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Optimization of the anisotropic Cheeger constant with respect to the anisotropy

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Abstract. Given an open, bounded set Ω in $\mathbb{R}^N$, we consider the minimization of the anisotropic Cheeger constant $h_K(\Omega)$ with respect to the anisotropy K, under a volume constraint on the associated unit ball. In the planar case, under the assumption that K is a convex, centrally symmetric body, we prove the existence of a minimizer. Moreover, if Ω is a ball, we show that the optimal anisotropy K is not a ball and that, among all regular polygons, the square provides the minimal value.

1 Introduction

Given an open, bounded set Ω in $\mathbb{R}^N$, the *Cheeger problem* amounts to finding sets attaining the *Cheeger constant* $h(\Omega)$, which is defined as

$$h(\Omega) := \inf\left\{ \frac{\mathrm{Per}(E)}{|E|} \,:\, E \subset \Omega,\ |E| > 0 \right\}, \tag{1.1}$$

where $\mathrm{Per}(E)$ denotes the variational perimeter of E, and $|E|$ its N-dimensional Lebesgue measure. This problem has been first introduced in the Riemannian setting in [5], and it has been deeply studied ever since, given its many applications in different problems. We refer the interested reader to the two surveys [12, 22]. The computation of the constant and the geometric characterization of minimizers is by now well understood in the two-dimensional case for convex sets [10], strips [14], and more general sets [13, 15].

A generalization of this problem can be given in the anisotropic Euclidean space, that is, the Euclidean space endowed with a norm induced by a convex, central-symmetric set. In turns, this underlying anisotropic metric induces a notion of anisotropic perimeter, that can be used to define an anisotropic analogous of (1.1).

More precisely, let $\mathcal{K}_N$ be the class of (nonempty) open, bounded and centrally symmetric (with respect to the origin) convex sets in $\mathbb{R}^N$. Given any $K \in \mathcal{K}_N$, its *polar*

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set K° is defined as

$$K^\circ := \{x \in \mathbb{R}^N \mid x \cdot y < 1 \text{ for every } y \in K\}.$$

For every $K \in \mathcal{K}_N$, its polar set K° belongs to $\mathcal{K}_N$, and $(K^\circ)^\circ = K$ (we refer to [27] for these standard facts). In particular, given any such K, the map Φ_K° defined as

$$\Phi_K^\circ(x) := \sup\{x \cdot y \colon y \in K\} \tag{1.2}$$

is a norm over $\mathbb{R}^N$, called *polar norm* of K. By definition,

$$K^\circ = \{x \in \mathbb{R}^N \mid \Phi_K^\circ(x) < 1\},$$

that is, K° is the unit ball with respect to the metric induced by Φ°.

By means of the polar norm, it is possible to define an anisotropic perimeter for any Borel set E as

$$\operatorname{Per}_K(E) := \int_{\partial^* E} \Phi_K^\circ(\nu_E(x))\, \mathrm{d}\mathcal{H}^{N-1}(x), \tag{1.3}$$

where $\partial^* E$ denotes the reduced boundary of E (see [16]).

Notice that such a perimeter is invariant under translation but, in general, not under the effect of the rotation group $\mathrm{SO}(N)$. The *anisotropic isoperimetric inequality* [29] states that, among sets E of fixed volume, the *unique* anisotropic perimeter minimizer (up to translations) is given by a dilation of the convex set K, which is called *Wulff shape* associated with Φ°, i.e.,

$$\operatorname{Per}_K(E) \geq \operatorname{Per}_K(K_E) = N|K|^{\frac{1}{N}}|K_E|^{\frac{N-1}{N}}, \tag{1.4}$$

where K_E is the dilation of K such that $|E| = |K_E|$.

With this notion of anisotropic perimeter (1.3) at our disposal, we can define the *K-Cheeger constant* of a set Ω analogously to (1.1) as

$$h_K(\Omega) := \inf\left\{\frac{\operatorname{Per}_K(E)}{|E|} \colon E \subset \Omega\,, |E| > 0\right\}. \tag{1.5}$$

Sets attaining the infimum are called *K-Cheeger sets* of Ω, and it is well known that they exist for any Ω regular enough with finite measure [11, 26]. The constant $h_K(\Omega)$, whenever Ω is Lipschitz regular, can be thought of as the first eigenvalue of the anisotropic 1-Laplacian [11]; it is related to anisotropic capillarity problems [1], and it is relevant for applications to image reconstruction [4].

A usual problem for shape functionals is to determine which shapes Ω minimize the functional $\Omega \mapsto h_K(\Omega)$ under a volume constraint on Ω. The anisotropic isoperimetric inequality (1.4) immediately implies that the minimizing shape is a dilation of K itself, and, in particular, we have

$$\inf\{\, h_K(\Omega) \,:\, |\Omega| = 1\,\} = \frac{\operatorname{Per}_K(K)}{|K|} = N. \tag{1.6}$$

While in (1.6) the set $K \in \mathcal{K}_N$ providing the metric is fixed and one minimizes among Ω, in the present paper, we fix Ω and want to minimize among the metrics K—under suitable constraints.

