Serrin's Problem and Alexandrov's Soap Bubble Theorem: Enhanced Stability via Integral Identities

ROLANDO MAGNANINI & GIORGIO POGGESI

ABSTRACT. We consider Serrin's overdetermined problem for the torsional rigidity, and Alexandrov's Soap Bubble Theorem. We present new integral identities that show a strong analogy between the two problems and help to obtain better (in some cases optimal) quantitative estimates for the radially symmetric configuration. The estimates for the Soap Bubble Theorem benefit from those of Serrin's problem.

1. INTRODUCTION

The pioneering symmetry results obtained by A. D. Alexandrov [Al1, Al2] and J. Serrin [Se] are now classical but still influential. The former—the well-known Soap Bubble Theorem—states that a compact hypersurface, embedded in $\mathbb{R}^N$, that has constant mean curvature must be a sphere. The latter—Serrin's symmetry result—has to do with certain overdetermined problems for partial differential equations. In its simplest formulation, it states that the overdetermined boundary value problem

$$\begin{cases} \Delta u = N & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma, \end{cases} \tag{1.1}$$

$$u_\nu = R \quad \text{on } \Gamma, \tag{1.2}$$

admits a solution for some positive constant R if and only if Ω is a ball of radius R and, up to translations, $u(x) = (|x|^2 - R^2)/2$. Here, Ω denotes a bounded domain in $\mathbb{R}^N$, $N \geq 2$, with sufficiently smooth boundary Γ, say C^2, and u_ν is the outward normal derivative of u on Γ. This result inaugurated a new and fruitful field in mathematical research at the confluence of Analysis and Geometry and has many applications to other areas of mathematics and natural sciences. To be sure, that same result was actually motivated by two concrete problems in mathematical physics regarding the torsion of a straight solid bar and the tangential stress of a fluid on the walls of a rectilinear pipe.

The two problems share several common features. To prove his result, Alexandrov introduced his reflection principle, an elegant geometric technique that also works for other symmetry results concerning curvatures. Serrin's proof hinges on his method of moving planes, an adaptation and refinement of the reflection principle. That method proves to be a very flexible tool, since it allows us to prove radial symmetry for positive solutions of a far more general class of non-linear equations that includes the semi-linear equation

$$\Delta u = f(u), \tag{1.3}$$

where f is a locally Lipschitz continuous non-linearity.

Also, alternative proofs of both symmetry results can be given, based on certain integral identities and inequalities, and the maximum principle. H. F. Weinberger's proof ([We]) of symmetry, even if it is known to work only for problem (1.1)– (1.2) and other few instances, leaves open the option of considering less regular settings. These ideas can be extended to the Soap Bubble Theorem, using R. C. Reilly's argument ([Re]), in which the relevant hypersurface is regarded as the zero level surface of the solution of (1.1).

In this paper, we further analyze the analogies in Weinberger's and Reilly's arguments to obtain quantitative estimates of the desired radial symmetries. Roughly speaking, we address the problem of estimating how close to a sphere is a hypersurface Γ, if either its mean curvature H is close to be constant or, alternatively, if the normal derivative on Γ of the solution of (1.1) is close to be constant.

In both problems, the radial symmetry of Γ will follow from that of the solution u of (1.1). In fact, we will show that in Newton's inequality

$$(\Delta u)^2 \le N |\nabla^2 u|^2, \tag{1.4}$$

which holds pointwise in Ω by Cauchy-Schwarz inequality, the equality sign is identically attained in Ω. Such equality holds if and only if u is a quadratic polynomial q of the form

$$q(x) = \frac{1}{2}(|x-z|^2 - a), \tag{1.5}$$

for some choice of $z \in \mathbb{R}^N$ and $a \in \mathbb{R}$. The boundary condition in (1.1) will then tell us that Γ must be a sphere centered at z.

The starting points of our analysis are the following two integral identities:

$$\int_\Omega (-u) \left\{ |\nabla^2 u|^2 - \frac{(\Delta u)^2}{N} \right\} dx = \frac{1}{2} \int_\Gamma (u_\nu^2 - R^2)(u_\nu - q_\nu)\, dS_x \tag{1.6}$$

and

$$\frac{1}{N-1}\int_\Omega \left\{|\nabla^2 u|^2 - \frac{(\Delta u)^2}{N}\right\} dx + \frac{1}{R}\int_\Gamma (u_\nu - R)^2 \, dS_x \\ = \int_\Gamma (H_0 - H)(u_\nu - q_\nu) u_\nu \, dS_x + \int_\Gamma (H_0 - H)(u_\nu - R) q_\nu \, dS_x. \tag{1.7}$$

The two identities hold regardless of how the point z or the constant a are chosen in (1.5). In (1.6) and (1.7), R and H_0 are reference constants given by

$$R = \frac{N|\Omega|}{|\Gamma|}, \quad H_0 = \frac{1}{R} = \frac{|\Gamma|}{N\,|\Omega|}. \tag{1.8}$$

If u_ν is constant on Γ, that constant must be equal to R, by the identity

$$\int_\Gamma u_\nu \, dS_x = N|\Omega|, \tag{1.9}$$

and hence we obtain symmetry by using (1.6), since the equality sign must hold in (1.4) (in fact, $-u > 0$ in Ω by the maximum principle). If, on the other hand, H is constant on Γ, then Minkowski's identity

$$\int_\Gamma H q_\nu \, dS_x = |\Gamma|, \tag{1.10}$$

implies that $H \equiv H_0$ on Γ, and hence once again symmetry follows by an inspection of (1.7), instead.

Identity (1.6), which appears without proof in [MP, Remark 2.5], puts together Weinberger's identities and some remarks of L. E. Payne and P. W. Schaefer [PS]; (1.7) is a slight modification of one that was proved in [MP, Theorem 2.2] and will turn out to be useful to improve our desired quantitative estimates. For the reader's convenience, we will present the proofs of (1.6)–(1.7) in Section 2.

Identity (1.6) certainly holds under Serrin's smoothness assumptions ($\Gamma \in C^2$), but it is clear that it can be easily extended by approximation to the case of $\Gamma \in C^{1,\alpha}$, since in that case u_ν is continuous on Γ. Nevertheless, it should be noticed that, to infer the radial symmetry of Ω, it suffices to show that the surface integral on the right-hand side of (1.6) is zero.

The main motivation of this paper is to investigate how the use of (1.6) and (1.7) benefits the study of the stability of the radial configuration in the Soap Bubble Theorem and Serrin's problem. Technically speaking, one may look for two concentric balls $B_{\rho_i}(z)$ and $B_{\rho_e}(z)$, centered at $z \in \Omega$ with radii ρ_i and ρ_e, such that

$$B_{\rho_i}(z) \subset \Omega \subset B_{\rho_e}(z) \tag{1.11}$$

and

$$\rho_e - \rho_i \le \psi(\eta), \tag{1.12}$$

where $\psi : [0, \infty) \to [0, \infty)$ is a continuous function vanishing at 0 and η is a suitable measure of the deviation of u_ν or H from being a constant.

