GIOELE'S ATTEMPT TO INCORPORATE THE "SOLVE IT" RITUAL IN HIS MEANINGFUL DISCOURSE ON EQUATIONS

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Over the past decades research has highlighted many difficulties in the domain of algebra teaching and learning, especially for low achieving students. With the advent of new technologies, findings have highlighted ways of using technological tools to overcome some of such difficulties. This study, part of a greater research project, explores the case of Gioele, a low achieving student who participated to our intervention and developed meaningful narratives when using the Dynamic Interactive Mediators (DIMs) in the context of equations and inequalities. In particular, through a commognitive lens, we analyze how he attempts to incorporate his previously learned "solve it" ritual, into his DIM-based discourse on equations and their solutions.

INTRODUCTION

Over the past decades research has highlighted many difficulties in the domain of school algebra teaching and learning, that include giving meaning to algebraic symbols, unknown and variables; viewing the equal sign not necessarily as a signal to compute an answer but also as a relational symbol of equivalence, overcoming the transition to the letter-symbolic form of equations for which students need to interpret algebraic expressions as mathematical objects as well as computational processes, and accept unclosed expressions such as $2x+5$ as valid responses, without thinking that they should do something with them (e.g., Kieran, 2020; Arcavi et al., 2017). For low achieving students, algebra can be particularly daunting (Xin et al., 2022).

Research findings suggest that with appropriately designed tasks, digital means turn out to be particularly helpful to students with a history of low achievement in mathematics or "with special educational needs" (e.g., Baccaglini-Frank, 2021; Palmas et al., 2020). This study is part of a greater funded research project that, through a design-based methodology, is conducting case studies of second year high school students with a history of low achievement in mathematics. These students, volunteering from different Italian high schools, participate to an intervention conducted by researchers during which they engage in a set of newly designed digital activities in the context of algebra. In this study we explore the case of Gioele, a low achieving student who during the proposed activities developed meaningful narratives in the context of equations and inequalities, and that the researcher tried to push to incorporate his previously learned "solve it" ritual (for equations), into his new meaningful discourse on equations and their solutions.

THEORETICAL FRAMEWORK AND RESEARCH QUESTIONS

We will take a commognitive perspective to mathematics teaching-learning (Sfard, 2008) and refer to the digital tools used by participants in their discourse as *Dynamic Interactive Mediators* (DIMs) (Baccaglini-Frank, 2021; Antonini et al., 2020). Previous studies have shown how DIMs can support secondary students' learning, if appropriately integrated into the teaching-learning of high school algebra, by offering "protagonists" for the development of meaningful narratives, specifically in the contexts of equations and inequalities (Baccaglini-Frank, 2021) and of functions and variables (Lisarelli, 2022; Antonini et al., 2020). In such studies, the *discursive*, or *commognitive*, approach allowed to capture sense-making processes through a fine-grained analysis of students' discourses, with particular attention to their *routines*.

A routine is composed of a *task* – as understood by a person in a given task situation (any setting in which a person considers herself bound to do something), is the set of all the characteristics of the precedent events (all that happened in a precedent task situation) that she considers as requiring replication – and a *procedure* – i.e., all the features of what was done in a previous task situation that the person believes should be replicated (Baccaglini-Frank, 2021; Lavie et al., 2019). Students' participation in mathematical discourse can be *ritualistic* if it consists mainly in the implementation of memorized routines for the sake of themselves, with the performer never attending to any product of this performance that could later be used independently of the procedure that produced it; or *explorative*, if it is aimed at constructing a *meaningful* narrative about *abstract objects*, in order to *make sense* of a particular task situation (Sfard, 2008). In discourse an abstract object is expressed through different *realizations* (e.g., an algebraic expression can be read as an indeterminate number or as a function). On the other hand, *unrealized symbols* are *concrete objects* that appear in the discourse alone and can only be manipulated in well-defined ways (Baccaglini-Frank, 2021). In the analyses we use these elements to capture the student's sense-making.

