha^2\Delta\partial_t u$ in the momentum equation of the fluid, where $\alpha > 0$ is

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a length-scale parameter. This is the so-called Voigt term, and in general we refer to equations with this extra piece as Voigt-regularized models (see, e.g., [1, 9, 24, 25] and references therein).

In recent years various regularization models have been proposed in the context of numerical simulations of turbulent flows and in the particular case of incompressible turbulent fluids (see, e.g., [11, 12, 14, 19, 20]), and among these we find the so-called α-models. These models are based on a smoothing procedure obtained through the application of the inverse of the Helmholtz operator A_α, which is defined as follows

$$A_\alpha := I - \alpha^2 \Delta, \text{ with } \alpha > 0. \tag{2}$$

Loosely speaking, the effect of applying the inverse of the Helmholtz operator to a vector field v (thus obtaining $\overline{v} := (A_\alpha)^{-1} v$), is to gain two additional space-derivatives for the considered quantity.

There is a wide range of α-approximation models for the Navier-Stokes equations and for other related systems in fluid dynamics (see, for instance, [1, 9, 12, 17, 23], see also [6]). Further details can be found in [8, 9].

From (1), when $\alpha = 0$, we formally recover the inviscid Boussinesq equations, which are often used for the determination of the coupled flow and temperature field in natural convection (see, e.g., [22]). These equations are also used, for instance, as a mathematical scheme to describe Newtonian fluids whenever salinity concentration or density stratification –according to the meaning of θ– play a significant role (see, e.g., [3–5, 7] for some recent papers on this subject). Moreover, they are extensively employed in studying oceanographic and atmospheric phenomena (see [26, 27, 30]). The problem related to the uniqueness and global regularity of the weak solutions of the 3D Boussinesq equations, strictly related to the one of the 3D Navier-Stokes equations, is a relevant open issue. These facts highlight the importance of this system of equations in a number of different contexts.

The main result of the paper, i.e. Theorem 2, provides the mentioned regularity criterion for $\partial_t u$ in L^p-norm, $4 \leq p < 9$, in terms of $\partial_3 \pi$ in L^λ-norm, $\lambda = \lambda(p)$ (see (6) below), within reasonably low regularity requirements on the initial data (u_0, θ_0).

We also mention that, recently, regularity criteria of the type considered in the present paper have been successfully used to study (in Sobolev and Besov spaces) 3D Navier-Stokes equations [10, 32], 2D and 3D viscous MHD equations [15, 16], and the 3D Ericksen-Leslie system [34].

Preliminaries

Denote by $x := (x_1, x_2, x_3)$ a generic point in $\mathbb{R}^3$ and L a number in $\mathbb{R}^+_* := (0, +\infty)$. We consider the case of periodic boundary conditions and the equations in (1) are set in a 3D torus $\mathbb{T}^3$ of size L: We set $\mathcal{T}_3 := 2\pi\mathbb{Z}^3/L$, and $\mathbb{T}^3$ as the torus given by the quotient $\mathbb{T}^3 = \mathbb{R}^3/\mathcal{T}_3$.

We use classical Lebesgue spaces $\mathrm{L}^p := \mathrm{L}^p(\mathbb{T}^3)$, $p \geq 1$, Sobolev spaces $W^{k,p} := W^{k,p}(\mathbb{T}^3)$, k non-negative integer, p as before, and $H^k := W^{k,2}$ with mean value equal to zero (essentially, for having at disposal the Poincaré inequality). We denote by $\|\cdot\|$ the L^2-norm and, similarly, $\|\cdot\|_p := \|\cdot\|_{\mathrm{L}^p}$ denotes the L^p-norm. Moreover, given X a real Banach space with norm $\|\cdot\|_X$, we will use the customary Bochner spaces $\mathrm{L}^q(0,T;X)$, $q \in \mathbb{N}$, with norm denoted by $\|\cdot\|_{\mathrm{L}^q(0,T;X)}$, and we will also make use of the spaces $C([0, T]; X)$ and $C^1([0,T];X)$.

Because we deal with divergence-free velocity field, we also define for a general exponent $s \geq 0$, the following spaces

$$H_s := \left\{ w : \mathbb{T}^3 \to \mathbb{R}^3 \mid w \in (H^s)^3,\ \nabla \cdot w = 0,\ \int_{\mathbb{T}^3} w\, dx = 0 \right\},$$

and $H := H_0$. Let us also recall the notation for dual spaces, i.e. $(H^s)' = H^{-s}$ and $(H_s)' = H_{-s}$.