Our main result for Serrin's problem is the following theorem. Here and in the sequel, d_Ω denotes the diameter of Ω, r_i and r_e are the relevant radii in the interior and exterior uniform sphere conditions for Ω, and $\delta_\Gamma(z)$ is the distance of z to Γ.

Theorem 1.1 (Stability for Serrin's problem). *Let $\Omega \subset \mathbb{R}^N$, $N \ge 2$, be a bounded domain with boundary Γ of class C^2, and R be the constant defined in* (1.8). *Let u be the solution of problem* (1.1) *and $z \in \Omega$ be any of its critical points.*

Then, (1.11) *holds with ρ_i and ρ_e such that*

$$\rho_e - \rho_i \le C \|u_\nu - R\|_{2,\Gamma}^{2/(N+2)} \quad \text{if } \|u_\nu - R\|_{2,\Gamma} < \varepsilon. \tag{1.13}$$

The constants C and ε depend on N, d_Ω, r_i, r_e, and $\delta_\Gamma(z)$.

The problem of stability for Serrin's problem was considered for the first time in [ABR]. There, for a $C^{2,\alpha}$-regular domain Ω, it is considered a positive solution u of semilinear equation (1.3), such that $u = 0$ on Γ, and it is proved that (1.11) and (1.12) hold with $\psi(\eta) = C\,|\log \eta|^{-1/N}$ and $\eta = \|u_\nu - c\|_{C^1(\Gamma)}$. (Here and in the remaining paragraphs of this introduction, for the dependence of the constants C on the geometric and regularity parameters associated with Ω and, when applicable, f, we refer the reader to the cited references.) The proof in [ABR] is based on a quantitative study of the method of moving planes. In [CMV], in the same general framework and with a similar proof, that stability estimate has been improved. It is in fact shown that (1.11) and (1.12) hold with $\psi(\eta) = C\eta^\tau$ and

$$\eta = \sup_{x,y\in\Gamma,\, x\ne y} \frac{|u_\nu(x) - u_\nu(y)|}{|x-y|}.$$

The exponent $\tau \in (0,1)$ can be computed for a general setting, and is (if Ω is convex) arbitrarily close to $1/(N+1)$.

Theorem 1.1 improves on a different technique, based on Weinberger's integral identities and first employed in [BNST]. There, a quantitative estimate of Hölder type was obtained for the first time, by using the weaker deviation $\eta = \|u_\nu - c\|_\infty$ of u_ν from some reference constant. In [BNST], it is also considered the deviation in L^1-norm $\eta = \|u_\nu - c\|_{1,\Gamma}$ and, by assuming an additional uniform bound on u_ν, it is shown that Ω can be approximated in measure by a finite number of mutually disjoint balls B_i. The error in the approximation is $\psi(\eta) = C\eta^{1/(4N+9)}$. Recently, the approach of [BNST] has been greatly improved in [Fe]. There, rather than by (1.11) and (1.12), the closeness of Ω to a ball is measured by the following slight modification of the so-called Fraenkel asymmetry:

$$\mathcal{A}(\Omega) = \inf\left\{ \frac{|\Omega \Delta B^x|}{|B^x|} : x \text{ center of a ball } B^x \text{ with radius } R \right\}. \tag{1.14}$$

Here, $\Omega \Delta B^x$ denotes the symmetric difference of Ω and B^x, and R is the constant defined in (1.8). In [Fe], the deviation of u_ν from a constant is measured in L^2-norm and the estimate obtained is of Lipschitz type: $\mathcal{A}(\Omega) \le C\|u_\nu - R\|_{2,\Gamma}$.

The proof of Theorem 1.1 simplifies and improves arguments of Feldman [Fe] by the use of our first identity (1.6). The obtained estimate (1.13) makes better than [CMV], even if we replace the Lipschitz semi-norm by an L^2-deviation. In addition, Feldman's inequality is improved with a stronger control of symmetry at the cost of a slight decrease of the continuity exponent. As pointed out in Remarks 3.8 and 3.9, the dependence on the point z of the constants in (1.13) can (so far) be removed when Ω is convex or by assuming some additional requirement (see Remark 3.9).

In passing, we will also prove the inequality

$$\rho_e - \rho_i \le C||u_\nu - R||_{1,\Gamma}^{1/(N+2)}, \tag{1.15}$$

which clearly improves that obtained in [BNST] (see Theorem 3.6).

Another important result in our paper is a quantitative control of symmetry in the Soap Bubble Theorem. That is obtained as a benefit from the analysis employed to derive (1.13).

Theorem 1.2 (Stability for the Soap Bubble Theorem). *Let $N \ge 2$ and let Γ be the connected boundary of class C^2 of a bounded domain $\Omega \subset \mathbb{R}^N$. Denote by H the mean curvature at points of Γ, and let H_0 be the constant defined in* (1.8).

Then, (1.11) *holds for some point $z \in \Omega$, and we have the following:*

(i) *If $N = 2$ or $N = 3$, then*

$$\rho_e - \rho_i \le C\|H_0 - H\|_{2,\Gamma}. \tag{1.16}$$

(ii) *If $N \ge 4$, then*

$$\rho_e - \rho_i \le C||H_0 - H||_{2,\Gamma}^{2/(N+2)} \quad \text{if } \|H_0 - H\|_{2,\Gamma} < \varepsilon. \tag{1.17}$$

The constants C and ε depend on N, d_Ω, r_i, r_e, and $\delta_\Gamma(z)$.

Theorem 1.2 enhances in all dimensions estimates obtained in [CM] for strictly mean convex surfaces with the uniform deviation $\|H_0 - H\|_{\infty,\Gamma}$. Also, it improves by a factor of 2 the results derived by the authors in [MP]. More importantly, it gains the (optimal) Lipschitz stability for the cases $N = 2, 3$ and for a general class of hypersurfaces. That kind of stability was already obtained in

[CV] and [KM, Theorem 1.8], but with a uniform deviation and for strictly mean convex hypersurfaces.

As a final important achievement, by arguments similar to those of [Fe], we also get in Theorem 4.6 the (optimal) inequality for the asymmetry (1.14):

$$\mathcal{A}(\Omega) \le C \, \|H_0 - H\|_{2,\Gamma}.$$

In the remaining paragraphs of this introduction, we shall pinpoint the main remarks that lead us to the proof of Theorems 1.1 and 1.2.