In this study we designed 3 digital artifacts (dynagraphs, two-pan balances with expressions, two-pan balances with weights – see Figure 1) that, for an expert, can be considered realizations of equations and inequalities, as well as their solutions. Used in students' discourse, these DIMs are designed to foster transitions between what an expert sees as different realizations of the same mathematical object; hence, they should foster students' construction of the mathematical objects equation, inequality and solution or set of solutions. We focus on the case of Gioele (pseudonym), who during the initial interview spoke of "solving" (an equation) and performed a(n incorrect) ritual involving symbolic manipulation of letters and numbers, and whom the researcher proposing the intervention (second author) tried to push to incorporate such ritual in the meaningful discourse he had developed in the context of the DIMs. To guide Gioele' s case study, we ask the following questions:

- RQ1: What are the characteristics of Gioele's discourse in the initial interview?

- RQ2: What are the characteristics of Gioele's DIM-based discourse by the end of the teaching intervention?
- RQ3: How does Gioele's discourse come to include the "solving an equation" routine recalled by the interviewer during the intervention?

METHODOLOGY

Gioele volunteered as a participant for the study, recognizing himself as a low achieving (in mathematics) student in 10th grade (15 years old). He was enrolled at a technical-professional high school and agreed to come to our research center 5 times in 2 months. During the first meeting he was interviewed by one of the researchers for 45 minutes. During the activity sessions he worked with two other researchers. He worked individually with the researchers in a quiet room with non-invasive recording devices. During the interview, Gioele had at his disposal a tablet where he could write, as with paper and pencil, and a computer displaying the questions as the interviewer asked them. During the activity sessions, Gioele had at his disposal one or two tablets showing the different digital artifacts and another tablet for writing. Since Gioele uses such digital artifacts as mediators of his discourse we will, for brevity, refer to them hereon as DIMs (Figure 1). DIM_A is a dynagraph with three arrows that realizes an independent and two dependent variables. The tick at the end of arrow x on the number line realizes the value "x" appearing in the expressions above and can be directly manipulated by dragging. The two arrows above realize the two expressions depending on x, and they move indirectly. DIM_B consists of a two-pan balance with expressions that "weigh" as much as the value of the draggable x-tick ("$x=2$" in Figure 1a,b). DIM_C has no symbolic inscriptions and it consists of a two-pan balance with weights (some known and some unknown, the triangles in Figure 1c) together with a dynagraph through which values can be assigned to the unknown weights.
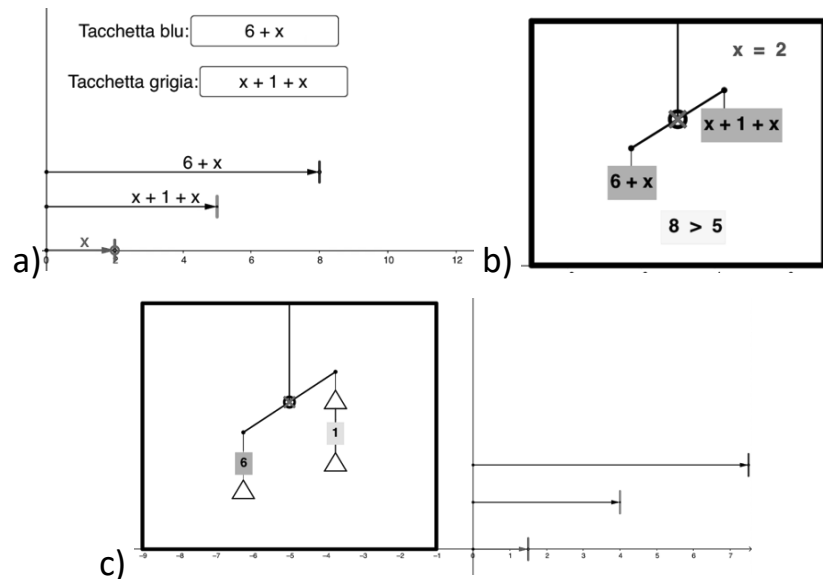

Figure 1: realizations of the inequality $6 + x > x + 1 + x$ with
a) dynagraphs, b) two-pan balance with expressions (for the value $x = 2$), and
c) two-pan balance with weights and associated dynagraphs (for the value $x = 1.5$).