For $v \in H^s$, we can expand such a field in Fourier series as $v(x) = \sum_{k \in \mathcal{T}_3^*} \widehat{w}_k e^{ik\cdot x}$ where $k \in \mathcal{T}_3^*$ is the wave number, and the Fourier coefficients are defined by $\widehat{w}_k = 1/|\mathbb{T}^3| \int_{\mathbb{T}^3} w(x) e^{-ik\cdot x} dx$. The magnitude of k is given by $|k| := \left((k_1)^2 + (k_2)^2 + (k_3)^2\right)^{\frac{1}{2}}$. We mention only the scalar case for $v \in H^s$, the notation translate accordingly in the vector case $(v \in H_s)$. The H^s norms are defined by $\|v\|_{s,2}^2 := \sum_{k \in \mathcal{T}_3^*} |k|^{2s} |\widehat{v}_k|^2$ where $\|v\|_{0,2}^2 := \|v\|^2$. The inner products associated to these norms are $(w, v)_{H^s} := \sum_{k \in \mathcal{T}_3^*} |k|^{2s} \widehat{w}_k \cdot \overline{\widehat{v}_k}$ where $\overline{\widehat{v}_k}$ denotes the complex conjugate of $\widehat{v}_k$. To have real valued vector fields, we impose $\widehat{w}_{-k} = \overline{\widehat{w}_k}$ for any $k \in \mathcal{T}_3^*$ and for any field denoted by w.

Observe that, given $w = \sum_{k \in \mathcal{T}_3^*} \widehat{w}_k e^{ik\cdot x} \in H^s$, the inverse of the Helmholtz operator $G_\alpha = (A_\alpha)^{-1}$ can be expressed, in terms of a Fourier series, as follows

$$G_\alpha(w) = \sum_{k \in \mathcal{T}_3^*} \frac{1}{1 + \alpha^2 |k|^2} \widehat{w}_k e^{ik\cdot x}. \tag{3}$$

Loosely speaking, the action of G_α makes to gain two derivatives to w (for more details see, e.g., [2, 5]).

In the sequel, we will use the same notation for scalar and vector-valued functions, as well as for related spaces, since no ambiguity occurs. Also, we will denote by c, or by C, a generic constant which may change from line to line. As a further matter of notation, for the remainder of the paper we will denote $\partial_t u$ by u_t, and $\partial_t \theta$ by θ_t.

Weak Solutions

Let us recall the following definition (see [25])

Definition 1 Let $u_0 \in H_1$, $\theta_0 \in \mathrm{L}^2$. For a given $T > 0$, we say that (u, θ) is a weak solution of the problem (1), on the interval [0, T], if it satisfies Eq. $(1)_1$ in the sense of $\mathrm{L}^2(0, T; H_1)$, and Eq. $(1)_2$ in the sense of $\mathrm{L}^2(0, T; H^{-1})$. Moreover, $u \in C^1([0, T]; H_1)$, and $\theta \in \mathrm{L}^\infty(0, T; \mathrm{L}^2) \cap \mathrm{L}^2(0, T; H^1)$, with $\theta_t \in \mathrm{L}^2(0, T; H^{-1})$.

The following result provides an existence criterion for the considered system.

Theorem 1 (Theorem 6.3 [25]) *Let $u_0 \in H_1$, $\theta_0 \in \mathrm{L}^2$, and let $T > 0$ be arbitrary. Then, there exists a unique solution to the system* (1) *in the sense of the Definition* 1. *Furthermore, if $\theta_0 \in \mathrm{L}^p$ for some $p \geq 2$, then $\theta \in \mathrm{L}^\infty(0, T; \mathrm{L}^p)$.*

In reference [25], this result is proved starting from initial data $u_0 \in H_3$. However, an analogous existence result can be obtained starting from $u_0 \in H_i$, $i \geq 1$ (see [9], see also [1, Theorem 2.1] for the case of the 3D Euler–Voigt equations).

A Priori Estimates

In the following we proceed formally and establish a priori estimates. The procedure actually goes through the use of a suitable Galerkin approximation scheme (see, e.g., [5]), which however we do not report here for the sake of conciseness.