We start to simplify matters by noticing that, by (1.6), the harmonic function $h = q - u$ satisfies

$$\int_\Omega (-u)|\nabla^2 h|^2 \, dx = \frac{1}{2} \int_\Gamma (R^2 - u_\nu^2) h_\nu \, dS_x. \tag{1.18}$$

Also, notice that $h = q$ on Γ, and hence

$$\max_\Gamma h - \min_\Gamma h = \frac{1}{2}(\rho_e^2 - \rho_i^2) \ge \frac{1}{2} \left(\frac{|\Omega|}{|B|} \right)^{1/N} (\rho_e - \rho_i). \tag{1.19}$$

Now, observe that (1.18) holds regardless of the choice of the parameters z and a defining q. We will thus complete the first step of our proof by choosing $z \in \Omega$ in a way that the oscillation of h on Γ on the lefthand side of (1.19) (which is indeed the oscillation of h on $\bar{\Omega}$) can be bounded in terms of the lefthand side of (1.18).

To carry out this plan, we use three ingredients. First, we choose $z \in \Omega$ as a minimum (or any critical) point of u and, as done in [MP, Lemma 3.3], we show that

$$\max_\Gamma h - \min_\Gamma h \le C \Big(\int_\Omega h^2 \, dx \Big)^{1/(N+2)}.$$

Second, we observe that, depending on the regularity of Γ, we can easily obtain the bound

$$C \delta_\Gamma(x)^\alpha \le -u \quad \text{on } \bar{\Omega},$$

where $\alpha = 1$ or 2. Third, we apply two integral inequalities to h and its first (harmonic) derivatives. One is the Hardy-Poincaré-type inequality

$$\int_\Omega v(x)^2 \, dx \le C \int_\Omega \delta_\Gamma(x)^\alpha |\nabla v(x)|^2 \, dx,$$

which is applied to the first (harmonic) derivatives. It holds for fixed $\alpha \in [0, 1]$ and for any harmonic function $v \in W^{1,2}(\Omega)$ that is zero at some given point in Ω (in our case that point will be z, since $\nabla h(z) = 0$). The other one is applied to h and is the Poincaré-type inequality

$$\int_\Omega v^2 \, dx \le C \int_\Omega |\nabla v|^2 \, dx,$$

which holds for any harmonic function $v \in W^{1,2}(\Omega)$ with zero mean value on Ω.

Thus, summing up the last five inequalities gives

$$\max_{\Gamma} h - \min_{\Gamma} h \le C\left(\int_{\Omega} (-u) |\nabla^2 h|^2 \, dx \right)^{1/(N+2)}. \tag{1.20}$$

Next, we work on the righthand side of (1.18). The important observation is that, if $u_\nu - R$ tends to 0, then h_ν does also. Quantitatively, this fact can be expressed by the inequality

$$\|h_\nu\|_{2,\Gamma} \le C\|u_\nu - R\|_{2,\Gamma},$$

which can be derived from [Fe]. Thus, (1.13) will follow by using this inequality, after an application of Hölder's inequality to the righthand side of (1.18).

In order to prove Theorem 1.2, we use the new identity (1.7). In fact, discarding the first summand at its lefthand side and applying Hölder's and the last inequality to its righthand side yield that

$$\|u_\nu - R\|_{2,\Gamma} \le C\|H_0 - H\|_{2,\Gamma}.$$

Theorem 1.2 then follows again from (1.7) and the following estimate already obtained in [MP]:

$$\max_{\Gamma} h - \min_{\Gamma} h \le C\left(\int_{\Omega} |\nabla^2 h|^2 \, dx \right)^{\tau_N/2},$$

where $\tau_N = 1$ for $N = 2, 3$ and $\tau_N = 2/(N+2)$ for $N \ge 4$.

The paper is organized as follows. In Section 2, we collect the relevant identities on which our results are based. Section 3 is dedicated to the stability of Serrin's problem. Subsection 3.2 contains the estimates on harmonic functions that are instrumental in deriving (1.15) and (1.13). Then, in Subsection 3.3, we prove Theorems 1.1 and 3.6 by assembling all the obtained relevant identities and inequalities. In Section 4, we prove the new stability for the Soap Bubble Theorem (Theorem 1.2).

2. Identities for Serrin's Problem and the Soap Bubble Theorem

We begin by setting some relevant notation. By $\Omega \subset \mathbb{R}^N$, $N \ge 2$, we shall denote a bounded domain that is a connected bounded open set, and call Γ its boundary. By $|\Omega|$ and $|\Gamma|$, we will denote indifferently the N-dimensional Lebesgue measure of Ω and the surface measure of Γ. When Γ is of class C^1, ν will denote the (exterior) unit normal vector field to Γ and, when Γ is a hypersurface of class C^2, $H(x)$ will denote its mean curvature (with respect to $-\nu(x)$) at $x \in \Gamma$.

We will also use the letter q to denote the quadratic polynomial defined in (1.5), where z is any point in $\mathbb{R}^N$ and a is any real number; furthermore, we will always use the letter h to denote the harmonic function

$$h = q - u.$$

Finally, as already mentioned in the Introduction, for a point $z \in \Omega$ to be determined, ρ_i and ρ_e shall denote the radius of the largest ball contained in Ω and that of the smallest ball that contains Ω, both centered at z; in formulas,

$$\rho_i = \min_{x \in \Gamma} |x - z| \quad \text{and} \quad \rho_e = \max_{x \in \Gamma} |x - z|. \tag{2.1}$$

To start, we provide the proof of identity (1.6) appearing in Remark 2.5 of [MP]. The proof is as minimal as possible. It follows the tracks of and improves on the work of Weinberger [We] and its modification due to Payne and Schaefer [PS].

Identity (1.6) is a consequence of two formulas involving the solution u of (1.1). One is the differential identity

$$|\nabla^2 u|^2 - \frac{(\Delta u)^2}{N} = \Delta P, \tag{2.2}$$

which associates the Cauchy-Schwarz deficit on the lefthand side with the P-function

$$P = \frac{1}{2}|\nabla u|^2 - u.$$

The other one is the Rellich-Pohozaev identity (see [Po]):

$$(N+2)\int_\Omega |\nabla u|^2 \, dx = \int_\Gamma (u_\nu)^2 q_\nu \, dS_x. \tag{2.3}$$

Notice that (2.2) also implies that P is subharmonic, since the lefthand side is non-negative by Cauchy-Schwarz inequality.

Theorem 2.1 (Fundamental identity for Serrin's problem). *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with boundary Γ of class $C^{1,\alpha}$, $0 < \alpha \le 1$, and R be the positive constant defined in* (1.8). *Then, the solution u of* (1.1) *satisfies identity* (1.6), *that is, it holds that*

$$\int_\Omega (-u)\left\{|\nabla^2 u|^2 - \frac{(\Delta u)^2}{N}\right\} dx = \frac{1}{2}\int_\Gamma (u_\nu^2 - R^2)(u_\nu - q_\nu)\, dS_x.$$

In particular, if the righthand side of (1.6) *is non-positive, Γ must be a sphere* (*and hence Ω a ball*) *of radius R. The same conclusion clearly holds if u_ν is constant on Γ.*

Proof. First, suppose that Γ is of class $C^{2,\alpha}$, so that $u \in C^{2,\alpha}(\bar{\Omega})$. Integration by parts then gives

$$\int_\Omega (u\Delta P - P\Delta u)\,\mathrm{d}x = \int_\Gamma (uP_\nu - u_\nu P)\,\mathrm{d}S_x.$$

Thus, since u satisfies (1.1), we have that

$$\int_\Omega (-u)\Delta P\,\mathrm{d}x = -N\int_\Omega P\,\mathrm{d}x + \frac{1}{2}\int_\Gamma u_\nu^3\,\mathrm{d}S_x, \tag{2.4}$$

since $P = |\nabla u|^2/2 = u_\nu^2/2$ on Γ.