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There are two possible reasonable choices for the volume constraint: either on the volume of the Wulff shape $|K|$, or on the volume of the unit ball $|K^\circ|$, leading to the study of the two (scaling invariant) functionals

$$\mathcal{F}_\Omega[K] := h_K(\Omega)|K|^{-\frac{1}{N}}, \tag{1.7}$$

$$\mathcal{J}_\Omega[K] := h_K(\Omega)|K^\circ|^{\frac{1}{N}}. \tag{1.8}$$

Since we are interested in the metric, it feels more natural to impose a constraint on the volume of the unit ball K°, that is, to consider (1.8). It is noteworthy that not only it is the more natural choice, but also the minimization of (1.7) is a trivial task, whenever Ω is fixed in $\mathcal{K}_N$. Indeed, by the anisotropic isoperimetric inequality (1.4), for every $E \subset \Omega$, one has

$$\mathrm{Per}_K(E)|K|^{-\frac{1}{N}} \geq \mathrm{Per}_K(K_E)|K|^{-\frac{1}{N}} = N|K_E|^{\frac{N-1}{N}} = N|E|^{\frac{N-1}{N}}$$

with equality holding if and only if E equals K_E up to a translation. Therefore,

$$\frac{\mathrm{Per}_K(E)}{|E|}|K|^{-\frac{1}{N}} \geq N|E|^{-\frac{1}{N}} \geq N|\Omega|^{-\frac{1}{N}}.$$

Passing to the infimum on all $E \subset \Omega$, we obtain

$$h_K(\Omega)|K|^{-\frac{1}{N}} \geq N|\Omega|^{-\frac{1}{N}}, \tag{1.9}$$

for all possible $K \in \mathcal{K}_N$. In particular, equality holds if and only if K coincides with Ω (up to translations and dilations), and Ω is the associated Cheeger set. Summing up, if $\Omega \in \mathcal{K}_N$, one has

$$\inf_{K\in\mathcal{K}_N} \mathcal{F}_\Omega[K] = \mathcal{F}_\Omega[\Omega] = N|\Omega|^{-\frac{1}{N}}, \tag{1.10}$$

and $K = \Omega$ is the unique minimizer.

On the other hand, the minimization of (1.8) is far from being trivial. Notice that we can immediately rewrite $\mathcal{J}_\Omega$ as

$$\mathcal{J}_\Omega[K] = \mathcal{F}_\Omega[K](|K||K^\circ|)^{\frac{1}{N}}. \tag{1.11}$$

We have already observed that the first factor of the above product is minimized in $\mathcal{K}_N$ by the choice $K = \Omega$, provided that $\Omega \in \mathcal{K}_N$. The second factor is known as the *Mahler volume* of K, a shape functional which is invariant under invertible affine transformations, and which is well known to be maximized by balls (in general, ellipsoids [25]). Regarding its minimization, it is conjectured to be minimized by (affine transformations of) hypercubes (in general, Hanner's polytopes). This holds true in dimension $N = 2$ (see [17]) and under some additional assumptions in higher dimension (see [24]). We refer to the expository article [28, Chapter 3.3] for further details. Therefore, when $N = 2$ and Ω is a parallelogram, the minimization is trivial, while otherwise a competition between the two terms arises.

Similar problems of finding the best metric have been studied, for instance, for the eigenvalues of the Laplace–Beltrami operator on the sphere $\mathbb{S}^2$ (see [8, 9, 19, 20]). In the same spirit, one could consider minimization problems with respect to the underlying metric for different shape functionals, e.g., depending on the eigenvalues of the anisotropic p-Laplacian, the K-capacity, or K-torsion. We leave these problems as further research directions, as they have an intrinsic higher difficulty due to the fact that they involve nongeometrical quantities.

In the present paper, we focus on the planar case. Moreover, we will always suppose that the set $\Omega \subset \mathbb{R}^2$ is convex, since in this case we can exploit the structure of K-Cheeger sets granted by [11, Theorem 5.1]. In Section 3, we prove that there exist minimizers of $\mathcal{J}_\Omega$, while its supremum is $+\infty$. In Section 4, we show by means of an example that the problem is indeed nontrivial: in the case $\Omega = B$, we prove that the square is the shape that yields the lowest energy among all regular n-gons, and we conjecture it to be the best shape among all possible centrally symmetric anisotropies.