Multiplying the Eq. $(1)_1$ by u and the equation $(1)_2$ by θ, and integrating over $\mathbb{T}^3$, we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\|u\|^2+\alpha^2\|\nabla u\|^2\right)=\left(\theta e_3,u\right),$$

and

$$\frac{1}{2}\frac{d}{dt}\|\theta\|^2+\kappa\|\nabla\theta\|^2=0.$$

From the second equation it follows that $\|\theta(t)\|^2\le\|\theta_0\|^2$ for $0\le t\le T$. Then, for the first equation, we have

$$\frac{1}{2}\frac{d}{dt}\left(\|u\|^2+\alpha^2\|\nabla u\|^2\right)\le\frac{1}{2}\|\theta_0\|^2+\frac{1}{2}\|u\|^2,$$

and the fact that $u\in \mathrm{L}^\infty(0,T;H_1)$ follows by a direct application of Gronwall's lemma.

Now, assuming $\theta_0\in\mathrm{L}^p$, $p\ge 2$, and taking product of $(1)_2$ against $|\theta|^{p-2}\theta$ and integrating over $\mathbb{T}^3$, we obtain

$$\frac{1}{p}\frac{d}{dt}\|\theta(t)\|_p^p+\kappa(p-1)\int_{\mathbb{T}^3}|\nabla\theta(t,x)|^2|\theta(t,x)|^{p-2}dx=0, \tag{4}$$

where we used the identity

$$\int_{\mathbb{T}^3}(u\cdot\nabla)\theta\cdot|\theta|^{p-2}\theta dx=0.$$

Therefore, we have the following bound (which is still valid when $\kappa=0$) for θ, i.e.

$$\|\theta(t)\|_p^p\le\left\|\theta_0\right\|_p^p, \tag{5}$$

with $0\le t\le T$, and $T>0$.

Regularity Criterion for the Inviscid Boussinesq–Voigt Model

This section is devoted to the proof of the following regularity result

Theorem 2 *Let $\left(u_0,\theta_0\right)\in H_1\times\mathrm{L}^p$, with $4\le p<9$. Given $T>0$, let (u,θ) be the weak solution of (1). If the temperature θ, and the partial derivative of the pressure $\partial_3\pi$ verify the condition*

$$\int_0^T \frac{\|\partial_3\pi(s)\|_\lambda^{\frac{p}{3}}}{1+\ln(1+\|\theta(s)\|_p)}ds < +\infty, \quad \text{with } \lambda = \frac{3p}{9-p}, \tag{6}$$

then the solution (u, θ) *of the problem* (1) *is such that*

$$\sup_{0\le t\le T} \|\theta(t)\|_p^p + \int_0^T \|u_t(s)\|_p^p ds < +\infty, \tag{7}$$

and in particular $u_t \in C([0,T];H_1) \cap \mathrm{L}^p(0,T;\mathrm{L}^p)$. *Moreover, for any* $\varepsilon \in (0,1)$, *it holds that* $u_t \in \mathrm{L}^p(0,T;W^{\frac{2}{p}-\varepsilon,p})$.

Remark 1 Since we already have at disposal the global regularity stated in Theorem 1 (in particular $u \in C^1([0,T];H_1)$), the above result goes in a quite different direction than that of other papers making use of similar logarithmic criteria. In fact, here we show some further properties for u_t not coming directly from the definition of weak solution, under the assumptions of initial data $(u_0, \theta_0) \in H_1 \times \mathrm{L}^p$, $4 \le p < 9$. When $p > 6$, the regularity of $u_t \in C([0,T];H_1)$ is, in principle, no longer sufficient to control the time-integral term in the inequality (7), and it does not even produce the last inclusion.

In proving Theorem 2, we will use the following estimate (see, e.g., [13, 18]):

$$\|f\|_\gamma \le C\|\partial_1 f\|_\sigma^{\frac{1}{3}}\|\partial_2 f\|_\sigma^{\frac{1}{3}}\|\partial_3 f\|_\lambda^{\frac{1}{3}}, \quad 1 \le \gamma < +\infty, \tag{8}$$

where the parameters σ, λ and γ satisfy the relations

$$1 \le \sigma, \lambda < +\infty, \quad \frac{1}{\lambda} + \frac{2}{\sigma} > 1, \quad \text{and} \quad \frac{1}{\lambda} + \frac{2}{\sigma} = 1 + \frac{3}{\gamma}. \tag{9}$$