In the activity we analyze Gioele uses a DIM that we call $DIM_{(A,B)}$ because it embeds DIM_A and DIM_B and later he also uses DIM_C. We note that the tick at the end of each arrow is not labeled, because we were interested in words students would use to speak of what for an expert is a "value of the unknown", "variable" or, in some positions, "solution". The symbols "$x=2$" and "$8>5$" in DIM_B change as the value of x changes and they change color (and the inequality changes to an equality) when the two-pan balance is balanced off.

During the activity sessions Gioele always had access to at least one DIM and he was asked to make predictions about when one expression would be greater, less than or equal to another, and then to manipulate the DIMs and explain his observations confirming or disproving his conjectures. The researcher sometimes would ask additional questions on-the-fly to gather more information about the student's reasoning. In the case of Gioele such questions often asked for more predictions related to changes of x's position, after an initial prediction and manipulation. The recordings of Gioele and the researcher were merged with the recordings of the screens of the tablets. They were then anonymized and transcribed by members of the research team. The analyses make use of the theoretical constructs introduced to reach answers to our RQS.

ANALYSIS OF SELECTED EXCERPTS AND ANSWERS TO THE RQS

Excerpt 1 - interview. This excerpt exemplifies Gioele's approach to solving equations during the interview before the activity sessions.

7	Int:	What comes to mind if you see this, what would you do? [Shows the equation 13–A=13+11].
8	Stud:	First I would find the A
[…]		
12	Stud:	It would occur to me to do… first group all the numbers together and then afterwards do like… do in parentheses… 13 minus 13 plus 11. In parentheses, A. [He writes on the tablet the expression (13–3+11)A]
13	Int:	Ok.
14	Stud:	And do everything, so 13 minus 13, zero, plus 11, 11 and that the result of that would be 11A. [He writes 11A]
15	Int:	Ok, so if I ask do that have any solutions?
16	Stud:	In my opinion yes, 11A.
17	Int:	So, what can they be? 11A. Now the question is still the same, just change the writing. Three plus A equals A plus 3. Does it have any solutions? If any, what are they?
18	Stud:	Yes, I mean, you have to group the numbers and on the other side group the letters, so... 3... 3... 3 plus 3, is equal to A plus A. And the result would be this. Although, being 3 plus 3, you could do like… Add 3 plus 3, that is 6, and A plus A raise it to the second power. [He writes 3+3=A+A. Then he writes in the line below 6=A^2]
19	Int:	So, would the solution be 6=A^2?
20	Stud:	Yes

Gioele's discourse here seems to be purely *ritualistic,* focused on performing (meaningless) procedures for their own sake. In this excerpt he performs his "find the A" ritual (in other excerpts he says "solve it" so we refer to this as his "solve it" ritual) twice. So, in front of the equation, without being asked to solve anything, he recognizes a familiar task situation, to which he responds to satisfy the interviewer. He uses verbs and impersonal forms like "do" (in [12] where it recurs 3 times, [14] where it seems to be synonym of "add up", [18] where it seems a synonym of "raise to the second power"), "group" (in [12], and "you have to" in [18]), "add" (in [18]). The objects of the discourse are mainly "numbers" and "letters" (in [12], [18]) and "A" ([8]) but there are no references to other realizations of these (concrete) objects, which therefore remain *unrealized symbols*. The only signifier that Gioele connects with different realizations is "result" (in [14], [18]), realized by 11A (in [16]) and $6=A^2$ (in [19]). Gioele performs manipulations solely to please the interviewer, without any apparent aim of creating meaningful (to him) stories. Moreover, Gioele's symbolic manipulation shows that he has no expectation about the outcome: he talks about "grouping" ([12], [18]) in the procedures he applies for both tasks, even if the two outcomes he obtained, for an expert, refer to two different mathematical objects, a literal expression and an equation. The "result" for Gioele thus seems to be whatever he finds at the end of his "solve it" ritual.

