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We consider the ferromagnetic q-state Potts model on a finite grid with non-zero external field and periodic boundary conditions. The system evolves according to Glauber-type dynamics described by the Metropolis algorithm, and we focus on the low temperature asymptotic regime. We analyze the case of negative external magnetic field. In this scenario there are $q-1$ stable configurations and a unique metastable state. We describe the asymptotic behavior of the first hitting time from the metastable state to the set of the stable states as $\beta \to \infty$ in probability, in expectation, and in distribution. We also identify the exponent of the mixing time and find an upper and a lower bound for the spectral gap. We identify the minimal gates for the transition from the metastable state to the set of the stable states and for the transition from the metastable state to a fixed stable state. Furthermore, we identify the tube of typical trajectories for these two transitions. The detailed description of the energy landscape that we develop allows us to give precise asymptotics for the expected transition time from the unique metastable state to the set of the stable configurations.

Keywords: Potts model, Ising Model, Glauber dynamics, metastability, tunnelling behaviour, critical droplet, tube of typical trajectories, gate, large deviations, potential theory.
MSC2020: 60K35, 82C20, *secondary*: 60J10, 82C22.

I. INTRODUCTION

Metastability is a phenomenon that is observed when a physical system is close to a first–order phase transition, and it remains stuck for a long time in a state which is different from the equilibrium state. This is known as a *metastable state*. After a long (random) time, the system performs a sudden transition from the metastable state to the stable state. In the models for metastable behavior a suitable stochastic dynamics is chosen and three main issues are typically investigated. The first is the study of the *first hitting time* of the stable state(s) for the process started in the metastable state. The second issue is the study of the *critical configurations* visited by the process with probability close to one during the transition from the metastable state to the stable state(s). The final issue is the study of the *tube of typical paths* of the process during the transition from the metastable state to the stable state(s).

In this paper we study the metastable behavior of the q-state Potts model with non-zero external magnetic field on a finite two-dimensional discrete torus Λ. On each site i of Λ lies a spin with value $\sigma(i) \in \{1,\dots,q\}$, hence the q-state Potts model is an extension of the classical Ising model from $q=2$ to an arbitrary number q of spins with $q>2$. To each configuration σ is associated an energy $H(\sigma)$ that depends on the ferromagnetic interaction between nearest-neighbor spins, and on an external magnetic field h which favors a specific spin value. We focus on the regime of large inverse temperature $\beta \to \infty$. The stochastic evolution is described by a *Glauber-type dynamics*, which is a Markov chain given by the Metropolis algorithm that only allows single spin flip updates. This dynamics is reversible with respect to the so-called *Gibbs measure*, see (2).

Our analysis focuses on the case of negative external magnetic field. In this scenario there are one metastable state and $q-1$ stable states. Without loss of generality, in the metastable configuration all spins are equal to 1. The remaining constant configurations are stable states. The goal of this paper is to investigate all three issues of metastability introduced above for this model. We focus on two classes of transitions: transitions from the metastable state to the set of stable states (briefly denoted

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$\mathbf{1} \to \mathscr{X}^s$) and transitions from the metastable state to any *fixed* stable state (briefly denoted $\mathbf{1} \to \mathbf{s}$). For both transitions, we investigate the transition time, the minimal gates and the tube of typical trajectories. Finally, we identify the prefactor of the expected transition time.

Let us now briefly describe our approach. First we prove that the only metastable configuration is the configuration with all spins equal to 1. For the transition $\mathbf{1} \to \mathscr{X}^s$, we are able to obtain the expected value and the distribution of the transition time. This is more complicated for the transition $\mathbf{1} \to \mathbf{s}$. Indeed, in this case with strictly positive probability the optimal path visits a stable state different from $\mathbf{s}$ before hitting $\mathbf{s}$. We prove that the energy barrier between two stable states is strictly larger than the energy barrier between a stable state and any other (non–stable) state. In view of this, we prove that the lower and the upper asymptotic bounds for the transition time have different exponents, see Remark III.1. Moreover, we characterize the behavior of the *mixing time* in the low-temperature regime and give an estimate of the *spectral gap*, see (25) and (26) for the formal definitions. Next, we identify the set of all minimal gates. In particular, we prove that this set is given by those configurations in which all spins are 1 except for a quasi-square of spins $s \in \{2,\dots,q\}$ with a unit protuberance on one of the longest sides. Using the so-called *potential-theoretic approach*, we give sharp estimates on the expected transition time by computing the prefactor explicitly. This requires a detailed knowledge of the critical configurations and the configurations connected to them. Finally, we give a geometric characterization of the configurations that belong to the tube of typical paths for both transitions.

a. Literature on the Potts model All grouped citations here and henceforth are in chronological order of publication. The Potts model is one of the most studied statistical physics models, as the vast literature on the subject, both on the mathematics side and the physics side, attests. The tunneling behaviour for the Potts model with zero external magnetic field has been studied in [1–3]. In this energy landscape there are q stable states and there is not any relevant metastable state. In [1], the authors derive the asymptotic behavior of the first hitting time for the transition between stable configurations, and give results in probability, in expectation and in distribution. They also characterize the behavior of the mixing time and give a lower and an upper bound for the spectral gap. In [2], the authors study the tunneling from a stable state to the other stable configurations and between two stable states. In both cases, they give a geometric characterization of the union of all minimal gates and the tube of typical trajectories. Finally, in [3], the authors study the model in dimensions two and three. They give a description of the so-called *gateway configurations* in order to compute the prefactor. We note that these gateway configurations are quite different from the minimal gates in [2]. The q-Potts model with *positive* external magnetic field has been studied in [4]. In this scenario there are $q-1$ multiple degenerate metastable states and a unique stable configuration. The authors answer all the three issues of the metastability introduced above for the transition from any metastable to the stable state. Finally, metastability for the Potts model with *three* colors, but general coupling constants, has been studied in [5].

b. Literature on metastability In this paper we adopt the framework known as *pathwise approach*, which was initiated in 1984 by Cassandro, Galves, Olivieri, Vares in [6] and it was further developed in [7] and independently in [8], see also references therein. The pathwise approach requires a detailed knowledge of the energy landscape to give quantitative answers to the three issues of metastability in the form of ad hoc large deviations estimates. This approach was further developed in [9, 10] (see also references therein) by separating the study of the transition time and of critical configurations from that of the tube of typical trajectories. Indeed, it was recognized that the latter requires more detailed model-dependant inputs. The pathwise approach has been recently used in [11] to tackle the three issues of metastability for Ising-like models with Glauber dynamics, and in [12–14] for Kawasaki dynamics. For more details, see the recent review [15]. The so-called *potential-theoretical approach* exploits a suitable Dirichlet form and spectral properties of the transition matrix to give sharp asymptotics for the hitting time. More precisely, this method estimates the leading order of the expected value of the transition time including its *prefactor*, see [16, 17] and references therein. The potential theoretical approach was applied to find the prefactor for Ising-like models and the hard-core model in [17–21] for Glauber dynamics and in [22, 23] for parallel dynamics. Recently, other approaches have been formulated in [24–26] and in [27] and they are particularly suited to estimating the prefactor when dealing with the tunnelling between two or more stable states.

c. Outline In Section II we define the ferromagnetic q-state Potts model and the associated Hamiltonian. We state our main results in Section III. In Section IV we analyse the energy landscape and give the proofs of some useful model-dependent results that are used throughout all the next sections. In Subsections V B and V C we give the explicit proofs of the main results on the critical configurations and on the tube of typical paths, respectively. Finally, in Section VI we compute the prefactor and refine the estimate on the expected transition time.

II. MODEL DESCRIPTION

In the q-state Potts model each spin lies on a vertex of a finite two-dimensional rectangular lattice $\Lambda = (V,E)$, where $V = \{0,\dots,K-1\} \times \{0,\dots,L-1\}$ is the vertex set and E is the edge set, namely the set of the pairs of vertices whose spins interact with each other. We identify each pair of vertices lying on opposite sides of the rectangular lattice, so that we obtain a two-dimensional torus. We denote by S the set of spin values, i.e., $S := \{1,\dots,q\}$ and assume $q > 2$. To each vertex $v \in V$ is associated a spin value $\sigma(v) \in S$, and $\mathscr{X} := S^V$ denotes the set of spin configurations.

We denote by $\mathbf{1},\dots,\mathbf{q} \in \mathscr{X}$ those configurations in which all the vertices have spin value $1,\dots,q$, respectively.

To each configuration $\sigma \in \mathscr{X}$ we associate the energy $H(\sigma)$ given by

$$H(\sigma) = -J \sum_{(v,w)\in E} \mathbb{1}_{\{\sigma(v)=\sigma(w)\}} + h \sum_{u\in V} \mathbb{1}_{\{\sigma(u)=1\}}, \tag{1}$$

where J is the *coupling constant* and h is the *negative external magnetic field*. We call h *negative* since there is a minus in front of H. In this paper we consider the ferromagnetic Potts model and set $J = 1$.

The *Gibbs measure* for the q-state Potts model on Λ is a probability distribution on the state space $\mathscr{X}$ given by

$$\mu_\beta(\sigma) := \frac{e^{-\beta H(\sigma)}}{Z}, \tag{2}$$

where $\beta > 0$ is the inverse temperature and where $Z := \sum_{\sigma'\in\mathscr{X}} e^{-\beta H(\sigma')}$.

The spin system evolves according to a Glauber-type dynamics. This dynamics is described by a single-spin update Markov chain $\{X_t^\beta\}_{t\in\mathbb{N}}$ on the state space $\mathscr{X}$ with the following transition probabilities: for $\sigma, \sigma' \in \mathscr{X}$,

$$P_\beta(\sigma,\sigma') := \begin{cases} Q(\sigma,\sigma') e^{-\beta[H(\sigma')-H(\sigma)]^+}, & \text{if } \sigma \neq \sigma', \\ 1 - \sum_{\eta\neq\sigma} P_\beta(\sigma,\eta), & \text{if } \sigma = \sigma', \end{cases} \tag{3}$$

where $[n]^+ := \max\{0,n\}$ is the positive part of n and

$$Q(\sigma,\sigma') := \begin{cases} \frac{1}{q|V|}, & \text{if } |\{v \in V : \sigma(v) \neq \sigma'(v)\}| = 1, \\ 0, & \text{if } |\{v \in V : \sigma(v) \neq \sigma'(v)\}| > 1, \end{cases} \tag{4}$$

for any $\sigma, \sigma' \in \mathscr{X}$. Q is the so-called *connectivity matrix* and it is symmetric and irreducible, i.e., for all $\sigma, \sigma' \in \mathscr{X}$, there exists a finite sequence of configurations $\omega_1, \dots, \omega_n \in \mathscr{X}$ such that $\omega_1 = \sigma$, $\omega_n = \sigma'$ and $Q(\omega_i, \omega_{i+1}) > 0$ for $i = 1, \dots, n-1$. Hence, the resulting stochastic dynamics defined by (3) is reversible with respect to the Gibbs measure (2). The triplet $(\mathscr{X}, H, Q)$ is called the *energy landscape*. The dynamics defined above belongs to the class of Metropolis dynamics. In particular, at each step the update of vertex v depends on the neighboring spins of v and on the following energy difference

$$H(\sigma^{v,s}) - H(\sigma) = \begin{cases} \sum_{w\sim v}(\mathbb{1}_{\{\sigma(v)=\sigma(w)\}} - \mathbb{1}_{\{\sigma(w)=s\}}) - h, & \text{if } \sigma(v) = 1,\ s \neq 1, \\ \sum_{w\sim v}(\mathbb{1}_{\{\sigma(v)=\sigma(w)\}} - \mathbb{1}_{\{\sigma(w)=s\}}), & \text{if } \sigma(v) \neq 1,\ s \neq 1, \\ \sum_{w\sim v}(\mathbb{1}_{\{\sigma(v)=\sigma(w)\}} - \mathbb{1}_{\{\sigma(w)=s\}}) + h, & \text{if } \sigma(v) \neq 1,\ s = 1, \end{cases} \tag{5}$$

where $\sigma^{v,s}$ is the configuration obtained from σ by updating the spin in the vertex v to s, i.e., $\sigma^{v,s}(w) = \sigma(w)$ if $w \neq v$, $\sigma^{v,s}(w) = s$ if $w = v$.

III. MAIN RESULTS ON THE q-STATE POTTS MODEL WITH NEGATIVE EXTERNAL MAGNETIC FIELD

In this section we state our main results. Note that we give the proof of the main results by considering the condition

$$L \geq K \geq 3\ell^*, \tag{6}$$

where $\ell^* := \lceil \frac{2}{h} \rceil$ is the *critical length*. It is possible to extend the results to the case $K > L$ by interchanging the role of rows and columns in the proof.

In order to state our main results on the Potts model with Hamiltonian as in (1), we assume as follows.

Assumption III.1 *We assume that the following conditions are verified:*
(i) the absolute value of the magnetic field h is such that $0 < h < 1$;
(ii) $2/h$ is not integer.

A. Energy landscape

The first result that we give is the identification of the set of the global minima of the Hamiltonian (1). This follows by simple algebraic calculations.

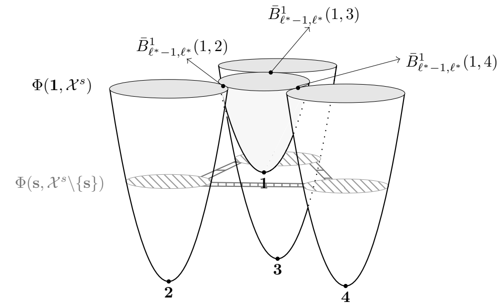


FIG. 1: Schematic picture of the energy landscape below $\Phi(\mathbf{1},\mathscr{X}^s)$ of the 4-state Potts model with negative external magnetic field with $S=\{1,2,3,4\}$, $\mathscr{X}^s=\{\mathbf{2},\mathbf{3},\mathbf{4}\}$. We have not represented the cycles (valleys) that contain configurations with stability level smaller than or equal to 2 (see Proposition III.2).

Proposition III.1 (Identification of $\mathscr{X}^s$) *If the external magnetic field is negative, then the set of the global minima $\mathscr{X}^s$ of the Hamiltonian* (1) *is given by $\mathscr{X}^s=\{\mathbf{2},\dots,\mathbf{q}\}$.*

Next, we prove that the q-state Potts model with Hamiltonian H defined in (1) has only one metastable state and we give an estimate of the stability level of this configuration. Formally, we call *path* a finite sequence ω of configurations $\omega_0,\dots,\omega_n\in\mathscr{X}$, $n\in\mathbb{N}$, such that $Q(\omega_i,\omega_{i+1})>0$ for $i=0,\dots,n-1$. Let $\Omega_{\sigma,\sigma'}$ be the set of all paths between σ and σ'. Given a path $\omega=(\omega_0,\dots,\omega_n)$, we define the *height* of ω as

$$\Phi_\omega := \max_{i=0,\dots,n} H(\omega_i). \tag{7}$$

For any pair $\sigma,\sigma'\in\mathscr{X}$, the *communication height* $\Phi(\sigma,\sigma')$ between σ and σ' is the minimal energy across all paths $\omega:\sigma\to\sigma'$, i.e.,

$$\Phi(\sigma,\sigma') := \min_{\omega:\sigma\to\sigma'} \Phi_\omega = \min_{\omega:\sigma\to\sigma'} \max_{\eta\in\omega} H(\eta). \tag{8}$$

We define the set of *optimal paths* between $\sigma,\sigma'\in\mathscr{X}$ as

$$\Omega^{opt}_{\sigma,\sigma'} := \{\omega\in\Omega_{\sigma,\sigma'}:\ \max_{\eta\in\omega} H(\eta)=\Phi(\sigma,\sigma')\}. \tag{9}$$

For any $\sigma\in\mathscr{X}$, let

$$\mathscr{I}_\sigma := \{\eta\in\mathscr{X}:\ H(\eta)<H(\sigma)\} \tag{10}$$

be the set of states with energy strictly smaller than $H(\sigma)$. We define *stability level* of σ the energy barrier

$$V_\sigma := \Phi(\sigma,\mathscr{I}_\sigma)-H(\sigma). \tag{11}$$

If $\mathscr{I}_\sigma=\varnothing$, we set $V_\sigma:=\infty$. Finally, we define the set of *metastable states* as

$$\mathscr{X}^m := \{\eta\in\mathscr{X}: V_\eta = \max_{\sigma\in\mathscr{X}\backslash\mathscr{X}^s} V_\sigma\}. \tag{12}$$

Furthermore, for any $\sigma\in\mathscr{X}$ and any $\varnothing\neq\mathscr{A}\subset\mathscr{X}$, we set

$$\Gamma(\sigma,\mathscr{A}) := \Phi(\sigma,\mathscr{A})-H(\sigma). \tag{13}$$

We refer to Figure 1 for an illustration of the 4-Potts model.

Theorem III.1 (Identification of $\mathscr{X}^m$) *If the external magnetic field is negative, then $\mathscr{X}^m=\{\mathbf{1}\}$ and*

$$\Gamma^m := \Gamma(\mathbf{1},\mathscr{X}^s) = 4\ell^* - h(\ell^*(\ell^*-1)+1). \tag{14}$$

Proof. To prove this, we apply [28, Theorem 2.4]. The first assumption on the identification of the communication height follows by Lemma IV.1 and Lemma IV.3. The second assumption, the estimate of the stability level of any $\sigma\in\mathscr{X}\backslash\{\mathbf{1},\dots,\mathbf{q}\}$, is proved in Proposition III.2. □

In the following proposition, which we prove in Subsection IV B, we give a uniform estimate of the stability level for any configuration $\eta\in\mathscr{X}\backslash\{\mathbf{1},\dots,\mathbf{q}\}$.

Proposition III.2 (Estimate on the stability level) *If the external magnetic field is negative, then for any $\eta \in \mathscr{X}\backslash\{\mathbf{1},\dots,\mathbf{q}\}$, $V_\eta \le 2 < \Gamma(\mathbf{1},\mathscr{X}^s)$.*

We define *metastable set at level* V the set of all the configurations with stability level larger than V, i.e.,

$$\mathscr{X}_V := \{\sigma \in \mathscr{X} : V_\sigma > V\}. \tag{15}$$

Moreover, given a non-empty subset $\mathscr{A} \subset \mathscr{X}$ and a configuration $\sigma \in \mathscr{X}$, we define

$$\tau^\sigma_{\mathscr{A}} := \inf\{t > 0 :\ X^\beta_t \in \mathscr{A}\} \tag{16}$$

as the *first hitting time* of the subset $\mathscr{A}$ for the Markov chain $\{X^\beta_t\}_{t\in\mathbb{N}}$ starting from σ at time $t = 0$. Exploiting the estimate of the stability level in Proposition III.2, we obtain the following result on a recurrence property to metastable and stable states, i.e., $\{\mathbf{1},\dots,\mathbf{q}\}$. Intuitively, the result can be understood as follows. Any path out of a valley of depth two will involve the creation of one protuberance along an interface between two stable states or a stable state and a metastable state. In fact, a protuberance is created by a single spin flip occurring with probability roughly proportional to $\exp(-2\beta)$. Assuming that every other possible spin flip results in a configuration that is still in the same local valley and with the same energy, the number of attempts T for the system to escape the valley is distributed as a geometric random variable with parameter roughly $\exp(-2\beta)$. Hence, the tail of the (rescaled) distribution of T can be estimated as

$$\mathbb{P}(T \ge e^{2\beta}x) = (1-e^{-2\beta})^{e^{2\beta}x} \le e^{-x}. \tag{17}$$

Choosing $x = \exp(\beta\varepsilon)$ gives $\mathbb{P}(T \ge e^{(2+\varepsilon)\beta}) \le \exp(-\exp(\varepsilon\beta))$. This heuristic suggests that the (rescaled) time to exit a local minima has sub-exponential tails. In fact, a stronger result holds, and this is the subject of the following theorem.

Theorem III.2 (Recurrence property) *If the external magnetic field is negative, then for any $\sigma \in \mathscr{X}$ and for any $\varepsilon > 0$ there exists $k > 0$ such that for β sufficiently large*

$$\mathbb{P}(\tau^\sigma_{\{\mathbf{1},\dots,\mathbf{q}\}} > e^{\beta(2+\varepsilon)}) \le e^{-e^{k\beta}}. \tag{18}$$

Proof. Apply [29, Theorem 3.1] with $V = 2$ and use (15) and Proposition III.2 to get $\mathscr{X}_2 = \{\mathbf{1},\dots,\mathbf{q}\} = \mathscr{X}^s \cup \mathscr{X}^m$, where the last equality follows by Proposition III.1 and Theorem III.1. □

From Theorem III.2 follows that the function $\beta \to f(\beta) := \mathbb{P}(\tau^\sigma_{\{\mathbf{1},\dots,\mathbf{q}\}} > e^{\beta(2+\varepsilon)})$ satisfies $\lim_{\beta\to\infty} \frac{\log f(\beta)}{\beta} = -\infty$ and such a function is known as *super-exponentially small*.

From Proposition III.1, we have that when $q > 2$ the energy landscape $(\mathscr{X}, H, Q)$ has multiple stable states. We are interested in studying the transition from the metastable state $\mathbf{1}$ to $\mathscr{X}^s$ and also the transition from $\mathbf{1}$ to a fixed stable configuration $\mathbf{s} \in \mathscr{X}^s$. To this end, it is useful to compare the communication energy between two different stable states and the communication energy between the metastable state and a stable configuration, see Theorem III.3. Furthermore, for any $\mathbf{s} \in \mathscr{X}^s$ in order to find the asymptotic upper bound in probability for $\tau^{\mathbf{1}}_{\mathbf{s}}$, we estimate the maximum energy barrier that the process started from $\mathbf{r} \in \mathscr{X}^s\backslash\{\mathbf{s}\}$ has to overcome to reach $\mathbf{s}$, see Theorem III.4. These are the goals of the next result. In order to state it, we need some further definitions. A non-empty subset $\mathscr{C} \subset \mathscr{X}$ is called *cycle* if it is either a singleton or a connected set such that

$$\max_{\sigma\in\mathscr{C}} H(\sigma) < H(\mathscr{F}(\partial\mathscr{C})). \tag{19}$$

When $\mathscr{C}$ is a singleton, it is said to be a *trivial cycle*. Let $\mathscr{C}(\mathscr{X})$ be the set of cycles of $\mathscr{X}$.
The *depth* of a cycle $\mathscr{C}$ is given by

$$\Gamma(\mathscr{C}) := H(\mathscr{F}(\partial\mathscr{C})) - H(\mathscr{F}(\mathscr{C})). \tag{20}$$

If $\mathscr{C}$ is a trivial cycle we set $\Gamma(\mathscr{C}) = 0$.
Given a non-empty set $\mathscr{A} \subset \mathscr{X}$, we denote by $\mathscr{M}(\mathscr{A})$ the *collection of maximal cycles* $\mathscr{A}$, i.e., $\mathscr{M}(\mathscr{A}) := \{\mathscr{C} \in \mathscr{C}(\mathscr{X}) |\ \mathscr{C}$ maximal by inclusion under constraint $\mathscr{C} \subseteq \mathscr{A}\}$. For any $\mathscr{A} \subset \mathscr{X}$, we define the *maximum depth of* $\mathscr{A}$ as the maximum depth of a cycle contained in $\mathscr{A}$, i.e.,

$$\widetilde{\Gamma}(\mathscr{A}) := \max_{\mathscr{C}\in\mathscr{M}(\mathscr{A})} \Gamma(\mathscr{C}). \tag{21}$$

In [9, Lemma 3.6] the authors give an alternative characterization of (21) as the maximum initial energy barrier that the process started from a configuration $\eta \in \mathscr{A}$ possibly has to overcome to exit from $\mathscr{A}$, i.e., $\widetilde{\Gamma}(\mathscr{A}) = \max_{\eta\in\mathscr{A}} \Gamma(\eta, \mathscr{X}\backslash\mathscr{A})$.
Finally, for any $\sigma \in \mathscr{X}$, if $\mathscr{A}$ is a non-empty target set, we define the *initial cycle* for the transition from σ to $\mathscr{A}$ as $\mathscr{C}^\sigma_{\mathscr{A}}(\Gamma) := \{\sigma\} \cup \{\eta \in \mathscr{X} :\ \Phi(\sigma,\eta) - H(\sigma) < \Gamma = \Phi(\sigma,\mathscr{A}) - H(\sigma)\}$. Note that if $\sigma \notin \mathscr{A}$, then $C^\sigma_{\mathscr{A}}(\Gamma) \cap \mathscr{A} = \varnothing$.

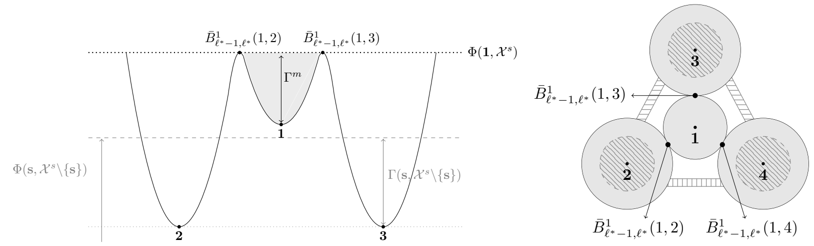


FIG. 2: On the left, we give a side view (vertical section) of the energy landscape depicted in Figure 1. We colour light gray the initial cycle $\mathscr{C}^{\mathbf{1}}_{\mathscr{X}^s}(\Gamma^m)$. On the right, viewpoint from above of the energy landscape depicted in Figure 1 cut to the energy level $\Phi(\mathbf{1},\mathbf{s})$, for some $\mathbf{s}\in\mathscr{X}^s$. The dashed part denotes the energy landscape whose energy value is smaller than $\Phi(\mathbf{1},\mathbf{s})$. The cycles whose bottom is a stable state are deeper than the cycle $\mathscr{C}^{\mathbf{1}}_{\mathscr{X}^s}(\Gamma^m)$ of the metastable state, hence we depict them with circles whose diameter is larger than the one related to the metastable state **1**.

Theorem III.3 *Consider the q-state Potts model on a $K\times L$ grid Λ, with periodic boundary conditions and with negative external magnetic field. For any $\mathbf{s}\in\mathscr{X}^s$, we have*

$$\Phi(\mathbf{1},\mathscr{X}^s) > \Phi(\mathbf{s},\mathscr{X}^s\backslash\{\mathbf{s}\}), \tag{22}$$

$$\Gamma(\mathbf{1},\mathscr{X}^s) < \Gamma(\mathbf{s},\mathscr{X}^s\backslash\{\mathbf{s}\}), \tag{23}$$

$$\widetilde{\Gamma}(\mathscr{X}\backslash\{\mathbf{s}\}) = \Gamma(\mathbf{r},\mathscr{X}^s\backslash\{\mathbf{r}\}),\ \textit{with}\ \mathbf{r}\in\mathscr{X}^s. \tag{24}$$

Proof. For the details we refer to the Appendix A. □

We refer to Figure 1 and Figure 2 for a schematic representation of the energy landscape and of the quantities of Theorem III.3 for the 4-state Potts model with negative magnetic field. An intuitive way to understand the result is the following. Consider the operator $\mathscr{P}_{1s}:\mathscr{X}\to\mathscr{X}$ that acts on a configuration by switching all spins 1 to s. Let $\omega:=(\omega_0,\omega_1,\dots,\omega_n)$ be a path from $\mathbf{1}$ to $\mathscr{X}^s$ that realizes the communication height and such that, say, $\omega_n\neq\mathbf{s}$. Then, the path $\mathscr{P}_{1s}(\omega):=(\mathscr{P}_{1s}(\omega_0),\dots,\mathscr{P}_{1s}(\omega_n))$ also realizes the communication height from $\mathbf{s}$ to $\mathscr{X}^s\backslash\{\mathbf{s}\}$. However, the energy cost of each step of ω that removes a 1 spin is smaller than the cost of the corresponding step in $\mathscr{P}_{1s}(\omega)$. In fact, the difference between the two costs is exactly h. In other words, each step of ω that switches a 1 spin happens with a probability that is larger by a factor $\exp(\beta h)$ than the corresponding step in $\mathscr{P}_{1s}(\omega)$ (due to the interaction contribution to the energy, this probability is still exponentially small). This explains why the height of the optimal path from 1 to $\mathscr{X}^s$ is lower than the height of the path from s to $\mathscr{X}^s\setminus\{\mathbf{s}\}$, that is (23). The result (22) requires more care. On the one hand, as we have argued, the height of the path from $\mathbf{1}$ to $\mathscr{X}^s$ is lower than the height of the path from $\mathbf{s}$ to $\mathscr{X}^s\backslash\{\mathbf{s}\}$. On the other hand, the energy of the starting configuration $\mathbf{1}$ is larger than $H(\mathbf{s})$. Intuitively, (22) can be explained as follows: the difference of the energy of the starting configurations $H(\mathbf{1})-H(\mathbf{s})=hKL$ is larger than the difference of the path heights. Detailed computations are required to better understand this trade-off, and we present these in the proof in Appendix A..