Notice that we can write $u_\nu^3 = (u_\nu^2 - R^2)(u_\nu - q_\nu) + R^2(u_\nu - q_\nu) + u_\nu^2 q_\nu$. Next, by the divergence theorem and (2.3) we compute

$$\begin{aligned} N\int_\Omega P\,\mathrm{d}x &= \frac{N}{2}\int_\Omega |\nabla u|^2\,\mathrm{d}x - \int_\Omega u\Delta u\,\mathrm{d}x \\ &= \left(\frac{N}{2}+1\right)\int_\Omega |\nabla u|^2\,\mathrm{d}x = \frac{1}{2}\int_\Gamma u_\nu^2 q_\nu\,\mathrm{d}S_x. \end{aligned}$$

Thus, this identity, (2.4), and (2.2) give (1.6), since

$$\int_\Gamma (u_\nu - q_\nu)\,\mathrm{d}S_x = 0,$$

with $u - q$ harmonic in Ω.

If Γ is of class $C^{1,\alpha}$, then $u \in C^{1,\alpha}(\bar{\Omega}) \cap C^2(\Omega)$. Thus, by a standard approximation argument, we conclude that (1.6) holds also in this case.

Now, if the righthand side of (1.6) is non-positive, then the integrand at the lefthand side must be zero, as it is non-negative by (1.4) and the maximum principle for u. Then, (1.4) must hold with the equality sign, since $u < 0$ on Ω, by the strong maximum principle. Since $\Delta u = N$, we infer that $\nabla^2 u$ coincides with the identity matrix I. Thus, u must be a quadratic polynomial q of the form (1.5), for some $z \in \mathbb{R}^N$ and $a \in \mathbb{R}$.

Since $u = 0$ on Γ, we have $|x - z|^2 = a$ for $x \in \Gamma$; that is, a must be positive and

$$\sqrt{a}|\Gamma| = \int_\Gamma |x - z|\,\mathrm{d}S_x = \int_\Gamma (x - z)\cdot\nu(x)\,\mathrm{d}S_x = N|\Omega|.$$

In conclusion, Γ must be a sphere centered at z with radius R.

Finally, if $u_\nu \equiv c$ on Γ for some constant c, then

$$c|\Gamma| = \int_\Gamma u_\nu\,\mathrm{d}S_x = N|\Omega|;$$

that is, $c = R$, and hence we can apply the previous argument. $\square$

Corollary 2.2. *Let u be the solution of (1.1) and set $h = q - u$. Then, h is harmonic in Ω and (1.18) holds true, that is,*

$$\int_\Omega (-u)|\nabla^2 h|^2 \, dx = \frac{1}{2}\int_\Gamma (R^2 - u_\nu^2) h_\nu \, dS_x.$$

Moreover, if the center z of the polynomial q in (1.5) is chosen in Ω, then the oscillation of h on Γ can be bounded below as in (1.19).

Proof. Simple computations give that $|\nabla^2 h|^2 = |\nabla^2 u|^2 - (\Delta u)^2/N$ and $h_\nu = q_\nu - u_\nu$, and hence (1.18) easily follows from the fundamental identity (1.6) for Serrin's problem.

Notice that $h = q$ on Γ. Thus, the equality in (1.19) follows from (2.1), by choosing z in Ω. The inequality in (1.19) is implied by $\rho_e + \rho_i \ge \rho_e \ge (|\Omega|/|B|)^{1/N}$, since $B_{\rho_e} \supseteq \Omega$. ◻

Remark 2.3. The assumptions on the regularity of Γ can further be weakened. For instance, if Ω is a (bounded) convex domain, then inequality (3.3) below imply that u_ν is essentially bounded on Γ with respect to the $(N-1)$-dimensional Hausdorff measure on Γ. Thus, an approximation argument again gives that (1.6) holds true.

We now present the proof of (1.7), which is a modification of formula (2.6) in [MP].

Theorem 2.4 (Fundamental identity for the Soap Bubble Theorem). *Let Ω be a bounded domain with boundary Γ of class C^2. Then, (1.7) holds true, that is,*

$$\frac{1}{N-1}\int_\Omega \left\{ |\nabla^2 u|^2 - \frac{(\Delta u)^2}{N} \right\} dx + \frac{1}{R}\int_\Gamma (u_\nu - R)^2 \, dS_x$$
$$= \int_\Gamma (H_0 - H)(u_\nu - q_\nu) u_\nu \, dS_x + \int_\Gamma (H_0 - H)(u_\nu - R) q_\nu \, dS_x.$$

Proof. We proceed similarly to the proof of Theorem 2.1, but with two main differences: in place of Pohozaev's identity, we use Minkowski's identity; we use the following well-known formula for the Laplacian of u:

$$\Delta u = u_{\nu\nu} + (N-1) H u_\nu.$$

This last identity holds pointwise on any regular level surface of u if we agree to still denote by ν the vector field $\nabla u / |\nabla u|$ (it is clear that, on Γ, this coincides with the normal).

We begin with the divergence theorem:

$$\int_\Omega \Delta P \, dx = \int_\Gamma P_\nu \, dS_x.$$

To compute P_ν, we observe that ∇u is parallel to ν on Γ, that is $\nabla u = (u_\nu)\nu$ on Γ. Thus,

$$\begin{aligned} P_\nu &= \langle \nabla^2 u \nabla u, \nu\rangle - u_\nu = u_\nu \langle (\nabla^2 u)\nu, \nu\rangle - u_\nu = u_{\nu\nu} u_\nu - u_\nu \\ &= u_\nu [\Delta u - (N-1) H u_\nu] - u_\nu = (N-1)(1 - H u_\nu) u_\nu, \end{aligned}$$

where we have used (1.1). Therefore,

$$\frac{1}{N-1} \int_\Omega \Delta P \, \mathrm{d}x = \int_\Gamma (1 - H u_\nu) u_\nu \, \mathrm{d}S_x. \tag{2.5}$$

Now, straightforward calculations that use (1.8), (1.10), and (1.9) tell us that

$$\begin{aligned} &\int_\Gamma (H_0 - H)(u_\nu - q_\nu) u_\nu \, \mathrm{d}S_x + \int_\Gamma (H_0 - H)(u_\nu - R) q_\nu \, \mathrm{d}S_x \\ &\quad = H_0 \int_\Gamma u_\nu^2 \, \mathrm{d}S_x - \int_\Gamma H u_\nu^2 \, \mathrm{d}S_x, \end{aligned}$$

while

$$\frac{1}{R} \int_\Gamma (u_\nu - R)^2 \, \mathrm{d}S_x = H_0 \int_\Gamma u_\nu^2 \, \mathrm{d}S_x - \int_\Gamma u_\nu \, \mathrm{d}S_x.$$