To answer RQ1, Gioele seems to recognize a familiar task concerning solving an equation; his discourse is characterized by ritualist manipulations of *unrealized symbols*; there are no references to other realizations of these objects and thus no transitions between realizations. In general, there is no evidence of sense-making concerning the "solution of an equation" in Gioele's discourse in the initial interview.

Excerpt 2 - last activity session. During this session Gioele's discourse always involves DIMs and the construction of meaningful narratives. In this excerpt, the interviewer asks Gioele to use a file with $DIM_{(A,B)}$ and to set the two-pan balance with the expressions $5+x$ on the left and $2x+1$ on the right (the default value of x is 2).

39	Int:	Ok. So now before you [...] imagine you put x on 4 – don't do that, wait a minute – and try to tell me everything that's going to change in this figure when you do that
40	Stud:	So, x plus 5 will change, which will be 4 plus 5 so 9, and 2 times 1... $2x$... 2 times 4 so 8 plus 1
41	Int:	Yes
42	Stud:	9 and 9
43	Int:	Uh, ok
44	Stud:	So putting it [he refers to the arrow "x"] on 4 it should be in balance
[…]		
47	Int:	Great, what about these [pointing to the arrows in DIM_A] What do you think will change about these things here? If anything changes
48	Stud:	They will stretch, because x will stretch further by 5

In excerpt 2, Gioele's discourse includes DIM-based narratives, such as "putting it on 4 it should be in balance" in [44] referring to balance in DIM_B and "They will stretch" in [45] referring to arrows in DIM_A. Gioele also expresses narratives about objects, such as "*x*" or expressions involving *x*, such as "*x* will stretch" in [48] and "*x* plus 5 which will be 4 plus 5" in [40].

Excerpt 3 - last activity session. In one of the next tasks, the interviewer asks Gioele to reconstruct in a $DIM_{(A,B)}$ the two-pan balance shown in a DIM_C (realizing the inequality $7+2+x>3x$). Gioele solves the task rapidly and explains:

> Looking here [pointing to DIM_C], since there are blanks I have to add them. Like on $7+2+x$, I mean, the blank one [pointing to the white triangle under the "weights" 7 and 2], and instead here there is 3, 3 blanks, and therefore 3 unknowns, $3x$.

Now Gioele has linked the object "*x*" to at least 3 different realizations, namely the symbol "*x*", the "blank", and the term "unknown".

In response to RQ2, excerpts 2 and 3 show that Gioele's DIM-based discourse is characterized by objects (perhaps in the DIMs themselves) and by meaningful narratives around these objects. Indeed, these narratives make sense with respect to the new task situation Gioele has learned to make sense of. There are also several realizations of objects such as "*x*" (the arrow, the "blank" triangle and the "unknown"). These are indicators of an ongoing sense-making process related to mathematical objects "unknown" and "solution of an equation", albeit still in DIM-based contexts.

Excerpt 4 - last activity session. Since Gioele had come to set up what looked like equations using the expressions in DIM_B, the researcher decides to intervene, reminding him of the correct ritual for solving an equation through symbolic manipulation, so that he could "make better predictions of what x might work". Then she asks him to set up a $DIM_{(A,B)}$ with the expressions $6+x$ and $x+1+x$, and to try to predict what value of x will balance it off. Gioele correctly sets up the balance and makes a prediction that the solution will be 4, then without checking it on the DIMs he writes down $x=4$, the equation $6+x=x+1+x$, and carries out the solution procedure, obtaining $5=x$.