B. Asymptotic behavior of $\tau^{\mathbf{1}}_{\mathscr{X}^s}$ and $\tau^{\mathbf{1}}_{\mathbf{s}}$ and mixing time

In the following theorem we give asymptotic bounds in probability for both $\tau^{\mathbf{1}}_{\mathscr{X}^s}$ and $\tau^{\mathbf{1}}_{\mathbf{s}}$, identify the order of magnitude of the expected value of $\tau^{\mathbf{1}}_{\mathscr{X}^s}$ and prove that the asymptotic rescaled distribution of $\tau^{\mathbf{1}}_{\mathscr{X}^s}$ is exponential. Furthermore, we also identify the mixing time and give an upper and a lower bound for the spectral gap. Formally, let $\{X^\beta_t\}_{t\in\mathbb{N}}$ be the Markov chain with transition probabilities (3) and stationary distribution (2). For every $\varepsilon\in(0,1)$, we define the *mixing time* $t^{\text{mix}}_\beta(\varepsilon)$ by

$$t^{\text{mix}}_\beta(\varepsilon) := \min\{n\geq 0|\ \max_{\sigma\in\mathscr{X}}||P^n_\beta(\sigma,\cdot)-\mu_\beta(\cdot)||_{\text{TV}}\leq\varepsilon\}, \tag{25}$$

where the total variance distance is defined by $||\nu-\nu'||_{\text{TV}}:=\frac{1}{2}\sum_{\sigma\in\mathscr{X}}|\nu(\sigma)-\nu'(\sigma)|$ for every two probability distribution ν,ν' on $\mathscr{X}$. Furthermore, we define *spectral gap* as

$$\rho_\beta := 1-\lambda^{(2)}_\beta, \tag{26}$$

where $1=\lambda^{(1)}_\beta>\lambda^{(2)}_\beta\geq\dots\geq\lambda^{(|\mathscr{X}|)}_\beta\geq -1$ are the eigenvalues of the matrix $P_\beta(\sigma,\eta))_{\sigma,\eta\in\mathscr{X}}$.

Theorem III.4 (Asymptotic behavior of $\tau^{\mathbf{1}}_{\mathscr{X}^s}$ and $\tau^{\mathbf{1}}_{\mathbf{s}}$ and mixing time) *If the external magnetic field is negative, then for any $\mathbf{s}\in\mathscr{X}^s$, the following statements hold:*

(a) *for any* $\varepsilon > 0$, $\lim_{\beta\to\infty}\mathbb{P}_\beta(e^{\beta(\Gamma^m-\varepsilon)} < \tau^{\mathbf{1}}_{\mathscr{X}^s} < e^{\beta(\Gamma^m+\varepsilon)}) = 1$*;*

(b) *for any* $\varepsilon > 0$, $\lim_{\beta\to\infty}\mathbb{P}_\beta(e^{\beta(\Gamma^m-\varepsilon)} < \tau^{\mathbf{1}}_{\mathbf{s}} < e^{\beta(\Gamma(\mathbf{s},\mathscr{X}^s\setminus\{\mathbf{s}\})+\varepsilon)}) = 1$*;*

(c) $\lim_{\beta\to\infty}\frac{1}{\beta}\log\mathbb{E}[\tau^{\mathbf{1}}_{\mathscr{X}^s}] = \Gamma^m$*;*

(d) $\frac{\tau^{\mathbf{1}}_{\mathscr{X}^s}}{\mathbb{E}[\tau^{\mathbf{1}}_{\mathscr{X}^s}]} \xrightarrow{d} \text{Exp}(1)$*;*

(e) *for every* $\varepsilon\in(0,1)$ *and* $\mathbf{s}\in\mathscr{X}^s$, $\lim_{\beta\to\infty}\frac{1}{\beta}\log t^{mix}_\beta(\varepsilon) = \Gamma(\mathbf{s},\mathscr{X}^s\setminus\{\mathbf{s}\})$ *and there exist two constants* $0 < c_1 \le c_2 < \infty$ *independent of* β *such that, for any* $\beta > 0$, $c_1 e^{-\beta\Gamma(\mathbf{s},\mathscr{X}^s\setminus\{\mathbf{s}\})} \le \rho_\beta \le c_2 e^{-\beta\Gamma(\mathbf{s},\mathscr{X}^s\setminus\{\mathbf{s}\})}$.

Proof. Item (a) holds in view of Theorem III.1 and [29, Theorem 4.1]. The lower bound of item (b) follows by Theorem III.1 and [9, Propositions 3.4], while the upper bound by (24) and [9, Propositions 3.7]. Item (c) follows from Theorem III.1 and [29, Theorem 4.9]. Lastly, item (d), i.e., the asymptotic exponentiality of $\tau^{\mathbf{1}}_{\mathscr{X}^s}$, follows from Theorem 24 and [29, Theorem 4.15]. For this last item, we refer also to [9, Theorem 3.19, Example 3]. Item (e) follows by (24) and by [9, Proposition 3.24]. □

Remark III.1 *Note that the lower and upper bounds for* $\tau^{\mathbf{1}}_{\mathbf{s}}$ *in item (b) have different exponents. Indeed, the presence of a subset of the optimal paths, that the process follows with probability strictly positive, going from* $\mathbf{1}$ *to* $\mathbf{s}$ *without crossing* $\mathscr{X}^s\setminus\{\mathbf{s}\}$*, implies that the lower bound is sharp. Moreover, the presence of a subset of the optimal paths going from* $\mathbf{1}$ *to* $\mathbf{s}$ *crossing* $\mathscr{X}^s\setminus\{\mathbf{s}\}$*, ensures that the process, with probability strictly positive, enters at least a cycle* $\mathscr{C}^{\mathbf{r}}_{\mathbf{s}}(\Gamma(\mathbf{r},\mathbf{s}))$ *for any given* $\mathbf{r}\in\mathscr{X}^s\setminus\{\mathbf{s}\}$ *which is deeper than the initial cycle* $\mathscr{C}^{\mathbf{1}}_{\mathbf{s}}(\Gamma^m)$*. This implies that the maximum depth of the cycles crossed by these paths is* $\Gamma(\mathbf{r},\mathbf{s})$*, thus the upper is sharp. Finally, we remark that in [4, Theorem 4.3] items (a) and (b) coincide since in that scenario there is a unique stable state.*

C. Minimal gates for the metastable transitions

We also identify the set of minimal gates for the transition $\mathbf{1}\to\mathscr{X}^s$ and also for the transition $\mathbf{1}\to\mathbf{s}$ for some fixed $\mathbf{s}\in\mathscr{X}^s$. To this end, we need some further definitions. The set of *minimal saddles* between $\sigma,\sigma'\in\mathscr{X}$ is defined as

$$\mathscr{S}(\sigma,\sigma') := \{\xi\in\mathscr{X} : \exists\omega\in\Omega^{opt}_{\sigma,\sigma'},\ \xi\in\omega :\ \max_{\eta\in\omega} H(\eta) = H(\xi)\}. \tag{27}$$

We say that $\eta\in\mathscr{S}(\sigma,\sigma')$ is an *essential saddle* if there exists $\omega\in\Omega^{opt}_{\sigma,\sigma'}$ such that either

- $\{\arg\max_\omega H\} = \{\eta\}$ or
- $\{\arg\max_\omega H\} \supset \{\eta\}$ and $\{\arg\max_{\omega'} H\} \not\subseteq \{\arg\max_\omega H\}\setminus\{\eta\}$ for all $\omega'\in\Omega^{opt}_{\sigma,\sigma'}$.

A saddle $\eta\in\mathscr{S}(\sigma,\sigma')$ that is not essential is said to be *unessential*.
Given $\sigma,\sigma'\in\mathscr{X}$, we say that $\mathscr{W}(\sigma,\sigma')$ is a *gate* for the transition from σ to σ' if $\mathscr{W}(\sigma,\sigma')\subseteq\mathscr{S}(\sigma,\sigma')$ and $\omega\cap\mathscr{W}(\sigma,\sigma')\neq\varnothing$ for all $\omega\in\Omega^{opt}_{\sigma,\sigma'}$. We say that $\mathscr{W}(\sigma,\sigma')$ is a *minimal gate* for the transition from σ to σ' if it is a minimal (by inclusion) subset of $\mathscr{S}(\sigma,\sigma')$ that is visited by all optimal paths. More in detail, it is a gate and for any $\mathscr{W}'\subset\mathscr{W}(\sigma,\sigma')$ there exists $\omega'\in\Omega^{opt}_{\sigma,\sigma'}$ such that $\omega'\cap\mathscr{W}'=\varnothing$. We denote by $\mathscr{G}=\mathscr{G}(\sigma,\sigma')$ the union of all minimal gates for the transition $\sigma\to\sigma'$.
In our scenario, we define

$$\mathscr{W}(\mathbf{1},\mathscr{X}^s) := \bigcup_{t=2}^{q}\bar{B}^1_{\ell^*-1,\ell^*}(1,t) \ \text{ and }\ \mathscr{W}'(\mathbf{1},\mathscr{X}^s) := \bigcup_{t=2}^{q}\bar{B}^1_{\ell^*,\ell^*-1}(1,t), \tag{28}$$

where $\bar{B}^l_{a,b}(r,s)$ denotes the set of those configurations in which all the vertices have spins r, except those, which have spins s, in a rectangle $a\times b$ with a bar $1\times l$ adjacent to one of the sides of length b, with $1\le l\le b-1$.
We refer to Figure 16(b)–(c) for an example of configurations belonging respectively to $\mathscr{W}'(\mathbf{1},\mathscr{X}^s)$ and to $\mathscr{W}(\mathbf{1},\mathscr{X}^s)$. These sets are investigated in Subsection V A where we study the gate for the transition from the metastable state $\mathbf{1}$ to $\mathscr{X}^s$. The characterization of the set of minimal gates for the transition $\mathbf{1}\to\mathscr{X}^s$ can be intuitively understood as follows. Any element of the minimal gate necessarily contains only spins of two colors (1 and, say, s). Then, any minimal gate for the 2-colors Potts model (i.e., the Ising model) is also a minimal gate for the q-colors Potts model with $q<2$. On the other hand, any optimal path from $\mathbf{1}$ to a specific $\mathbf{s}\in\mathscr{X}^s$ must necessarily hit $\mathscr{X}^s$. Therefore, any minimal gate for the transition $\mathbf{1}\to\mathscr{X}^s$ is also a minimal gate for the transition $\mathbf{1}\to\mathbf{s}$. The proofs of the results given in Subsection V B formalize these intuitions.

Theorem III.5 (Minimal gates for the transition $\mathbf{1} \to \mathscr{X}^s$) *If the external magnetic field is negative, then $\mathscr{W}(\mathbf{1}, \mathscr{X}^s)$ is a minimal gate for the transition from the metastable state $\mathbf{1}$ to $\mathscr{X}^s$. Moreover,*

$$\mathscr{G}(\mathbf{1}, \mathscr{X}^s) = \mathscr{W}(\mathbf{1}, \mathscr{X}^s). \tag{29}$$

We refer to Figures 1 and 2 for illustrations of the set $\mathscr{W}(\mathbf{1}, \mathscr{X}^s)$ when $q = 4$.

Finally, in Theorem III.6 we establish the set of all minimal gates for the transition from the metastable state $\mathbf{1}$ to a fixed stable configuration $\mathbf{s} \in \mathscr{X}^s$. In particular, starting from $\mathbf{1}$ the process may visit some stable states in $\mathscr{X}^s \backslash \{\mathbf{s}\}$ before hitting $\mathbf{s}$. Thanks to Theorem III.3, we get that along any optimal path between two different stable states the process only visits states with energy value lower than $\Phi(\mathbf{1}, \mathscr{X}^s)$ and so it does not visit any other gate. See for instance Figure 1 and Figure 2, where we indicate with a dashed gray line the communication energy among the stable states.

Theorem III.6 (Minimal gates for the transition $\mathbf{1} \to \mathbf{s} \in \mathscr{X}^s$) *If the external magnetic field is negative, then for any $\mathbf{s} \in \mathscr{X}^s$, $\mathscr{W}(\mathbf{1}, \mathbf{s}) \equiv \mathscr{W}(\mathbf{1}, \mathscr{X}^s)$ is a minimal gate for the transition from the metastable state $\mathbf{1}$ to $\mathbf{s}$ and*

$$\mathscr{G}(\mathbf{1}, \mathbf{s}) \equiv \mathscr{G}(\mathbf{1}, \mathscr{X}^s). \tag{30}$$

We defer to Subsection V B for the proof of the theorem above. Finally, in the next corollary we prove that in both the transitions, i.e., $\mathbf{1} \to \mathscr{X}^s$ and $\mathbf{1} \to \mathbf{s} \in \mathscr{X}^s$, the process typically intersects the gates.

Corollary III.1 *If the external magnetic field is negative, then for any $\mathbf{s} \in \mathscr{X}^s$,*

(a) $\lim_{\beta \to \infty} \mathbb{P}_\beta(\tau^{\mathbf{1}}_{\mathscr{W}(\mathbf{1}, \mathscr{X}^s)} < \tau^{\mathbf{1}}_{\mathscr{X}^s}) = 1$;

(b) $\lim_{\beta \to \infty} \mathbb{P}_\beta(\tau^{\mathbf{1}}_{\mathscr{W}(\mathbf{1}, \mathbf{s})} < \tau^{\mathbf{1}}_{\mathbf{s}}) = 1$.

Proof. The corollary follows from Theorems III.5 and III.6 and from [29, Theorem 5.4]. □

Finally, we remark that in [4, Theorem 4.5] the authors identify the union of all minimal gates for the metastable transition for the q-Potts model with *positive* external magnetic field. These minimal gates have the same geometric definition of those of our scenario, the main difference is that in the positive scenario the spins inside the quasi-square union a unit protuberance and in the sea are fixed, while in the negative case we have to take the union on all $t \in S \backslash \{1\}$, see (28).

D. Sharp estimate on the mean transition time

Exploiting the model-dependent results given in Subsections III A and III C and some model-independent results by [16, 30] and from [12], in Subsection VI A we prove the following theorem in which we refine the result of Theorem III.4(c) by identifying the precise scaling of the prefactor multipling the exponential.

Theorem III.7 (Mean crossover time) *If the external magnetic field is negative, then there exists a constant $K \in (0, \infty)$ such that*

$$\lim_{\beta \to \infty} e^{-\beta \Gamma^m} \mathbb{E}_{\mathbf{1}}(\tau_{\mathscr{X}^s}) = K. \tag{31}$$

In particular, the constant K is the so-called prefactor and it is given by

$$K = \frac{3}{4} \frac{1}{2\ell^* - 1} \frac{1}{q-1} \frac{1}{|\Lambda|}. \tag{32}$$

Remark III.2 *In order to prove Theorem III.4(c) the only model-independent inputs are the identification of $\mathscr{X}^m$, the recurrence property given in Theorem III.2, and the computation of the energy barrier $\Gamma(\mathbf{1}, \mathscr{X}^s)$ for the transition from the metastable state to the stable configurations, see* (14). *On the other hand, in order to prove Theorem III.7 we need of a more accurate knowledge of the energy landscape. Indeed, it is necessary to know the geometrical identification of the critical configurations and of the configurations connected to them by a single step of the dynamics for the transition $\mathbf{1} \to \mathscr{X}^s$, that we give in Theorem III.5.*

E. Tube of typical trajectories of the metastable transitions

In this subsection we give the results on the tube of typical trajectories $\mathfrak{T}_{\mathscr{X}^s}(\mathbf{1})$ and $\mathfrak{T}_{\mathbf{s}}(\mathbf{1})$ for both the transitions $\mathbf{1} \to \mathscr{X}^s$ and $\mathbf{1} \to \mathbf{s}$ for any fixed $\mathbf{s} \in \mathscr{X}^s$. The tube $\mathfrak{T}_{\mathscr{X}^s}(\mathbf{1})$ (resp. $\mathfrak{T}_{\mathbf{s}}(\mathbf{1})$) can be characterized, and indeed identified, by only relying on the geometrical structure of the energy landscape. Once this is done it follows from standard model-independent considerations [7, 9] that the dynamics leaves $\mathfrak{T}_{\mathscr{X}^s}(\mathbf{1})$ (resp. $\mathfrak{T}_{\mathbf{s}}(\mathbf{1})$) through its non-principal boundary before reaching $\mathbf{1}$ with exponentially small probability. In particular, the non-principal boundary are all those configurations on the boundary that do not minimize the energy. From this follows that $\mathfrak{T}_{\mathscr{X}^s}(\mathbf{1})$ (resp. $\mathfrak{T}_{\mathbf{s}}(\mathbf{1})$) contains those configurations which are visited with positive probability before hitting $\mathscr{X}^s$ (resp. $\mathbf{s}$) as $\beta \to \infty$. Formally, for any $\mathscr{C} \in \mathscr{C}(\mathscr{X})$, we define as

$$\mathscr{B}(\mathscr{C}) := \begin{cases} \mathscr{F}(\partial\mathscr{C}) & \text{if } \mathscr{C} \text{ is a non-trivial cycle,} \\ \{\eta \in \partial\mathscr{C} : H(\eta) < H(\sigma)\} & \text{if } \mathscr{C} = \{\sigma\} \text{ is a trivial cycle,} \end{cases} \tag{33}$$

the *principal boundary* of $\mathscr{C}$. Furthermore, let $\partial^{np}\mathscr{C}$ be the *non-principal boundary* of $\mathscr{C}$, i.e., $\partial^{np}\mathscr{C} := \partial\mathscr{C} \setminus \mathscr{B}(\mathscr{C})$.

The tube is defined in terms of unions of $\bar{B}^l_{a,b}(r,s)$, defined below Theorem 28, and of the following sets.

- $\bar{R}_{a,b}(r,s)$ denotes the set of those configurations in which all the vertices have spins equal to r, except those, which have spins s, in a rectangle $a \times b$. Note that when either $a = L$ or $b = K$, $\bar{R}_{a,b}(r,s)$ contains those configurations which have an r-strip and an s-strip. In particular, a configuration σ has an *s-strip* if it has a cluster of spins s which is a rectangle that wraps around Λ. For any $r,s \in S$, we say that an s-strip is *adjacent* to an r-strip if they are at lattice distance one from each other. For instance, in Figure 11(a) there are depicted vertical adjacent strips.
- For any $s \neq 1$, we define

$$\mathscr{S}^v(1,s) := \{\sigma \in \mathscr{X}(1,s) : \sigma \text{ has a vertical } s\text{-strip of thickness at least } \ell^* \text{ with possibly a bar of length } l = 1,\dots,K \text{ on one of the two vertical edges}\}, \tag{34}$$

$$\mathscr{S}^h(1,s) := \{\sigma \in \mathscr{X}(1,s) : \sigma \text{ has a horizontal } s\text{-strip of thickness at least } \ell^* \text{ with possibly a bar of length } l = 1,\dots,L \text{ on one of the two horizontal edges}\}, \tag{35}$$

where $\mathscr{X}(r,s) = \{\sigma \in \mathscr{X} : \sigma(v) \in \{r,s\} \text{ for any } v \in V\}$.

Theorem III.8 (Tube of typical paths for the transition $\mathbf{1} \to \mathscr{X}^s$) *If the external magnetic field is negative, then the tube of typical trajectories for the transition $\mathbf{1} \to \mathscr{X}^s$ is*

$$\begin{aligned} \mathfrak{T}_{\mathscr{X}^s}(\mathbf{1}) := \bigcup_{s=2}^{q} \Bigg[& \bigcup_{\ell=1}^{\ell^*-1} \bar{R}_{\ell-1,\ell}(1,s) \cup \bigcup_{\ell=1}^{\ell^*} \bar{R}_{\ell-1,\ell-1}(1,s) \cup \bigcup_{\ell=1}^{\ell^*-1} \bigcup_{l=1}^{\ell-1} \bar{B}^l_{\ell-1,\ell}(1,s) \cup \bigcup_{\ell=1}^{\ell^*} \bigcup_{l=1}^{\ell-2} \bar{B}^l_{\ell-1,\ell-1}(1,s) \cup \bar{B}^1_{\ell^*-1,\ell^*}(1,s) \\ & \cup \bigcup_{\ell_1=\ell^*}^{K-1} \bigcup_{\ell_2=\ell^*}^{K-1} \bar{R}_{\ell_1,\ell_2}(1,s) \cup \bigcup_{\ell_1=\ell^*}^{K-1} \bigcup_{\ell_2=\ell^*}^{K-1} \bigcup_{l=1}^{\ell_2-1} \bar{B}^l_{\ell_1,\ell_2}(1,s) \cup \bigcup_{\ell_1=\ell^*}^{L-1} \bigcup_{\ell_2=\ell^*}^{L-1} \bar{R}_{\ell_1,\ell_2}(1,s) \cup \bigcup_{\ell_1=\ell^*}^{L-1} \bigcup_{\ell_2=\ell^*}^{L-1} \bigcup_{l=1}^{\ell_2-1} \bar{B}^l_{\ell_1,\ell_2}(1,s) \cup \mathscr{S}^v(1,s) \cup \mathscr{S}^h(1,s) \Bigg]. \end{aligned} \tag{36}$$

Furthermore, there exists $k > 0$ such that for β sufficiently large

$$\mathbb{P}_\beta(\tau^{\mathbf{1}}_{\partial^{np}\mathfrak{T}_{\mathscr{X}^s}(\mathbf{1})} \leq \tau^{\mathbf{1}}_{\mathscr{X}^s}) \leq e^{-k\beta}. \tag{37}$$

Note that in [4, Theorem 4.7] the authors identify the tube of typical trajectories for the metastable transition for the q-Potts model with *positive* external magnetic field. This tube has the a similar geometric definition of the tube (36) of our scenario, the main difference is that in this negative scenario we have to take the union on all $t \in S \setminus \{1\}$.

Remark III.3 *In [2] the authors study the q-state Potts model with zero external magnetic field. Since in this energy landscape there are q stable configurations and no relevant metastable states, the authors study the transitions between stable states. More precisely, they identify the union of all minimal gates and the tube of typical paths for the transition between two fixed stable states and these results hold also in the current scenario for the transition $\mathbf{r} \to \mathbf{s}$ for any $\mathbf{r},\mathbf{s} \in \mathscr{X}^s$, $\mathbf{r} \neq \mathbf{s}$. Indeed, the communication height computed in Subsection IV C is equal to the one given in [1] and its value is strictly lower than $\Phi(\mathbf{1}, \mathscr{X}^s)$ as we prove in Theorem III.3. It follows that for any $\mathbf{r},\mathbf{s} \in \mathscr{X}^s$, $\mathbf{r} \neq \mathbf{s}$, any optimal path for the transition $\mathbf{r} \to \mathbf{s}$ does not visit the metastable state $\mathbf{1}$ and for this type of transition the identification of the union of all minimal gates and of the tube of typical trajectories is given by [2, Theorem 3.6] and [2, Theorem 4.3], respectively.*

Using Remark III.3, the tube of typical paths for the transition from the metastable to any fixed stable state is

$$\mathfrak{T}_{\mathbf{s}}(\mathbf{1}) := \mathfrak{T}_{\mathscr{X}^s}(\mathbf{1}) \cup \bigcup_{\mathbf{r}\in\mathscr{X}^s\setminus\{\mathbf{s}\}} \mathfrak{T}_{\mathbf{s}}^{\text{zero}}(\mathbf{r}), \tag{38}$$

where $\mathfrak{T}_{\mathbf{s}}^{\text{zero}}(\mathbf{r})$ is given by [2, Equation 4.23].

Theorem III.9 (Tube of typical paths for the transition $\mathbf{1} \to \mathbf{s}$) *If the external magnetic field is negative, then for any $\mathbf{s} \in \mathscr{X}^s$ the tube of typical trajectories for the transition $\mathbf{1} \to \mathbf{s}$ is by* (38)*. Furthermore, there exists $k > 0$ such that for β sufficiently large*

$$\mathbb{P}_\beta(\tau^{\mathbf{1}}_{\partial^{np}\mathfrak{T}_{\mathbf{s}}(\mathbf{1})} \leq \tau^{\mathbf{1}}_{\mathbf{s}}) \leq e^{-k\beta}. \tag{39}$$

We defer the explicit proof of Theorem 39 in Subsection V C.

IV. ENERGY LANDSCAPE ANALYSIS

We devote this section to analyse the energy landscape of the q-state Potts model with negative external magnetic field.

A. Disagreeing edges, bridges and crosses

In the following list we introduce the notions of *disagreeing edges*, *bridges* and *crosses* of a Potts configuration on a grid-graph Λ. These definitions are taken from [1].

- We call $e = (v,w) \in E$ a *disagreeing edge* if it connects two vertices with different spin values, i.e., $\sigma(v) \neq \sigma(w)$.
- For any $i = 0,\dots,K-1$, let

$$d_{r_i}(\sigma) := \sum_{(v,w)\in r_i} \mathbb{1}_{\{\sigma(v)\neq\sigma(w)\}} \tag{40}$$

be the total number of disagreeing horizontal edges on row r_i. Furthermore, for any $j = 0,\dots,L-1$ let

$$d_{c_j}(\sigma) := \sum_{(v,w)\in c_j} \mathbb{1}_{\{\sigma(v)\neq\sigma(w)\}}, \tag{41}$$

be the total number of disagreeing vertical edges on column c_j.
- We define $d_h(\sigma)$ as the total number of disagreeing horizontal edges and $d_v(\sigma)$ as the total number of disagreeing vertical edges.
Since we may partition the edge set E in the two subsets of horizontal edges E_h and of vertical edges E_v, such that $E_h \cap E_v = \varnothing$, the total number of disagreeing edges is given by

$$\sum_{(v,w)\in E_v} \mathbb{1}_{\{\sigma(v)\neq\sigma(w)\}} + \sum_{(v,w)\in E_h} \mathbb{1}_{\{\sigma(v)\neq\sigma(w)\}} = d_v(\sigma) + d_h(\sigma). \tag{42}$$

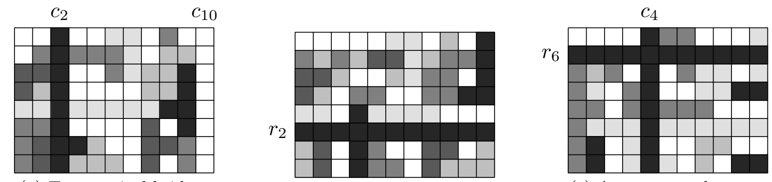
- We say that σ has a *horizontal bridge* on row r if $d_r(\sigma) = 0$.
- We say that σ has a *vertical bridge* on column c if $d_c(\sigma) = 0$.
- We say that $\sigma \in \mathscr{X}$ has a *cross* if it has at least one vertical and one horizontal bridge. If σ has a bridge of spins $s \in S$, then we say that σ has an s-bridge. Similarly, if σ has a cross of spins s, we say that σ has an s-cross.
- For any $s \in S$, the total number of s-bridges of the configuration σ is denoted by $B_s(\sigma)$. Note that if a configuration $\sigma \in \mathscr{X}$ has an s-cross, then $B_s(\sigma)$ is at least 2.

B. Metastable state and stability level of the metastable state

In this subsection we find the unique metastable state and we compute its stability level. Furthermore, we find the set of the local minima and the set of the stable plateaux of the Hamiltonian (1). First we define a reference path from $\mathbf{1}$ to $\mathbf{s}$, for any $\mathbf{s} \in \mathscr{X}^s$. The energy value of any configuration σ in this path, using (42) can be computed as

$$H(\sigma) - H(\mathbf{1}) = d_v(\sigma) + d_h(\sigma) - h\sum_{u\in V} \mathbb{1}_{\{\sigma(u)\neq 1\}}, \tag{43}$$

We say that a path $\omega \in \Omega_{\sigma,\sigma'}$ is the *concatenation* of the L paths $\omega^{(i)} = (\omega_0^{(i)},\dots,\omega_{n_i}^{(i)})$, for some $n_i \in \mathbb{N}$, $i = 1,\dots,L$ if $\omega = (\omega_0^{(1)} = \sigma,\dots,\omega_{n_1}^{(1)},\omega_0^{(2)},\dots,\omega_{n_2}^{(2)},\dots,\omega_0^{(L)},\dots,\omega_{n_L}^{(L)} = \sigma')$.