The conclusion then follows by a simple inspection, from the two formulas (2.5) and (2.2). ◻

Corollary 2.5. *Let Ω be a bounded domain with boundary Γ of class C^2, and set $h = q - u$. Then, it holds that*

$$\begin{aligned} &\frac{1}{N-1} \int_\Omega |\nabla^2 h|^2 \, \mathrm{d}x + \frac{1}{R} \int_\Gamma (u_\nu - R)^2 \, \mathrm{d}S_x \\ &\quad = - \int_\Gamma (H_0 - H) h_\nu u_\nu \, \mathrm{d}S_x + \int_\Gamma (H_0 - H)(u_\nu - R) q_\nu \, \mathrm{d}S_x. \end{aligned} \tag{2.6}$$

3. Stability for Serrin's Overdetermined Problem

3.1. Notation. As already mentioned in the Introduction, the diameter of Ω is indicated by d_Ω, while $\delta_\Gamma(x)$ denotes the distance of a point x to the boundary Γ.

Even if the fundamental identity for Serrin's problem, (1.6), holds for less regular domains, in order to consider the stability issue, we shall assume that Ω is a bounded domain with boundary Γ of class C^2. In fact, under this assumption, Ω has the properties of the uniform interior and exterior sphere condition, whose respective radii we have designated by r_i and r_e. In other words, there exists $r_e > 0$ (respectively, $r_i > 0$) such that for each $p \in \Gamma$ there exists a ball contained in $\mathbb{R}^N \setminus \bar{\Omega}$ (respectively, contained in Ω) of radius r_e (respectively, r_i) such that its closure intersects Γ only at p. We shall later see that, when Ω is a convex domain we can remove the assumption on the regularity of Γ.

The assumed regularity of Ω ensures that the unique solution of (1.1) is of class at least $C^{1,\alpha}(\bar{\Omega})$. Thus, we can define

$$M = \max_{\bar{\Omega}} |\nabla u| = \max_{\Gamma} u_\nu. \tag{3.1}$$

As shown in [MP, Theorem 3.10], the following bound holds for M:

$$M \le c_N \, \frac{d_\Omega (d_\Omega + r_e)}{r_e}, \tag{3.2}$$

where $c_N = \frac{3}{2}$ for $N = 2$ and $c_N = N/2$ for $N \ge 3$. Notice that, when Ω is convex, we can choose $r_e = +\infty$ in (3.2) and obtain

$$M \le c_N \, d_\Omega. \tag{3.3}$$

For other similar estimates present in the literature, see [MP, Remark 3.11].

3.2. Some estimates for harmonic functions. As already sketched in the Introduction, the desired stability estimate for the spherical symmetry of Ω will be obtained by linking the oscillation of h on Γ to the integral

$$\int_\Omega (-u) |\nabla^2 h|^2 dx,$$

by means of identity (1.18).

To this end, we start by relating the factor $-u$ appearing in that quantity to the function $\delta_\Gamma(x)$; we do this in the following lemma.

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^N$, $N \ge 2$, be a bounded domain such that Γ is made of regular points for the Dirichlet problem, and let u be the solution of* (1.1)*. Then,*

$$-u(x) \ge \frac{1}{2} \delta_\Gamma(x)^2 \quad \text{for every } x \in \bar{\Omega}.$$

Moreover, if Γ is of class C^2, then it holds that

$$-u(x) \ge \frac{r_i}{2} \delta_\Gamma(x) \quad \text{for every } x \in \bar{\Omega}. \tag{3.4}$$

Proof. If every point of Γ is regular, then a unique solution $u \in C^0(\bar{\Omega}) \cap C^2(\Omega)$ exists for (1.1). Now, for $x \in \Omega$, let $r = \delta_\Gamma(x)$ and consider the ball $B = B_r(x)$. Let w be the solution of (1.1) in B, that is, $w(y) = (|y - x|^2 - r^2)/2$. By comparison we have that $w \ge u$ on $\bar{B}$ and hence, in particular, $w(x) \ge u(x)$. Thus, we infer the first inequality in the lemma.

If Γ is of class C^2, (3.4) certainly holds if $\delta_\Gamma(x) \ge r_i$. If $\delta_\Gamma(x) < r_i$, instead, let z be the closest point in Γ to x, and call B the ball of radius r_i touching Γ at z and containing x. Up to a translation, we can always suppose that the center of the ball B is the origin 0. If w is the solution of (1.1) in B, that is $w(y) = (|y|^2 - r_i^2)/2$, by comparison we have that $w \ge u$ in B, and hence

$$-u(x) \ge \frac{1}{2}\,(|x|^2 - r_i^2) = \frac{1}{2}\,(r_i + |x|)(r_i - |x|) \ge \frac{1}{2}\,r_i\,(r_i - |x|).$$

This implies (3.4), since $r_i - |x| = \delta_\Gamma(x)$. $\square$

By the last lemma, we can estimate the righthand side of (1.18) from below in terms of the integral

$$\int_\Omega |\nabla h|^2 \delta_\Gamma^{2\alpha}\, dx,$$

with $\alpha = 1$ or $\frac{1}{2}$. For this kind of integral, useful estimates are present in the literature. We shall briefly report on some of them.

Lemma 3.2 (Hardy-Poincaré-type inequalities). *Let $\Omega \subset \mathbb{R}^N$, $N \ge 2$, be a bounded domain with boundary Γ of class $C^{0,\alpha}$, and let z be a point in Ω. Then, we have the following:*

(i) *There exists a positive constant $\mu_\alpha(\Omega, z)$, such that*

$$\int_\Omega v^2\, dx \le \mu_\alpha(\Omega, z)^{-1} \int_\Omega |\nabla v|^2 \delta_\Gamma^{2\alpha}\, dx, \tag{3.5}$$

for every function v which is harmonic in Ω and such that $v(z) = 0$.

(ii) *There exists a positive constant $\bar\mu_\alpha(\Omega)$ such that*

$$\int_\Omega v^2\, dx \le \bar\mu_\alpha(\Omega)^{-1} \int_\Omega |\nabla v|^2 \delta_\Gamma^{2\alpha}\, dx, \tag{3.6}$$

for every function v which is harmonic in Ω and has mean value zero on Ω.

In particular, if Γ has a Lipschitz boundary, the number α can be replaced by any exponent in $(0, 1]$.

Proof. The assertions (i) and (ii) are easy consequences of a general result of Boas and Straube (see [BS]) that improves a work of Ziemer's ([Zi]). In case (i), we apply [BS, Example 2.5]). In case (ii), [BS, Example 2.1] is appropriate.