Let us also recall, for $w \in H_p$, $p \ge 2$, the Poincaré inequality

$$\|w\|_p \le C\|\nabla w\|_p.$$

Proof of Theorem 2

Multiplying Eq. $(1)_1$ by $|u_t|^{p-2}u_t$ and integrating over $\mathbb{T}^3$, after an integration by parts, we have

$$\begin{aligned}
&\|u_t\|_p^p + \alpha^2\int_{\mathbb{T}^3}|\nabla u_t|^2|u_t|^{p-2}dx + \alpha^2\frac{p-2}{4}\int_{\mathbb{T}^3}\left|\nabla|u_t|^2\right|^2|u_t|^{p-4}dx\\
&\quad = \int_{\mathbb{T}^3}\theta e_3\cdot|u_t|^{p-2}u_t dx + \int_{\mathbb{T}^3}\pi\nabla\cdot(|u_t|^{p-2}u_t) - \int_{\mathbb{T}^3}(u\cdot\nabla)u\cdot|u_t|^{p-2}u_t dx.
\end{aligned}$$

By using Hölder's and Young's inequalities for the first term on the right-hand side of the above relation, and recalling the divergence-free nature of u_t, we get

$$\begin{aligned}
&\|u_t\|_p^p + \alpha^2\int_{\mathbb{T}^3}|\nabla u_t|^2|u_t|^{p-2}dx + \alpha^2\frac{p-2}{4}\int_{\mathbb{T}^3}\left|\nabla|u_t|^2\right|^2|u_t|^{p-4}dx\\
&\quad \le C_\varepsilon\|\theta\|_p^p + \varepsilon\|u_t\|_p^p + \left|\int_{\mathbb{T}^3}\pi u_t\cdot\nabla|u_t|^{p-2}dx\right| + \left|\int_{\mathbb{T}^3}(u\cdot\nabla)u\cdot|u_t|^{p-2}u_t dx\right|\\
&\quad = C_\varepsilon\|\theta\|_p^p + \varepsilon\|u_t\|_p^p + I_1 + I_2.
\end{aligned} \tag{10}$$

Here, we require $p \geq 4$. Now, we have to estimate the two integral terms in the right-hand side of (10). Let us start with I_2. We have that

$$
\begin{aligned}
I_2 &= \left| \int_{\mathbb{T}^3} (u \cdot \nabla)\big(|u_t|^{p-2} u_t\big) \cdot u dx \right| \\
&\leq \int_{\mathbb{T}^3} |u|^2 |\nabla u_t| |u_t|^{p-2} dx + \frac{p-2}{2} \int_{\mathbb{T}^3} |u|^2 \big|\nabla |u_t|^2\big| |u_t|^{p-3} dx \\
&\leq (p-1) \int_{\mathbb{T}^3} |u|^2 |\nabla u_t| |u_t|^{p-2} dx \\
&= (p-1) \int_{\mathbb{T}^3} |u|^2 \Big(|\nabla u_t| |u_t|^{\frac{p-2}{2}} \Big) |u_t|^{\frac{p-2}{2}} dx \\
&\leq C \|u\|_6^2 \bigg(\int_{\mathbb{T}^3} \Big(|\nabla u_t| |u_t|^{\frac{p-2}{2}} \Big)^{\frac{3}{2}} \Big(|u_t|^{\frac{p-2}{2}} \Big)^{\frac{3}{2}} dx \bigg)^{\frac{2}{3}} \\
&\leq C \|u\|_6^2 \bigg(\int_{\mathbb{T}^3} |\nabla u_t|^2 |u_t|^{p-2} dx \bigg)^{\frac{1}{2}} \| |u_t|^{\frac{p-2}{2}} \|_6,
\end{aligned}
$$

where we used twice Hölder's inequality, first with exponents $p = 3$ and $q = 3/2$, and subsequently with exponents $p = 4/3$ and $q = 4$. In particular, the following inequality emerges

$$
I_2 \leq C \|u\|_6^2 \bigg(\int_{\mathbb{T}^3} |\nabla u_t|^2 |u_t|^{p-2} dx \bigg)^{\frac{1}{2}} \| |u_t|^{\frac{p}{2}} \|_6^{\frac{p-2}{p}}. \tag{11}
$$