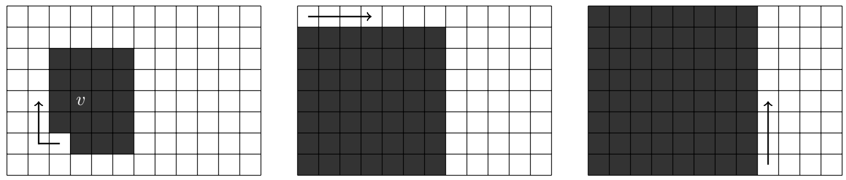


(a) Two vertical bridges on columns c_2 and c_{10}.

(b) An horizontal bridge on row r_2.

(c) A cross on column c_4 and on row r_6.

FIG. 3: Example of configurations on a 8×11 grid graph displaying a vertical s-bridge (a), a horizontal s-bridge (b) and a s-cross (c). We color black the spins s.



(a) (b) (c)

FIG. 4: (a) First steps of path $\hat{\omega}^{(2)}$ on a 10×12 grid Λ starting from the vertex $v=(3,3)$. We color white the vertices with spin 1, black those with spin s. The arrow indicates the order in which the spins are flipped from 1 to s. (b) Illustration of $\hat{\omega}_{(K-1)^2}$. (c) Illustration of $\hat{\omega}_{(K-1)^2+K-1}$.

Definition IV.1 (Reference path) *For any* $\mathbf{s}\in\mathscr{X}^s$*, we define a reference path* $\hat{\omega}:\mathbf{1}\to\mathbf{s}$*,* $\hat{\omega}:=(\hat{\omega}_0,\dots,\hat{\omega}_{KL})$ *as the concatenation of the two paths* $\hat{\omega}^{(1)}:=(\mathbf{1}=\hat{\omega}_0,\dots,\hat{\omega}_{(K-1)^2})$ *and* $\hat{\omega}^{(2)}:=(\hat{\omega}_{(K-1)^2+1},\dots,\mathbf{s}=\hat{\omega}_{KL})$*. The path* $\hat{\omega}^{(1)}$ *is defined as follows. We set* $\hat{\omega}_0:=\mathbf{1}$*. Then, we set* $\hat{\omega}_1:=\hat{\omega}_0^{(i,j),s}$ *where* (i,j) *is any vertex in* Λ*. Sequentially, we flip clockwise from* 1 *to* s *all the vertices that sourround* (i,j) *in order to construct a* 3×3 *square. We iterate this construction until we get* $\hat{\omega}_{(K-1)^2}\in\bar{R}_{K-1,K-1}(1,s)$*. See Figure 4(a). The path* $\hat{\omega}^{(2)}$ *is defined as follows. Without loss of generality, assume that* $\hat{\omega}_{(K-1)^2}\in\bar{R}_{K-1,K-1}(1,s)$ *has the cluster of spin* s *in the first* $c_0,\dots,c_{K-2}$ *columns, see Figure 4(b). We define* $\hat{\omega}_{(K-1)^2+1},\dots,\hat{\omega}_{(K-1)^2+K-1}$ *as a sequence of configurations in which the cluster of spins* s *grows gradually by flipping the spins* 1 *on the vertices* $(K-1,j)$*, for* $j=0,\dots,K-2$*. Thus,* $\hat{\omega}_{(K-1)^2+K-1}\in\bar{R}_{K-1,K}(1,s)$ *as depicted in Figure 4(c). Finally, we define the configurations* $\hat{\omega}_{(K-1)^2+K},\dots,\hat{\omega}_{KL}$ *as a sequence of states in which the cluster of spin* s *grows gradually column by column. More precisely, starting from* $\hat{\omega}_{(K-1)^2+K-1}\in\bar{R}_{K-1,K}(1,s)$*,* $\hat{\omega}^{(2)}$ *passes through configurations in which the spins* 1 *on columns* $c_K,\dots,c_{L-1}$ *become* s*. The procedure ends with* $\hat{\omega}_{KL}=\mathbf{s}$*. Note that the energy value of the configurations in the reference path is independent of the first flipped spin* (i,j)*.*

Next we show that any configuration belonging to $\bigcup_{t=2}^q\bar{R}_{\ell^*-1,\ell^*}(1,t)$ is connected to the metastable configuration $\mathbf{1}$ by a path that does not overcome the energy value $4\ell^*-h(\ell^*(\ell^*-1)+1)+H(\mathbf{1})$. For any $s\in S$, we define as

$$N_s(\sigma):=|\{v\in V:\ \sigma(v)=s\}| \tag{44}$$

the number of vertices with spin s in $\sigma\in\mathscr{X}$.

Lemma IV.1 *If the external magnetic field is negative, then for any* $\sigma\in\bigcup_{t=2}^q\bar{R}_{\ell^*-1,\ell^*}(1,t)$ *there exists a path* $\gamma:\sigma\to\mathbf{1}$ *such that the maximum energy along* γ *is bounded as*

$$\max_{\xi\in\gamma}H(\xi)<4\ell^*-h(\ell^*(\ell^*-1)+1)+H(\mathbf{1}). \tag{45}$$

Proof. Let $s\in\{2,\dots,q\}$ be such that $\sigma\in\bar{R}_{\ell^*-1,\ell^*}(1,s)$. Hence, the proof proceeds as in the Ising case [16, Lemma 17.7]. See Appendix B for the detailed proof. □

In the next lemma we prove that any configuration in $\bigcup_{t=2}^q\bar{B}^2_{\ell^*-1,\ell^*}(1,t)$ is connected to the stable set $\mathscr{X}^s$ by a path that does not overcome the energy value $4\ell^*-h(\ell^*(\ell^*-1)+1)+H(\mathbf{1})$.

Lemma IV.2 *If the external magnetic field is negative, then for any* $\sigma\in\bar{B}^2_{\ell^*-1,\ell^*}(1,s)$*, then there exists a path* $\gamma:\sigma\to\mathbf{s}$ *such that the maximum energy along* γ *is bounded as*

$$\max_{\xi\in\gamma}H(\xi)<4\ell^*-h(\ell^*(\ell^*-1)+1)+H(\mathbf{1}). \tag{46}$$

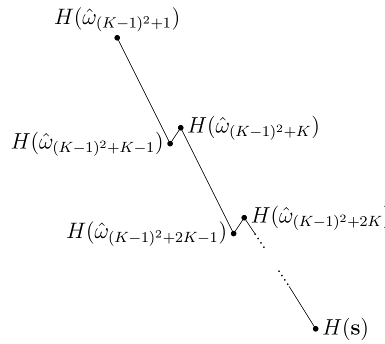


FIG. 5: Qualitative illustration of the energy of the configurations belonging to $\hat{\omega}^{(2)}$.

Furthermore, we get the following upper bound for the communication height

$$\Phi(\mathbf{1},\mathscr{X}^s)-H(\mathbf{1})\leq 4\ell^*-h(\ell^*(\ell^*-1)+1). \tag{47}$$

Proof. Consider the reference path of Definition IV.1 and assume that this path is constructed in such a way that $\hat{\omega}_{\ell^*(\ell^*-1)+2}:=\sigma$. Let $\gamma:=(\hat{\omega}_{\ell^*(\ell^*-1)+2}=\sigma,\hat{\omega}_{\ell^*(\ell^*-1)+3},\dots,\hat{\omega}_{KL-1},\mathbf{s})$. We claim that $\max_{\xi\in\gamma}H(\xi)<4\ell^*-h(\ell^*(\ell^*-1)+1)+H(\mathbf{1})$. Since γ is defined as a subpath of $\hat{\omega}$, we prove this claim by showing that $\max_{\xi\in\hat{\omega}}H(\xi)=4\ell^*-h(\ell^*(\ell^*-1)+1)+H(\mathbf{1})$ and that γ does not intersect the unique configuration in which this maximum is reached. Indeed, for $\ell\leq K-2$, note that the path $\hat{\omega}^{(1)}$ is defined by a sequence of configurations in which all the spins are equal to 1 except those, which are s, in either a square $\ell\times\ell$ or a quasi-square $\ell\times(\ell-1)$ possibly with one of the longest sides not completely filled. For some $\ell\leq K-2$, if $\hat{\omega}_{\ell(\ell-1)}\in\bar{R}_{\ell-1,\ell}(1,s)$ and $\hat{\omega}_{\ell^2}\in\bar{R}_{\ell,\ell}(1,s)$, then $\max_{\sigma\in\{\hat{\omega}_{\ell(\ell-1)},\hat{\omega}_{\ell(\ell-1)+1},\dots,\hat{\omega}_{\ell^2}\}}H(\sigma)=H(\hat{\omega}_{\ell(\ell-1)+1})=4\ell-h\ell^2+h\ell-h+H(\mathbf{1})$. Otherwise, if $\hat{\omega}_j:=\hat{\omega}_{\ell^2}\in\bar{R}_{\ell,\ell}(1,s)$ and $\hat{\omega}_{\ell(\ell+1)}\in\bar{R}_{\ell,\ell+1}(1,s)$, then $\max_{\sigma\in\{\hat{\omega}_{\ell^2},\hat{\omega}_{\ell^2+1},\dots,\hat{\omega}_{\ell(\ell+1)}\}}H(\sigma)=H(\hat{\omega}_{\ell^2+1})=4\ell-h\ell^2+2-h+H(\mathbf{1})$. Let $k^*:=\ell^*(\ell^*-1)+1$. By recalling the condition $\frac{2}{h}\notin\mathbb{N}$ of Assumption III.1(ii) and by studying the maxima of H as a function of ℓ, we have $\arg\max_{\hat{\omega}^{(1)}}H=\{\hat{\omega}_{k^*}\}$.

Let us now study the maximum energy value reached along $\hat{\omega}^{(2)}$. This path is constructed by a sequence of configurations whose clusters of spins s wrap around Λ. Moreover the maximum of the energy is reached by the first configuration, see Figure 5 for a qualitative representation of the energy of the configurations in $\hat{\omega}^{(2)}$. Indeed, using (5), we have

$$\begin{aligned}
&H(\hat{\omega}_{(K-1)^2+j})-H(\hat{\omega}_{(K-1)^2+j-1})=-2-h,\ j=2,\dots,K-1,\\
&H(\hat{\omega}_{(K-1)^2+K})-H(\hat{\omega}_{(K-1)^2+K-1})=2-h,\\
&H(\hat{\omega}_{(K-1)^2+j})-H(\hat{\omega}_{(K-1)^2+j-1})=-h,\ j=K+1,\dots,2K-1,\\
&H(\hat{\omega}_{(K-1)^2+2K})-H(\hat{\omega}_{(K-1)^2+2K-1})=2-h.
\end{aligned}$$

Using $K\geq 3\ell^*>3$, note that $H(\hat{\omega}_{(K-1)^2+1})-H(\hat{\omega}_{(K-1)^2+K})=2K-6+h(K-1)>0$. Moreover, $H(\hat{\omega}_{(K-1)^2+K})-H(\hat{\omega}_{(K-1)^2+2K})=2K+2-h((K-1)^2+K)-(2K+2-h((K-1)^2+2K))=hK>0$. By iterating the analysis of the energy gap between two consecutive configurations along $\hat{\omega}^{(2)}$, we conclude that $\arg\max_{\hat{\omega}^{(2)}}H=\{\hat{\omega}_{(K-1)^2+1}\}$. In particular,

$$H(\hat{\omega}_{(K-1)^2+1})<H(\hat{\omega}_{k^*})=4\ell^*-h(\ell^*(\ell^*-1)+1)+H(\mathbf{1}) \tag{48}$$

and, we refer to Appendix B for the explicit calculation. Hence, $\arg\max_{\hat{\omega}}H=\{\hat{\omega}_{k^*}\}$. Since γ is defined as the subpath of $\hat{\omega}$ which goes from $\hat{\omega}_{\ell^*(\ell^*-1)+2}=\sigma$ to $\mathbf{s}$, γ does not visit the configuration $\hat{\omega}_{k^*}$. Hence, $\max_{\xi\in\gamma}H(\xi)<4\ell^*-h(\ell^*(\ell^*-1)+1)+H(\mathbf{1})$ and the claim is proved.

The upper bound (47) follows by (46). □

Next, we give a lower bound for $\Phi(\mathbf{1},\mathscr{X}^s)-H(\mathbf{1})$.

Lemma IV.3 (Lower bound for the communication height) *If the external magnetic field is negative, then*

$$\Phi(\mathbf{1},\mathscr{X}^s)-H(\mathbf{1})\geq 4\ell^*-h(\ell^*(\ell^*-1)+1). \tag{49}$$

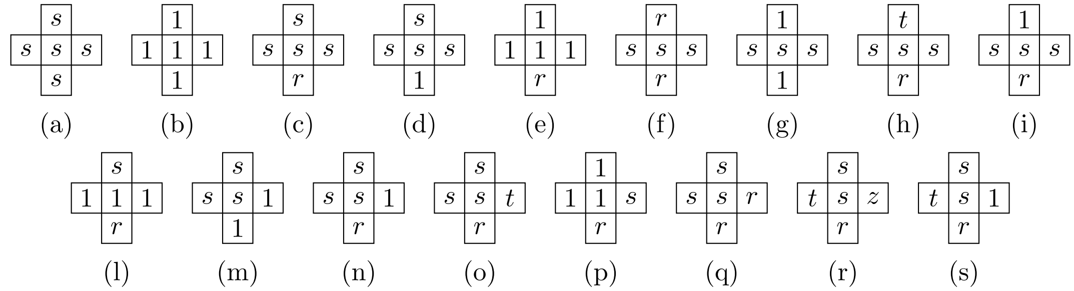


FIG. 6: Stable tiles centered in any $v \in V$ for a q-Potts configuration σ on Λ for any $r,s,t,z \in S\backslash\{1\}$ different from each other. The tiles are depicted up to a rotation of $\alpha\frac{\pi}{2}$, $\alpha \in \mathbb{Z}$.

Proof. For any $\sigma \in \mathscr{X}$, we set $N(\sigma) := \sum_{t=2}^{q} N_t(\sigma)$, where $N_t(\sigma)$ is defined in (44). Moreover, for all $k = 1,\dots,|V|$, we define $\mathscr{V}_k := \{\sigma \in \mathscr{X} : N(\sigma) = k\}$. Note that every path $\omega \in \Omega_{\mathbf{1},\mathscr{X}^s}$ has to cross $\mathscr{V}_k$ for every $k = 0,\dots,|V|$. In particular it has to intersect the set $\mathscr{V}_{k^*}$ with $k^* := \ell^*(\ell^*-1)+1$. We prove (49) by showing that $H(\mathscr{F}(\mathscr{V}_{k^*})) = 4\ell^* - h(\ell^*(\ell^*-1)+1) + H(\mathbf{1})$. Note that because of the definition of H and of (43), the presence of disagreeing edges increases the energy. Thus, in order to describe the bottom $\mathscr{F}(\mathscr{V}_{k^*})$ we have to consider those configurations in which the k^* spins different from 1 belong to a unique s-cluster for some $s \neq 1$ inside a sea of spins 1. For this reason, the proof is similar to the one of the second part of [29, Theorem 4.29]. For the detailed proof see Appendix B. □

Lemma IV.4 *If the external magnetic field is negative, then any $\omega \in \Omega^{opt}_{\mathbf{1},\mathscr{X}^s}$ is such that $\omega \cap \bigcup_{t=2}^{q} \bar{R}_{\ell^*-1,\ell^*}(1,t) \neq \varnothing$.*

Proof. The proof proceeds as in the Ising case [16, Lemma 17.9]. For the detailed proof see Appendix B. □

Let $\sigma \in \mathscr{X}$ and let $v \in V$. We define the *tile* centered in v, denoted by v-tile, as the set of five sites consisting of v and its four nearest neighbors. See for instance Figure 6. A v-tile is said to be *stable* for σ if by flipping the spin on vertex v from $\sigma(v)$ to any $s \in S$ the energy difference $H(\sigma^{v,s}) - H(\sigma)$ is greater than or equal to zero.
Our next aim is to prove a recurrence property in Proposition III.2, which will be useful to prove that $\mathbf{1} \in \mathscr{X}^m$ as stated in Theorem III.1. In order to do this, in Lemma IV.5 for any configuration $\sigma \in \mathscr{X}$ we describe all the possible stable v-tiles induced by the Hamiltonian (1) and we exploit this result to prove Proposition IV.1. For any $\sigma \in \mathscr{X}$, $v \in V$ and $s \in S$, we define $n_s(v)$ as the number of nearest neighbors to v with spin s in σ, i.e.,

$$n_s(v) := |\{w \in V : \; w \sim v, \; \sigma(w) = s\}|. \tag{50}$$

Lemma IV.5 (Characterization of stable v-tiles for a configuration σ) *Let $\sigma \in \mathscr{X}$ and let $v \in V$. If the external magnetic field is negative, then the tile centered in v is stable for σ if and only if it satisfies one of the following conditions.*

(1) *If $\sigma(v) = s \neq 1$, v has at least two nearest neighbors with spin s, see Figure* 6(a),(c),(d),(f)–(i),(m)–(o), (q), *or one nearest neighbor s and three nearest neighbors with spin $r,t,z \in S\backslash\{s\}$, different from each other, see Figure* 6(r)–(s).

(2) *If $\sigma(v) = 1$, v has either at least three nearest neighbors with spin* 1 *or two nearest neighbors with spin* 1 *and two nearest neighbors with spin $r,s \neq 1$, $r \neq s$, see Figure* 6(b),(e),(l), (p).

In particular, if $\sigma(v) = s$, then

$$H(\sigma^{v,r}) - H(\sigma) = n_s(v) - n_r(v) - h\mathbb{1}_{\{s=1\}} + h\mathbb{1}_{\{r=1\}}. \tag{51}$$

Proof. (1) and (2) follow by simple algebraic computations of the energy difference (5). See Appendix B for the explicit proof. □

We define the set $C^s(\sigma) \subseteq \mathbb{R}^2$ as the union of unit closed squares centered at the vertices $v \in V$ such that $\sigma(v) = s$. We define *s-clusters* the maximal connected components $C_1^s,\dots,C_n^s$, $n \in \mathbb{N}$, of $C^s(\sigma)$.
For any $s \in S$, we say that a configuration $\sigma \in \mathscr{X}$ has an s-rectangle if it has a rectangular cluster in which all the vertices have spin s.
Let R_1 an r-rectangle and R_2 an s-rectangle. They are said to be *interacting* if either they intersect (when $r = s$) or are disjoint but there exists a site $v \notin R_1 \cup R_2$ such that $\sigma(v) \neq r,s$ and v has two nearest-neighbor w,u lying inside R_1,R_2 respectively. For instance, in Figure 11(b) the gray rectangles are not interacting. Furthermore, we say that R_1 and R_2 are *adjacent* when they are at lattice distance one from each other, see for instance Figure 11(c) and (e). We are now able to describe precisely the set of the local minima $\mathscr{M}$ and the set of the stable plateaux $\bar{\mathscr{M}}$ of the energy function (1). More precisely, the set of local minima $\mathscr{M}$ is the set of stable points, i.e., $\mathscr{M} := \{\sigma \in \mathscr{X} : \; H(\mathscr{F}(\partial\{\sigma\})) > H(\sigma)\}$. While, a plateau $D \subset \mathscr{X}$, namely a maximal connected

set of equal energy states, is said to be *stable* if $H(\mathscr{F}(\partial D)) > H(D)$. Note that $\mathscr{M} \cup \bar{\mathscr{M}} \subset \hat{\mathscr{X}} := \{\sigma \in \mathscr{X} :$ for any $v \in V$ the tile centered in v is stable$\} \subset \mathscr{X}$. In Proposition IV.1, we prove that $\mathscr{M} \cup \bar{\mathscr{M}}$ is given by the union of the following sets. See also Figure 11.

$\mathscr{M}^1 := \{\mathbf{1}, \mathbf{2}, \dots, \mathbf{q}\}$;

$\mathscr{M}^2 := \{\sigma \in \hat{\mathscr{X}} :\ \sigma$ has strips of any spin $s \in S$ of thickness larger than or equal to one such that for any s an s-strip of thickness one is in between strips of spins different from each other$\}$;

$\mathscr{M}^3 := \{\sigma \in \hat{\mathscr{X}} :\ \sigma$ has one or more s-rectangles for some $s \neq 1$, with minimum side-length larger than or equal to two, either in a sea of spins 1 or inside a 1-strip such that rectangles with the same spins are not interacting$\}$;

$\mathscr{M}^4 := \{\sigma \in \hat{\mathscr{X}} :\ \sigma$ has one or more s-rectangles for some $r,s \neq 1$, with minimum side-length larger than or equal to two, inside a 1-strip adjacent to an r-strip$\} \cup \{\sigma \in \hat{\mathscr{X}} :\ \sigma$ is covered by interacting s-rectangles such that each spin on the corners has outside the rectangle two nearest neighbors with different spins from each other and from the one inside the rectangle$\}$

$\bar{\mathscr{M}}^1 := \{\sigma \in \hat{\mathscr{X}}$: for any $r,s \neq 1$, σ has an s-cluster with two consecutive sides next either to a connected r-cluster or to two r-cluster and the sides on the interfaces are of different length$\}$.

Remark IV.1 *The set $\bar{\mathscr{M}}^1$ is defined by fixing a representative configuration σ and implicitly it includes also all the configurations connected to σ via a path along which the energy is constant, see Figure 8.*

A path $\omega = (\omega_0, \dots, \omega_n)$ is said to be *downhill* (*strictly downhill*) if $H(\omega_{i+1}) \leq H(\omega_i)$ ($H(\omega_{i+1}) < H(\omega_i)$) for $i = 0, \dots, n-1$.

Proposition IV.1 (Sets of local minima and of stable plateaux) *If the external magnetic field is negative, then*

$$\mathscr{M} \cup \bar{\mathscr{M}} = \mathscr{M}^1 \cup \mathscr{M}^2 \cup \mathscr{M}^3 \cup \mathscr{M}^4 \cup \bar{\mathscr{M}}^1. \tag{52}$$

Proof. A configuration $\sigma \in \mathscr{X}$ is a local minimum when, for any $v \in V$ and $s \in S$, the energy difference (5) is strictly positive. On the other hand, σ belongs to a stable plateau when, for any $v \in V$ and $s \in S$, the energy difference (5) is larger than or equal to zero. Since a local minimum and a stable plateau are the union of stable tiles, we obtain *all* the local minima and *all* the stable plateaux by considering all the possible ways in which the stable tiles may be combined. We do this in various steps. First we consider all configurations which can be obtained from combining tiles (a)–(b). Then, we progressively add more tile types and construct all the possible resulting configurations. To refer to a tile type, we will use its corresponding lett in Figure 6.

Step 1. If σ has only stable tiles as in Figure 6(a) and (b), then there are no interfaces and $\sigma \in \mathscr{M}^1$.

Step 2. Let us assume now that the only stable tiles in σ are (a)–(l). Note that if σ contains a tile of type (f), then σ does not belong to $\mathscr{M} \cup \bar{\mathscr{M}}$. Indeed, if σ contains at least an (f) tile, then it also contains an s-strip of thickness one in between two r-strips and there exists a downhill path of two steps. First, flip from s to r the central spin s and this does not change the energy, then flip from s to r a spin, which, has now three spin r neighbor. This flip reduces the energy by 2. On the other hand, any spin update on the central vertex of the tiles (a)–(e) and (g)–(l) strictly increases the energy of σ. By considering these, we obtain that σ may contain horizontal (resp. vertical) interfaces of length L (resp. K). In particular, for any $s \in S$, an s-strip of thickness one must be either in between strips with different spins, using (h)–(l) tiles, or in between two 1-strips if $s \neq 1$, using (g) tiles. We conclude that if σ is obtained by a combination of the stable tiles (a)–(l), then $\sigma \in \mathscr{M}^1 \cup \mathscr{M}^2$, see Figure 11(a).

Step 3. Next we consider those σ that are defined as the combination of the stable tiles (a)–(e), (g)–(p). Any spin update on the central vertex of the tiles (m)–(p). Since the central spin $s \neq 1$ of these tiles has at least two nearest neighbors with the same spin, the admissible shapes of an s-cluster are either strips or rectangles. It follows that the local minima containing only tiles (a)–(p) may additionally contain the following shapes.

(i) One or more s-rectangles ($s \neq 1$) with minimum side length two either in a sea of spins 1 or inside a 1-strip under the condition that rectangles with the same spins are not interacting, see Figure 11(b).

(ii) One or more s-rectangles ($s \neq 1$) with minimum side length two, inside a 1-strip, with a side adjacent to an r-strip ($r \neq 1$), see Figure 11(d).

(iii) Alternatively, σ is covered by interacting rectangles under the condition that each spin on the corners is the centre of a tile of type (n)–(o), see Figure 11(e).

We conclude that if σ is defined by the combination of the stable tiles as in Figure 6(a)–(p), then $\sigma \in \mathscr{M}^1 \cup \mathscr{M}^2 \cup \mathscr{M}^3 \cup \mathscr{M}^4$.

Step 4. Next we consider those σ that are obtained as the combination of the stable tiles (a)–(e), (g)–(q). Combining the tiles of type (q) with all the previous ones, we obtain that for any $r,s \neq 1, r \neq s$, an s-cluster may have two consecutive sides adjacent either to a connected cluster or to two clusters with spins r and the sides on the interface may have either the same or different length, see Figure 7. We claim that in a stable configuration σ these are no clusters as in Figure 7(a)–(b), in which an r-cluster has a side longer than or equal to the side of the the s-cluster on the interface. Indeed, the path $(\omega_1 = \sigma, \dots, \omega_\ell)$ that flips from s to r all the spins s on the interface of length ℓ visits states $\omega_1, \dots, \omega_{\ell-1}$ with constant energy and $H(\omega_\ell) < H(\omega_{\ell-1})$.

Let us now focus on the case in Figure 7(c), and let σ be a configuration with such clusters. We prove that $\sigma \in \bar{\mathscr{M}}$. In particular, all configurations connected to σ via a path along which the energy does not change also belong to $\bar{\mathscr{M}}$. In order to see this, consider the path which flips from s to r the spins s adjacent to an r-rectangle (see for instance the path depicted in

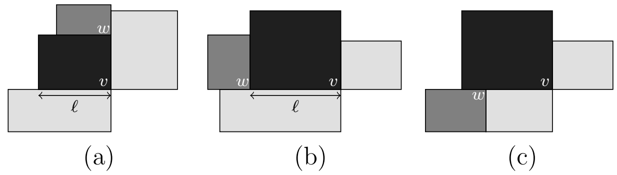


FIG. 7: Illustration of an s-rectangle, that we color black, adjacent to two r-rectangles, that we color light gray. Furthermore, we color gray those t-rectangles with $t \in S \setminus \{r, s\}$.

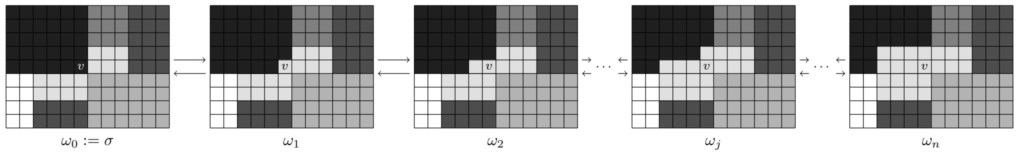


FIG. 8: Example of a path $\omega := (\omega_0, \dots, \omega_n)$ started in a configuration $\omega_0 := \sigma$ with a cluster as the one depicted in Figure 7(c) and such that $H(\omega_i) = H(\omega_j)$, for any $i, j = 0, \dots, n$. Since all the configurations depicted have the same energy value and they are connected by means a path, they belong to a stable plateau.

Figure 8). Note that at any step the energy does not change. Hence, combining all the stable tiles (a)–(q), we conclude that $\sigma \in \mathscr{M}^1 \cup \mathscr{M}^2 \cup \mathscr{M}^3 \cup \mathscr{M}^4 \cup \bar{\mathscr{M}}^1$.

Step 5. Finally, assume that σ may be obtained by a combination of tiles (a)–(s).