The variational problems

$$\mu_\alpha(\Omega, z) = \min\left\{ \int_\Omega |\nabla v|^2 \delta_\Gamma^{2\alpha}\, dx \,:\, \int_\Omega v^2\, dx = 1,\ \Delta v = 0 \text{ in } \Omega,\ v(z) = 0 \right\} \tag{3.7}$$

and

$$\bar\mu_\alpha(\Omega) = \min\left\{ \int_\Omega |\nabla v|^2 \delta_\Gamma^{2\alpha}\, dx \,:\, \int_\Omega v^2\, dx = 1,\ \Delta v = 0 \text{ in } \Omega,\ \int_\Omega v\, dx = 0 \right\} \tag{3.8}$$

then characterize the two constants. $\square$

Remark 3.3. (i). Notice that in the special case where $\alpha = 0$ from (3.5) and (3.6), we recover the Poincaré-type inequalities that we proved and used in [MP]. Also, in the case $\alpha = 0$, inequality (3.5) directly follows from the result in [Zi]. Of course, in place of (3.8), the constant in the classical Poincaré inequality would work as well. The addition of the harmonicity of v in (3.8) clearly gives a better constant.

(ii) We have that $\mu_\alpha(\Omega, z) \le \bar{\mu}_\alpha(\Omega)$, as one can verify by using the function

$$v_0 = \frac{v - v(z)}{1 + |\Omega| v(z)^2},$$

where v is a minimizer for (3.8).

(iii) In the sequel, we will choose $\alpha = \frac{1}{2}$ in (3.7) and $\alpha = 0$ in (3.8) and use the simplified notation

$$\mu(\Omega, z) = \mu_{1/2}(\Omega, z) \ \text{ and } \ \bar{\mu}(\Omega) = \bar{\mu}_0(\Omega).$$

The next lemma, which modifies for our purposes an idea of W. Feldman [Fe], will be useful to bound the righthand side of (1.18).

Lemma 3.4 (Trace inequality). *Let $\Omega \subset \mathbb{R}^N$, $N \ge 2$, be a bounded domain with boundary Γ of class C^2 and z be any critical point in Ω of the solution u of* (1.1).

The following inequality holds for $h = q - u$, where q is given by (1.5)*:*

$$\int_\Gamma |\nabla h|^2 \, dS_x \le \frac{2}{r_i}\left(1 + \frac{N}{r_i \mu(\Omega, z)}\right) \int_\Omega (-u) |\nabla^2 h|^2 \, dx. \tag{3.9}$$

Proof. It is clear that $h \in C^1(\bar{\Omega}) \cap C^2(\Omega)$. We begin with the following differential identity:

$$\operatorname{div}\{v^2 \nabla u - u \nabla(v^2)\} = v^2 \Delta u - u \Delta(v^2) = N v^2 - 2u|\nabla v|^2,$$

which holds for any v harmonic function in Ω, if u satisfies (1.1). Next, we integrate on Ω and, by the divergence theorem, we get

$$\int_\Gamma v^2 u_\nu \, dS_x = N \int_\Omega v^2 \, dx + 2 \int_\Omega (-u) |\nabla v|^2 \, dx.$$

We use this identity with $v = h_{x_i}$, and hence we sum up over $i = 1, \dots, N$ to obtain

$$\int_\Gamma |\nabla h|^2 u_\nu \, dS_x = N \int_\Omega |\nabla h|^2 \, dx + 2 \int_\Omega (-u) |\nabla^2 h|^2 \, dx.$$

This formula, together with (3.5) and (3.4), gives us

$$\int_\Gamma |\nabla h|^2 u_\nu \, dS_x \le 2\left(1 + \frac{N}{r_i \mu(\Omega, z)}\right) \int_\Omega (-u)|\nabla^2 h|^2 \, dx.$$

The term u_ν at the lefthand side of this last inequality can be bounded from below by r_i, by an adaptation of Hopf's lemma (see also [MP, Theorem 3.10]). Therefore, (3.9) follows at once. ◻

The crucial step in our analysis is Theorem 3.5 below, in which we associate the oscillation of h, and hence $\rho_e - \rho_i$, with a weighted L^2-norm of its Hessian matrix.

Before stating that, we recall from [MP] the following estimate that links $\rho_e - \rho_i$ with the L^2-norm of h. In fact, if Ω is a bounded domain with boundary of class C^2, we have that

$$\rho_e - \rho_i \le a_N M^{N/(N+2)} |\Omega|^{-1/N} \, ||h||_2^{2/(N+2)}, \tag{3.10}$$

for

$$\|h\|_2 \le \alpha_N M r_i^{(N+2)/2}. \tag{3.11}$$

The values of the constants a_N and α_N can be found in [MP, Lemma 3.3].

Theorem 3.5. *Let $\Omega \subset \mathbb{R}^N$, $N \ge 2$, be a bounded domain with boundary Γ of class C^2, and z be any critical point in Ω of the solution u of (1.1).*

Consider the function $h = q - u$, with q given by (1.5), where the constant a is chosen such that h has mean value zero on Ω. Then, we have that

$$\rho_e - \rho_i \le C\left\{\int_\Omega |\nabla^2 h|^2 \delta_\Gamma \, dx\right\}^{1/(N+2)}, \quad \text{if } \int_\Omega |\nabla^2 h|^2 \delta_\Gamma \, dx < \varepsilon^2.$$

Here,

$$C = \frac{a_N M^{N/(N+2)}}{|\Omega|^{1/N} \, [\bar\mu(\Omega)\mu(\Omega, z)]^{1/(N+2)}},$$
$$\varepsilon = \alpha_N M r_i^{(N+2)/2} \sqrt{\bar\mu(\Omega)\mu(\Omega, z)},$$

and the constants a_N and α_N are those in (3.10) and (3.11).

Proof. We apply (3.5) with and $\alpha = \frac{1}{2}$ to each first derivative of h (since $\nabla h(z) = 0$), and obtain that

$$\int_\Omega |\nabla h|^2 \, dx \le \mu(\Omega, z)^{-1} \int_\Omega |\nabla^2 h|^2 \delta_\Gamma \, dx.$$

Hence, we apply (3.6) with $\alpha = 0$ to h and get

$$\int_\Omega h^2 \, dx \le \bar{\mu}(\Omega)^{-1} \int_\Omega |\nabla h|^2 \, dx.$$

Thus,

$$\int_\Omega h^2 \, dx \le [\bar{\mu}(\Omega)\mu(\Omega, z)]^{-1} \int_\Omega |\nabla^2 h|^2 \delta_\Gamma \, dx,$$

and the conclusion follows from (3.10) and (3.11). □

3.3. Stability for Serrin's problem. We collect here our results on the stability of the spherical configuration by putting together the identities of Section 2 and the estimates in the previous subsection.

Theorem 3.5 above gives an estimate from below of the lefthand side of (1.18). In this subsection, we will take care of its righthand side and prove our main result for Serrin's problem.