Exploiting the embedding $H_1 \hookrightarrow \mathrm{L}^6$ along with Poincaré's inequality, the last factor in the above formula can be estimated as follows

$$
\begin{aligned}
\| |u_t|^{\frac{p}{2}} \|_6^{\frac{p-2}{p}} &\leq C \| \nabla |u_t|^{\frac{p}{2}} \|^{\frac{p-2}{p}} \\
&\leq C \bigg(\int_{\mathbb{T}^3} \Big(\frac{p}{4} |\nabla |u_t|^2| |u_t|^{\frac{p-4}{2}} \Big)^2 dx \bigg)^{\frac{p-2}{2p}} \\
&\leq C \bigg(\int_{\mathbb{T}^3} |\nabla u_t|^2 |u_t|^{p-2} dx \bigg)^{\frac{p-2}{2p}}.
\end{aligned}
$$

By plugging such a relation into (11) and using Hölder's inequality, we reach

$$
\begin{aligned}
I_2 \leq & C \|u\|_6^2 \bigg(\int_{\mathbb{T}^3} |\nabla u_t|^2 |u_t|^{p-2} \bigg)^{\frac{p-1}{p}} \\
\leq & C_\varepsilon \|u\|_6^{2p} + \varepsilon \int_{\mathbb{T}^3} |\nabla u_t|^2 |u_t|^{p-2} dx \\
\leq & C_\varepsilon \|\nabla u\|^{2p} + \varepsilon \int_{\mathbb{T}^3} |\nabla u_t|^2 |u_t|^{p-2} dx,
\end{aligned}
\tag{12}
$$

where in the last step we used again $H_1 \hookrightarrow \mathrm{L}^6$ along with Poincaré's inequality.

Now, consider I_1. We have

$$
\begin{aligned}
I_1 &\le (p-2)\int_{\mathbb{T}^3} |\pi||\nabla u_t||u_t|^{p-2}dx \\
&\le (p-2)\int_{\mathbb{T}^3} \Big(|\pi||u_t|^{\frac{p-2}{2}}\Big)\Big(|\nabla u_t||u_t|^{\frac{p-2}{2}}\Big)dx \\
&\le (p-2)\Big(\int_{\mathbb{T}^3} |\pi|^2|u_t|^{p-2}dx\Big)^{\frac{1}{2}}\Big(\int_{\mathbb{T}^3} |\nabla u_t|^2|u_t|^{p-2}dx\Big)^{\frac{1}{2}}.
\end{aligned}
$$

Hence, we reach

$$
\begin{aligned}
I_1 &\le \varepsilon\int_{\mathbb{T}^3} |\nabla u_t|^2|u_t|^{p-2}dx + C_\varepsilon\int_{\mathbb{T}^3} |\pi|^2|u_t|^{p-2}dx \\
&\le \varepsilon\int_{\mathbb{T}^3} |\nabla u_t|^2|u_t|^{p-2}dx + C_\varepsilon\|\pi\|_p^2\|u_t\|_p^{p-2} \\
&\le \varepsilon\int_{\mathbb{T}^3} |\nabla u_t|^2|u_t|^{p-2}dx + C_{\varepsilon,\delta}\|\pi\|_p^p + \delta\|u_t\|_p^p,
\end{aligned}
\tag{13}
$$

where we employed Hölder's and Young's inequalities.

Now, we can use this last inequality to make explicit the terms, involving the space-derivatives of π, needed to reach (6). To do so, we follow the same line of reasoning as in the proof of [28, Theorem 2.2]. Taking the operator div (*i.e.*$\mathrm{div}v = \nabla\cdot v$) on both sides of Eq. $(1)_1$, we have

$$
-\Delta\pi = \partial_i\partial_j(u_iu_j) - \partial_3\theta.
$$

As a consequence, we get

$$
-\Delta\nabla\pi = \partial_i\partial_j(u_i\nabla u_j + u_j\nabla u_i) - \partial_3\nabla\theta, \quad \text{for } i,j = 1,2,3.
$$

Applying the inverse of the Helmholtz operator, i.e. G_α, to both sides of the above relation, we get

$$
G_\alpha\big(\Delta\nabla\pi\big) = -G_\alpha\big(\partial_i\partial_j(u_i\nabla u_j + u_j\nabla u_i)\big) + G_\alpha\big(\partial_3\nabla\theta\big),
\tag{14}
$$

then, taking the L^q-norm, $q>1$, to both sides of (14), we reach

$$
\|\nabla\pi\|_q \le C\big(\|u\cdot\nabla u\|_q + \|\theta\|_q\big).
\tag{15}
$$