2 Preliminary results

2.1 The anisotropic Cheeger problem

Let $K \in \mathcal{K}_N$, and let $\Omega \subset \mathbb{R}^N$ be an open, bounded (nonempty) set. A K-Cheeger set is a non-negligible measurable set $C_K \subset \Omega$ such that

$$h_K(\Omega) = \frac{\mathrm{Per}_K(C_K)}{|C_K|}.$$

Whenever Ω is a convex, two-dimensional set, there exists a unique K-Cheeger set $C_K \subset \Omega$. Moreover, it is possible to give a complete geometrical characterization of C_K, which we recall in Theorem 2.1. This has been first proved in the isotropic Euclidean setting [10] and later extended to the anisotropic Euclidean setting in [11]. For the ease of the reader, before stating the result, we recall the definition of Minkowski addition and difference between two sets E and F. Given any $x \in \mathbb{R}^2$, if we denote by $E + x$, the translation of the set E by x, we have

$$E \oplus F := \cup_{x \in F}(E + x),$$
$$E \ominus F := \cap_{x \in F}(E - x).$$

When F is chosen as a ball B, the Minkowski addition $E \oplus B$ can be thought as an outward regularization of the set E, and the Minkowski difference $E \ominus B$ as an inward regularization. We remark that, in general, these operations do not commute and one only has the set inclusion $(E \ominus B) \oplus B \subset E$.

Theorem 2.1 [11, Theorem 5.1] *Let $\Omega \subset \mathbb{R}^2$ be an open, bounded, convex set. Then, there exists a unique K-Cheeger set C_K of Ω. Moreover, C_K is convex and we have*

$$C_K = \Omega^\rho \oplus \rho K,$$

where $\Omega^\rho := \Omega \ominus \rho K$ and ρ is the inverse of the K-Cheeger constant. Moreover, ρ is the unique value such that $|\Omega^\rho| = \rho^2 |K|$.

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2.2 The Mahler volume

Let $K \subset \mathbb{R}^N$ be a convex set. The *Mahler volume* of K is the quantity

$$V(K) := |K||K^\circ|,$$

where K° is the polar set of K. It can be proven that the Mahler volume is invariant under invertible affine transformations. The following result holds true.

Proposition 2.2 *Let K be any convex set in $\mathcal{K}_2$, and let $V(K)$ be its Mahler volume. Then*

$$8 = V(Q) \leq V(K) \leq V(B) = \pi^2,$$

where Q is a square and B is a ball.

The upper bound is known as *Blaschke–Santaló inequality*, whose proof can be found in [27, Section 10.5]. The lower bound was proven in [17], and an accessible proof can be found in [30].

2.3 Uniform convergence of polar norms

The following result provides a link between convergence of the metrics in the Hausdorff distance of convex sets, and local uniform convergence of the associated polar norms.

Proposition 2.3 *Let $K \in \mathcal{K}_N$ and $\{K_n\}_{n\in\mathbb{N}} \subset \mathcal{K}_N$ be a sequence such that $K_n \to K$ in the Hausdorff topology. Then, $\Phi^\circ_{K_n} \to \Phi^\circ_K$ locally uniformly.*

Proof Let $\varepsilon > 0$, for n sufficiently big, $(1-\varepsilon)K \subset K_n \subset (1+\varepsilon)K$. If $y \in \mathbb{S}^{N-1}$, it holds

$$\Phi^\circ_{(1-\varepsilon)K}(y) \leq \Phi^\circ_{K_n}(y) \leq \Phi^\circ_{(1+\varepsilon)K}(y),$$

which implies

$$(1-\varepsilon)\Phi^\circ_K(y) \leq \Phi^\circ_{K_n}(y) \leq (1+\varepsilon)\Phi^\circ_K(y).$$

By the 1-homogeneity of Φ°_K and $\Phi^\circ_{K_n}$, we obtain the claim. ■

3 Main results

Throughout this section, we will restrict to the two-dimensional case, and we will suppose that Ω is an open, bounded, convex planar set. We consider the functional $\mathcal{J}_\Omega$ introduced in (1.8), that is,

$$\mathcal{J}_\Omega[K] := h_K(\Omega)|K^\circ|^{\frac{1}{2}},$$

where the multiplicative factor $|K^\circ|^{\frac{1}{2}}$ appears in order to make it scale invariant.

Proposition 3.1 *Let $\Omega \subset \mathbb{R}^2$ be an open, bounded, convex set. Let $V > 0$, and let $\{K_n\}_{n\in\mathbb{N}} \subset \mathcal{K}_2$ be a sequence such that $|K_n| = V$ for every $n \in \mathbb{N}$, with* $\operatorname{diam}(K_n) \to$

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$+\infty$. *Then,*

$$\lim_{n\to+\infty} \mathcal{J}_\Omega[K_n] = +\infty.$$

Proof By Theorem 2.1, the Cheeger set C_{K_n} of Ω associated with the anisotropy K_n is given by

$$C_{K_n} = \bigcup \rho_n K_n,$$

where the union is taken among all dilations of the Wulff shape K_n by ρ_n that are contained in Ω. Since $\operatorname{diam}(K_n) \to +\infty$ as $n \to +\infty$, it must necessarily hold $\rho_n \to 0$. Recalling that $h_{K_n}(\Omega) = \rho_n^{-1}$, one infers that $h_{K_n}(\Omega) \to +\infty$ as $n \to +\infty$. By the reverse Mahler inequality contained in Proposition 2.2, the quantities $|K_n^\circ|$ are uniformly bounded from below. Therefore, $\lim_{n\to+\infty} \mathcal{J}_\Omega[K_n] = +\infty$. ∎

Corollary 3.2 *Let $\Omega \subset \mathbb{R}^2$ be an open, bounded, convex set. Then,*

$$\sup_{K\in\mathcal{K}_2} \mathcal{J}_\Omega[K] = +\infty.$$

We now prove that our shape functional $K \mapsto \mathcal{J}_\Omega[K]$ has a minimizer.