For this step, we refer to Figure 9, where we represent r, s, t, z respectively by ○,●,●,● and where we take $r, t \notin \{s, 1\}$ and $z \neq s$. Let us assume that this type of tile belongs to a configuration σ and consider the following cases.

Step 5.1. If $n_s(v_1) = 4$, then $\sigma(w_1) = \sigma(w_2) = s$. If both the v_2-tile and v_4-tile are of type (m), then v_3 would be the central vertex of a unstable tile. Thus, at least one of them is of type (q). Proceeding as in Step 4 we show that σ is either unstable or it belongs to a stable plateau.

Step 5.2. Assume $n_s(v_1) = 3$. If $\sigma(w_1) = \sigma(w_2) = s$, then again σ is either unstable or belongs to a stable plateau. If $\sigma(w_1) = s$ and $\sigma(w_2) \neq s$, then v_1 must be the central vertex of a tile of type (q) and again σ is either unstable or belongs to a stable plateau. Otherwise, v_3 would be the central vertex of a unstable tile.

Step 5.3. We now consider the case $n_s(v_1) = 1$. This will be useful to study the case $n_s(v_1) = 2$ in the next step. Along the path $\omega := (\sigma, \sigma^{v,r}, (\sigma^{v,r})^{v_1,r})$ the energy decreases. Indeed,

$$H(\sigma^{v,r}) - H(\sigma) = 0, \tag{53}$$

$$H((\sigma^{v,r})^{v_1,r}) - H(\sigma^{v,r}) = \begin{cases} -2, & \text{if } n_r(v_1) = 1; \\ -3, & \text{if } n_r(v_1) = 2; \\ -4, & \text{if } n_r(v_1) = 3. \end{cases} \tag{54}$$

It follows that the tiles as (r)–(s) with $n_s(v_1) = 1$ do not belong to any configuration in $\mathscr{M} \cup \bar{\mathscr{M}}$.

Step 5.4. Lastly, let us consider the case $n_s(v_1) = 2$. Without loss of generality, assume that the spin s nearest neighbors of v_1 lie on the same row. Consider the following two cases:

- v_1 has at least one nearest neighbor with a spin among $r, t, z \notin \{1, s\}$, say r, then along the path $(\sigma, \sigma^{v,r}, (\sigma^{v,r})^{v_1,r})$ the energy decreases. Indeed, we have

$$H(\sigma^{v,r}) - H(\sigma) = 0 \quad \text{and} \quad H((\sigma^{v,r})^{v_1,r}) - H(\sigma^{v,r}) \leq -1. \tag{55}$$

Thus, there are no such tiles in configurations in $\mathscr{M} \cup \bar{\mathscr{M}}$.

- v_1 has two nearest neighbors with spin s on vertices v and v_5 and two nearest neighbors with spins $r'_1, r'_2 \notin \{r, t, z\}$. If $r'_1 = r'_2 = r$, then the path $(\sigma, \sigma^{v,r}, (\sigma^{v,r})^{v_1,r'})$ is downhill. Indeed, $H(\sigma^{v,r}) - H(\sigma) = 0$ and $H((\sigma^{v,r})^{v_1,r'}) - H(\sigma^{v,r}) = -1 + h\mathbb{1}_{\{r'=1\}}$.

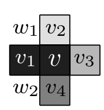


FIG. 9: Example of a v-tile equal to the one depicted in Figure 6(r)–(s). We do not color the vertices w_1 and w_2 since in the proof they assume different value in different steps.

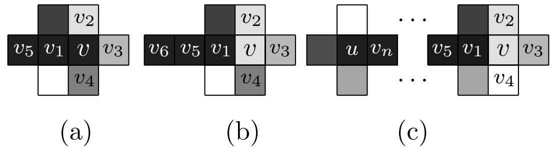


FIG. 10: Illustration of the Step 5.4.

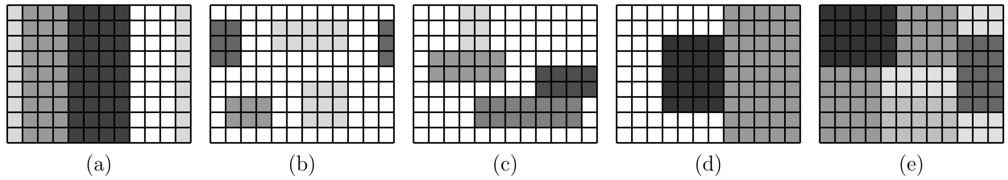


FIG. 11: Examples of local minima of the Hamiltonian (1). We color white the vertices with spin 1 and we use the other colors to denote the other spins $2,\dots,q$.

Assume now that $r_1' \neq r_2'$, as in Figure 10(a) where we represent r_1' by ● and r_2' by ○. We may repeat the discussion above by considering the tile centered in v_1, and performing, if possible, another zero-cost flip, and so on. This procedure necessarily ends, see, e.g., Figure 10(c). Note that the vertex u may coincide with v_3. This concludes the proof of Step 5.4.

Finally, in view of the discussion above, the stable tiles (h) and (i) belong to a stable configuration only when they belong to a strip of thickness one and the stable tiles of type (f) does not belong to any stable configuration. □

We are now ready to prove Proposition III.2.

Proof of Proposition III.2(Estimate on the stability level). In order to prove the recurrence property it is enough to focus on the configurations belonging to $\widetilde{\mathscr{M}} := (\mathscr{M}\backslash\{\mathbf{1},\dots,\mathbf{q}\}) \cup \bar{\mathscr{M}}$. For any $\eta \in \widetilde{\mathscr{M}}$ we prove that V_η is smaller than or equal to $V^* := 2 < \Gamma(\mathbf{1},\mathscr{X}^s)$. Let us first give an outline of the proof. First, we estimate of the stability level of those configurations in $\widetilde{\mathscr{M}}$ that have at least two adjacent strips of different spins, see Figure 11(a) and (d). Second, we estimate the stability level of those configurations in $\widetilde{\mathscr{M}}$ that have at least an s-rectangle ($s \neq 1$) either in a sea of spins 1 or inside a cluster of spins 1, as well as those configurations in which there is at least an s-rectangle ($s \neq 1$) having a side such that on the corners there are stable tiles of type (m) and elsewhere there are stable tiles of type (d). See Figure 11(b),(c) and (d). Third, we consider those local minima in which at least an r-cluster has a side completely adjacent to a side of an s-cluster, see for instance Figure 11(c)–(e) and Figure 13. Finally, we focus on those local minima that do not belong to any of the cases above, that is, those local minima with at least an s-rectangle with each side adjacent both to an r-cluster ($r \neq s$) and to a 1-rectangle, see for instance 11(e).

Case 1. Let us begin by assuming that η has either at least two horizontal or vertical strips. Consider the case depicted in Figure 11(a). Assume that η has an r-strip $a \times K$ adjacent to an s-strip $b \times K$, $a,b \in \mathbb{Z}$, $a,b \geq 1$. Assume that $r,s \in S$, $s \neq 1$. Let $\bar{\eta}$ be the configuration obtained from η by flipping from r to s all the spins r belonging to the r-strip. We define a path $\omega : \eta \to \bar{\eta}$ as the concatenation of a paths $\omega^{(1)},\dots,\omega^{(a)}$. Let $\omega : \eta \to \bar{\eta}$ be the path that flips the spins in the r-strip to s, column by column, starting from the column adjacent to the s-strip. Number the columns of the r-strip in order of flipping, and let $\omega^{(i)} := (\omega_0^{(i)} = \eta_{i-1}, \omega_1^{(i)},\dots,\omega_K^{(i)} = \eta_i)$ be the path that flips the r spins in the i-th column. Then, for $i = 1,\dots,a-1$,

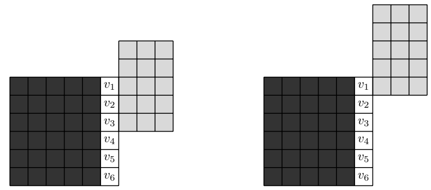
$$H(\omega_j^{(i)}) - H(\omega_{j-1}^{(i)}) = \begin{cases} 2 - h\mathbb{1}_{\{r=1\}}, & \text{if } j = 1; \\ -h\mathbb{1}_{\{r=1\}}, & \text{if } j = 2,\dots,K-1; \\ -2 - h\mathbb{1}_{\{r=1\}}, & \text{if } j = K. \end{cases} \tag{56}$$

For any $i = 1,\dots,a-1$, the maximum energy value along $\omega^{(i)}$ is reached at the first step. Computing the energy values along the sub-path $\omega^{(a)}$, that flips the last r-column, requires more care. Denoting by v_i the vertex whose spin is flipping at the step i,

$$H(\omega_1^{(a)}) - H(\omega_{K-1}^{(a-1)}) = \begin{cases} 1 - h\mathbb{1}_{\{r=1\}}, & \text{if } n_s(v_1) = 1, \\ -h\mathbb{1}_{\{r=1\}} & \text{if } n_s(v_1) = 2, \end{cases} \tag{57}$$

and, if $i = 2,\dots,K$,

$$H(\omega_i^{(a)}) - H(\omega_{i-1}^{(a)}) = \begin{cases} -1 - h\mathbb{1}_{\{r=1\}}, & \text{if } n_s(v_i) = 2, \\ -2 - h\mathbb{1}_{\{r=1\}}, & \text{if } n_s(v_i) = 3. \end{cases} \tag{58}$$



(a) (b)

FIG. 12: Examples of interacting rectangles in $\tilde{\eta}$ when $\ell^*=5$. We color gray the r-rectangle $\hat{R}$ and black the s-rectangle.

In view of the above construction, $H(\eta)>H(\bar{\eta})$ and, by comparing (56)–(58), $V_\eta \le 2 = V^*$.

Case 2. Let us now consider η characterized by a sea of spins 1 with some non-interacting s-rectangles ($s\neq 1$). We distinguish the following cases:

(i) η has at least a rectangle $R_{\ell_1\times\ell_2}$ of spins s, for some $s\in\{2,\dots,q\}$, with its minimum side of length $\ell:=\min\{\ell_1,\ell_2\}$ larger than or equal to ℓ^*;

(ii) η has only rectangles $R_{\ell_1\times\ell_2}$ of spins s, for some $s\in\{2,\dots,q\}$, with a side of length ℓ smaller than ℓ^*.

In case (i), we construct a path $\omega=(\omega_0,\dots,\omega_{\ell-1})$, where $\omega_0=\eta$ and $\omega_{\ell-1}=:\tilde{\eta}$, that flips consecutively from 1 to s those spins adjacent to a side of length $\ell\ge\ell^*$. We have

$$H(\omega_1)-H(\eta)=2-h, \tag{59}$$
$$H(\omega_i)-H(\omega_{i-1})=-h, \text{ for } i=2,\dots,\ell-2. \tag{60}$$

It follows that $H(\tilde{\eta})-H(\eta)=2-h\ell$. If $\ell>\ell^*=\lceil\frac{2}{h}\rceil$, then $2-h\ell<0$. Therefore the maximum energy is reached at the first step and by (59) we get $V_\eta=2-h<V^*$. Otherwise, if η has only rectangles $R_{\ell^*\times\ell^*}$ of spins s, then $\tilde{\eta}$ has a rectangle $R_{\ell^*\times(\ell^*+1)}$ of spins s. Now, either this s-rectangle does not interact with the other rectangles of $\tilde{\eta}$ or it interacts with another rectangle $\hat{R}$. In the former case we conclude by arguing as previously since $\ell^*+1>\ell^*$. In the latter case, we have the following two possibilities

(1) $\hat{R}$ is an s-rectangle,

(2) $\hat{R}$ is an r-rectangle with $r\notin\{1,s\}$.

In case (1), we define a configuration $\hat{\eta}$ from $\tilde{\eta}$ by flipping a spin 1 to s in the interaction interface. In particular,

$$H(\hat{\eta})-H(\tilde{\eta})=h. \tag{61}$$

Hence, the maximum energy along $(\eta,\omega_1,\dots,\omega_{\ell-2},\tilde{\eta},\hat{\eta})$ is reached at the first step and we conclude that $V_\eta=2-h<V^*$.

Let us now focus on case (2). We have to consider the two cases depicted in Figure 12. Let $v_1,\dots,v_{\ell^*+1}$ be the vertices next to the side of length ℓ^*+1 of the s-rectangle such that v_1 has two nearest neighbors with spin 1, one nearest neighbor s and one nearest neighbor inside the r-rectangle $\hat{R}$. In the case depicted in Figure 12(a), we define $\hat{\eta}_1:=\tilde{\eta}^{(v_1,s)}$ and $\hat{\eta}_2:=\hat{\eta}_1^{(v_2,s)}$. In particular,

$$H(\hat{\eta}_1)-H(\tilde{\eta})=1-h, \tag{62}$$
$$H(\hat{\eta}_2)-H(\hat{\eta}_1)=-1-h. \tag{63}$$

Hence, from (59)–(60) and (62)–(63), we have that $H(\hat{\eta}_2)-H(\eta)=2-h(\ell^*+2)<2-h\ell^*\le 0$. Moreover, in view of (59) and (62), along the path $(\eta,\omega_1,\dots,\omega_{\ell-2},\tilde{\eta},\hat{\eta}_1,\hat{\eta}_2)$, we get that the maximum energy is reached at the first step. Hence, $V_\eta=2-h<V^*$.

On the other hand, in the case depicted in Figure 12(b) we define $\hat{\eta}_1:=\tilde{\eta}^{(v_1,s)}$ and $\hat{\eta}_i:=\hat{\eta}_{i-1}^{(v_i,s)}$ for any $i=2,\dots,\ell^*+1$. Note that

$$H(\hat{\eta}_1)-H(\tilde{\eta})=1-h, \tag{64}$$
$$H(\hat{\eta}_{i+1})-H(\hat{\eta}_i)=-h,\ i=1,\dots,\ell^*. \tag{65}$$

Hence, from (59), (60), (64) and (65), we have $H(\hat{\eta}_{\ell^*+1})-H(\eta)=3-h(2\ell^*+1)<0$. Moreover, by comparing (59) and (64) along the path $(\eta,\omega_1,\dots,\omega_{\ell-2},\tilde{\eta},\hat{\eta}_1,\dots,\hat{\eta}_{\ell^*+1})$ the maximum energy is reached at the first step. Hence, $V_\eta=2-h<V^*$.

Now, we focus on the case (ii). We define a path $\omega=(\omega_0,\dots,\omega_{\ell-1})$ that flips consecutively from s to 1 those spins s next to a side of length $\ell<\ell^*$. We get:

$$H(\omega_i)-H(\omega_{i-1})=h, \text{ for } i=1,\dots,\ell-2; \tag{66}$$
$$H(\omega_{\ell-1})-H(\omega_{\ell-2})=-(2-h). \tag{67}$$

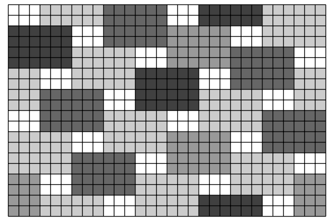


FIG. 13: Local minimum on a 30×20 grid graph in which there are not any s-rectangle with at least a side neither completely adjacent to an r-cluster nor completely sourrounded by spins 1.

Hence the maximum energy is achieved after $\ell-1$ steps and $H(\omega_{\ell-1})-H(\omega_0)=h(\ell-1)<2-h<V^*$.

Case 3. Let us now assume that η has an s-rectangle $\bar{R}:=R_{a\times b}$ and an r-rectangle $\tilde{R}:=R_{c\times d}$ such that $\bar{R}$ has a side of length a adjacent to a side of $\tilde{R}$ of length $c\geq a$, see for instance Figure 11(e). The case $c<a$ may be studied by interchanging the role of spins s and r. Given $\bar{\eta}$ the configuration obtained from η by flipping to r all the spins s belonging to $\bar{R}$, we construct a path $\omega:\eta\to\bar{\eta}$ as the concatenation of b paths $\omega^{(1)},\dots,\omega^{(b)}$. Let $\omega:\eta\to\bar{\eta}$ be the path that flips the spins in the r-rectangle $\tilde{R}$ to s, side by side, starting from the side adjacent to the s-rectangle $\bar{R}$. Number the sides of $\tilde{R}$ in order of flipping, and let $\omega^{(i)}:=(\omega_0^{(i)}=\eta_{i-1},\omega_1^{(i)},\dots,\omega_a^{(i)}=\eta_i)$ be the path that flips the r spins in the i-th side. Then, for $i=1,\dots,b-1$,

$$H(\omega_j^{(i)})-H(\omega_{j-1}^{(i)})=\begin{cases}1, & \text{if } j=1;\\ 0, & \text{if } j=2,\dots,a-1;\\ -1, & \text{if } j=a.\end{cases} \tag{68}$$

For any $i=1,\dots,b-1$, $H(\eta)=H(\eta_i)$ and the maximum energy value along $\omega^{(i)}$ is reached at the first step. Computing the energy values along the sub-path $\omega^{(b)}$, that flips the last r-side of the initial $\tilde{R}$, requires more care. Denoting by v_i the vertex whose spin is flipping at the step i

$$H(\omega_1^{(b)})-H(\eta_{b-1})=\begin{cases}0, & \text{if } n_r(v_1)=1,\\ -1, & \text{if } n_r(v_1)=2,\end{cases} \tag{69}$$

$$H(\omega_i^{(b)})-H(\omega_{i-1}^{(b)})=\begin{cases}-1, & \text{if } n_r(v_i)=2,\\ -2, & \text{if } n_r(v_i)=3,\end{cases} \tag{70}$$

for all $i=2,\dots,a$. In view of the above construction, $\Phi_\omega=H(\eta)+1$. Furthermore since $a\geq 2$, $H(\eta)>H(\bar{\eta})$ and, by comparing (68)–(70) we have $V_\eta=1<V^*$.

Case 4 Finally, let us consider η with at least an s-rectangle, say $\hat{R}$, with each side adjacent both to an r-cluster, $r\neq s$, and to a 1-rectangle. Let ℓ be the length of the interface between $\hat{R}$ and the 1-rectangle and let $\omega=(\omega_0=\eta,\dots,\omega_\ell)$ be the path that flips from 1 to s all the spins 1 on the ℓ vertices that lie on the interaface between $\hat{R}$ and the 1-rectangle. We have that

$$H(\omega_i)-H(\omega_{i-1})=\begin{cases}1-h, & \text{if } i=1;\\ -h, & \text{if } i=2,\dots,\ell-1;\\ -1-h, & \text{if } i=\ell.\end{cases} \tag{71}$$

Since $H(\omega_\ell)-H(\eta)=-h\ell<0$ and $\Phi_\omega=H(\eta)+1-h$, we get $V=1-h<V^*$. $\square$

C. Communication height between stable configurations

In order to study the hitting time $\tau_{\mathbf{s}}^{\mathbf{1}}$ of a stable configuration $\mathbf{s}\in\mathscr{X}^s$, we first estimate the communication height $\Phi(\mathbf{r},\mathbf{s})$ between two stable configurations $\mathbf{r},\mathbf{s}\in\mathscr{X}^s$, $\mathbf{r}\neq\mathbf{s}$. Indeed, during the transition $\mathbf{1}\to\mathbf{s}$, the process may visit a stable state $\mathbf{r}\neq\mathbf{s}$ before hitting $\mathbf{s}$. Using (42), the energy difference between any $\sigma\in\mathscr{X}$ and any $\mathbf{s}\in\mathscr{X}^s$ reads

$$H(\sigma)-H(\mathbf{s})=d_v(\sigma)+d_h(\sigma)+h\sum_{u\in V}\mathbb{1}_{\{\sigma(u)=1\}}. \tag{72}$$

In [1, Proposition 2.4] the authors define the so-called *expansion algorithm*. We rewrite this procedure in the proof of the next proposition by adapting it to our scenario. Indeed, it is different from [1] since in our setting there is a non-zero external magnetic field.

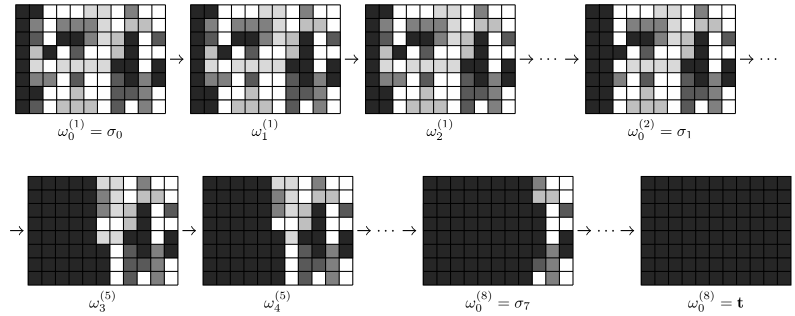


FIG. 14: Illustration of some particular configurations belonging to the path $\omega : \sigma \to \mathbf{t}$ of Proposition IV.2. We color black those vertices whose spin is t.

Proposition IV.2 (Expansion algorithm) *If the external magnetic field is negative and if $\sigma \in \mathscr{X}$ has a t-bridge for some $t \in \{2, \dots, q\}$, then there exists a path $\omega : \sigma \to \mathbf{t}$ such that $\Phi_\omega - H(\sigma) \leq 2$.*

Proof. Without loss of generality we assume that the first column c_0 is the t-bridge. Following an iterative procedure, we define a path $\omega : \sigma \to \mathbf{t}$ that flips all spins to t column-by-column starting with column c_0. Formally, ω is the concatenation of L paths $\omega^{(1)}, \dots, \omega^{(L)}$ with $\omega^{(i)} := (\omega_0^{(i)} = \sigma_{i-1}, \dots, \omega_K^{(i)} = \sigma_i)$ and $\omega_j^{(i)} := (\omega_{j-1}^{(i)})^{(u,t)}$, for $u := (i, j-1)$ and $j = 1, \dots, K$. In particular, $\sigma_0 := \sigma$, $\sigma_L := \mathbf{t}$ and the configurations σ_i, $i = 0, \dots, L$, are given by

$$\sigma_i(v) := \begin{cases} t & \text{if } v \in \bigcup_{j=0}^{i} c_j, \\ \sigma(v) & \text{if } v \in V \setminus \bigcup_{j=0}^{i} c_j. \end{cases} \tag{73}$$

See Figure 14 for an illustration of the construction above. Let us now study the energy difference $H(\omega_j^{(i)}) - H(\omega_{j-1}^{(i)})$ for $j = 1, \dots, K$. It is immediate to see that if $\sigma(u) = t$, then $H(\omega_j^{(i)}) - H(\omega_{j-1}^{(i)}) = 0$. Hence, assume that $\sigma(u) \neq t$. Using (5) and counting the number of spins s neighbors of u, we get

$$H(\omega_j^{(i)}) - H(\omega_{j-1}^{(i)}) \leq \begin{cases} 2 - h\mathbb{1}_{\{\omega_{j-1}^{(i)}(v)=1\}}, & \text{if } j = 1; \\ -h\mathbb{1}_{\{\omega_{j-1}^{(i)}(v)=1\}}, & \text{if } 1 < j < K; \\ -2 - h\mathbb{1}_{\{\omega_{j-1}^{(i)}(v)=1\}}, & \text{if } j = K. \end{cases} \tag{74}$$

For every $i = 1, \dots, L-1$, the inequalities (74) imply that $\Phi_{\omega^{(i)}} - H(\sigma_{i-1}) \leq 2$. Hence, the path $\omega : \sigma \to \mathbf{t}$ is such that $\Phi_\omega - H(\sigma) \leq 2$. □

Thanks to Proposition IV.2 we are able to obtain an upper bound on $\Gamma(\mathbf{r}, \mathbf{s}) := \Phi(\mathbf{r}, \mathbf{s}) - H(\mathbf{r})$, for any $\mathbf{r}, \mathbf{s} \in \mathscr{X}^s$, $\mathbf{r} \neq \mathbf{s}$.

Proposition IV.3 (Upper bound for the stability level between two stable configurations) *If the external magnetic field is negative, then for any $\mathbf{r}, \mathbf{s} \in \mathscr{X}^s$, $\mathbf{r} \neq \mathbf{s}$, we have*

$$\Phi(\mathbf{r}, \mathbf{s}) - H(\mathbf{r}) \leq 2\min\{K, L\} + 2. \tag{75}$$

Proof. The proof is analogous to the one of [1, Proposition 2.5] by replacing the role of [1, Proposition 2.4] with Proposition IV.2. For the details we refer to Appendix B. □

Now let us estimate a lower bound for $\Gamma(\mathbf{r}, \mathbf{s})$, for any $\mathbf{r}, \mathbf{s} \in \mathscr{X}^s$, $\mathbf{r} \neq \mathbf{s}$. The following proposition is an adaptation of [1, Proposition 2.7] to the case of Potts model with external magnetic field. Recall that $B_s(\sigma)$ denotes the total number of vertical and horizontal s-bridges in $\sigma \in \mathscr{X}$, see Subsection IV A.

Proposition IV.4 (Lower bound for the stability level between two stable configurations) *If the external magnetic field is negative, then for every $\mathbf{r}, \mathbf{s} \in \mathscr{X}^s$, the following inequality holds*

$$\Phi(\mathbf{r}, \mathbf{s}) - H(\mathbf{r}) \geq 2\min\{K, L\} + 2. \tag{76}$$

Proof. We show that along every path $\omega : \mathbf{r} \to \mathbf{s}$ in $\mathscr{X}$ there exists a configuration η such that $H(\eta) - H(\mathbf{r}) \geq 2K + 2$. Consider a path $\omega = (\omega_1, \dots, \omega_n)$ with $\omega_1 = \mathbf{r}$ and $\omega_n = \mathbf{s}$. Obviously, $B_s(\mathbf{r}) = 0$ and $B_s(\mathbf{s}) = K + L$. Let $\omega_{\bar{k}}$ be the configuration along the path ω that is the first to have at least two s-bridges, i.e., $\bar{k} := \min\{k \le n \mid B_s(\omega_k) \ge 2\}$. We claim that the configuration $\omega_{\bar{k}-1}$ is such that

$$H(\omega_{\bar{k}-1}) - H(\mathbf{r}) \ge 2K + 2. \tag{77}$$

Let us prove this claim by studying separately the following three cases:

(i) $\omega_{\bar{k}}$ has only vertical s-bridges,

(ii) $\omega_{\bar{k}}$ has only horizontal s-bridges,

(iii) $\omega_{\bar{k}}$ has at least one s-cross.

We study scenarios (i) and (iii), since scenario (ii) may be studied similarly as (i). Let us begin by assuming that (i) holds. From the definition of $\bar{k}$, it follows that $B_s(\omega_{\bar{k}-1}) = 1$ and $B_s(\omega_{\bar{k}}) = 2$. Otherwise $\omega_{\bar{k}}$ would have an s-cross in view of [1, Lemma 2.6] and it would be a contradiction with (i). Let us assume that $\omega_{\bar{k}}$ has the two vertical s-bridges on columns c and $\hat{c}$ and, without loss of generality, $\omega_{\bar{k}-1}$ has only one s-bridge on column c. In particular, in $\omega_{\bar{k}-1}$ all spins in $\hat{c}$ are s, except one which is different from s. Thus, in view of [1, Lemma 2.3(d)] we have

$$d_{\hat{c}}(\omega_{\bar{k}-1}) = 2. \tag{78}$$

Moreover, it is easy to see that there are no horizontal bridges. Thanks to this fact and to [1, Lemma 2.3(c)], we have $d_{r_i}(\omega_{\bar{k}-1}) \ge 2$ for every row r_i, $i = 0, \dots, K-1$. Then,

$$d_h(\omega_{\bar{k}-1}) = \sum_{i=0}^{K-1} d_{r_i}(\omega_{\bar{k}-1}) \ge 2K. \tag{79}$$

From (72), (78) and (79) we get that

$$H(\omega_{\bar{k}-1}) - H(\mathbf{r}) \ge 2 + 2K + h \sum_{u \in V} \mathbb{1}_{\{\omega_{\bar{k}-1}(u)=1\}} \ge 2 + 2K. \tag{80}$$

Let us now focus on (iii). In this case $\omega_{\bar{k}}$ has at least one s-cross and, by definition of $\bar{k}$, $B_s(\omega_{\bar{k}-1})$ is either 0 or 1 and we study these two cases separately.