Remark 2 Here, the procedure to get the estimate (15) is formal, but after properly selecting $q = 3/2$ (see (17) below), we obtain the actual control on $\nabla\pi$ in $\mathrm{L}^{3/2}$. Indeed, since $u\in\mathrm{L}^\infty(0,T;\mathrm{H}_1)$, $T>0$, we have that $\nabla\cdot\big((u\cdot\nabla)u\big)\in\mathrm{L}^\infty(0,T;\mathrm{W}^{-1,3/2})$, where $\mathrm{W}^{-1,3/2} := \big(W^{1,3}\big)'$, clearly $\partial_3\theta\in\mathrm{L}^\infty(0,T;\mathrm{W}^{-1,3/2})$ and even more, and consequently, by using the classical regularity theory for the Poisson equation in the periodic setting, we obtain $\nabla\pi\in\mathrm{L}^\infty(0,T;\mathrm{L}^{3/2})$ (see, e.g., [1]).

Inequality (8), with $\gamma = p$ and $\frac{1}{\lambda} + \frac{2}{\sigma} = 1 + \frac{3}{p}$, implies

$$
\begin{aligned}
\|\pi\|_p &\le C\|\partial_1\pi\|_\sigma^{\frac{1}{3}}\|\partial_2\pi\|_\sigma^{\frac{1}{3}}\|\partial_3\pi\|_\lambda^{\frac{1}{3}} \\
&\le C\|\nabla\pi\|_\sigma^{\frac{2}{3}}\|\partial_3\pi\|_\lambda^{\frac{1}{3}}.
\end{aligned}
\tag{16}
$$

Furthermore, by using (15) with $q = \sigma$, $1\le\sigma<2$, along with Hölder's inequality, we get

$$\begin{aligned}\|\nabla\pi\|_\sigma &\le c\big(\|u\cdot\nabla u\|_\sigma + \|\theta\|_\sigma\big)\\ &\le c\big(\|u\|_{\frac{2\sigma}{2-\sigma}}\|\nabla u\| + \|\theta\|_\sigma\big),\end{aligned}$$

and for $\sigma = 3/2$ we find

$$\begin{aligned}\|\nabla\pi\|_{\frac{3}{2}} &\le c\big(\|u\|_6\|\nabla u\| + \|\theta\|_{\frac{3}{2}}\big)\\ &\le c\big(\|\nabla u\|^2 + \|\theta\|_p\big)\\ &\le c\|\nabla u\|^2 + C,\end{aligned}\tag{17}$$

after using inequality (5) and taking $C = C(\|\theta_0\|_p)$. Thus, from (16) and (17) we get

$$\|\pi\|_p^p \le c\big(\|\nabla u\|^2 + C\big)^{\frac{2p}{3}}\|\partial_3\pi\|_\lambda^{\frac{p}{3}},$$

and plugging such a control into (13), we obtain

$$\begin{aligned}I_1 &\le C_{\varepsilon,\delta}\big(\|\nabla u\|^2 + C\big)^{\frac{2p}{3}}\|\partial_3\pi\|_\lambda^{\frac{p}{3}} + \delta\|u_t\|_p^p + \varepsilon\int_{\mathbb{T}^3}|\nabla u_t|^2|u_t|^{p-2}dx\\ &\le C_{\varepsilon,\delta}\big(\|\nabla u\|^{\frac{4p}{3}} + C\big)\|\partial_3\pi\|_\lambda^{\frac{p}{3}} + \delta\|u_t\|_p^p + \varepsilon\int_{\mathbb{T}^3}|\nabla u_t|^2|u_t|^{p-2}dx.\end{aligned}\tag{18}$$

Remark 3 With the above choices for the positive parameters σ, λ, and γ, the requirements in (9) are satisfied. Indeed, for $\gamma = p$, with $p \ge 4$, due to (10), we have

$$\frac{1}{\lambda} + \frac{2}{\sigma} = \frac{3+p}{p} > 1.$$

In particular, we also have

$$1 \ge \frac{1}{\lambda} = \frac{3+p}{p} - \frac{2}{\sigma} = \frac{9-p}{3p} > 0 \quad\text{and}\quad \lambda = \frac{3p}{9-p},\tag{19}$$

where $\sigma = 3/2$ as done in (17). Taking $4 \le p < 9$ the above condition is verified.