Proposition 3.3 *There exists $\hat{K} \in \mathcal{K}_2$ such that*

$$\mathcal{J}_\Omega[\hat{K}] = \min_{K\in\mathcal{K}_2} \mathcal{J}_\Omega[K].$$

Proof Let $\{K_n\}_{n\in\mathbb{N}}$ be a minimizing sequence. Without loss of generality, we can suppose that $|K_n| = V$ for some $V > 0$. By Mahler's inequality, the volumes of the polar sets K_n° are uniformly bounded from above. If $\operatorname{diam}(K_n) \to +\infty$ as $n \to +\infty$, by Proposition 3.1, it would hold $\mathcal{J}_\Omega[K_n] \to +\infty$, a contradiction. Therefore, $\operatorname{diam}(K_n)$ is uniformly bounded. By Blaschke's Selection Theorem [27, Theorem 1.8.7], there exists $\hat{K} \in \mathcal{K}_2$ such that $K_n \to \hat{K}$ in the Hausdorff topology. By Proposition 2.3,

$$\Phi^\circ_{K_n} \to \Phi^\circ_K \quad \text{locally uniformly,}$$

from which we also infer the L^1_{loc} convergence of $K_n^\circ \to \hat{K}^\circ$, and thus $|\hat{K}^\circ| \le \liminf_n |K_n^\circ|$. Let C_{K_n} be the Cheeger sets associated with $h_{K_n}(\Omega)$. By Theorem 2.1, all these sets are convex, and they are contained in the bounded set Ω. Again by Blaschke's Selection Theorem, up to a subsequence which we do not relabel, there exists a set $\hat{C} \subset \Omega$ that is the Hausdorff limit of the sequence. In particular, C_{K_n} converge to $\hat{C}$ in the sense of characteristic functions. By [21, Proposition 2.1] and the local uniform convergence of $\Phi^\circ_{K_n}$ to Φ°_K, it holds

$$\operatorname{Per}_{\hat{K}}(\hat{C}) \le \liminf_{n\to+\infty} \operatorname{Per}_{K_n}(C_{K_n}).$$

Moreover, $|C_{K_n}| \to |\hat{C}|$ and $|\hat{C}| > 0$ as otherwise, also taking into account that $|K_n^\circ|$ is uniformly bounded from above, the anisotropic isoperimetric inequality would

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contradict the fact that K_n is a minimizing sequence for $\mathcal{J}_\Omega$. Therefore,

$$\begin{aligned} h_{\hat{K}}(\Omega)|\hat{K}^\circ|^{\frac{1}{2}} &\leq \frac{\mathrm{Per}_{\hat{K}}(\hat{C})}{|\hat{C}|}|\hat{K}^\circ|^{\frac{1}{2}} \\ &\leq \liminf_{n\to+\infty} \frac{\mathrm{Per}_{K_n}(C_{K_n})}{|C_{K_n}|}|K_n^\circ|^{\frac{1}{2}} = \liminf_{n\to+\infty} h_{K_n}(\Omega)|K_n^\circ|^{\frac{1}{2}}, \end{aligned}$$

which means

$$\mathcal{J}_\Omega[\hat{K}] \leq \liminf_{n\to+\infty} \mathcal{J}_\Omega[K_n].$$

Since $\hat{K} \in \mathcal{K}_2$ and $\{K_n\}_{n\in\mathbb{N}}$ is a minimizing sequence, the claim follows. ∎

4 Examples

In this section, we provide a few examples in order to highlight how the problem is far from being trivial. We have already made the useful observation that one can rewrite the functional $\mathcal{J}_\Omega[K]$ in terms of $\mathcal{F}_\Omega[K]$ as

$$\mathcal{J}_\Omega[K] = \mathcal{F}_\Omega[K](|K||K^\circ|)^{\frac{1}{2}} = \mathcal{F}_\Omega[K]V(K)^{\frac{1}{2}}. \tag{4.1}$$

This easily allows to infer that there cannot be a set $K \in \mathcal{K}_2$ that always minimizes $\mathcal{J}_\Omega[K]$ independently of the choice of Ω.