Assume $B_s(\omega_{\bar{k}-1}) = 0$. $\omega_{\bar{k}-1}$ has no s-bridges, then, by [1, Lemma 2.6], $B_s(\omega_{\bar{k}}) = 2$ and $\omega_{\bar{k}}$ has exactly one s-cross. Let us assume that this s-cross lies on row $\hat{r}$ and on column $\hat{c}$. The horizontal and vertical s-bridges of $\omega_{\bar{k}}$ must have then been created simultaneously by updating the spin on the vertex $\hat{v} := \hat{r} \cap \hat{c}$. Hence, we have $\omega_{\bar{k}-1}(\hat{v}) \ne s$, $\omega_{\bar{k}-1}(v) = s$, for all $v \in \hat{r} \cup \hat{c}$, $v \ne \hat{v}$, and $\omega_{\bar{k}}(\hat{v}) = s$. Since there is a spin equal to s in every row and in every column, $\omega_{\bar{k}-1}$ has no t-bridges ($t \ne s$). Since by assumption $B_s(\omega_{\bar{k}-1}) = 0$, $\omega_{\bar{k}-1}$ has no bridges of any spin. Therefore, from [1, Lemma 2.3(c)–(d)] follows that

$$d_h(\omega_{\bar{k}-1}) = \sum_{i=0}^{K-1} d_{r_i}(\omega_{\bar{k}-1}) \ge 2K \quad \text{and} \quad d_v(\omega_{\bar{k}-1}) = \sum_{j=0}^{L-1} d_{c_j}(\omega_{\bar{k}-1}) \ge 2L. \tag{81}$$

Plugging (81) in (72), we conclude that

$$H(\omega_{\bar{k}-1}) - H(\mathbf{r}) \ge 2L + 2K > 2\min\{K, L\} + 2 = 2K + 2. \tag{82}$$

Assume now $B_s(\omega_{\bar{k}-1}) = 1$. In this case, $\omega_{\bar{k}-1}$ has an unique s-bridge and we assume that such a bridge is vertical and lies on column $\tilde{c}$. In view of [1, Lemma 2.2], there are no horizontal t-bridges in $\omega_{\bar{k}-1}$ ($t \ne s$). Hence, $\omega_{\bar{k}-1}$ has no horizontal bridges and by [1, Lemma 2.3(c)] we get

$$d_h(\omega_{\bar{k}-1}) = \sum_{i=0}^{K-1} d_{r_i}(\omega_{\bar{k}-1}) \ge 2K. \tag{83}$$

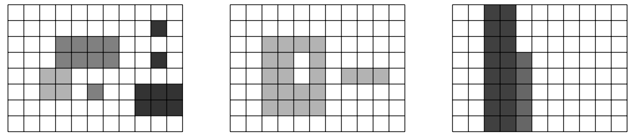
Moreover, $\omega_{\bar{k}}$ has a unique horizontal s-bridge, say on row $\hat{r}$. Hence, if $\hat{v}$ is the vertex where $\omega_{\bar{k}-1}$ and $\omega_{\bar{k}}$ differ, $\hat{v}$ must lie in $\hat{r}$ and $\omega_{\bar{k}-1}(\hat{v}) \ne s$ and $\omega_{\bar{k}-1}(v) = s$, $\forall v \in \hat{r}$, $v \ne \hat{v}$, and $\omega_{\bar{k}}(\hat{v}) = s$. Let $\hat{c}$ be the column where $\hat{v}$ lies. [1, Lemma 2.3(d)] implies that $d_c(\omega_{\bar{k}-1}) \ge 2$ for any column $c \ne \tilde{c}$, $\hat{c}$. Then,

$$d_v(\omega_{\bar{k}-1}) = \sum_{j=0}^{L-1} d_{c_j}(\omega_{\bar{k}-1}) \ge 2L - 4. \tag{84}$$

In view of (72), (83) and (84) it follows that

$$H(\omega_{\bar{k}-1}) - H(\mathbf{r}) \ge 2L + 2K - 4 > 2\min\{K, L\} + 2 = 2K + 2, \tag{85}$$

where the second inequality holds because $L \ge K \ge 3\ell^* > 3$. □



(a) (b) (c)

FIG. 15: Illustration of three examples of $\sigma \in \mathscr{D}$ when $\ell^* = 5$. In (a) the $\ell^*(\ell^*-1)+1 = 21$ spins different from 1 have not all the same spin value and they belong to more clusters. In (b) we consider the same number of spins with value $s \neq 1$ that belong to two different clusters. In (c) we consider the same number of spins different from 1 that are different between each other and that belong to two adjacent clusters.

V. MINIMAL GATES AND TUBE OF TYPICAL TRAJECTORIES

In this section we give a geometrical characterization of the critical configurations and the tube of typical paths for both metastable transitions $\mathbf{1} \to \mathscr{X}^s$ and $\mathbf{1} \to \mathbf{s}$ for any fixed $\mathbf{s} \in \mathscr{X}^s$.

A. Identification of critical configurations for the transition from the metastable configuration to the set of stable states

This subsection is devoted to a more accurate study of the energy landscape $(\mathscr{X}, H, Q)$. From a technical point of view, the proofs are a generalization of the corresponding results for the Blume Capel model [28, Section 6].

Let $\mathscr{D} \subset \mathscr{X}$ be the set

$$\mathscr{D} := \{\sigma \in \mathscr{X} : N_1(\sigma) = |\Lambda| - [\ell^*(\ell^*-1)+1]\}. \tag{86}$$

Furthermore, let $\mathscr{D}^+$ and $\mathscr{D}^-$ be the sets

$$\mathscr{D}^+ := \{\sigma \in \mathscr{X} : N_1(\sigma) > |\Lambda| - [\ell^*(\ell^*-1)+1]\}, \tag{87}$$

$$\mathscr{D}^- := \{\sigma \in \mathscr{X} : N_1(\sigma) < |\Lambda| - [\ell^*(\ell^*-1)+1]\}. \tag{88}$$

Note that $\mathbf{1} \in \mathscr{D}^+$. For any $\sigma \in \mathscr{D}$, we remark that σ has $\ell^*(\ell^*-1)+1$ spins different from 1 and they may have all the same spin value and may belong to one or more clusters, see Figure 15.

A *two dimensional polyomino* on $\mathbb{Z}^2$ is a finite union of unit squares. The area of a polyomino is the number of its unit squares, while its perimeter is the cardinality of its boundary, namely, the number of interfaces on $\mathbb{Z}^2$ between the sites inside the polyomino and those outside. The polyominoes with minimal perimeter among those with the same area are said to be *minimal polyominoes*.

Lemma V.1 *If the external magnetic field is negative, then the minimum of the energy in $\mathscr{D}$ is achieved by those configurations in which all the spins are equal to* 1 *except those, which have the same value $t \neq 1$, in a unique cluster of perimeter $4\ell^*$. More precisely,*

$$\mathscr{F}(\mathscr{D}) = \bigcup_{t=2}^{q} \mathscr{D}^t, \tag{89}$$

where

$$\mathscr{D}^t := \{\sigma \in \mathscr{D} : \sigma \text{ has all spins } 1 \text{ except those in a unique cluster } C^t(\sigma) \text{ of spins } t \text{ of perimeter } 4\ell^*\}. \tag{90}$$

Moreover,

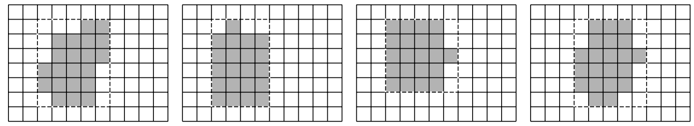
$$H(\mathscr{F}(\mathscr{D})) = H(\mathbf{1}) + \Gamma(\mathbf{1}, \mathscr{X}^s) = \Phi(\mathbf{1}, \mathscr{X}^s). \tag{91}$$

Proof. Since the presence of disagreeing edges increases the energy, in the configurations in $\mathscr{F}(\mathscr{D})$, all $\ell^*(\ell^*-1)+1$ spins different from 1 are equal to t (say) and belong to a unique cluster $C^t(\sigma)$. As we have illustrated in the second part of the proof of Lemma IV.3, the minimal perimeter of a polyomino of area $\ell^*(\ell^*-1)+1$ is $4\ell^*$. Thus, (89) is verified and we get that $\mathscr{W}(\mathbf{1}, \mathscr{X}^s) \subset \mathscr{F}(\mathscr{D})$. Hence, $H(\mathscr{W}(\mathbf{1}, \mathscr{X}^s)) = H(\mathscr{F}(\mathscr{D}))$ and, since for any $\eta \in \mathscr{W}(\mathbf{1}, \mathscr{X}^s)$

$$H(\eta) - H(\mathbf{1}) = 4\ell^* - h(\ell^*(\ell^*-1)+1) = \Gamma(\mathbf{1}, \mathscr{X}^s), \tag{92}$$

(91) is satisfied. □

In the next corollary we prove that $\mathscr{F}(\mathscr{D})$ is a gate for the transition $\mathbf{1} \to \mathscr{X}^s$.



(a) (b) (c) (d)

FIG. 16: Examples of $\sigma \in \tilde{\mathscr{D}}^t$ in (a) and of $\sigma \in \hat{\mathscr{D}}^t$ in (b) and (c) when $\ell^* = 5$. We associate the color gray to the spin t, the color white to the spin 1. The dashed rectangle represents the smallest surrounding rectangle of $C^t(\sigma)$. Figure (d) is an example of configuration that does not belong to $\hat{\mathscr{D}}^t$.

Corollary V.1 *If the external magnetic field is negative, then for any* $\omega \in \Omega^{opt}_{\mathbf{1},\mathscr{X}^s}$, $\omega \cap \mathscr{F}(\mathscr{D}) \neq \varnothing$. *In other words,* $\mathscr{F}(\mathscr{D})$ *is a gate for the transition from* $\mathbf{1}$ *to* $\mathscr{X}^s$.

Proof. For any path $\omega \in \Omega_{\mathbf{1},\mathscr{X}^s}$, $\omega = (\omega_0, \dots, \omega_n)$, there exists $i \in \{0, \dots, n\}$ such that $\omega_i \in \mathscr{D}$. Indeed, given $N(\sigma) := \sum_{t=2}^q N_t(\sigma)$, any path has to pass through the set $\mathscr{V}_k := \{\sigma \in \mathscr{X} : N(\sigma) = k\}$, for any $k = 0, \dots, |V|$, at least once and $\mathscr{V}_{\ell^*(\ell^*-1)+1} \equiv \mathscr{D}$. Since from (91) we get that the energy value of any configuration belonging to the bottom of $\mathscr{D}$ is equal to the min-max reached by any optimal path from $\mathbf{1}$ to $\mathscr{X}^s$, we get that $\omega_i \in \mathscr{F}(\mathscr{D})$. □

In the next proposition, we show that $\mathscr{W}(\mathbf{1}, \mathscr{X}^s)$ is a gate for the transition from $\mathbf{1}$ to $\mathscr{X}^s$. We define $R_{\ell_1 \times \ell_2}$ be the set of the rectangles in $\mathbb{R}^2$ with sides of length ℓ_1 and ℓ_2.

Proposition V.1 (Gate for the transition from the metastable state to the stable set) *If the external magnetic field is negative, then any path* $\omega \in \Omega^{opt}_{\mathbf{1},\mathscr{X}^s}$ *visits* $\mathscr{W}(\mathbf{1}, \mathscr{X}^s)$. *Hence,* $\mathscr{W}(\mathbf{1}, \mathscr{X}^s)$ *is a gate for the transition from* $\mathbf{1}$ *to* $\mathscr{X}^s$.

Proof. For any $t \neq 1$, let $\tilde{\mathscr{D}}^t$ be the set of configurations $\sigma \in \mathscr{D}^t$ such that the boundary of $C^t(\sigma)$ intersects each side of the boundary of its smallest surrounding rectangle $R(C^t(\sigma))$ on a set of the dual lattice $\mathbb{Z}^2 + (1/2, 1/2)$ made by at least two consecutive unit segments, see Figure 16(a). Furthermore, let $\hat{\mathscr{D}}^t$ be the set of configurations $\sigma \in \mathscr{D}^t$ such that the boundary of the polyomino $C^t(\sigma)$ intersects at least one side of the boundary of $R(C^t(\sigma))$ in a single unit segment, see for instance Figure 16(b) and (c). Hence, $\mathscr{F}(\mathscr{D}) = \tilde{\mathscr{D}} \cup \hat{\mathscr{D}}$, where $\tilde{\mathscr{D}} := \bigcup_{t=2}^q \tilde{\mathscr{D}}^t$ and $\hat{\mathscr{D}} := \bigcup_{t=2}^q \hat{\mathscr{D}}^t$. The proof proceeds in five steps.

Step 1. Our first aim is to prove that

$$\hat{\mathscr{D}} = \mathscr{W}(\mathbf{1}, \mathscr{X}^s) \cup \mathscr{W}'(\mathbf{1}, \mathscr{X}^s). \tag{93}$$

From (28), we have $\mathscr{W}(\mathbf{1}, \mathscr{X}^s) \cup \mathscr{W}'(\mathbf{1}, \mathscr{X}^s) \subseteq \hat{\mathscr{D}}$. Thus, we are left to prove the reverse inclusion $\sigma \in \hat{\mathscr{D}}$. The boundary of the cluster $C^t(\sigma)$ could intersect the other three sides of the boundary of $R(C^t(\sigma))$ in proper subsets of each side, see Figure 16(d). Assume $R(C^t(\sigma)) \in R_{(\ell^*+a)\times(\ell^*+b)}$ for some $a, b \in \mathbb{Z}$. $C^t(\sigma)$ is a minimal polyomino and so it is also convex and monotone by [28, Lemma 6.16]. Hence, the perimeter of $C^t(\sigma)$ is equal to the perimeter of $R(C^t(\sigma))$, which implies $4\ell^* = 4\ell^* + 2(a+b)$, and so $a = -b$. Now, let $\hat{C}^t(\sigma)$ be the polyomino obtained by removing the unit protuberance from $C^t(\sigma)$ and let $\hat{R}$ be its smallest surrounding rectangle. ,If $C^t(\sigma)$ has the unit protuberance adjacent to a side of length $\ell^* + a$, then $\hat{R}$ is a rectangle $(\ell^* + a) \times (\ell^* - a - 1)$. Since the area of $\hat{R}$ must be larger than or equal to the area of $\hat{C}^t(\sigma)$, we have

$$\text{Area}(\hat{R}) = (\ell^* + a)(\ell^* - a - 1) \geq \text{Area}(\hat{C}^t(\sigma)) = \ell^*(\ell^* - 1) \iff -a^2 - a \geq 0.$$

Since $a \in \mathbb{Z}$, $-a^2 - a \geq 0$ is satisfied only if either $a = 0$ or $a = -1$. On the other hand, if the unit protuberance of $C^t(\sigma)$ is adjacent to a side of length $\ell^* - a$, the same argument gives

$$\text{Area}(\hat{R}) = (\ell^* + a - 1)(\ell^* - a) \geq \text{Area}(\hat{C}^t(\sigma)) = \ell^*(\ell^* - 1) \iff -a^2 + a \geq 0.$$

Again this is satisfied only if either $a = 0$ or $a = 1$. In both cases we get that $\hat{R} \in R_{\ell^* \times (\ell^*-1)}$. Thus, if the protuberance is attached to one of the longest sides of $\hat{R}$, then $\sigma \in \mathscr{W}(\mathbf{1}, \mathscr{X}^s)$, otherwise $\sigma \in \mathscr{W}'(\mathbf{1}, \mathscr{X}^s)$. Then, (93) is verified.

Step 2. For any $\omega = (\omega_0, \dots, \omega_n) \in \Omega^{opt}_{\mathbf{1},\mathscr{X}^s}$ and any $t \in \{2, \dots, q\}$, let

$$f_t(\omega) := \{k \in \mathbb{N} : \omega_k \in \mathscr{F}(\mathscr{D}),\ N_1(\omega_{k-1}) = |\Lambda| - \ell^*(\ell^* - 1),\ N_t(\omega_{k-1}) = \ell^*(\ell^* - 1)\}. \tag{94}$$

We claim that the set $f(\omega) := \bigcup_{t=2}^q f_t(\omega)$ is not empty. Let $\omega = (\omega_0, \dots, \omega_n) \in \Omega^{opt}_{\mathbf{1},\mathscr{X}^s}$ and let $\bar{k} \leq n$ be the smallest integer such that $(\omega_{\bar{k}}, \dots, \omega_n) \cap \mathscr{D}^+ = \varnothing$. Since $\omega_{\bar{k}-1}$ is the last configuration in $\mathscr{D}^+$ along ω, it follows that $\omega_{\bar{k}} \in \mathscr{D}$ and, by the proof of Corollary V.1 we have that $\omega_{\bar{k}} \in \mathscr{F}(\mathscr{D})$. Thus, there exists $t \neq 1$ such that $\omega_{\bar{k}} \in \mathscr{D}^t$. Furthermore, $N_1(\omega_{\bar{k}-1}) = |\Lambda| - \ell^*(\ell^* - 1)$ and $\omega_{\bar{k}}$ is obtained from $\omega_{\bar{k}-1}$ by flipping a spin 1 to $s \neq 1$. Note that $N_1(\omega_{\bar{k}-1}) = |\Lambda| - \ell^*(\ell^* - 1)$ implies that $N_t(\omega_{\bar{k}-1}) \leq \ell^*(\ell^* - 1)$. Since by Lemma V.1 we have that $N_t(\omega_{\bar{k}}) = \ell^*(\ell^* - 1) + 1$, we conclude that $N_t(\omega_{\bar{k}-1}) = \ell^*(\ell^* - 1)$ since in a single spin flip the number of spins t changes by at most one. Thus, $N_t(\omega_{\bar{k}-1}) = \ell^*(\ell^* - 1)$ and $\bar{k} \in f(\omega)$.

Step 3. We claim that for any path $\omega \in \Omega^{opt}_{\mathbf{1},\mathscr{X}^s}$ one has $\omega_i \in \hat{\mathscr{D}}$ for all $i \in f(\omega)$. We argue by contradiction. Assume that there exists $i \in f(\omega)$ such that $\omega_i \notin \hat{\mathscr{D}}$ and $\omega_i \in \tilde{\mathscr{D}}$. Since $i \in f(\omega)$, there exists $t \neq 1$, such that $i \in f_t(\omega)$. Furthermore, ω_{i-1} is obtained from ω_i by flipping a spin t from t to 1. In view of the definition of $\tilde{\mathscr{D}}$, every spin equal to $t \neq 1$ has at least two nearest neighbors with spin t. Hence,

$$H(\omega_{i-1}) - H(\omega_i) \geq (2-2) + h = h > 0. \tag{95}$$

From (95) we get a contradiction since

$$\Phi_\omega \geq H(\omega_{i-1}) > H(\omega_i) = H(\mathbf{1}) + \Gamma(\mathbf{1},\mathscr{X}^s) = \Phi(\mathbf{1},\mathscr{X}^s),$$

where the first identity follows from Lemma V.1. Then the claim is proved.

Step 4. Now we claim that for any path $\omega \in \Omega^{opt}_{\mathbf{1},\mathscr{X}^s}$, $\omega_i \in \mathscr{F}(\mathscr{D})$ implies $\omega_{i-1}, \omega_{i+1} \notin \mathscr{D}$. Using Corollary V.1, there exists a positive integer i such that $\omega_i \in \mathscr{F}(\mathscr{D})$. Thus, there exists $t \neq 1$ such that $\omega_i \in \mathscr{D}^t$. Assume by contradiction that $\omega_{i+1} \in \mathscr{D}$. Then ω_{i+1} must be obtained by ω_i by flipping a spin t to $s \neq t$, since $N_1(\omega_i) = N_1(\omega_{i+1})$. In particular, this spin-update increases the energy and so, using Lemma V.1, we obtain $\Phi_\omega \geq H(\omega_{i+1}) > H(\omega_i) = H(\mathbf{1}) + \Gamma(\mathbf{1},\mathscr{X}^s) = \Phi(\mathbf{1},\mathscr{X}^s)$, which is a contradiction. Hence $\omega_{i+1} \notin \mathscr{D}$ and similarly we may also prove that $\omega_{i-1} \notin \mathscr{D}$.

Step 5. Our final aim is to show that for any path $\omega \in \Omega^{opt}_{\mathbf{1},\mathscr{X}^s}$, we have that $\omega \cap \mathscr{W}(\mathbf{1},\mathscr{X}^s) \neq \varnothing$. Given a path $\omega = (\omega_0,\dots,\omega_n) \in \Omega^{opt}_{\mathbf{1},\mathscr{X}^s}$, assume by contradiction that $\omega \cap \mathscr{W}(\mathbf{1},\mathscr{X}^s) = \varnothing$. From step 4 we know that along ω the configurations which belong to $\mathscr{F}(\mathscr{D})$ are not consecutive and they are separated by a subpath which belongs either to $\mathscr{D}^+$ or to $\mathscr{D}^-$. Let $j \in \{1,\dots,n\}$ be the smallest integer such that $\omega_j \in \mathscr{F}(\mathscr{D})$ and such that $(\omega_j,\dots,\omega_n) \cap \mathscr{D}^+ = \varnothing$. In particular, $j \in f(\omega)$ since j plays the same role of $\bar{k}$ in Step 2. Note that using (93), Step 2 and the assumption $\omega \cap \mathscr{W}(\mathbf{1},\mathscr{X}^s) = \varnothing$, we have $\omega_j \in \mathscr{W}'(\mathbf{1},\mathscr{X}^s)$. Furthermore, by (91) the energy along the path from $\omega_j \in \mathscr{F}(\mathscr{D})$ to ω_n decreases. Let $t \neq 1$ be such that $\omega_j \in \mathscr{D}^t$. Then the only moves that decrease the energy are

(i) flipping the spin in the unit protuberance from t to 1,

(ii) flipping a spin 1 with two nearest neighbors with spin t from 1 to t.

Since $\omega_{j+1} \notin \mathscr{D}^+$, (i) is not feasible. Hence, necessarily $H(\omega_{j+1}) = H(\mathbf{1}) + \Gamma(\mathbf{1},\mathscr{X}^s) - h$ and starting from ω_{j+1} we consider a spin-update that either decreases the energy or increases the energy of at most h. Hence the only feasible moves are

(iii) flipping a spin 1, with two nearest neighbors with spin t, from 1 to t,

(iv) flipping a spin t, with two nearest neighbors with spin 1, from t to 1.

Note that by (iii) and (iv), the process reaches a configuration σ with all spins equal to 1 except those, which are t, in a polyomino $C^t(\sigma)$ that is convex and such that $R(C^t(\sigma)) = R_{(\ell^*+1)\times(\ell^*-1)}$. Note that we cannot iterate move (iv) since otherwise we would find a configuration that does not belong to $\mathscr{D}$. On the other hand, applying once (iv) and iteratively (iii), until we fill the rectangle $R_{(\ell^*+1)\times(\ell^*-1)}$ with spins t, we find a set of configurations in which the one with the smallest energy is σ such that $C^t(\sigma) \equiv R(C^t(\sigma))$. Starting from any configuration of this set, the smallest energy increase is $2-h$ and it is achieved by flipping from 1 to t a spin 1 with three nearest neighbors with spin 1 and a neighbor of spin t inside $C^t(\sigma)$. It follows that

$$\Phi_\omega - H(\mathbf{1}) \geq 4\ell^* - h(\ell^*+1)(\ell^*-1) + 2 - h > \Gamma(\mathbf{1},\mathscr{X}^s), \tag{96}$$

where the last inequality holds because $2 > h(\ell^*-1)$ since $0<h<1$ and $\ell^* := \lceil \frac{2}{h} \rceil$, see Assumption III.1. Since in (96) we obtained a contradiction, we conclude that any path $\omega \in \Omega^{opt}_{\mathbf{1},\mathscr{X}^s}$ must visit $\mathscr{W}(\mathbf{1},\mathscr{X}^s)$. □

B. Minimal gates: proof of the main results

We are now able to prove Theorems III.5 and III.6.

Proof of Theorem III.5. By Proposition V.1 we get that $\mathscr{W}(\mathbf{1},\mathscr{X}^s)$ is a gate for the transition from the metastable state $\mathbf{1}$ to $\mathscr{X}^s$. In order to show that $\mathscr{W}(\mathbf{1},\mathscr{X}^s)$ is a minimal gate, we exploit [29, Theorem 5.1] and we show that any $\eta \in \mathscr{W}(\mathbf{1},\mathscr{X}^s)$ is an essential saddle. In order to do this, in view of the definition of an essential saddle given in Subsection III C, for any $\eta \in \mathscr{W}(\mathbf{1},\mathscr{X}^s)$ we construct an optimal path from $\mathbf{1}$ to $\mathscr{X}^s$ passing through η and reaching its maximum energy only there. Since $\eta \in \mathscr{W}(\mathbf{1},\mathscr{X}^s)$, there exists $s \neq 1$ such that $\eta \in \bar{B}_{\ell^*-1,\ell^*}(s,1)$ and the optimal path above is defined by modifying the reference path $\hat{\omega}$ of Definition IV.1 in a such a way that $\hat{\omega}_{\ell^*(\ell^*-1)+1} = \eta$ in which $C^s(\eta)$ is a quasi-square $\ell^* \times (\ell^*-1)$ with a unit protuberance. It follows that $\hat{\omega} \cap \mathscr{W}(\mathbf{1},\mathscr{X}^s) = \{\eta\}$ and $\arg\max_{\hat{\omega}} H = \{\eta\}$ by the proof of Lemma IV.2. To conclude, we prove that $\mathscr{W}(\mathbf{1},\mathscr{X}^s)$ is the only minimal gate. Note that the above reference path $\hat{\omega}$ reaches the energy $\Phi(\mathbf{1},\mathscr{X}^s)$ only in $\mathscr{W}(\mathbf{1},\mathscr{X}^s)$. It follows that, for any $\eta_1 \in \mathscr{W}(\mathbf{1},\mathscr{X}^s)$, the set $\mathscr{W}(\mathbf{1},\mathscr{X}^s) \setminus \{\eta_1\}$ is not a gate for the transition $\mathbf{1} \to \mathscr{X}^s$. Indeed, from the above discussion we get that there exists an optimal path $\hat{\omega}$ such that $\hat{\omega} \cap \mathscr{W}(\mathbf{1},\mathscr{X}^s) \setminus \{\eta_1\} = \varnothing$. Note that the uniqueness of the minimal gate follows by the condition $2/h \notin \mathbb{N}$, see Assumption III.1. □

Proof of Theorem III.6. For any $\mathbf{s} \in \mathscr{X}^s$, the min-max energy value that is reached by any path $\omega : \mathbf{1} \to \mathbf{s}$ is $\Phi(\mathbf{1},\mathbf{s}) \equiv \Phi(\mathbf{1},\mathscr{X}^s)$. Furthermore, Theorem III.3 implies that when a path $\omega : \mathbf{1} \to \mathbf{s}$ visits some $\mathbf{r} \in \mathscr{X}^s \setminus \{\mathbf{s}\}$, the min-max energy value that the path reaches is still $\Phi(\mathbf{1},\mathscr{X}^s)$. Indeed, for instance in the case in which the path ω may be decomposed in two paths $\omega_1 : \mathbf{1} \to \mathbf{r}$ and $\omega_2 : \mathbf{r} \to \mathbf{s}$, we have $\Phi_\omega = \max\{\Phi_{\omega_1}, \Phi_{\omega_2}\} = \Phi(\mathbf{1},\mathscr{X}^s)$ where we used (22). Hence, the saddles visited by the process are only the ones crossed during the transition between $\mathbf{1}$ and the first stable state. This fact, together with Theorem III.5, allows us to state that the set $\mathscr{W}(\mathbf{1},\mathscr{X}^s)$ is the unique minimal gate for the transition from $\mathbf{1}$ to $\mathbf{s}$, for any fixed $\mathbf{s} \in \mathscr{X}^s$. Thus, (30) is satisfied. □

C. Tube of typical trajectories: proof of the main results

In order to give the proofs of Theorems III.8 and III.9, first we prove the following lemmas.