Finally, by inserting (12) and (18) into (10), and taking $\delta = \varepsilon$, we get

$$\begin{aligned}(1-2\varepsilon)\|u_t\|_p^p + (\alpha^2 - 2\varepsilon)\int_{\mathbb{T}^3}|\nabla u_t|^2|u_t|^{p-2}dx + \alpha^2\frac{p-2}{4}\int_{\mathbb{T}^3}\big|\nabla|u_t|^2\big|^2|u_t|^{p-4}dx\\ \le C\|\theta\|_p^p + C\|\nabla u\|^{2p} + C\Big(\|\nabla u\|^{\frac{4p}{3}} + C\Big)\|\partial_3\pi\|_\lambda^{\frac{p}{3}},\end{aligned}$$

and hence

$$\|u_t\|_p^p + (\alpha^2 - 2\varepsilon)\int_{\mathbb{T}^3}|\nabla u_t|^2|u_t|^{p-2}dx \le c\Big[\|\theta\|_p^p + \|\nabla u\|^{2p} + \Big(\|\nabla u\|^{\frac{4p}{3}} + C\Big)\|\partial_3\pi\|_\lambda^{\frac{p}{3}}\Big].\tag{20}$$

Now, let us rewrite and improve this control in order to get the claimed regularity criterion. As a consequence of relation (4) and inequality (20), we have

$$\begin{aligned}
&\frac{d}{dt}\left(\|\theta(t)\|_p^p + \int_0^t \|u_t(s)\|_p^p ds\right)\\
&\quad\le c\|\theta\|_p^p + c\|\nabla u\|^{2p} + c\left(\|\nabla u\|^{\frac{4p}{3}} + C\right)\|\partial_3\pi\|_\lambda^{\frac{p}{3}}\\
&\quad\le c\left(\|\theta\|_p^p + \int_0^t \|u_t(s)\|_p^p ds\right) + c\|\nabla u\|^{2p} + c\left(\|\nabla u\|^{\frac{4p}{3}} + C\right)\|\partial_3\pi\|_\lambda^{\frac{p}{3}}\\
&\quad\le c\left(e + \|\theta\|_p^p + \int_0^t \|u_t(s)\|_p^p ds\right) + c\left(e + \|\theta\|_p^p + \int_0^t \|u_t(s)\|_p^p ds\right)\|\nabla u\|^{2p}\\
&\qquad + c\left(e + \|\theta\|_p^p + \int_0^t \|u_t(s)\|_p^p ds\right)\left(\|\nabla u\|^{\frac{4p}{3}} + C\right)\|\partial_3\pi\|_\lambda^{\frac{p}{3}},
\end{aligned} \tag{21}$$

where, in the right-hand side, to keep the notation compact we omit the explicit dependence on t for the various terms taken in space-norm.

Define the auxiliary quantity L as

$$L := \|\partial_3\pi\|_{\mathrm{L}^\lambda}^{\frac{p}{3}}\left(e + \|\theta\|_p^p + \int_0^t \|u_t(s)\|_p^p ds\right).$$

By Young's inequality for 1 and $\|\theta\|_p$, with exponents p and $p/(p-1)$, we have

$$\begin{aligned}
1 + \ln\left(1 + \|\theta\|_p\right) &\le 1 + \ln\left(1 + \frac{\|\theta\|_p^p}{p} + \frac{p-1}{p}\right)\\
&\le 1 + \ln\left(e + \|\theta\|_p^p\right),
\end{aligned}$$

and L can be rewritten as follows

$$\begin{aligned}
L =& \frac{\|\partial_3\pi\|_{\mathrm{L}^\lambda}^{\frac{p}{3}}}{1 + \ln(1 + \|\theta\|_p)}\left(e + \|\theta\|_p^p + \int_0^t \|u_t(s)\|_p^p ds\right)\left[1 + \ln(1 + \|\theta\|_p)\right]\\
\le& \frac{\|\partial_3\pi\|_{\mathrm{L}^\lambda}^{\frac{p}{3}}}{1 + \ln(1 + \|\theta\|_p)}\left(e + \|\theta\|_p^p + \int_0^t \|u_t(s)\|_p^p ds\right)\left[1 + \ln(e + \|\theta\|_p^p)\right]\\
\le& \frac{\|\partial_3\pi\|_{\mathrm{L}^\lambda}^{\frac{p}{3}}}{1 + \ln(1 + \|\theta\|_p)} R[1 + \ln R],
\end{aligned}$$