4.1 Ω is a parallelogram

Let T be an invertible affine transformation, and consider the parallelogram given by $T(Q)$, where Q is the unit square in $\mathbb{R}^2$. By Proposition 2.2 and the fact that the Mahler volume is invariant with respect to invertible affine transformations,

$$V(T(Q)) \leq V(K)$$

for every $K \in \mathcal{K}_2$. Moreover, as noticed in the introduction (see (1.10)), $T(Q)$ is the unique minimizer of $\mathcal{F}_{T(Q)}[K]$. Hence,

$$\mathcal{J}_{T(Q)}[T(Q)] = \mathcal{F}_{T(Q)}[T(Q)] \cdot V(T(Q))^{\frac{1}{2}} \leq \mathcal{F}_{T(Q)}[K] \cdot V(K)^{\frac{1}{2}} = \mathcal{J}_{T(Q)}[K]$$

for every $K \in \mathcal{K}_2$, the strict inequality holding whenever $K \neq T(Q)$.

4.2 Ω is a ball

Let Ω be the Euclidean ball B of unit radius. In this section, we show that the minimizing convex body $K \in \mathcal{K}_2$ for $\mathcal{J}_B[K]$ can not be the ball itself. More in general, the square Q provides the lowest energy among all possible regular n-gons.

Using (4.1), Proposition 2.2, and equality (1.10), we have as benchmark

$$\inf_{K\in\mathcal{K}_2} \mathcal{J}_B[K] \leq \mathcal{F}_B[B] \cdot V(B)^{\frac{1}{2}} = 2\sqrt{\pi}.$$

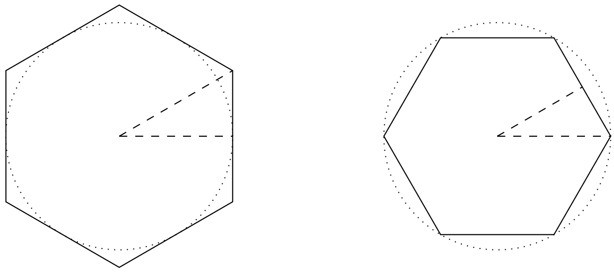

Figure 1: On the left, the Wulff shape P_n^*, and on the right, its polar body $(P_n^*)^\circ$ inducing the metric $\Phi^\circ_{P_n^*}$, for $n = 6$. The unit radius disk appears dotted.

We let P_n^* be a regular n-gon, with $n \geq 4$ even, circumscribed to B. With this choice, we have the following:

$$|P_n^*| = n\tan(\pi/n)\,, \qquad |(P_n^*)^\circ| = n\sin(\pi/n)\cos(\pi/n)\,, \tag{4.2}$$

where the first one is a well known fact of Euclidean geometry, while the second equality comes from [2, Corollary 4]. We remark that the polar body of P_n^* coincides with a rotation and a dilation by $\cos(\pi/n)$ of P_n^*. In particular, we have the following information on the side length $\mathrm{s}(\cdot)$, the apothem $\mathrm{a}(\cdot)$, and the circumradius $\mathrm{R}(\cdot)$ of P_n^* and its polar body $(P_n^*)^\circ$:

$$\begin{aligned} &\mathrm{s}(P_n^*) = 2\tan(\pi/n)\,, && \mathrm{s}((P_n^*)^\circ) = 2\sin(\pi/n)\,,\\ &\mathrm{a}(P_n^*) = 1\,, && \mathrm{a}((P_n^*)^\circ) = \cos(\pi/n)\,,\\ &\mathrm{R}(P_n^*) = \cos(\pi/n)^{-1}\,, && \mathrm{R}((P_n^*)^\circ) = 1\,, \end{aligned} \tag{4.3}$$

that we shall use through the next computations.

Given the symmetry of B, any choice of K yields the same anisotropic constant of $T(K)$, for any rotation $T \in \mathrm{SO}(2)$. Hence, without loss of generality, we can suppose P_n^* to be rotated in such a way that it has two sides parallel to the y-axis, and thus its polar body $(P_n^*)^\circ$ has one diagonal on the x-axis (see Figure 1).

By the symmetry of B, of P_n^* and by Theorem 2.1, the boundary of the minimizer is made of n straight sides parallel to those of P_n^* with endpoints on ∂B and n circular and symmetric arcs of ∂B, as in Figure 2. We consider the one-parameter family of competitors E_x that have this particular structure, being the parameter $x = x(n)$ half the length of one of the straight sides. Given the symmetric nature of our setting, we can divide the plane $\mathbb{R}^2$ in $2n$ symmetric circular sectors and compute the area and the anisotropic perimeter of these candidates in just one of these.

The area in the sector S_i is given by the area of a triangle with base $\sqrt{1-x^2}$ and height x plus the area of a circular sector of radius 1 and angle $\pi/n - \arcsin(x)$, yielding

$$|E_x| = 2n\left(\frac{1}{2}x\sqrt{1-x^2} + \frac{\pi}{2n} - \frac{1}{2}\arcsin(x)\right). \tag{4.4}$$

Concerning the perimeter, it is immediate that the Euclidean one is x plus $\pi/n - \arcsin(x)$, but we should take into account the presence of the anisotropy. The straight

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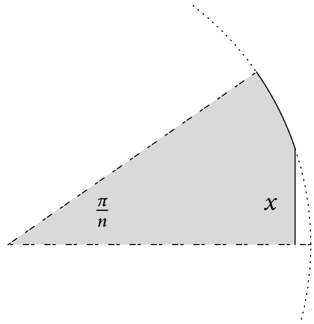

Figure 2: The shape of the Cheeger set in a sector of width π/n w.r.t. the anisotropy given by the regular n-gon.