Proof of Theorem 1.1. We have that

$$\int_\Gamma (R^2 - u_\nu^2) h_\nu \, dS_x \le (M + R) \|u_\nu - R\|_{2,\Gamma} \|h_\nu\|_{2,\Gamma},$$

after an application of Hölder's inequality. Thus, by Lemma 3.4, (1.18), and this inequality, we infer that

$$\begin{aligned} \|h_\nu\|_{2,\Gamma}^2 &\le \frac{2}{r_i}\left(1 + \frac{N}{r_i \mu(\Omega, z)}\right) \int_\Omega (-u) |\nabla^2 h|^2 \, dx \\ &\le \frac{M + R}{r_i}\left(1 + \frac{N}{r_i \mu(\Omega, z)}\right) \|u_\nu - R\|_{2,\Gamma} \|h_\nu\|_{2,\Gamma}, \end{aligned}$$

and hence

$$\|h_\nu\|_{2,\Gamma} \le \frac{M + R}{r_i}\left(1 + \frac{N}{r_i \mu(\Omega, z)}\right) \|u_\nu - R\|_{2,\Gamma}. \tag{3.12}$$

Therefore,

$$\begin{aligned} \int_\Omega |\nabla^2 h|^2 \delta_\Gamma(x) \, dx &\le \frac{2}{r_i} \int_\Omega (-u) |\nabla^2 h|^2 \, dx \\ &\le \left(\frac{M + R}{r_i}\right)^2 \left(1 + \frac{N}{r_i \mu(\Omega, z)}\right) \|u_\nu - R\|_{2,\Gamma}^2, \end{aligned}$$

by Lemma 3.1. These inequalities and Theorem 3.5 then give (1.13). □

If we want to measure the deviation of u_ν from R in L^1-norm, we get a smaller stability exponent.

Theorem 3.6 (Stability in L^1-norm). *Under the same assumptions of Theorem 1.1, we have that*

$$\rho_e - \rho_i \le C\|u_\nu - R\|_{1,\Gamma}^{1/(N+2)} \quad \text{if } \|u_\nu - R\|_{1,\Gamma} < \varepsilon, \tag{3.13}$$

for some positive constants C and ε.

Proof. Instead of applying Hölder's inequality to the righthand side of (1.18), we just use the rough bound:

$$\int_\Omega (-u)|\nabla^2 h|^2 \, dx \le \frac{1}{2}(M+R)(M+d_\Omega)\int_\Gamma |u_\nu - R| \, dS_x,$$

since $(u_\nu + R)|h_\nu| \le (M+R)(M+d_\Omega)$ on Γ. The conclusion then follows from similar arguments. ◻

Remark 3.7 (On the constants C and ε). For the sake of clarity, we did not display the values of the constants C and ε in Theorems 1.1, 3.6, and the relevant theorems in the sequel. However, their computation will be clear by following the steps of the proofs.

For instance, an inspection of the proof of Theorem 1.1 informs us that the constants in (1.13) are

$$C = a_N \frac{M^{N/(N+2)}(M+R)^{2/(N+2)}}{|\Omega|^{1/N} r_i^{3/(N+2)}} \left\{ \frac{N + r_i \mu(\Omega, z)}{\bar\mu(\Omega)\mu(\Omega, z)^2} \right\}^{1/(N+2)}$$

and

$$\varepsilon = \alpha_N \frac{M}{M+R} \frac{\sqrt{\bar\mu(\Omega)}\,\mu(\Omega, z)}{\sqrt{N + r_i \mu(\Omega, z)}} r_i^{(N+5)/2}.$$

Here, the constants a_N and α_N are those appearing in (3.10) and (3.11).

The constants C and ε can be shown to depend only on some geometric parameters of Ω. In fact, we can use (3.2) to bound M in terms of d_Ω and r_e. To estimate the ratio R, we can use the isoperimetric inequality to bound $|\Gamma|$ from below in terms of $|\Omega|^{1-1/N}$, and then a trivial inequality to bound $|\Omega|^{1/N}$ in terms of d_Ω.

Remark 3.8 (Estimating $\bar\mu(\Omega)$ and $\mu(\Omega, z)$). For simplicity, in what follows, k_N denotes a positive number that only depends on N, whose value may change from line to line.

(i) A lower bound of $\bar\mu(\Omega)$ can be obtained as follows. In Theorem 1.3 of [HS] a more general form of inequality (3.6) (without the assumption of harmonicity) is proved for the class of the b_0-John domains (see [HS] for the definition), and the constants are explicitly computed; in particular, with the aid of [HS], we can easily deduce that

$$\bar\mu(\Omega)^{-1} \le k_N |\Omega|^{2/N} b_0^{2N},$$

for some constant k_N only depending on N. A domain of class C^2 is obviously a b_0-John domain, and it is not difficult to show that

$$b_0 \le \frac{d_\Omega}{r_i},$$

and hence obtain that

$$\bar{\mu}(\Omega)^{-1} \le k_N |\Omega|^{2/N} \left(\frac{d_\Omega}{r_i}\right)^{2N}. \tag{3.14}$$

An alternative way to estimate $\bar{\mu}(\Omega)$ can be found in [MP, Remark 3.8 (ii)].

A lower bound for $\mu(\Omega, z)$ can also be obtained, but it may depend on the choice of the particular critical point of u. In fact, by following the argument of [Fe, Lemma 8], one can adapt the result contained in [HS, Theorem 1.3] to the case of harmonic functions vanishing at a given point z (i.e., the case of (3.5)). To explicitly carry out the computations, the definition of bounded L_0-John domain with given base point is appropriate (see [Fe]). In fact, for that class of domains, by means of [Fe, Lemma 8], we can deduce that

$$\mu(\Omega, z)^{-1} \le k_N |\Omega|^{1/N} L_0^{2N} \tag{3.15}$$

and, by the definition, it is not difficult to prove the following bound:

$$L_0 \le \frac{d_\Omega}{\min[r_i, \delta_\Gamma(z)]}. \tag{3.16}$$

Thus, we obtain

$$\mu(\Omega, z)^{-1} \le k_N |\Omega|^{1/N} \left(\frac{d_\Omega}{\min[r_i, \delta_\Gamma(z)]}\right)^{2N}. \tag{3.17}$$

Finally, it is clear that we can eliminate the dependence on $|\Omega|$ in (3.14) and (3.17) in favour of d_Ω again.

(ii) The use of $\delta_\Gamma(z)$ gives an estimate of the number L_0 in terms of an explicit geometrical quantity. Moreover, in case Ω is convex, by using $\delta_\Gamma(z)$ we are able to completely remove the dependence of the constants on z, and we can affirm that C and ε are controlled by r_i and d_Ω, only.