where

$$R(t) := e + \|\theta(t)\|_p^p + \int_0^t \|\partial_s u_t(s)\|_p^p ds.$$

Hence, from (21) we obtain

$$\frac{dR}{dt} \le c\frac{\left(\|\nabla u\|^{\frac{4p}{3}} + C\right)\|\partial_3\pi\|_{\mathrm{L}^\lambda}^{\frac{p}{3}}}{1 + \ln(1 + \|\theta\|_p)}[1 + \ln R]R + c\left(1 + \|\nabla u\|^{2p}\right)R,$$

for all $t \in [0, T]$. By Gronwall's lemma, we have

$$R(t) \le R(0) \exp\left\{ c \int_0^t \frac{\left(\|\nabla u\|^{\frac{4p}{3}} + C\right) \|\partial_3 \pi\|_\lambda^{\frac{p}{3}}}{1 + \ln(1 + \|\theta(s)\|_p)} [1 + \ln R(s)] ds \right\}$$
$$\times \exp\left\{ c \int_0^t \left(1 + \|\nabla u(s)\|^{2p}\right) ds \right\}.$$

As a consequence of the regularity of $u \in \mathrm{L}^\infty(0, T, H_1)$, we find

$$c \int_0^T \left(1 + \|\nabla u(s)\|^{2p}\right) ds \le \widehat{C}(\|u_0\|_{1,2}, T) =: \widehat{C},$$

and hence, up multiply both sides by e and then *log* them, we obtain

$$1 + \ln(R(t)) \le \widehat{C} + \ln\left(R(0)\right) + c \int_0^t \frac{\left(\|\nabla u\|^{\frac{4p}{3}} + C\right) \|\partial_3 \pi(s)\|_\lambda^{\frac{p}{3}}}{1 + \ln(1 + \|\theta(s)\|_p)} [1 + \ln(R(s))] ds$$
$$\le \widehat{C} + \ln\left(R(0)\right) + C(T) \int_0^t \frac{\|\partial_3 \pi(s)\|_\lambda^{\frac{p}{3}}}{1 + \ln(1 + \|\theta(s)\|_p)} [1 + \ln(R(s))] ds,$$

where in the last inequality we used again the uniform control on $\|\nabla u\|$ in $[0, T]$ and we actually have that $C(T) = C(\|u_0\|_{1,2}, \|\theta_0\|, T)$. Moreover, by Gronwall's lemma, we obtain

$$\ln(R(t)) \le \bar{C} \exp\left\{ \int_0^T \frac{\|\partial_3 \pi(s)\|_\lambda^{\frac{p}{3}}}{1 + \ln(1 + \|\theta(s)\|_p)} ds \right\},$$

where $\bar{C} = \bar{C}(\|u_0\|_{1,2}, \|\theta_0\|_p, T)$. This relation, together with (6), (5) and (19), implies

$$\int_0^T \|u_t(s)\|_p^p ds < +\infty,$$

which gives (7).

Now, set

$$J_p(u_t) := \int_{\mathbb{T}^3} |\nabla u_t|^2 |u_t|^{p-2} dx.$$

The above calculations, along with (20), also provide

$$\int_0^T J_p(u_t) ds = \int_0^T \int_{\mathbb{T}^3} |\nabla u_t|^2 |u_t|^{p-2} dx ds < +\infty. \tag{22}$$

In particular, the boundedness of $J_p(u_t)$ implies that u_t belongs also to a specific type of Besov space, i.e. the Nikol'skiĭ space $\mathcal{N}^{\frac{2}{p},p} := B_\infty^{\frac{2}{p},p}$ (see [21]), and that

$$\|u_t\|_{\mathcal{N}^{\frac{2}{p},p}}^p \le c J_p(u_t), \tag{23}$$

where $c > 0$ is depending only on the exponent p and on the space-domain. For any $\varepsilon \in (0, 1)$, we have the following embeddings (see [21, 29])

$$\mathcal{N}^{\frac{2}{p},p} \hookrightarrow W^{\frac{2}{p}-\varepsilon,p} \hookrightarrow \mathcal{N}^{\frac{2}{p}-\varepsilon,p},$$

and hence, using (22) and (23), we infer

$$\int_0^T \|u_t\|^p_{W^{\frac{2}{p}-\varepsilon,p}} ds \leq c \int_0^T J_p(u_t) ds < +\infty,$$

which concludes the proof of the theorem.

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