129　Int.:　So, you had imagined $x=4$, here you obtained $x=5$, ok? So, who do you think is right, let's say, this prediction of yours or your calculations [...]?

130　Stud.:　calculations

131　Int.:　calculations

132　Stud.:　Because the calculations, I mean… here they are saying a different thing from what I am saying

[…]

143　Int:　Okay...but what do you think $x=5$ meant?

144　Stud:　That x… I don't know.

[…]

146 Int: So, you predicted x=4 [and] actually still it didn't work, and instead calculations gave you x equals 5 [she points to the inscription "5=x"]

147 Stud: Ah, maybe it could be 5 then!

148 Int: Mm, how come?

149 Stud: Because here I did all this and it tells me that it is so

Gioele opens a new text file, starts writing 6+x again, then writes x=5 above, and completes the equality with 6+x=x+1+x, then solves it exactly as before.

155 Int: Excuse me, how come you wrote up here at the top your initial idea?

156 Stud: x equals 5?

157 Int: uhm

158 Stud: I mean, x=5 I wrote it here to remind me here that x equals 5

159 Int: Ah, and so like you wanted to "think it first" before you got it from here

160 Stud: Yeah, it's like… I mean, that is like, how can I say, test everything out

Now the object "x", previously the protagonist of *meaningful* DIM-based narratives, seems to return to being an *unrealized symbol*, like "A" in the initial interview – see in [144] when Gioele states that he does not know what x=5 means. The inscription "5=x" for Gioele now has an unclear relationship with his initial narrative "x=4", a conjecture about the balance position. Although he seems to be trying to make sense of it: in [132] he states "calculations […] are saying a different thing from what I am saying" However, Gioele does not conclude that the value of the unknown should be 5. "5=x" seems to just falsify his initial conjecture, as a test that failed (in [160]). Moreover, Gioele spontaneously constructs a narrative about "x" ("it could be 5 then!" in [147]) and performs the same procedure again ("all this [...] tells me that it is so" in [149]), constructing a new (for him) narrative for the hypothesis "x=5".

In response to RQ3, there is no evidence in Gioele's discourse indicating that the symbol "x" involved in the procedure recalled by the interviewer was recognized as a different realization of the object he was talking about in the interaction with DIMs. For Gioele, the inscriptions "x=4" and "x=5" seem to be meaningful narratives about x only in his DIM-discourse, but not when he is performing the "solve it" ritual. He seems to relate the symbolic equations he writes (invited by the researcher) to the situations realized by the two-pan balances, but not the output of the "solve it" ritual applied to such equations. The only element that may indicate a possible seed of a meaningful link that eventually could be established might be when he says "all this [...] tells me that it is so" in reference to the symbolic manipulation ending with "5=x".

CONCLUSION

Gioele's case highlights some still-problematic issues to be considered when teaching with DIMs such as the ones proposed in this study. Gioele's discourse on equations starts off as purely *ritualistic*, with no evidence that he has constructed the meaning of abstract objects such as "equation" or "solution of an equation". With the DIM-based activities he constructs meaningful narratives about "x" and the DIMs themselves, as

protagonists of the discourse, confirming previous findings (Baccaglini-Frank, 2021). However, this brief session of activities with DIMs was not sufficient for the researcher's attempt to meaningfully incorporate his previously learned "solve it" ritual into his DIM-based discourse on equations and their solutions to succeed. This finding points to obstacles to the eventually necessary transition from DIM-based discourse to formal mathematical discourse, especially when students have constructed extremely strong rituals detached from mathematical objects, and from any meaning. In Gioele's case, the researcher's attempt actually led him to interpret his revisited "solve it" ritual as a "test": the attempt to re-incorporate the symbolic manipulation into meaningful discourse led to a distortion from its endorsed use.

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