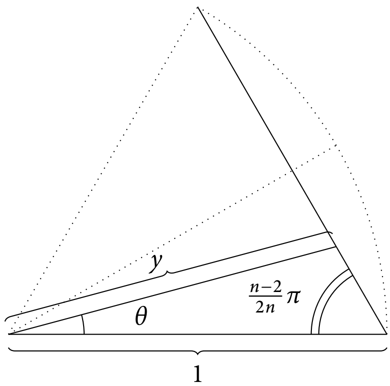

Figure 3: Close up of the unit ball in the metric $\Phi^{\circ}_{P_n^*}$. The dots individuate the sectors of the Euclidean unit ball B with angles π/n and $2\pi/n$.

side, of Euclidean length x, has constant normal given by the horizontal direction e_1. By the choices we made (see (4.3)), the greatest $y \in \mathbb{R}_+$ such that $ye_1 \in (P_n^*)^{\circ}$ is $y = 1$, and thus $\Phi^{\circ}_{P_n^*}(e_1) = 1$. Hence, this straight side has anisotropic perimeter equal to the Euclidean one. Concerning the circular arc, we can parameterize it as $\gamma(\theta) = (\cos\theta, \sin\theta)$ for $\theta \in [\arcsin(x), \pi/n]$, and thus

$$\int_{\partial B\cap\gamma} \Phi^{\circ}_{P_n^*}(\nu_B(y))\mathrm{d}\mathcal{H}^1(y) = \int_{\arcsin(x)}^{\frac{\pi}{n}} \Phi^{\circ}_{P_n^*}(\theta)\|\dot{\gamma}(\theta)\|\mathrm{d}\theta = \int_{\arcsin(x)}^{\frac{\pi}{n}} \Phi^{\circ}_{P_n^*}(\theta)\mathrm{d}\theta\,.$$

Hence, it is just a matter of computing the anisotropy $\Phi^{\circ}_{P_n^*}(\theta)$. Given our initial assumptions, the point e_1 is a vertex of $(P_n^*)^{\circ}$, and the x-axis splits the interior angle with vertex in e_1 in two equal angles of width $(n-2)\pi/2n$. Let $y\theta$ belong to the boundary $(P_n^*)^{\circ}$, for $\theta \in (0, \pi/n)$. Using the law of sines (see also Figure 3), we obtain

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the following equality:

$$y = \frac{\sin\left(\frac{n-2}{2n}\pi\right)}{\sin\left(\pi - \frac{n-2}{2n}\pi - \theta\right)} = \frac{\cos\left({}^{\pi}/_{n}\right)}{\cos\left({}^{\pi}/_{n} - \theta\right)},$$

and, since $\Phi^{\circ}_{P_n^*}(\theta) = y^{-1}$, we eventually get

$$\Phi^{\circ}_{P_n^*}(\theta) = \frac{\cos\left({}^{\pi}/_{n} - \theta\right)}{\cos\left({}^{\pi}/_{n}\right)}.$$

Hence,

$$\begin{aligned} \mathrm{Per}_{P_n^*}(E_x) &= 2n\left(x + \frac{1}{\cos\left(\frac{\pi}{n}\right)}\int_{\arcsin(x)}^{\frac{\pi}{n}} \cos\left(\frac{\pi}{n} - \theta\right) \mathrm{d}\theta\right) \\ &= 2n\left(x + \frac{\sin\left(\frac{\pi}{n} - \arcsin(x)\right)}{\cos\left(\frac{\pi}{n}\right)}\right). \end{aligned} \tag{4.5}$$

By (4.4) and (4.5), it follows that the Cheeger set C_n of B, in the metric for which P_n^* is the Wulff shape, coincides with E_x, for the value $\bar{x}_n$ that minimizes the ratio

$$\frac{2}{\cos\left(\frac{\pi}{n}\right)} \frac{x\cos\left(\frac{\pi}{n}\right) + \sin\left(\frac{\pi}{n} - \arcsin(x)\right)}{x\sqrt{1-x^2} + \frac{\pi}{n} - \arcsin(x)}. \tag{4.6}$$