Lemma V.2 *Let $\mathscr{C}(\eta)$ and $\mathscr{C}(\zeta)$ be the non-trivial cycles whose bottom are $\eta \in \bar{R}_{\ell,\ell-1}(1,s)$ and $\zeta \in \bar{R}_{\ell,\ell}(1,s)$ with $\ell \le \ell^* - 1$ and $s \neq 1$, respectively. Then,*

$$\mathscr{B}(\mathscr{C}(\eta)) = \bar{B}^1_{\ell-1,\ell-1}(1,s); \tag{97}$$

$$\mathscr{B}(\mathscr{C}(\zeta)) = \bar{B}^1_{\ell-1,\ell}(1,s). \tag{98}$$

Proof. For any $s \neq 1$, let $\eta_1 \in \bar{R}_{\ell,\ell-1}(1,s)$ with $\ell \le \ell^*$. By Proposition IV.1, $\eta_1 \in \mathscr{M}^3$ is a local minimum for the Hamiltonian H. Using (33), our aim is to prove the following

$$\bar{B}^1_{\ell-1,\ell-1}(1,s) = \mathscr{F}(\partial\mathscr{C}(\eta_1)). \tag{99}$$

In η_1, for any $v \in V$ the corresponding v-tile (see before Lemma IV.5 for the definition) is of type (a), (b), (d), (e) and (h), see Figure 6. Starting from η_1, by flipping to 1 (resp. s) the spin s (resp. 1) on a vertex whose tile is of type (a), (d) (resp. (b), (e)), the process visits a configuration σ_1 such that

$$H(\sigma_1) - H(\eta_1) \ge 2 - h. \tag{100}$$

Thus, the smallest energy increase is given by h by flipping to 1 a spin s on a vertex v_1 centered in a tile of type (h). Let $\eta_2 := \eta_1^{v_1,1} \in \bar{B}^{\ell-2}_{\ell-1,\ell-1}(1,s)$. In η_2, for any $v \in V$ the corresponding v-tile is one among those depicted in Figure 6(a), (b), (d), (e), (h) and (p) with $r = s$. Since $H(\eta_2) = H(\eta_1) + h$, the spin flips on a vertex whose tile is of type (a), (b), (d) and (e) lead to $H(\sigma_2) - H(\eta_1) \ge 2$. Thus, as in the previous case, the smallest energy increase is given by flipping to 1 a spin s on a vertex v_1 centered in a tile of type (h). Note that starting from η_2 the only spin flip which decreases the energy leads to the bottom of $\mathscr{C}(\eta_1)$, namely in η_1.
Let us now note that

$$H(\eta_{\ell-1}) - H(\eta_1) = h(\ell-2). \tag{101}$$

Since $\ell \le \ell^*$, comparing (100) with (101), we get that $\eta_{\ell-1} \in \mathscr{F}(\partial\mathscr{C}(\eta_1))$, and (99) is verified.
Let us now consider for any $s \neq 1$ the local minimum $\zeta_1 \in \bar{R}_{\ell,\ell}(1,s) \subset \mathscr{M}^3$ with $\ell \le \ell^* - 1$. Arguing similarly to the previous case, (98) may be verified by proving that $\bar{B}^1_{\ell-1,\ell}(1,s) = \mathscr{F}(\partial\mathscr{C}(\zeta_1))$. □

Lemma V.3 *Let $\mathscr{C}(\eta)$ be the non-trivial cycle whose bottom is $\eta \in \bar{R}_{\ell_1,\ell_2}(1,s)$ with $\min\{\ell_1,\ell_2\} \ge \ell^*$ and $s \neq 1$. Then,*

$$\mathscr{B}(\mathscr{C}(\eta)) = \bar{B}^1_{\ell_1,\ell_2}(1,s) \cup \bar{B}^1_{\ell_2,\ell_1}(1,s). \tag{102}$$

Proof. For any $s \neq 1$, let $\eta_1 \in \bar{R}_{\ell_1,\ell_2}(1,s)$ with $\ell^* \le \ell_1 \le \ell_2$. By Proposition IV.1, $\eta_1 \in \mathscr{M}^3$ is a local minimum for the Hamiltonian H. Using (33), our aim is to prove the following

$$\bar{B}^1_{\ell_1,\ell_2}(1,s) \cup \bar{B}^1_{\ell_2,\ell_1}(1,s) = \mathscr{F}(\partial\mathscr{C}(\eta_1)). \tag{103}$$

In η_1, for any $v \in V$ the corresponding v-tile is of type (a), (b), (d), (e) and (h). Let $v_1 \in V$ such that the v_1-tile is of type (e) with $r = s$, and let $\eta_2 := \eta_1^{v_1,s}$. Note that if v_1 is adjacent to a side of length ℓ_2, then $\eta_2 \in \bar{B}^1_{\ell_1,\ell_2}(1,s)$, otherwise $\eta_2 \in \bar{B}^1_{\ell_2,\ell_1}(1,s)$. Without loss of generality, let us assume that $\eta_2 \in \bar{B}^1_{\ell_1,\ell_2}(1,s)$. By simple algebraic calculation we obtain that

$$H(\eta_2) - H(\eta_1) = 2 - h. \tag{104}$$

In η_2 for any $v \in V$ the corresponding v-tile is of type (a), (b), (d), (e), (h) and (p) with $t = r = 1$. By flipping to s a spin 1 on a vertex w whose tile is of type (p) with $r = s$ the energy decreases by h and the process enters a cycle different from the previous one that is either the cycle $\bar{\mathscr{C}}$ whose bottom is a local minimum belonging to $\bar{R}_{\ell_1+1,\ell_2}(m,1)$, or a trivial cycle for which iterating this procedure the process enters $\bar{\mathscr{C}}$. Thus, $\bar{B}^1_{\ell_1,\ell_2}(1,s) \subseteq \partial\mathscr{C}(\eta_1)$. Similarly we prove that $\bar{B}^1_{\ell_2,\ell_1}(1,s) \subseteq \partial\mathscr{C}(\eta_1)$.

Let us now note that starting from η_1 the smallest energy increase is h, and it is given by flipping to 1 a spin s on a vertex whose tile is of type (h). Let us consider the uphill path ω started in η_1 and constructed by flipping to 1 all the spins s along a side of the rectangular $\ell_1 \times \ell_2$ s-cluster, say one of length ℓ_1. Using the discussion given in the proof of Lemma V.2 and the construction of ω, we get that the process intersects $\partial\mathscr{C}(\eta)$ in a configuration σ belonging to $\bar{B}^1_{\ell_2-1,\ell_1}(1,s)$. By algebraic computations, we obtain the following

$$H(\sigma) - H(\eta_1) = h(\ell_2 - 1). \tag{105}$$

Since $\ell_2 \geq \ell^*$, it follows that $H(\sigma) > H(\eta_2)$.
Since by flipping to 1 (resp. s) the vertex centered in a tile of type (a), (d) (resp. (e)), the energy increase is largest than or equal to $2+h$, it follows that (103) is satisfied. $\square$

In order to prove Theorems III.8 and III.9, we need some further definitions that are taken from [7, 9, 31]. Our goal is to give an equivalent definition of the tube that only relies on the energy landscape data. We call *cycle-path* a finite sequence $(\mathscr{C}_1,\dots,\mathscr{C}_m)$ of trivial or non-trivial cycles $\mathscr{C}_1,\dots,\mathscr{C}_m \in \mathscr{C}(\mathscr{X})$, such that $\mathscr{C}_i \cap \mathscr{C}_{i+1} = \varnothing$ and $\partial\mathscr{C}_i \cap \mathscr{C}_{i+1} \neq \varnothing$, for every $i = 1,\dots,m-1$. A cycle-path $(\mathscr{C}_1,\dots,\mathscr{C}_m)$ is said to be *downhill* (*strictly downhill*) if the cycles $\mathscr{C}_1,\dots,\mathscr{C}_m$ are pairwise connected with decreasing height, i.e., when $H(\mathscr{F}(\partial\mathscr{C}_i)) \geq H(\mathscr{F}(\partial\mathscr{C}_{i+1}))$ $(H(\mathscr{F}(\partial\mathscr{C}_i)) > H(\mathscr{F}(\partial\mathscr{C}_{i+1})))$ for any $i = 0,\dots,m-1$.
We denote the set of cycle-paths that lead from σ to $\mathscr{A}$ and consist of maximal cycles in $\mathscr{X}\setminus\mathscr{A}$ as

$$\mathscr{P}_{\sigma,\mathscr{A}} := \{\text{cycle-path } (\mathscr{C}_1,...,\mathscr{C}_m) | \mathscr{C}_1,...,\mathscr{C}_m \in \mathscr{M}(\mathscr{C}^+_{\mathscr{A}}(\sigma)\setminus A), \sigma \in \mathscr{C}_1, \partial\mathscr{C}_m \cap \mathscr{A} \neq \varnothing\}.$$

Given a non-empty set $\mathscr{A} \subset \mathscr{X}$ and $\sigma \in \mathscr{X}$, we constructively define a mapping $G : \Omega_{\sigma,A} \to \mathscr{P}_{\sigma,\mathscr{A}}$ in the following way. Given $\omega = (\omega_1,\dots,\omega_n) \in \Omega_{\sigma,A}$, we set $m_0 = 1$, $\mathscr{C}_1 = \mathscr{C}_{\mathscr{A}}(\sigma)$ and define recursively $m_i := \min\{k > m_{i-1} |\ \omega_k \notin \mathscr{C}_i\}$ and $\mathscr{C}_{i+1} := \mathscr{C}_{\mathscr{A}}(\omega_{m_i})$. We note that ω is a finite sequence and $\omega_n \in \mathscr{A}$, so there exists an index $n(\omega) \in \mathbb{N}$ such that $\omega_{m_{n(\omega)}} = \omega_n \in \mathscr{A}$ and there the procedure stops. By $(\mathscr{C}_1,\dots,\mathscr{C}_{m_{n(\omega)}})$ is a cycle-path with $\mathscr{C}_1,\dots,\mathscr{C}_{m_{n(\omega)}} \subset \mathscr{M}(\mathscr{X}\setminus\mathscr{A})$. Moreover, the fact that $\omega \in \Omega_{\sigma,A}$ implies that $\sigma \in \mathscr{C}_1$ and that $\partial\mathscr{C}_{n(\omega)} \cap \mathscr{A} \neq \varnothing$, hence $G(\omega) \in \mathscr{P}_{\sigma,\mathscr{A}}$ and the mapping is well-defined.
We say that a cycle-path $(\mathscr{C}_1,\dots,\mathscr{C}_m)$ is *connected via typical jumps* to $\mathscr{A} \subset \mathscr{X}$ or simply *vtj−connected* to $\mathscr{A}$ if

$$\mathscr{B}(\mathscr{C}_i) \cap \mathscr{C}_{i+1} \neq \varnothing, \ \ \forall i = 1,\dots,m-1, \ \text{ and } \ \mathscr{B}(\mathscr{C}_m) \cap \mathscr{A} \neq \varnothing. \tag{106}$$

Let $J_{\mathscr{C},\mathscr{A}}$ be the collection of all cycle-paths $(\mathscr{C}_1,\dots,\mathscr{C}_m)$ that are vtj-connected to $\mathscr{A}$ and such that $\mathscr{C}_1 = \mathscr{C}$. Given a non-empty set $\mathscr{A}$ and $\sigma \in \mathscr{X}$, we define $\omega \in \Omega_{\sigma,A}$ as a *typical path* from σ to $\mathscr{A}$ if its corresponding cycle-path $G(\omega)$ is vtj-connected to $\mathscr{A}$ and we denote by $\Omega^{\text{vtj}}_{\sigma,A}$ the collection of all typical paths from σ to $\mathscr{A}$, i.e.,

$$\Omega^{\text{vtj}}_{\sigma,\mathscr{A}} := \{\omega \in \Omega_{\sigma,\mathscr{A}} |\ G(\omega) \in J_{\mathscr{C}_{\mathscr{A}}(\sigma),\mathscr{A}}\}. \tag{107}$$

Finally, we define the *tube of typical paths* $T_{\mathscr{A}}(\sigma)$ from σ to $\mathscr{A}$ as the subset of states $\eta \in \mathscr{X}$ that can be reached from σ by means of a typical path which does not enter $\mathscr{A}$ before visiting η, i.e.,

$$T_{\mathscr{A}}(\sigma) := \{\eta \in \mathscr{X} |\ \exists\omega \in \Omega^{\text{vtj}}_{\sigma,\mathscr{A}} :\ \eta \in \omega\}. \tag{108}$$

Finally, we define $\mathfrak{T}_{\mathscr{A}}(\sigma)$ as the set of all maximal cycles that belong to at least one vtj-connected path from $\mathscr{C}^{\sigma}_{\mathscr{A}}(\Gamma)$ to $\mathscr{A}$, i.e.,

$$\mathfrak{T}_{\mathscr{A}}(\sigma) := \{\mathscr{C} \in \mathscr{M}(\mathscr{C}^+_{\mathscr{A}}(\sigma)\setminus\mathscr{A}) | \exists(\mathscr{C}_1,\dots,\mathscr{C}_n) \in J_{\mathscr{C}^{\sigma}_{\mathscr{A}}(\Gamma),\mathscr{A}}, \exists j \in \{1,...,n\} : \mathscr{C}_j = \mathscr{C}\}. \tag{109}$$

Note that $\mathfrak{T}_{\mathscr{A}}(\sigma) = \mathscr{M}(T_{\mathscr{A}}(\sigma)\setminus\mathscr{A})$. and that the boundary of $T_{\mathscr{A}}(\sigma)$ consists of states either in $\mathscr{A}$ or in the non-principal part of the boundary of some $\mathscr{C} \in \mathfrak{T}_{\mathscr{A}}(\sigma)$, i.e, $\partial T_{\mathscr{A}}(\sigma)\setminus\mathscr{A} \subseteq \bigcup_{\mathscr{C}\in\mathfrak{T}_{\mathscr{A}}(\sigma)}(\partial\mathscr{C}\setminus\mathscr{B}(\mathscr{C})) =: \partial^{np}\mathfrak{T}_{\mathscr{A}}(\sigma)$.

Proof of Theorem III.8. Following the same approach as [7, Section 6.7], we characterize the tube of typical trajectories using the so-called "standard cascades". See [7, Figure 6.3] for an example of a standard cascade. We describe these in terms of the paths that are started in $\mathbf{1}$ and are vtj-connected to $\mathscr{X}^s$. See (107) for the formal definition and see [9, Lemma 3.12] for an equivalent characterization of these paths. We remark that any typical path from $\mathbf{1}$ to $\mathscr{X}^s$ is also an optimal path for the same transition.

In order to give a geometrical description of these typical paths, we proceed similarly to [7, Section 7.4], where the authors apply the model-independent results given in Section 6.7 to identify the tube of typical paths in the context of the Ising model. We define a vtj-connected cycle-path that is the concatenation of both trivial and non-trivial cycles. Let η_1 be a configuration belonging to one of the minimal gates for the transition $\mathbf{1} \to \mathscr{X}^s$, see Theorem III.5 . We begin by studying the first descent from η_1 both to $\mathbf{1}$ and to $\mathscr{X}^s$. Then, we complete the description of $\mathfrak{T}_{\mathscr{X}^s}(\mathbf{1})$ by joining the time reversal of the first descent from η_1 to $\mathbf{1}$ with the first descent from η_1 to $\mathscr{X}^s$. In view of (28) we have that $\eta_1 \in \bar{B}^1_{\ell^*-1,\ell^*}(1,s)$ for some $s \neq 1$, and for the sake of semplicity we describe a vtj-connected path from $\mathbf{1}$ to $\mathscr{X}^s$ conditioned to hit $\mathscr{X}^s$ for the first time in $\mathbf{s}$.

Let us begin by studying the standard cascades from η_1 to $\mathbf{1}$. Since a spin flip from s to $t \notin \{1,s\}$ implies an increase of the energy value equal to the increase of the number of the disagreeing edges, we consider only the splin-flips from s to 1 on those vertices belonging to the s-cluster. Thus, starting from η_1 and given v_1 a vertex such that $\eta_1(v_1) = s$, since $H(\eta_1) = \Phi(\mathbf{1}, \mathscr{X}^s)$, we get

$$H(\eta_1^{v_1,1}) = \Phi(\mathbf{1}, \mathscr{X}^s) + n_s(v_1) - n_1(v_1) + h. \tag{110}$$

It follows that the only possibility for the path to be optimal is $n_s(v_1) = 1$ and $n_1(v_1) = 3$. Thus, along the first descent from η_1 to $\mathbf{1}$ the process visits η_2 in which all the vertices have spin 1 except those, which are s, in a rectangular cluster $\ell^* \times (\ell^* - 1)$, i.e., $\eta_2 \in \bar{R}_{\ell^*-1,\ell^*}(1,s)$. By Proposition IV.1 $\eta_2 \in \mathscr{M}^3$ is a local minimum, thus according to (106) we have to describe its non-trivial cycle and its principal boundary. Starting from η_2, the next configuration along a typical path is defined by flipping to 1 a spin s on a vertex v_2 on one of the four corners of the rectangular s-cluster. Indeed, since $H(\eta_2) = \Phi(\mathbf{1}, \mathscr{X}^s) - 2 + h$, we have

$$H(\eta_2^{v_2,1}) = \Phi(\mathbf{1}, \mathscr{X}^s) - 2 + 2h + n_s(v_2) - n_1(v_2), \tag{111}$$

and for the path to be optimal, we must have $n_s(v_2) = 2$ and $n_1(v_2) = 2$. The smallest energy increase for any single step of the dynamics is equal to h. Thus, a typical path towards $\mathbf{1}$ proceeds by eroding the $\ell^* - 2$ unit squares with spin s belonging to a side of length $\ell^* - 1$ that are corners of the s-cluster and that belong to the same side of v_2. Each of the first $\ell^* - 3$ spin flips increases the energy by h, and these uphill steps are necessary in order to exit from the cycle whose bottom is the local minimum η_2. After these $\ell^* - 3$ steps, the process hits the bottom of the boundary of this cycle in a configuration $\eta_{\ell^*} \in \bar{B}^1_{\ell^*-1,\ell^*-1}(1,s)$, see Lemma V.2. The last spin-update, that flips from s to 1 the spin s on the unit protuberance of the s-cluster, decreases the energy by $2 - h$. Thus, the typical path arrives in a local minimum $\eta_{\ell^*+1} \in \bar{R}_{\ell^*-1,\ell^*-1}(1,s)$, i.e., it enters a new cycle whose bottom is a configuration in which all the vertices have spin 1, except those, which are s, in a square $(\ell^* - 1) \times (\ell^* - 1)$ s-cluster. Summarizing the construction above, we have the following sequence of vtj-connected cycles

$$\{\eta_1\}, \mathscr{C}_{\mathbf{1}}^{\eta_2}(h(\ell^* - 2)), \{\eta_{\ell^*}\}, \mathscr{C}_{\mathbf{1}}^{\eta_{\ell^*+1}}(h(\ell^* - 2)). \tag{112}$$

Iterating this argument, we obtain that the first descent from $\eta_1 \in \mathscr{W}(\mathbf{1}, \mathscr{X}^s)$ to $\mathbf{1}$ is characterized by the concatenation of those vtj-connected cycle-subpaths between the cycles whose bottom is the local minima in which all the vertices have spin equal to 1, except those, which are s, in either a quasi-square $(\ell - 1) \times \ell$ or a square $(\ell - 1) \times (\ell - 1)$ for any $\ell = \ell^*, \dots, 1$, and whose depth is given by $h(\ell - 2)$. More precisely, from a quasi-square to a square, a typical path proceeds by flipping to 1 those spins s on one of the shortest sides of the s-cluster. On the other hand, from a square to a quasi-square, it proceeds by flipping to 1 those spins s belonging to one of the four sides of the square. Thus, a standard cascade from η_1 to $\mathbf{1}$ is characterized by the sequence of those configurations that belong to

$$\bigcup_{\ell=1}^{\ell^*} \left[\bigcup_{l=1}^{\ell-1} \bar{B}^l_{\ell-1,\ell}(1,s) \cup \bar{R}_{\ell-1,\ell}(1,s) \cup \bigcup_{l=1}^{\ell-2} \bar{B}^l_{\ell-1,\ell-1}(1,s) \cup \bar{R}_{\ell-1,\ell-1}(1,s) \right]. \tag{113}$$

Let us now consider the first descent from $\eta_1 \in \bar{B}^1_{\ell^*-1,\ell^*}(1,s)$ to $\mathbf{s} \in \mathscr{X}^s$. Since the path is optimal, we only consider flips from 1 to s. Thus, let w_1 be a vertex such that $\eta_1(w_1) = 1$. Flipping the spin 1 on the vertex w_1, we get

$$H(\eta_1^{w_1,s}) = \Phi(\mathbf{1}, \mathscr{X}^s) + n_1(w_1) - n_s(w_1) - h, \tag{114}$$

and the only feasible choice is $n_1(w_1) = 2$ and $n_s(w_1) = 2$. Thus, $\eta_1^{w_1,s} \in \bar{B}^2_{\ell^*-1,\ell^*}(1,s)$, namely the bar is now of length two. Arguing similarly, we get that along the descent to $\mathbf{s}$ a typical path proceeds by flipping from 1 to s the spins 1 with two nearest-neighbors with spin s and two nearest-neighbors with spin 1 belonging to the incomplete side of the s-cluster. More precisely, it proceeds downhill visiting $\bar{\eta}_i \in \bar{B}^i_{\ell^*-1,\ell^*}(1,s)$ for any $i = 2, \dots, \ell^* - 1$ and $\bar{\eta}_{\ell^*} \in \bar{R}_{\ell^*,\ell^*}(1,s)$, which is a local minimum by Proposition IV.1. In order to exit from the cycle whose bottom is $\bar{\eta}_{\ell^*}$, the process crosses the bottom of its boundary by creating a unit protuberance of spin s adjacent to one of the four edges of the s-square, i.e., visits $\{\bar{\eta}_{\ell^*+1}\}$ where $\bar{\eta}_{\ell^*+1} \in \bar{B}^1_{\ell^*,\ell^*}(1,s)$, see Lemma V.3. Starting from $\{\bar{\eta}_{\ell^*+1}\}$, a typical path towards $\mathbf{s}$ proceeds by enlarging the protuberance to a bar of length two to $\ell^* - 1$, thus it visits $\bar{\eta}_{\ell^*+i} \in \bar{B}^i_{\ell^*,\ell^*}(1,s)$ for any $i = 2, \dots, \ell^* - 1$. Each of these steps decreases the energy by h, and eventually the bottom of the cycle is reached, i.e., in the local minimum $\bar{\eta}_{2\ell^*} \in \bar{R}_{\ell^*,\ell^*+1}(1,s)$. Then, the process exits from this cycle through the bottom of its boundary by adding a unit protuberance of spin s on *any* one of the four edges of the rectangular $\ell^* \times (\ell^* + 1)$ s-cluster in $\bar{\eta}_{2\ell^*}$. Thus, it visits the trivial cycle $\{\bar{\eta}_{2\ell^*+1}\}$, where $\bar{\eta}_{2\ell^*+1} \in \bar{B}^1_{\ell^*,\ell^*+1}(1,s) \cup \bar{B}^1_{\ell^*+1,\ell^*}(1,s)$. Note that the resulting standard cascade is different from the one towards **1**. Thus, summarizing the construction above, we have defined the following sequence of vtj-connected cycles

$$\{\eta_1\}, \mathscr{C}_{\mathbf{s}}^{\bar{\eta}_{\ell^*}}(h(\ell^*-1)), \{\bar{\eta}_{\ell^*+1}\}, \mathscr{C}_{\mathbf{s}}^{\bar{\eta}_{2\ell^*}}(h(\ell^*-1)), \{\bar{\eta}_{2\ell^*+1}\}. \tag{115}$$

Note that if $\bar{\eta}_{2\ell^*} \in \bar{B}^1_{\ell^*,\ell^*+1}(1,s)$, then the process enters the cycle whose bottom is a configuration belonging to $\bar{R}_{\ell^*+1,\ell^*+1}(1,s)$. On the other hand, if $\bar{\eta}_{2\ell^*} \in \bar{B}^1_{\ell^*+1,\ell^*}(1,s)$, then the standard cascade enters the cycle whose bottom is a configuration belonging to $\bar{R}_{\ell^*,\ell^*+2}(1,s)$. In the first case the cycle has depth $h\ell^*$, in the second case the cycle has depth $h(\ell^*-1)$. Iterating this argument, we get that the first descent from η_1 to $\mathbf{s}$ is characterized by vtj-connected cycle-subpaths from $\bar{R}_{\ell_1,\ell_2}(1,s)$ to $\bar{R}_{\ell_1,\ell_2+1}(1,s)$ defined as the sequence of those configurations belonging to $\bar{B}^l_{\ell_1,\ell_2}(1,s)$ for any $l=1,\dots,\ell_2-1$. Eventually, a configuration in which this cluster is either a vertical or a horizontal strip is reached, i.e., it intersects one of the two sets defined in (34)–(35). If the descent arrives in $\mathscr{S}^v(1,s)$, then it proceeds by enlarging the vertical strip column by column. Otherwise, if it arrives in $\mathscr{S}^h(1,s)$, then it enlarges the horizontal strip row by row. In both cases, starting from a configuration with an s-strip, i.e., a local minimum in $\mathscr{M}^2$ by Proposition IV.1, the path exits from its cycle by adding a unit protuberance with a spin s adjacent to one of the two vertical (resp. horizontal) edges and increasing the energy by $2-h$. Starting from this trivial cycle, the standard cascade proceeds downhill in a new cycle by filling the column (resp. row) with spins s. More precisely, the standard cascade visits $K-1$ (resp. $L-1$) configurations such that each of them is defined by the previous one flipping from 1 to s a spin 1 with two nearest-neighbors with spin 1 and two nearest-neighbors with spin s. Each of these spin-updates decreases the energy by h. The process arrives in this way to the bottom of the cycle, i.e., in a configuration in which the thickness of the s-strip has been enlarged by a column (resp. row). Starting from this state with the new s-strip, we repeat the same arguments above until the standard cascade arrives in the trivial cycle of a configuration σ with an s-strip of thickness $L-2$ (resp. $K-2$) and with a unit protuberance. Starting from $\{\sigma\}$, the process enters the cycle whose bottom is $\mathbf{s}$ and it proceeds downhill either by flipping from 1 to s those spins 1 with two nearest-neighbors with spin 1 and two nearest-neighbors with spin s, or by flipping to s all the spins 1 with three nearest-neighbors with spin s and one nearest-neighbor with spin 1. The last step flips from 1 to s the last spin 1 with four nearest-neighbors with spin s. Thus, the first descent from η_1 to $\mathscr{X}^s$ conditioning to hit this set in $\mathbf{s}$ is characterized by the sequence of those configurations that belong to

$$\bigcup_{\ell_1=\ell^*}^{K-1}\bigcup_{\ell_2=\ell^*}^{K-1}\bar{R}_{\ell_1,\ell_2}(1,s)\cup\bigcup_{\ell_1=\ell^*}^{K-1}\bigcup_{\ell_2=\ell^*}^{K-1}\bigcup_{l=1}^{\ell_2-1}\bar{B}^l_{\ell_1,\ell_2}(1,s)\cup\bigcup_{\ell_1=\ell^*}^{L-1}\bigcup_{\ell_2=\ell^*}^{L-1}\bar{R}_{\ell_1,\ell_2}(1,s)$$
$$\cup\bigcup_{\ell_1=\ell^*}^{L-1}\bigcup_{\ell_2=\ell^*}^{L-1}\bigcup_{l=1}^{\ell_2-1}\bar{B}^l_{\ell_1,\ell_2}(1,s)\cup\mathscr{S}^v(1,s)\cup\mathscr{S}^h(1,s). \tag{116}$$

To conclude we need to find the standard cascade from **1** to $\mathscr{X}^s$. Using Theorem III.5 and the symmetry of the energy landscape with respect to the $q-1$ stable states, we complete the proof by taking the union of the standard cascades from **1** to all possible $\mathbf{s}\in\mathscr{X}^s$ given by (113)–(116). Finally, (37) follows by [9, Lemma 3.13]. □

Proof of Theorem III.9 Let us assume $q>2$, otherwise the result is proven in [7, Section 7.4]. Starting from the metastable state **1**, the process hits $\mathscr{X}^s$ in any stable state $\mathbf{r}$ with the same probability $\frac{1}{q-1}$. The set of typical paths $\Omega^{\text{vtj}}_{\mathbf{1},\mathbf{s}}$ may be partitioned in two subsets $\Omega^{\text{vtj},1}_{\mathbf{1},\mathbf{s}} := \{\omega\in\Omega^{\text{vtj}}_{\mathbf{1},\mathbf{s}}:\ \omega\cap\mathscr{X}^s\backslash\{\mathbf{s}\}=\varnothing\}$ and $\Omega^{\text{vtj},2}_{\mathbf{1},\mathbf{s}} := \{\omega\in\Omega^{\text{vtj}}_{\mathbf{1},\mathbf{s}}:\ \omega\cap\mathscr{X}^s\backslash\{\mathbf{s}\}\neq\varnothing\}$. Since the process follows a path belonging to $\Omega^{\text{vtj},2}_{\mathbf{1},\mathbf{s}}$ with probability $\frac{q-2}{q-1}>0$, these trajectories also belong to the tube of typical paths. Thus, the tube $\mathfrak{T}_{\mathbf{s}}(\mathbf{1})$ is comprised of those configurations that belong to all the typical paths that go from **1** to $\mathscr{X}^s$, i.e., those states belonging to $\mathfrak{T}_{\mathscr{X}^s}(\mathbf{1})$, and of those configurations that belong to all typical paths from any $\mathbf{r}\in\mathscr{X}^s\backslash\{\mathbf{s}\}$ to $\mathbf{s}$. Using Remark III.3, these last configurations belong to the tube $\mathfrak{T}^{\text{zero}}_{\mathbf{s}}(\mathbf{r})$ given by [2, Equation 4.25, Theorem 4.3]. Finally, we apply [9, Lemma 3.13] to prove (39). □

VI. SHARP ESTIMATE ON THE MEAN TRANSITION TIME FROM THE METASTABLE STATE TO THE SET OF THE STABLE STATES

In order to prove our main results on the computation of the prefactor and on the estimate of the expected value of the transition time from a metastable state to the stable set, we adopt the *potential theoretic approach*. In order to apply this method, let us give some further definitions and some known results taken from [16, 30] and from [12].