Furthermore, recall that Theorem 2.1 states that the Cheeger set is the union of ρP_n^*, where ρ is the inverse of the Cheeger constant. Thus, if $\bar{x}_n$ is minimizing (4.6), we also have the following equality:

$$2\bar{x}_n = \mathrm{s}(\rho P_n^*) = 2\rho\tan({}^{\pi}/_{n});$$

hence, the minimizing $\bar{x}_n$ is a solution of

$$\frac{1}{x}\tan\left(\frac{\pi}{n}\right) = \frac{2}{\cos\left(\frac{\pi}{n}\right)} \frac{x\cos\left(\frac{\pi}{n}\right) + \sin\left(\frac{\pi}{n} - \arcsin(x)\right)}{x\sqrt{1-x^2} + \frac{\pi}{n} - \arcsin(x)}$$

and

$$h_{P_n^*}(B) = \frac{1}{\bar{x}_n}\tan\left(\frac{\pi}{n}\right). \tag{4.7}$$

At this point, use the trigonometric identity

$$\sin\left(\frac{\pi}{n} - \arcsin(x)\right) = \sin\left(\frac{\pi}{n}\right)\sqrt{1-x^2} - x\cos\left(\frac{\pi}{n}\right)$$

to simplify the identity into

$$\frac{1}{x} = \frac{2\sqrt{1-x^2}}{x\sqrt{1-x^2} + \frac{\pi}{n} - \arcsin(x)},$$

and hence $\bar{x}_n$ solves

$$\arcsin(x) + x\sqrt{1-x^2} = \frac{\pi}{n}. \tag{4.8}$$

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Table 1: Values of the minimizing half-side $\bar{x}_n$ and of the functional $\mathcal{J}_B[P_n^*]$ for some choices of n.

	$\bar{x}_n$	$\mathcal{J}_B[P_n^*]$		$\bar{x}_n$	$\mathcal{J}_B[P_n^*]$
$n=4$	$0.4040\ldots$	$3.5008\ldots$	$n=50$	$0.0315\ldots$	$3.5443\ldots$
$n=6$	$0.2649\ldots$	$3.5126\ldots$	$n=100$	$0.0158\ldots$	$3.5448\ldots$
$n=8$	$0.1976\ldots$	$3.5246\ldots$	$n=200$	$0.0079\ldots$	$3.5449\ldots$
$n=10$	$0.1578\ldots$	$3.5313\ldots$	$n=+\infty$	0	$2\sqrt{\pi}$

Since the function on the LHS is increasing in x, there exists a unique solution, and the function $n \mapsto \bar{x}_n$ is decreasing, so that the maximum value of $\bar{x}_n$ is attained for $n=4$. Unfortunately this equation cannot be solved explicitly for x, but its unique solution for each n can be numerically computed: a plot of the map $n \mapsto \bar{x}_n$ is shown in Figure 4(a), and some values are collected in Table 1.

The lack of an explicit expression for $\bar{x}_n$ prevents us from having an explicit value of $\mathcal{J}_B[P_n^*]$. Nevertheless, we can easily give an upper and a lower bound to $\bar{x}_n$. On the one hand, as a consequence of the isoperimetric inequality, we can estimate $h_{P_n^*}(B)$ from below by $h_{P_n^*}(\rho P_n^*)$, where ρ is such that $|\rho P_n^*| = |B| = \pi$. This gives

$$\frac{1}{\bar{x}_n}\tan\left(\frac{\pi}{n}\right) = h_{P_n^*}(B) \geq h_{P_n^*}(\rho P_n^*) = \frac{2}{\rho}. \tag{4.9}$$

By the relations in (4.2), we have

$$\pi = |\rho P_n^*| = \rho^2 |P_n^*| = \rho^2 n \tan\left(\frac{\pi}{n}\right),$$

and solving for ρ and plugging in (4.9) gives

$$\bar{x}_n \leq \frac{\sqrt{\pi}}{2}\sqrt{\frac{\tan\left(\frac{\pi}{n}\right)}{n}}.$$

On the other hand, using as competitor $\cos(\pi/n)P_n^*$, that is the greatest Wulff shape contained in B, by definition of K-Cheeger constant, we obtain, as a lower bound,

$$\begin{aligned}\frac{1}{\bar{x}_n}\tan\left(\frac{\pi}{n}\right) = h_{P_n^*}(B) &\leq \frac{\mathrm{Per}_{P_n^*}\left(\cos\left(\frac{\pi}{n}\right)P_n^*\right)}{\left|\cos\left(\frac{\pi}{n}\right)P_n^*\right|} \\ &= \frac{1}{\cos\left(\frac{\pi}{n}\right)}\frac{\mathrm{Per}_{P_n^*}(P_n^*)}{|P_n^*|} = \frac{2}{\cos\left(\frac{\pi}{n}\right)}.\end{aligned}$$

Putting these two inequalities together, we get

$$\frac{1}{2}\sin\left(\frac{\pi}{n}\right) \leq \bar{x}_n \leq \frac{\sqrt{\pi}}{2}\sqrt{\frac{\tan\left(\frac{\pi}{n}\right)}{n}}. \tag{4.10}$$

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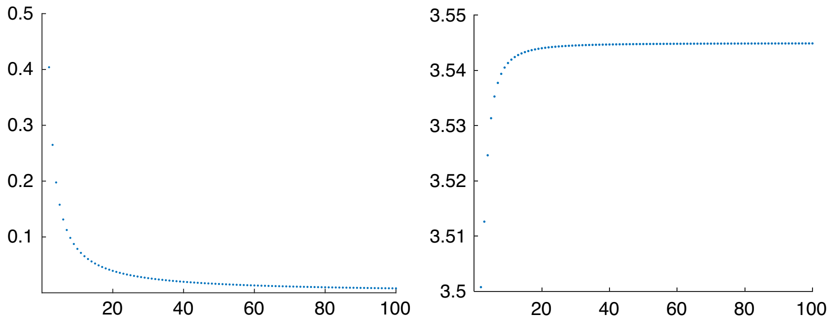

(a) (b)

Figure 4: Graphs of $\bar{x}_n$ (LHS) and of $\mathcal{J}_B[P_n^*]$ (RHS).