We begin by introducing some further model-independent definitions and results. Consider any energy landscape $(\mathscr{X},H,Q)$

and let $h : \mathscr{X} \to \mathbb{R}$. We define *Dirichlet form* as

$$\mathfrak{E}_\beta(h) := \frac{1}{2} \sum_{\sigma,\eta \in \mathscr{X}} \mu_\beta(\sigma) P_\beta(\sigma,\eta) [h(\sigma) - h(\eta)]^2$$
$$= \frac{1}{2} \sum_{\sigma,\eta \in \mathscr{X}} \frac{e^{-\beta H(\sigma)}}{Z} \frac{e^{-\beta [H(\eta)-H(\sigma)]_+}}{|\Lambda|} [h(\sigma) - h(\eta)]^2. \tag{117}$$

Given two non-empty disjoint sets $\mathscr{A}_1, \mathscr{A}_2 \subset \mathscr{X}$, the *capacity* of the pair $\mathscr{A}_1, \mathscr{A}_2$ is defined by

$$\text{CAP}(\mathscr{A}_1, \mathscr{A}_2) := \min_{\substack{h : \mathscr{X} \to [0,1] \\ h_{|\mathscr{A}_1} = 1, h_{|\mathscr{A}_2} = 0}} \mathfrak{E}_\beta(h). \tag{118}$$

Note that from (118) it follows immediately that the capacity is symmetric in $\mathscr{A}_1$ and $\mathscr{A}_2$. In particular, the right hand side of (118) has a unique minimizer $h^*_{\mathscr{A}_1,\mathscr{A}_2}$ known as *equilibrium potential* of $\mathscr{A}_1, \mathscr{A}_2$ and given by $h^*_{\mathscr{A}_1,\mathscr{A}_2}(\eta) = \mathbb{P}(\tau^\eta_{\mathscr{A}_1} < \tau^\eta_{\mathscr{A}_2})$, for any $\eta \in \mathscr{X}$. Finally, using what we have just defined, consider the following.

Definition VI.1 *A set $\mathscr{A} \subset \mathscr{X}$ is said to be p.t.a.-metastable if*

$$\lim_{\beta \to \infty} \frac{\max_{\sigma \notin \mathscr{A}} \mu_\beta(\sigma) [CAP_\beta(\sigma, \mathscr{A})]^{-1}}{\min_{\sigma \in \mathscr{A}} \mu_\beta(\sigma) [CAP_\beta(\sigma, \mathscr{A} \setminus \{\sigma\})]^{-1}} = 0. \tag{119}$$

The prefix p.t.a. stands for potential theoretic approach and it is used for distinguishing the Definition VI.1 from that of the metastable set $\mathscr{X}^m$. We remark that the idea of defining a set as in Definition VI.1 was introduced in [30], where the authors refer to it as set of metastable points. We refer to [30] and to [16, Chapter 8] for the study of the main properties of this set.

Since the identification of a p.t.a.-metastable set is quite difficult if one starts from the Definition VI.1, we recall [28, Theorem 3.6] where the authors give a constructive method for defining any p.t.a.-metastable set. In particular, for any $\sigma, \eta \in \mathscr{X}$, the authors introduced the following equivalence relation

$$\sigma \sim \eta \text{ if and only if } \Phi(\sigma,\eta) - H(\sigma) < \Gamma^m \text{ and } \Phi(\eta,\sigma) - H(\eta) < \Gamma^m. \tag{120}$$

Assumed $\mathscr{X} \setminus \mathscr{X}^s \neq \varnothing$, let $\mathscr{X}^m_{(1)}, \dots, \mathscr{X}^m_{(k_m)}$ and $\mathscr{X}^s_{(1)}, \dots, \mathscr{X}^s_{(k_s)}$ be the equivalence classes in which $\mathscr{X}^m$ and $\mathscr{X}^s$ are partitioned with respect to the relation $\sim$, respectively.

Theorem VI.1 [28, Theorem 3.6] *Assume that $\mathscr{X} \setminus \mathscr{X}^s \neq \varnothing$ and $\mathscr{X} \setminus (\mathscr{X}^s \cup \mathscr{X}^m) \neq \varnothing$. Choose arbitrarily $\sigma_{s,i} \in \mathscr{X}^s_{(i)}$ for any $i = 1, \dots, k_s$ and $\sigma_{m,j} \in \mathscr{X}^m_{(j)}$ for any $j = 1, \dots, k_m$. The set $\{\sigma_{s,1}, \dots, \sigma_{s,k_s}, \sigma_{m,1}, \dots, \sigma_{m,k_m}\}$ is a p.t.a.-metastable.*

Remark VI.1 *In [16, Chapters 8 and 16] the authors state the main metastability theorems for those energy landscapes in which the stable set $\mathscr{X}^s = \{\mathbf{s}\}$ and the metastable set $\mathscr{X}^m = \{\mathbf{m}\}$ are singletons. In particular, in [16, Lemma 16.13] the authors prove that the pair $\mathscr{A} = \{\mathbf{m}, \mathbf{s}\}$ is a p.t.a.-metastable set.*

A. Mean crossover time and computation of prefactor: proof of main results

In this subsection we prove Theorem III.7 by using the model independent results given in [12] and [16, Chapter 16], by exploiting the discussion given in [28, Subsection 3.1] and also by using some results given in [30],[18]. Let us begin by giving the following list of definitions and notations.

- With an abuse of notation we consider $\mathscr{X}$ as a graph whose vertices are the configurations. Given two configurations $\sigma, \eta \in \mathscr{X}$ there is an edge between the corresponding vertices if it is possible to move from σ to η (resp. η to σ) in one step of the dynamics.
- Let $\mathscr{X}^* \subset \mathscr{X}$ be the subgraph obtained by removing all the vertices corresponding to configurations $\sigma \in \mathscr{X}$ such that $H(\sigma) > \Gamma^m + H(\mathbf{1})$ and also removing all edges incident to these configurations.
- Let $\mathscr{X}^{**} \subset \mathscr{X}^*$ be the subgraph obtained by removing all the vertices corresponding to configurations σ such that $H(\sigma) = \Gamma^m + H(\mathbf{1})$ and also removing all edges incident to these configurations.

- Let $\mathscr{P}^*_{\rm PTA}(\mathbf{1},\mathscr{X}^s)$ be the *protocritical set* and let $\mathscr{C}^*_{\rm PTA}(\mathbf{1},\mathscr{X}^s)$ be the *critical set*. More precisely, we exploit [16, Definition 16.3] and define $(\mathscr{C}^*_{\rm PTA}(\mathbf{1},\mathscr{X}^s),\mathscr{P}^*_{\rm PTA}(\mathbf{1},\mathscr{X}^s))$ as the maximal subset of $\mathscr{X}\times\mathscr{X}$ such that: (1) for any $\sigma\in\mathscr{P}^*_{\rm PTA}(\mathbf{1},\mathscr{X}^s)$ there exists $\eta\in\mathscr{C}^*_{\rm PTA}(\mathbf{1},\mathscr{X}^s)$ such that $\sigma\sim\eta$ and for any $\eta\in\mathscr{C}^*_{\rm PTA}(\mathbf{1},\mathscr{X}^s)$ there exists $\sigma\in\mathscr{P}^*_{\rm PTA}(\mathbf{1},\mathscr{X}^s)$ such that $\eta\sim\sigma$;

  (2) for any $\sigma\in\mathscr{P}^*_{\rm PTA}(\mathbf{1},\mathscr{X}^s)$, $\Phi(\sigma,\mathbf{1})<\Phi(\sigma,\mathscr{X}^s)$;

  (3) for any $\eta\in\mathscr{C}^*_{\rm PTA}(\mathbf{1},\mathscr{X}^s)$ there exists a path $\omega:\eta\to\mathscr{X}^s$ such that $\max_{\zeta\in\omega}H(\zeta)-H(\mathbf{1})\le\Gamma^m$ and $\omega\cap\{\zeta\in\mathscr{X}:\Phi(\zeta,\mathbf{1})<\Phi(\zeta,\mathscr{X}^s)\}=\varnothing$.

Next, consider $\mathscr{W}(\mathbf{1},\mathscr{X}^s)=\mathscr{G}^1\cup\mathscr{G}^2$ where $\mathscr{G}^1$ and $\mathscr{G}^2$ are defined as follows.
- $\mathscr{G}^1:=\{\sigma\in\mathscr{W}(\mathbf{1},\mathscr{X}^s)$: the cluster of spins different from 1 has the unit protuberance on a corner of one of the longest sides of the quasi-square $\ell^*\times(\ell^*-1)\}$.
- $\mathscr{G}^2:=\{\sigma\in\mathscr{W}(\mathbf{1},\mathscr{X}^s)$: the cluster of spins different from 1 has the unit protuberance on one of the ℓ^*-2 vertices different from the corners of one of the longest sides of the quasi-square $\ell^*\times(\ell^*-1)\}$. Following the same strategy given in [12], let us consider the set

$$\mathscr{X}^{**}\backslash(\mathscr{C}^{\mathbf{1}}_{\mathscr{X}^s}(\Gamma^m)\cup\mathscr{C}^{\mathscr{X}^s}_{\mathbf{1}}(\Gamma(\mathscr{X}^s,\mathbf{1})))=\bigcup_{i=1}^{I}\mathscr{X}(i), \tag{121}$$

where each $\mathscr{X}(i)$ is a set of communicating states with energy strictly lower than $\Phi(\mathbf{1},\mathscr{X}^s)$ and with communication energy $\Phi(\mathbf{1},\mathscr{X}^s)$ with respect to both $\mathbf{1}$ and $\mathscr{X}^s$. Among these sets we find also the wells $\mathscr{Z}^{\mathbf{1}}_j$ (resp. $\mathscr{Z}^{\mathscr{X}^s}_j$) that are connected by one step of the dynamics with the unessential saddles that in [12, Definitions 3.2 and 3.4] are said to be "of the first type" (resp. "of the second type") and that are denoted by σ_j (resp. ζ_j). In view of the above discussion, let us define the following subsets of $\mathscr{X}^*$.

- $A:=\mathscr{C}^{\mathbf{1}}_{\mathscr{X}^s}(\Gamma^m)\cup\bigcup_{j=1}^{J_{meta}}(\{\sigma_j\}\cup\mathscr{Z}^{\mathbf{1}}_j)$.
- $B:=\mathscr{C}^{\mathscr{X}^s}_{\mathbf{1}}(\Gamma(\mathscr{X}^s,\mathbf{1}))\cup\bigcup_{j=1}^{J_{stab}}(\{\zeta_j\}\cup\mathscr{Z}^{\mathscr{X}^s}_j)$.

Before of the proof of Theorem III.7, it is useful to state the following results.

Lemma VI.1 *The cardinality of $\mathscr{G}^1$ and $\mathscr{G}^2$ are $|\mathscr{G}^1|=8|\Lambda|(q-1)$ and $|\mathscr{G}^2|=4|\Lambda|(\ell^*-2)(q-1)$, respectively.*

Proof. In $\mathscr{G}^1$ the protuberance lies at one of the two extreme ends of one of the side of length ℓ^*, hence there are four possible positions. On the other hand, in $\mathscr{G}^2$ there are $2(\ell^*-2)$ sites in which can place the unit protuberance. In both cases, the quantity $2|\Lambda|$ counts the number of locations and rotations of the cluster with spins different from 1. Indeed, the quasi-square with the unit protuberance may be located anywhere in Λ in two possible orientations. Furthermore, the factor $(q-1)$ counts the number of possible spins that may characterize this homogenous cluster. □

Lemma VI.2 *If the external magnetic field is negative, then the set $\{\mathbf{1},\mathscr{X}^s\}$ is p.t.a-metastable.*

Proof. Consider the equivalence relation $\sim$ given in (120). From Theorem III.1, we get that in the energy landscape $(\mathscr{X},H,Q)$ the metastable set is a singleton. Hence, there exists only one equivalence class with respect to $\sim$ given by $\mathscr{X}^m$ itself. On the other hand, $\mathscr{X}^s=\{\mathbf{2},\dots,\mathbf{q}\}$ and from Equation (23) of Theorem III.3 we get that $\mathscr{X}^s_{(1)}:=\{\mathbf{2}\},\dots,\mathscr{X}^s_{(q-1)}:=\{\mathbf{q}\}$ are the equivalence classes with respect to the relation $\sim$ that partition $\mathscr{X}^s$. Thus, by Theorem VI.1 we conclude that the set $\{\mathbf{1},\mathbf{2},\dots,\mathbf{q}\}=\{\mathbf{1},\mathscr{X}^s\}$ is p.t.a.-metastable. □

Proposition VI.1 *If the external magnetic field is negative, then*

$$\mathscr{C}^*_{\rm PTA}(\mathbf{1},\mathscr{X}^s)=\mathscr{W}(\mathbf{1},\mathscr{X}^s). \tag{122}$$

Proof. Following the same strategy of the proof of [16, Theorem 17.3], (122) follows by the definition of $\mathscr{C}^*_{\rm PTA}(\mathbf{1},\mathscr{X}^s)$, by Lemmas IV.1–IV.4 and by Proposition V.1. □

Lemma VI.3 *Let $\eta\in\mathscr{W}(\mathbf{1},\mathscr{X}^s)$ and let $\bar\eta\in\mathscr{X}$ such that $\bar\eta:=\eta^{v,t}$ for some $v\in V$ and $t\in S$, $t\neq\eta(v)$. If the external magnetic field is negative, then either $H(\eta)<H(\bar\eta)$ or $H(\eta)>H(\bar\eta)$.*

Proof. Since $\eta\in\mathscr{W}(\mathbf{1},\mathscr{X}^s)=\bigcup_{t=2}^q\bar B^1_{\ell^*-1,\ell^*}(1,t)$, there exists $s\neq1$ such that $\eta\in\bar B^1_{\ell^*-1,\ell^*}(1,s)$. This implies that η is characterized by all spins 1 except those, which are s, in a quasi-square $(\ell^*-1)\times\ell^*$ with a unit protuberance on one of the longest sides. In particular, for any $u\in V$, either $\eta(u)=1$ or $\eta(u)=s$. If $\eta(u)=1$, then for any $t\in S\backslash\{1\}$, depending on the distance between the vertex u and the s-cluster, we have

$$H(\bar\eta)-H(\eta)=\begin{cases}4-h\mathbb{1}_{\{t=s\}}, & \text{if } n_1(u)=4\,;\\ 3-\mathbb{1}_{\{t=s\}}-h\mathbb{1}_{\{t=s\}}, & \text{if } n_1(u)=3,\, n_s(u)=1;\\ 2-2\mathbb{1}_{\{t=s\}}-h\mathbb{1}_{\{t=s\}}, & \text{if } n_1(u)=2,\, n_s(u)=2.\end{cases} \tag{123}$$

Otherwise, if $\eta(u)=1$, for any $t\in S\backslash\{1\}$, depending on the distance between the vertex u and the boundary of the s-cluster, we get

$$H(\bar\eta)-H(\eta)=\begin{cases}4+h, & \text{if } n_s(u)=4;\\ 3-\mathbb{1}_{\{t=1\}}+h, & \text{if } n_1(u)=1,\, n_s(u)=3;\\ 2-2\mathbb{1}_{\{t=1\}}+h, & \text{if } n_1(u)=2,\, n_s(u)=2;\\ 1-3\mathbb{1}_{\{t=1\}}+h, & \text{if } n_1(u)=3,\, n_s(u)=1.\end{cases} \tag{124}$$

We conclude that $H(\eta)\neq H(\bar\eta)$. □

In [12, Definitions 3.2 and 3.4] the authors define two subsets of unessential saddles for the metastable transition and they call them respectively unessential saddles of the first type" and of the second type and in [12, Equations (3.16)–(3.17)] they define the sets K and $\tilde K$. Using these definitions and Lemma VI.3, we are now able to prove the following.

Lemma VI.4 *If the external magnetic field is negative, then the following properties are verified.*

(a) $K=\varnothing$, $\tilde K=\varnothing$.

(b) *Any $\sigma\in\mathscr{W}'(\mathbf{1},\mathscr{X}^s)$ is such that $\sigma\in\bigcup_{j=1}^{J_{meta}}(\{\sigma_j\}\cup\mathscr{Z}_j^{\mathbf{1}})$, namely there exist at least a unessential saddle σ_i "of the first type" and its well $\mathscr{Z}_i^{\mathbf{1}}$ is not empty.*

(c) *The set $\bigcup_{j=1}^{J_{stab}}(\{\zeta_j\}\cup\mathscr{Z}_j^{\mathscr{X}^s})$ is not empty, namely there exists at least a unessential saddle ζ_i "of the second type".*

Proof. By Lemma VI.3 we have that any $\eta\in\mathscr{W}(\mathbf{1},\mathscr{X}^s)$ that communicates with configurations in the cycles $\mathscr{C}^{\mathbf{1}}_{\mathscr{X}^s}(\Gamma^m)\cup\mathscr{C}_{\mathbf{1}}^{\mathscr{X}^s}(\Gamma(\mathscr{X}^s,\mathbf{1}))$, in $\mathscr{X}\backslash\mathscr{X}^*$, and it does not communicate by a single step of the dynamics with another saddle. This implies that for any $\bar\eta\in\mathscr{S}(\mathbf{1},\mathscr{X}^s)\backslash\mathscr{W}(\mathbf{1},\mathscr{X}^s)$, visited by the process before visiting the gate $\mathscr{W}(\mathbf{1},\mathscr{X}^s)$, it does not exist a path $\omega_1:\eta\to\bar\eta$ such that $\omega_1\cap\mathscr{C}^{\mathbf{1}}_{\mathscr{X}^s}(\Gamma^m)=\varnothing$, $\omega_1\cap\mathscr{W}(\mathbf{1},\mathscr{X}^s)=\{\eta\}$, and $\max_{\sigma\in\omega_1}H(\sigma)\le\Phi(\mathbf{1},\mathscr{X}^s)$. This concludes that $K=\varnothing$. Furthermore, for any $\bar\eta\in\mathscr{S}(\mathbf{1},\mathscr{X}^s)\backslash\mathscr{W}(\mathbf{1},\mathscr{X}^s)$, visited by the process after visiting the gate $\mathscr{W}(\mathbf{1},\mathscr{X}^s)$, there does not exist $\omega_1:\eta\to\bar\eta$ such that $\omega_1\cap\mathscr{C}_{\mathbf{1}}^{\mathscr{X}^s}(\Gamma(\mathscr{X}^s,\mathbf{1}))=\varnothing$, $\omega_1\cap\mathscr{W}(\mathbf{1},\mathscr{X}^s)=\{\eta\}$, and $\max_{\sigma\in\omega_1}H(\sigma)\le\Phi(\mathbf{1},\mathscr{X}^s)$. This concludes that $\tilde K=\varnothing$ and the proof of item (a).

Let us now prove item (b). Using Theorem III.5, we get that any saddle in which the protuberance is on one of the shortest sides: $\sigma_i\in\mathscr{W}'(\mathbf{1},\mathscr{X}^s)$, is an unessential saddle. Thus, σ_i satisfies [12, Definition 3.2] and it belongs to $\bigcup_{j=1}^{J_{meta}}(\{\sigma_j\}\cup\mathscr{Z}_j^{\mathbf{1}})$. Moreover, if $\sigma_i\in\bar B^1_{\ell^*,\ell^*-1}(1,s)$, and without loss of generality the protuberance is on the shortest side that is north, then it communicates by one step of the dynamics with a configuration in $\bar B^2_{\ell^*,\ell^*-1}(1,s)$ with a bar of length two on the north side. This belongs to $\mathscr{Z}_i^{\mathbf{1}}$ together with those configurations with a bar of length l on the north side belonging to $\bar B^l_{\ell^*,\ell^*-1}(1,s)$ for any $l=3,\dots,\ell^*-2$ and its bottom is a configuration belonging to $\bar R_{\ell^*-1,\ell^*+1}(1,s)$ with the shortest sides that are north and south. The same arguments hold by replacing north with south, east, west.

Let us now prove item (c) by illustrating an example of unessential saddle "of the second type". We choose this unessential saddles as the configuration $\zeta\in\partial\mathscr{C}_{\mathbf{1}}^{\mathscr{X}^s}(\Gamma(\mathscr{X}^s,\mathbf{1}))\cap(\mathscr{S}(\mathbf{1},\mathscr{X}^s)\backslash\mathscr{W}(\mathbf{1},\mathscr{X}^s))$ in which all the vertices have spin equal to 1 except those, which are all equal to s for some $s\neq1$, in a cluster that is a square $(\ell^*-1)\times(\ell^*-1)$ with a bar of length two on one of the four sides and a bar of length ℓ^*-2 on one of the two consecutive sides, see Figure 17. Note that $\zeta\in\mathscr{S}(\mathbf{1},\mathscr{X}^s)\backslash\mathscr{W}(\mathbf{1},\mathscr{X}^s)$ since the perimeter of the s-cluster is $4\ell^*$ and since its area is equal to $\ell^*(\ell^*-1)+1$, and so by (43) we get that $H(\zeta)=H(\mathbf{1})+4\ell^*-h(\ell^*(\ell^*-1)+1))=\Phi(\mathbf{1},\mathscr{X}^s)$. Furthermore, $\zeta\in\partial\mathscr{C}_{\mathbf{1}}^{\mathscr{X}^s}(\Gamma(\mathscr{X}^s,\mathbf{1}))$. Indeed, by flipping to s the spin 1 adjacent to the bar of length ℓ^*-2, the process intersects a configuration belonging to $\bar B^2_{\ell^*-1,\ell^*}(1,s)\subset\mathscr{C}_{\mathbf{1}}^{\mathscr{X}^s}(\Gamma(\mathscr{X}^s,\mathbf{1}))$. □

Now we are able to give the proof of Theorem III.7. Since our model is under Glauber dynamics, we exploit the proof of [16, Theorem 17.4].

Proof of Theorem III.7. Let us begin to compute the prefactor (32) by exploiting the variational formula for $\Theta=1/K$ given in [12, Lemma 10.7]. This variational problem is simplified because of our Glauber dynamics. Indeed, from the definition of A and B and from Proposition VI.1, we get that $\mathscr{X}^*\backslash(A\cup B)=\mathscr{C}^*_{\text{PTA}}(\mathbf{1},\mathscr{X}^s)$. It follows that there are no wells inside $\mathscr{C}^*_{\text{PTA}}(\mathbf{1},\mathscr{X}^s)$ and any critical configuration may not transform into each other via single spin-update. We proceed by computing a lower and un upper bound for Θ as follows.

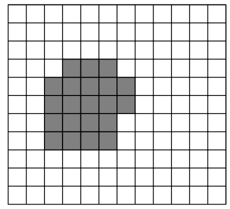


FIG. 17: Example of a unessential saddle ζ "of the second type" defined in [12] when $\ell^*=5$. We color white the vertices with spin 1 and gray the vertices with spin $s\neq 1$.

Upper bound. In order to estimate un upper bound for the capacity we choose a test function $h:\mathscr{X}^*\to\mathbb{R}$ defined as

$$h(\sigma):=\begin{cases}1, & \text{if } \sigma\in A,\\ 0, & \text{if } \sigma\in B,\\ c_i, & \text{if } \sigma\in\mathscr{G}^i, i=1,2,\end{cases} \tag{125}$$

where c_1,c_2 are two constants, see [12, Equation (10.17)]. Thus, we get

$$\begin{aligned}\Theta &\le (1+o(1))\min_{c_1,c_2\in[0,1]}\min_{\substack{h:\mathscr{X}^*\to[0,1]\\ h_{|_A}=1,h_{|_B}=0\\ h_{|_{\mathscr{G}^i}}=c_i,i=1,2}}\frac{1}{2}\sum_{\sigma,\eta\in\mathscr{X}^*}\mathbb{1}_{\{\sigma\sim\eta\}}[h(\sigma)-h(\eta)]^2\\ &=(1+o(1))\min_{c_1,c_2\in[0,1]}[\sum_{\substack{\sigma\in A\\ \eta\in\mathscr{G}^i,i=1,2\\ \sigma\sim\eta}}(1-h(\eta))^2+\sum_{\substack{\sigma\in B\\ \eta\in\mathscr{G}^i,i=1,2\\ \sigma\sim\eta}}h(\eta)^2]\\ &=(1+o(1))\min_{c_1,c_2\in[0,1]}[\sum_{\substack{\eta\in\mathscr{G}^i,i=1,2\\ \sigma\sim\eta}}N^-(\eta)(1-c_i)^2+\sum_{\substack{\eta\in\mathscr{G}^i,i=1,2\\ \sigma\sim\eta}}N^+(\eta)c_i^2]\end{aligned} \tag{126}$$

where $N^-(\eta):=|\{\xi\in\bigcup_{t=2}^q\bar{R}_{\ell^*-1,\ell^*}(1,t):\ \xi\sim\eta\}|$, and $N^+(\eta):=|\{\xi\in\bigcup_{t=2}^q\bar{B}^2_{\ell^*-1,\ell^*}(1,t):\ \xi\sim\eta\}|$. Let us note that

$$N^-(\eta)=1,\ \text{if } \eta\in\mathscr{G}^1\cup\mathscr{G}^2,\ \text{and } N^+(\eta)=\begin{cases}1, & \text{if } \eta\in\mathscr{G}^1,\\ 2, & \text{if } \eta\in\mathscr{G}^2.\end{cases} \tag{127}$$

Thus, we have

$$\begin{aligned}\Theta &\le (1+o(1))\min_{c_1,c_2\in[0,1]}[\sum_{\eta\in\mathscr{G}^1}(1-c_1)^2+c_1^2+\sum_{\eta\in\mathscr{G}^2}(1-c_2)^2+2c_2^2]\\ &=(1+o(1))\min_{c_1,c_2\in[0,1]}[|\mathscr{G}^1|(2c_1^2-2c_1+1)+|\mathscr{G}^2|(3c_2^2-2c_2+1)],\end{aligned}$$

where the equality follows by the fact that the sums are independent from $\eta\in\mathscr{G}^i$, $i=1,2$. Furthermore, since the minimum value of the function $g_1(c_1):=2c_1^2-2c_1+1$ is $\frac{1}{2}$ and the minimum value of the function $g_2(c_2):=3c_2^2-2c_2+1$ is $\frac{2}{3}$, we have

$$\Theta=|\mathscr{G}^1|\frac{1}{2}+|\mathscr{G}^2|\frac{2}{3}=\frac{1}{2}8|\Lambda|(q-1)+\frac{2}{3}4|\Lambda|(\ell^*-2)(q-1)=\frac{4}{3}|\Lambda|(2\ell^*-1)(q-1),$$

where the second equality follows by Lemma VI.1.
Lower bound. Since the variational formula for $\Theta=1/K$ given in [12, Lemma 10.7] is defined by a sum with only non-negative summands, we obtain a lower bound for Θ as follows

$$\Theta\ge\min_{c_1,c_2\in[0,1]}\min_{\substack{h:\mathscr{X}^*\to[0,1]\\ h_{|_A}=1,h_{|_B}=0\\ h_{|_{\mathscr{G}^i}}=c_i,i=1,2}}\frac{1}{2}\sum_{\sigma,\eta\in(\mathscr{C}^*_{\text{PTA}}(\mathbf{1},\mathscr{X}^s))^+}\mathbb{1}_{\{\sigma\sim\eta\}}[h(\sigma)-h(\eta)]^2$$

where $(\mathscr{C}^*_{\text{PTA}}(\mathbf{1},\mathscr{X}^s))^+:=\mathscr{C}^*_{\text{PTA}}(\mathbf{1},\mathscr{X}^s)\cup\partial\mathscr{C}^*_{\text{PTA}}(\mathbf{1},\mathscr{X}^s)$.