This is enough to prove that $\mathcal{J}_B[P_n^*]$ achieves its minimum on P_4^*. Indeed, recalling (4.7) and (4.2), one has

$$\mathcal{J}_B[P_n^*] = h_{P_n^*}(B)|(P_n^*)^\circ|^{\frac{1}{2}} = \frac{1}{\bar{x}_n}\tan\left(\frac{\pi}{n}\right)\sqrt{n\cos(\pi/n)\sin(\pi/n)}.$$

Using the bounds in (4.10), one has

$$\frac{2n}{\sqrt{\pi}}\sin\left(\frac{\pi}{n}\right) \le \mathcal{J}_B[P_n^*] \le 2\sqrt{n\tan\left(\frac{\pi}{n}\right)}.$$

Both the LHS and the RHS converge to $2\sqrt{\pi}$, and, in particular, the LHS is greater than $\mathcal{J}_B[P_4^*]$ as soon as $n \ge 12$. In order to conclude, it is enough to check a finite number of values $\mathcal{J}_B[P_n^*]$, corresponding to $n = 4, 6, 8, 10$. These, along with some other values, numerically obtained, are reported in Table 1, along with the value of $\bar{x}_n$ minimizing (4.6) subject to the constraint (4.10). Graphs depicting the behavior of $\bar{x}_n$ and $\mathcal{J}_B[P_n^*]$ for the first 100 even numbers $n \ge 4$ are shown in Figures 4(a),4(b). We conjecture that $\mathcal{J}_B[P_n^*]$ is actually increasing in n.

5 Final remarks and open questions

In this paper, we initiated the study of optimization of anisotropic shape functionals with respect to the anisotropy. We proved that the minimization problem for the functional $\mathcal{J}_\Omega$ is well posed, and we obtained some partial results in the case when Ω is a ball. We are left with several open questions.

- Is it possible to show existence of minimizers in the N-dimensional case? The difficulty lies in the fact that, apart from the planar case, there is no obvious characterization of K-Cheeger sets, which is a key ingredient of the proof of Proposition 3.1.
- If $\Omega = B$, is it true that $\mathcal{J}_B[P_n^*] \le \mathcal{J}_B[P]$ for any polygon with sides n? While this assertion might seem reasonable, we need to observe, bearing in mind equation (1.11), that the Mahler volume V is actually *maximized*, among all polygons with n sides, by the regular one (see also [18]).

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The functional that has been considered in this paper involves only purely geometrical quantities. In the literature, several other functionals, involving anisotropic differential operators, have been investigated; we can mention, among others, the first eigenvalue of the anisotropic p-Laplacian [11], which is defined, for $p \in (1, +\infty)$, as

$$\lambda_p^K(\Omega) := \inf_{u \in W_0^{1,p}(\Omega)} \frac{\int_\Omega (\Phi^\circ(|\nabla u|))^p}{\int_\Omega |u|^p}.$$

It would be interesting to carry a similar analysis for the shape optimization problem

$$\inf_{K \in \mathcal{K}_N} \lambda_p^K(\Omega)|K^\circ|^{\frac{p}{N}}.$$

However, this problem is considerably more difficult than the one considered in the present work, since nongeometrical quantities are involved. We remark that the results proved here, combined with some well known inequalities between the first eigenvalue of the anisotropic Dirichlet p-Laplacian and the anisotropic Cheeger constant allow us to say something in the two-dimensional case. Namely, we have the following.

Remark 5.1. The K-Cheeger constant provides a lower bound to the first eigenvalue of the anisotropic Dirichlet p-Laplacian $\lambda_p^K(\Omega)$ (Cheeger's inequality; see [11]). Then, from Proposition 3.1, it follows that, for the scaling invariant functional $K \mapsto \lambda_p^K(\Omega)|K^\circ|^{\frac{p}{2}}$, has infinite supremum in $\mathcal{K}_2$.

Remark 5.2. Besides the lower bound to $\lambda_p^K(\Omega)$ provided by Cheeger's inequality [11], the K-Cheeger constant also provides an upper bound to $\lambda_p^K(\Omega)$, known as Buser's inequality or reverse Cheeger's inequality (refer to [3, 6, 7, 23]). Hence, from Proposition 3.3, it follows that the shape functional $K \mapsto \lambda_p^K(\Omega)|K^\circ|^{\frac{p}{2}}$ has nonzero, finite infimum in $\mathcal{K}_2$.

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