Note that $\partial\mathscr{C}^*_{\text{PTA}}(\mathbf{1},\mathscr{X}^s)\cap\mathscr{X}^*=\bigcup_{s=2}^q(\bar{R}_{\ell^*,\ell^*-1}(1,s)\cup\bar{B}^2_{\ell^*,\ell^*-1}(1,s))$, with $\bigcup_{s=2}^q\bar{R}_{\ell^*,\ell^*-1}(1,s)\subset\mathscr{C}^{\mathbf{1}}_{\mathscr{X}^s}(\Gamma^m)$ and $\bigcup_{s=2}^q\bar{B}^2_{\ell^*,\ell^*-1}(1,s)\subset\mathscr{C}^{\mathscr{X}^s}_{\mathbf{1}}(\Gamma(\mathscr{X}^s,\mathbf{1}))$. Thus, we have

$$\begin{aligned}\Theta &\geq \min_{h:\mathscr{X}^*\to[0,1]}\sum_{\eta\in\mathscr{C}^*_{\text{PTA}}(\mathbf{1},\mathscr{X}^s)}\Bigg(\sum_{\substack{\sigma\in\bigcup_{s=2}^q\bar{R}_{\ell^*,\ell^*-1}(1,s),\\ \sigma\sim\eta}}[1-h(\eta)]^2+\sum_{\substack{\sigma\in\bigcup_{s=2}^q\bar{B}^2_{\ell^*,\ell^*-1}(1,s),\\ \sigma\sim\eta}}h(\eta)^2\Bigg)\\ &=\sum_{\sigma,\eta\in\mathscr{C}^*_{\text{PTA}}(\mathbf{1},\mathscr{X}^s)}\min_{h\in[0,1]}\Big(N^-(\eta)[1-h]^2+N^+(\eta)h^2\Big).\end{aligned}\tag{128}$$

Since the minimizer of the function $f(h):=N^-(\eta)[1-h]^2+N^+(\eta)h^2$ is $h_{\min}=\frac{N^-(\eta)}{N^-(\eta)+N^+(\eta)}$, we obtain

$$\Theta\geq\sum_{\sigma,\eta\in\mathscr{C}^*_{\text{PTA}}(\mathbf{1},\mathscr{X}^s)}\frac{N^-(\eta)N^+(\eta)}{N^-(\eta)+N^+(\eta)}=\frac{4}{3}|\Lambda|(2\ell^*-1)(q-1),\tag{129}$$

where the first equality follows by (127). Finally, (31) is proven following the strategy given in [16, Subsection 16.3.2] by taking into account the metastable set $\{\mathbf{1},\mathscr{X}^s\}$ by replacing the role of Lemma 16.17 with [12, Lemma 10.7], see Remark VI.1 and Lemma VI.2. □

Appendix A: Additional material for Subsection III A

a. *Proof of Theorem III.3*

Let us begin by recalling that for any $\mathbf{r},\mathbf{s}\in\mathscr{X}^s$, $\mathbf{r}\neq\mathbf{s}$, from Theorem III.1 we have

$$\Gamma(\mathbf{1},\mathscr{X}^s)=\Phi(\mathbf{1},\mathscr{X}^s)-H(\mathbf{1})=4\ell^*-h(\ell^*(\ell^*-1)+1),\tag{A1}$$

and, from Proposition IV.3 and Proposition IV.4,

$$\Gamma(\mathbf{r},\mathbf{s})=\Phi(\mathbf{r},\mathbf{s})-H(\mathbf{r})=2\min\{K,L\}+2.\tag{A2}$$

For any $\mathbf{r},\mathbf{s}\in\mathscr{X}^s$, $\mathbf{r}\neq\mathbf{s}$, first we show that $\Gamma(\mathbf{1},\mathscr{X}^s)<\Gamma(\mathbf{s},\mathbf{r})$. Indeed, given $0<h<1$ and $L\geq K\geq 3\ell^*$, we have

$$\begin{aligned}\Gamma(\mathbf{1},\mathscr{X}^s)-\Gamma(\mathbf{r},\mathbf{s})&=4\ell^*-h(\ell^*(\ell^*-1)+1)-(2K+2)\\ &\leq 4\ell^*-h(\ell^*(\ell^*-1)+1)-6\ell^*-2<-2\ell^*-h(\ell^*)^2+\ell^*-h-2<0\implies(23).\end{aligned}$$

Furthermore, by Assumption III.1, $\Phi(\mathbf{r},\mathbf{s})$ is smaller than or equal to $\Phi(\mathbf{1},\mathbf{s})$, for any $\mathbf{r},\mathbf{s}\in\mathscr{X}^s$, $\mathbf{r}\neq\mathbf{s}$. Indeed, since $|V|=KL$,

$$\begin{aligned}\Phi(\mathbf{r},\mathbf{s})-\Phi(\mathbf{1},\mathbf{s})&=2K+2+H(\mathbf{r})-4\ell^*+h(\ell^*)^2-h\ell^*+h-H(\mathbf{1})\\ &=2K+2-|E|-4\ell^*+h(\ell^*)^2-h\ell^*+h+|E|-h|V|\\ &=2K+2-4\ell^*+h(\ell^*)^2-h\ell^*+h-hKL\end{aligned}\tag{A3}$$

for any $\mathbf{r},\mathbf{s}\in\mathscr{X}^s$, $\mathbf{r}\neq\mathbf{s}$. Since $\ell^*=\lceil\frac{2}{h}\rceil$, we can write $\ell^*=\frac{2}{h}+1-\delta$ where $0<\delta<1$ denotes the fractional part of $2/h$, that is not integer in view of Assumption III.1(ii). Assume by contradiction that (22) is false, i.e.,

$$\Phi(\mathbf{r},\mathbf{s})\geq\Phi(\mathbf{1},\mathbf{s}).\tag{A4}$$

Using (A3), we have that (A4) is verified if and only if

$$\begin{aligned}&2K+2-4\ell^*+h(\ell^*)^2-h\ell^*+h\geq hKL\\ \iff&2K+2-4(\frac{2}{h}+1-\delta)+h(\frac{2}{h}+1-\delta)^2-h(\frac{2}{h}+1-\delta)+h\geq hKL\\ \iff&\frac{2}{h}K+\frac{2}{h}-\frac{4}{h}(\frac{2}{h}+1-\delta)+(\frac{2}{h}+1-\delta)^2-\frac{2}{h}-1+\delta+1\geq KL\\ \iff&\frac{2}{h}K+\frac{2}{h}-\frac{8}{h^2}-\frac{4}{h}+\frac{4}{h}\delta+\frac{4}{h^2}+1+\delta^2+\frac{4}{h}-\frac{4}{h}\delta-2\delta-\frac{2}{h}-1+\delta+1\geq KL\\ \iff&\frac{2}{h}K-\frac{4}{h^2}+1+\delta^2-\delta\geq KL.\end{aligned}\tag{A5}$$

Since $L \geq K \geq 3\ell^*$ and since $0 < \delta < 1$, we have that

$$KL \geq 3K\ell^* = 3K(\frac{2}{h} + 1 - \delta) = \frac{6}{h}K + 3K - 3K\delta > \frac{6}{h}K. \tag{A6}$$

Moreover, since $0 < \delta < 1$ implies that $\delta^2 - \delta < 0$, we have that

$$\frac{2}{h}K - \frac{4}{h^2} + 1 + \delta^2 - \delta < \frac{2}{h}K - \frac{4}{h^2} + 1. \tag{A7}$$

Combining (A5), (A6) and (A7), since $0 < \delta < 1$, approximately we get that (A4) is satisfied if and only if

$$\frac{2}{h}K - \frac{4}{h^2} + 1 > \frac{6}{h}K \iff -\frac{4}{h}K - \frac{4}{h^2} + 1 > 0, \tag{A8}$$

that is a contradiction. Indeed, the l.h.s. of (A8) is strictly negative since Assumption III.1(i), i.e., $0 < h < 1$, implies that $-\frac{4}{h^2} + 1 < 0$. Hence, we conclude that (22) is satisfied.

Finally, we prove (24). By [9, Lemma 3.6] we get that $\widetilde{\Gamma}(\mathscr{X} \setminus \{\mathbf{s}\})$ is the maximum energy that the process started in $\eta \in \mathscr{X} \setminus \{\mathbf{s}\}$ has to overcome in order to arrive in $\mathbf{s}$, i.e.

$$\widetilde{\Gamma}(\mathscr{X} \setminus \{\mathbf{s}\}) = \max_{\eta \in \mathscr{X} \setminus \{\mathbf{s}\}} \Gamma(\eta, \mathbf{s}). \tag{A9}$$

For any $\eta \in \mathscr{X} \setminus (\mathscr{X}^s \cup \{\mathbf{1}\})$ we have that

$$\begin{aligned}\Gamma(\eta, \mathbf{s}) = \Gamma(\eta, \mathscr{X}^s) &= \Phi(\eta, \mathscr{X}^s) - H(\eta) \\ &\leq \Phi(\mathbf{1}, \mathscr{X}^s) - H(\mathbf{1}) = \Gamma(\mathbf{1}, \mathscr{X}^s),\end{aligned}$$

where the inequality follows by the fact that $\mathbf{1}$ is the unique metastable configuration and this means that starting from $\eta \in \mathscr{X} \setminus \mathscr{X}^s$ there are not initial cycles $\mathscr{C}^{\eta}_{\{\mathbf{s}\}}(\Gamma(\eta, \mathbf{s}))$ deeper than $\mathscr{C}^{\mathbf{1}}_{\{\mathbf{s}\}}(\Gamma^m)$. Note that this fact holds since we are in the metastability scenario as in the [9, Subsection 3.5, Example 1]. Thus, using (23), since for any $\mathbf{r} \in \mathscr{X}^s \setminus \{\mathbf{s}\}$ we have $\Gamma(\mathbf{r}, \mathbf{s}) = \Gamma(\mathscr{X}^s \setminus \{\mathbf{s}\}, \mathbf{s})$, we conclude that

$$\max_{\eta \in \mathscr{X} \setminus \{\mathbf{s}\}} \Gamma(\eta, \mathbf{s}) = \max\{ \max_{\eta \in \mathscr{X} \setminus (\mathscr{X}^s \setminus \{\mathbf{s}\})} \Gamma(\eta, \mathbf{s}), \max_{\eta \in \mathscr{X}^s \setminus \{\mathbf{s}\}} \Gamma(\eta, \mathbf{s}) \} = \Gamma(\mathbf{r}, \mathbf{s}).$$

□

Appendix B: Additional material for Subsection IV B

a. *Proof of Lemma IV.1*

Proof. Let $\sigma \in \bigcup_{t=2}^{q} \bar{R}_{\ell^*-1,\ell^*}(1,t)$. Hence, there exists $s \in \{2, \dots, q\}$ such that $\sigma \in \bar{R}_{\ell^*-1,\ell^*}(1,s)$. Consider the reference path of Definition IV.1 and note that for any $i = 0, \dots, KL$, $N_s(\hat{\omega}_i) = i$. The reference path may be constructed in such a way that $\hat{\omega}_{\ell^*(\ell^*-1)} := \sigma$. Let $\gamma := (\hat{\omega}_{\ell^*(\ell^*-1)} = \sigma, \hat{\omega}_{\ell^*(\ell^*-1)-1}, \dots, \hat{\omega}_1, \hat{\omega}_0 = \mathbf{1})$ be the time reversal of the subpath $(\hat{\omega}_0, \dots, \hat{\omega}_{\ell^*(\ell^*-1)})$ of $\hat{\omega}$. We claim that $\max_{\xi \in \gamma} H(\xi) < 4\ell^* - h(\ell^*(\ell^*-1)+1) + H(\mathbf{1})$. Indeed, note that $\hat{\omega}_{\ell^*(\ell^*-1)} = \sigma, \dots, \hat{\omega}_1$ is a sequence of configurations in which all the spins are equal to 1 except those, which are s, in either a quasi-square $\ell \times (\ell-1)$ or a square $(\ell-1) \times (\ell-1)$ possibly with one of the longest sides not completely filled. For any $\ell = \ell^*, \dots, 2$, the path γ moves from $\bar{R}_{\ell,\ell-1}(1,s)$ to $\bar{R}_{\ell-1,\ell-1}(1,s)$ by flipping the $\ell - 1$ spins s on one of the shortest sides of the s-cluster. In particular, $\hat{\omega}_{\ell(\ell-1)-1}$ is obtained by $\hat{\omega}_{\ell(\ell-1)} \in \bar{R}_{\ell,\ell-1}(1,s)$ by flipping the spin on a corner of the quasi-square from s to 1 and this increases the energy by h. The next $\ell - 3$ steps are defined by flipping the spins on the incomplete shortest side from s to 1 where each step increases the energy by h. Finally, $\hat{\omega}_{(\ell-1)^2} \in \bar{R}_{\ell-1,\ell-1}(1,s)$ is defined by flipping the last spin s to 1 and this decreases the energy by $2-h$. For any $\ell = \ell^*, \dots, 2$, $h(\ell-2) < 2-h$. Indeed, $\ell^* = \lceil \frac{2}{h} \rceil$ and from Assumption III.1, we have $2 - h > h(\ell^*-2) \geq h(\ell-2)$. Hence, $\max_{\xi \in \gamma} H(\xi) = H(\sigma) = 4\ell^* - h(\ell^*(\ell^*-1)+1) - (2-h) + H(\mathbf{1})$ and the claim is verified. □

b. *Explicit calculation of the inequality* (48)

We have

$$\begin{aligned}&H(\hat{\omega}_{k^*}) - H(\mathbf{1}) = 4\ell^* - h(\ell^*(\ell^*-1)+1), \\ &H(\hat{\omega}_{(K-1)^2+1}) - H(\mathbf{1}) = 4K - 4 - h(K-1)^2 - h.\end{aligned}$$

Note that

$$H(\hat{\omega}_{k^*}) - H(\hat{\omega}_{(K-1)^2+1}) = 4\ell^* - h(\ell^*)^2 + h\ell^* - 4K + 4 + hK^2 - 2hK + h. \tag{B1}$$

Using the constraints of Assumption III.1 it follows that, we may write $\ell^* = \frac{2}{h} + 1 - \delta$ where $0 < \delta < 1$ denotes the fractional part of $2/h$. Hence, using (B1), we get

$$\begin{aligned} & H(\hat{\omega}_{k^*}) \le H(\hat{\omega}_{(K-1)^2+1}) \\ \iff & 4\ell^* - h(\ell^*)^2 + h\ell^* - 4K + 4 + hK^2 - 2hK + h \le 0 \\ \iff & -\frac{4}{h}(\frac{2}{h} + 1 - \delta) + (\frac{2}{h} + 1 - \delta)^2 - (\frac{2}{h} + 1 - \delta) + \frac{4}{h}K - \frac{4}{h} - K^2 + 2K - 1 \ge 0 \\ \iff & -\frac{8}{h^2} - \frac{4}{h} + \frac{4}{h}\delta + \frac{4}{h^2} + 1 + \delta^2 + \frac{4}{h} - \frac{4}{h}\delta - 2\delta - \frac{2}{h} - 1 + \delta + \frac{4}{h}K - \frac{4}{h} - 1 \ge K^2 - 2K \\ \iff & -\frac{4}{h^2} - \frac{6}{h} + \frac{4}{h}K + \delta^2 - \delta - 1 \ge K^2 - 2K. \end{aligned} \tag{B2}$$

Since $K \ge 3\ell^* = 3(\frac{2}{h} + 1 - \delta)$ and since $0 < \delta < 1$, it follows that

$$K^2 - 2K \ge K(3\ell^*) - 2K = 3K(\frac{2}{h} + 1 - \delta) - 2K = \frac{6}{h}K + K - 3K\delta > \frac{6}{h}K - 2K.$$

Moreover, since $0 < \delta < 1$ implies that $\delta^2 - \delta < 0$, we have that

$$-\frac{4}{h^2} - \frac{6}{h} + \frac{4}{h}K + \delta^2 - \delta - 1 < -\frac{4}{h^2} - \frac{6}{h} + \frac{4}{h}K. \tag{B3}$$

Hence, approximately we get that (B2) is verified if and only if

$$-\frac{4}{h^2} - \frac{6}{h} + \frac{4}{h}K > \frac{6}{h}K - 2K \iff -\frac{4}{h^2} - \frac{6}{h} - \frac{2}{h}K + 2K > 0,$$

that is an absurd because of the l.h.s. is strictly negative. Indeed, Assumption III.1(ii), i.e., $0 < h < 1$, implies that $-\frac{2}{h}K + 2K = 2K(1 - \frac{1}{h}) < 0$. Thus, (B2) is not verified and

$$H(\hat{\omega}_{k^*}) > H(\hat{\omega}_{(K-1)^2+1}). \tag{B4}$$

c. *Proof of Lemma IV.3*

Proof. For any $\sigma \in \mathscr{X}$, we set $N(\sigma) := \sum_{t=2}^{q} N_t(\sigma)$, where $N_t(\sigma)$ is defined in (44). Moreover, for all $k = 1, \dots, |V|$, we define $\mathscr{V}_k := \{\sigma \in \mathscr{X} : N(\sigma) = k\}$. Note that every path $\omega \in \Omega_{\mathbf{1},\mathscr{X}^s}$ has to cross $\mathscr{V}_k$ for every $k = 0, \dots, |V|$. In particular it has to intersect the set $\mathscr{V}_{k^*}$ with $k^* := \ell^*(\ell^* - 1) + 1$. We prove the lower bound given in (49) by computing that $H(\mathscr{F}(\mathscr{V}_{k^*})) = 4\ell^* - h(\ell^*(\ell^* - 1) + 1) + H(\mathbf{1})$. Note that beacuse of the definition of H and of (43), the presence of disagreeing edges increases the energy. Thus, in order to describe the bottom $\mathscr{F}(\mathscr{V}_{k^*})$ we have to consider those configurations in which the k^* spins different from 1 belong to a unique s-cluster for some $s \ne 1$ inside a sea of spins 1. Hence, consider the reference path $\hat{\omega}$ of Definition IV.1 whose configurations satisfy this characterization. Note that $\hat{\omega} \cap \mathscr{V}_{k^*} = \{\hat{\omega}_{k^*}\}$ with $\hat{\omega} \in \bar{B}^1_{\ell^*-1,\ell^*}(1,s)$. In particular,

$$H(\hat{\omega}_{k^*}) - H(\mathbf{1}) = 4\ell^* - h(\ell^*(\ell^* - 1) + 1), \tag{B5}$$

where $4\ell^*$ represents the perimeter of the cluster of spins different from 1. Our goal is to prove that it is not possible to find a configuration with k^* spins different from 1 in a cluster of perimeter smaller than $4\ell^*$. Since the perimeter is an even integer, we assume that there exists a configuration belonging in $\mathscr{V}_{k^*}$ such that for some $s \in S\backslash\{1\}$ the s-cluster has perimeter $4\ell^* - 2$. Since $4\ell^* - 2 < 4\sqrt{k^*}$, where $\sqrt{k^*}$ is the side-length of the square $\sqrt{k^*} \times \sqrt{k^*}$ of minimal perimeter among those of area k^* in $\mathbb{R}^2$, and since the square is the figure that minimizes the perimeter for a given area, we conclude that there does not exist a configuration with k^* spins different from 1 in a cluster with perimeter strictly smaller than $4\ell^*$. Hence, $\hat{\omega}_{k^*} \in \mathscr{F}(\mathscr{V}_{k^*})$ and (49) is satisfied thanks to (B5). □

d. Proof of Lemma IV.4

Proof. At the beginning of the proof of Lemma IV.3 we note that any path $\omega : \mathbf{1} \to \mathscr{X}^s$ has to visit $\mathscr{V}_k$ at least once for every $k = 0, \dots, |V|$. Consider $\mathscr{V}_{\ell^*(\ell^*-1)}$. From [32, Theorem 2.6] we get the unique configuration of minimal energy in $\mathscr{V}_{\ell^*(\ell^*-1)}$ is the one in which all spins are 1 except those that are s, for some $s \in \{2, \dots, q\}$, in a quasi-square $\ell^* \times (\ell^* - 1)$. In particular, this configuration has energy $\Phi(\mathbf{1}, \mathscr{X}^s) - (2-h) = 4\ell^* - 2 - h\ell^*(\ell^*-1) + H(\mathbf{1})$. Note that $4\ell^* - 2$ is the perimeter of its s-cluster, $s \neq 1$. Since the perimeter is an even integer, we have that the other configurations belonging to $\mathscr{V}_{\ell^*(\ell^*-1)}$ have energy that is larger than or equal to $4\ell^* - h\ell^*(\ell^*-1) + H(\mathbf{1})$. Thus, they are not visited by any optimal path. Indeed, $4\ell^* - h\ell^*(\ell^*-1) + H(\mathbf{1}) > \Phi(\mathbf{1}, \mathscr{X}^s)$. Hence, we conclude that every optimal path intersects $\mathscr{V}_{\ell^*(\ell^*-1)}$ in a configuration belonging to $\bigcup_{t=2}^q \bar{R}_{\ell^*-1,\ell^*}(1,t)$. □

e. Proof of Lemma IV.5

Proof. Let $\sigma \in \mathscr{X}$ and let $v \in V$. Assume that $\sigma(v) = s$, for some $s \in S$. To find if a v-tile is stable for σ we reduce ourselves to flip the spin on vertex v from s to a spin r such that v has at least one nearest neighbor r, i.e., $n_r(v) > 1$. Indeed, otherwise the energy difference (5) is for sure strictly positive. Let us divide the proof in several cases.
Case 1. Assume that $n_s(v) = 0$ in σ. Then the corresponding v-tile is not stable for σ. Indeed, in view of the energy difference (5), if $r \neq 1$, by flipping the spin on vertex v from s to r we have

$$H(\sigma^{v,r}) - H(\sigma) = -n_r(v) - h\mathbb{1}_{\{s=1\}}. \tag{B6}$$

Furthermore, for any $s \neq 1$, by flipping the spin on vertex v from s to 1 we have

$$H(\sigma^{v,1}) - H(\sigma) = -n_1(v) + h. \tag{B7}$$

Hence, for any $s \in S$, if v has spin s and it has four nearest neighbors with spins different from s, i.e., $n_s(v) = 0$, then the tile centered in v is not stable for σ.
Case 2. Assume that $v \in V$ has three nearest neighbors with spin value different from s in σ, i.e., $n_s(v) = 1$. Then, in view of the energy difference (5), for any $s \in S$ and $r \notin \{1, s\}$, by flipping the spin on vertex v from s to r we have

$$H(\sigma^{v,r}) - H(\sigma) = 1 - n_r(v) - h\mathbb{1}_{\{s=1\}}. \tag{B8}$$

Moreover, by flipping the spin on vertex v from $s \neq 1$ to 1 we have

$$H(\sigma^{v,1}) - H(\sigma) = 1 - n_1(v) + h. \tag{B9}$$

Hence, for any $s \in S$, if v has only one nearest neighbor with spin s, a tile centered in v is stable for σ only if $s \neq 1$ and v has nearest neighbors with spins different from each other, see Figure 6(r) and (s).
Case 3. Assume that $v \in V$ has two nearest neighbors with spin s, i.e., $n_s(v) = 2$. Then, in view of the energy difference (5), for any $s \in S$ and $r \notin \{1, s\}$, by flipping the spin on vertex v from s to r we have

$$H(\sigma^{v,r}) - H(\sigma) = 2 - n_r(v) - h\mathbb{1}_{\{s=1\}}. \tag{B10}$$

Moreover, by flipping the spin on vertex v from $s \neq 1$ to 1 we get

$$H(\sigma^{v,1}) - H(\sigma) = 2 - n_1(v) + h. \tag{B11}$$

Hence, for any $s \in S$, if v has two nearest neighbors with spin s in σ, a v-tile is stable for σ if v has the other two nearest neighbors with different spin, see Figure 6(m)–(q). Furthermore, if $s \neq 1$, a v-tile is stable for σ even if v has two nearest neighbors with spin s and the other two nearest neighbors with the same spin, see Figure 6(f)–(i).
Case 4. Assume that $v \in V$ has three nearest neighbors with spin s in σ, i.e., $n_s(v) = 3$, and that the fourth nearest neighbor has spin $r \neq s$. Then, for any $s \in S$ and $r \notin \{1, s\}$, we have

$$H(\sigma^{v,r}) - H(\sigma) = 2 - h\mathbb{1}_{\{s=1\}}. \tag{B12}$$

Furthermore, by flipping the spin on vertex v from $s \neq 1$ to 1 we get

$$H(\sigma^{v,1}) - H(\sigma) = 2 + h. \tag{B13}$$

Case 5. Assume that $v \in V$ has four nearest neighbors with spin s in σ, i.e., $n_s(v) = 4$. Then, for any $s \in S$ and $r \notin \{1, s\}$, we have

$$H(\sigma^{v,r}) - H(\sigma) = 4 - h\mathbb{1}_{\{s=1\}}. \tag{B14}$$

Furthermore, by flipping the spin on vertex v from $s \neq 1$ to 1 we get

$$H(\sigma^{v,1}) - H(\sigma) = 4 + h. \tag{B15}$$

From Case 4 and Case 5, for any $s \in S$, we get that a v-tile is stable for σ if v has at least three nearest neighbors with spin s, see Figure 6(a)–(e). Finally, note that (51) is satisfied in all the cases 1–5 above thanks to (B6)–(B15). □

f. Proof of Proposition IV.3

Proof. Our aim is to prove (75) by constructing a path $\omega : \mathbf{r} \to \mathbf{s}$ such that

$$\Phi_\omega - H(\mathbf{r}) = 2\min\{K, L\} + 2 = 2K + 2, \tag{B16}$$

where the last equality follows by our assumption $L \geq K$. Let $\sigma^* \in \mathscr{X}$ be the configuration defined as

$$\sigma^*(v) := \begin{cases} s, & \text{if } v \in c_0, \\ r, & \text{otherwise.} \end{cases} \tag{B17}$$

We define the path ω as the concatenation of the two paths $\omega^{(1)} : \mathbf{r} \to \sigma^*$ and $\omega^{(2)} : \sigma^* \to \mathbf{s}$ such that $\Phi_{\omega^{(1)}} = H(\mathbf{r}) + 2K$ and $\Phi_{\omega^{(2)}} = H(\mathbf{r}) + 2K + 2$. We define $\omega^{(1)} := (\omega_0^{(1)}, \dots, \omega_K^{(1)})$ where $\omega_0^{(1)} = \mathbf{r}$ and where for any $i = 1, \dots, K$ the state $\omega_i^{(1)}$ is obtained by flipping the spin on the vertex $(i-1, 0)$ from r to s. The energy difference at each step of the path is

$$H(\omega_i^{(1)}) - H(\omega_{i-1}^{(1)}) = \begin{cases} 4, & \text{if } i = 1, \\ 2, & \text{if } i = 2, \dots, K-1, \\ 0, & \text{if } i = K. \end{cases} \tag{B18}$$

Hence, $\arg\max_{\omega^{(1)}} = \{\omega_{K-1}^{(1)}, \omega_K^{(1)} = \sigma^*\}$. Indeed, in view of the periodic boundary conditions and of the (B18), we have

$$H(\omega_{K-1}^{(1)}) - H(\mathbf{r}) = 2K = H(\omega_K^{(1)}) - H(\mathbf{r}). \tag{B19}$$

Therefore, $\Phi_{\omega^{(1)}} = H(\mathbf{r}) + 2K$. Let us now define the path $\omega^{(2)}$. We note that σ^* has an s-bridge on column c_0 and so we apply to it the expansion algorithm introduced in Proposition IV.2. The algorithm gives a path $\omega^{(2)} : \sigma^* \to \mathbf{s}$ such that $\Phi_{\omega^{(2)}} = H(\sigma^*) + 2 = H(\mathbf{r}) + 2K + 2$, where the last equality follows by (B19). □

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