In fact, we notice that the dependence of M on r_e can be removed, thanks to (3.3). Moreover, we can further eliminate the dependence on $\delta_\Gamma(z)$ and $|\Omega|$ appearing in (3.17). Indeed, u has a unique minimum point z in Ω, since u is analytic and the level sets of u are convex by a result in [Ko] (see [MS], for a similar argument), and hence we have only one choice for the point z. Thus, an estimate of $\delta_\Gamma(z)$ from below can be obtained, first, by putting together arguments in [BMS, Theorem 2.7] and [BMS, Remark 2.5], to obtain that

$$\delta_\Gamma(z) \ge \frac{k_N}{|\Omega| d_\Omega^{N-1}} \max_{\bar{\Omega}}(-u).$$

Second, by a simple comparison argument, the maximum can be bounded from below by $r_i^2/2$, and hence we have that

$$\delta_\Gamma(z) \ge k_N \frac{r_i^{2N}}{|\Omega| d_\Omega^{N-1}}.$$

Again, $|\Omega|$ can be easily bounded in terms of d_Ω. By similar arguments, we can take care of ε.

Remark 3.9. The dependence on $\delta_\Gamma(z)$ of the constants in the relevant estimates can also be removed by choosing the point z appearing in (1.5) differently.

(i) As already done in [MP, Theorem 3.6 and item (ii) of Remark 4.2], we can choose z as the center of mass of Ω. In fact, we obtain that $\int_\Omega \nabla h(x)\, dx = 0$, and we can use (3.6), instead of (3.5), in Lemma 3.4 and Theorems 3.5, 1.1, and 3.6. In this way, we avoid the use of $\mu(\Omega, z)$ (and hence of $\delta_\Gamma(z)$).

However, in this case the extra assumption that $z \in \Omega$ is needed, since we want that the ball $B_{\rho_i}(z)$ (considered in Theorems 1.1, 1.2, and 3.6) be contained in Ω.

(ii) As pointed out by the anonymous referee and as done in [Fe], another possible way to choose z is $z = x_0 - \nabla u(x_0)$, where x_0 is the base point in the definition of L_0-John domain mentioned in Remark 3.8. In fact, we obtain that $\nabla h(x_0) = 0$, and we can thus use (3.5), with $\mu(\Omega, z)$ replaced by $\mu(\Omega, x_0)$, in Lemma 3.4 and Theorems 3.5, 1.1, and 3.6. Moreover, it is easy to check that every $x_0 \in \Omega$ such that $\delta_\Gamma(x_0) \ge r_i$ is an appropriate base point for the L_0-John condition for which it holds that

$$L_0 \le \frac{d_\Omega}{r_i},$$

since one can use (3.16) with z replaced by x_0. Hence, from (3.15) we have that

$$\mu(\Omega, x_0)^{-1} \le k_N |\Omega|^{1/N} \left(\frac{d_\Omega}{r_i}\right)^{2N}.$$

As in item (i), we should additionally require that $z \in \Omega$.

Since the estimates in Theorems 1.1 and 3.6 do not depend on the particular critical point chosen, as a corollary, we obtain results of closeness to a union of balls, similar to [MP, Corollary 4.3]: here, we just illustrate the instance of Theorem 1.1.

Corollary 3.10 (Closeness to an aggregate of balls). *Let Γ, R, and u be as in Theorem 1.1. Then, there exist points $z_1, \dots, z_n$ in Ω, $n \ge 1$, and corresponding numbers*

$$\rho_i^j = \min_{x \in \Gamma} |x - z_j| \text{ and } \rho_e^j = \min_{x \in \Gamma} |x - z_j|, \quad j = 1, \dots, n,$$

such that

$$\bigcup_{j=1}^{n} B_{\rho_i^j}(z_j) \subset \Omega \subset \bigcap_{j=1}^{n} B_{\rho_e^j}(z_j)$$

and

$$\max_{1 \le j \le n} (\rho_e^j - \rho_i^j) \le C \|u_\nu - R\|_{2,\Gamma}^{2/(N+2)} \quad \text{if } \|u_\nu - R\|_{2,\Gamma} < \varepsilon,$$

for some positive constants C and ε.

Proof. We pick one point z_j from each connected component of the set of local minimum points of u. By applying Theorem 1.1 to each z_j, the conclusion is then evident. ◻

4. ENHANCED STABILITY FOR ALEXANDROV'S SOAP BUBBLE THEOREM

In this section, we collect the benefits of the new estimate (1.13) that affect the stability issue for the Soap Bubble Theorem.

We begin by recalling a couple of inequalities concerning the harmonic function h that we obtained in [MP, Theorem 3.4] and, in the sequel, will play the role of those of Theorem 3.5. It holds that

$$\rho_e - \rho_i \le C_0 \Big(\int_\Omega |\nabla^2 h|^2 \, dx \Big)^{1/2}, \tag{4.1}$$

if $N = 2, 3$, and

$$\rho_e - \rho_i \le C_0 \Big(\int_\Omega |\nabla^2 h|^2 \, dx \Big)^{1/(N+2)} \quad \text{for } \int_\Omega |\nabla^2 h|^2 \, dx < \varepsilon_0^2, \tag{4.2}$$

for $N \ge 4$. The point z chosen in the definition (2.1) of ρ_i and ρ_e is any critical point of u, as usual.

The positive constants C_0 and ε_0 depend on N, $|\Omega|$, d_Ω, r_i r_e, and $\mu_0(\Omega, z)$ given by (3.7) with $\alpha = 0$. For their values, we refer to [MP, Theorem 3.4].

Remark 4.1. The parameter $\mu_0(\Omega, z)$ can be estimated by following the arguments used in item (i) of Remark 3.8 to estimate $\mu(\Omega, z)$. In fact, we deduce that

$$\mu_0(\Omega, z)^{-1} \le k_N |\Omega|^{2/N} L_0^{2N}.$$

Another way to estimate $\mu_0(\Omega, z)$ can be found in [MP, Remark 3.8 (iii)].

Next, we derive the following lemma, which parallels and is a useful consequence of Lemma 3.4.

Lemma 4.2. *Let $N \ge 2$ and let Γ be the connected boundary of class C^2 of a bounded domain $\Omega \subset \mathbb{R}^N$. Denote by H the mean curvature function for Γ, and let H_0 be the constant defined in* (1.8).

Then, the following inequality holds:

$$\|u_\nu - R\|_{2,\Gamma} \le R\left\{d_\Omega + \frac{M(M+R)}{r_i}\left(1 + \frac{N}{r_i\,\mu(\Omega,z)}\right)\right\}\|H_0 - H\|_{2,\Gamma}. \tag{4.3}$$

Proof. Discarding the first summand on the lefthand side of (2.6) and applying Hölder's inequality on its righthand side gives that

$$\frac{1}{R}||u_\nu - R||_{2,\Gamma}^2 \le \|H_0 - H\|_{2,\Gamma}(M\|h_\nu\|_{2,\Gamma} + d_\Omega\|u_\nu - R\|_{2,\Gamma}),$$

since $u_\nu \le M$ and $|q_\nu| \le d_\Omega$ on Γ. Thus, inequality (3.12) implies that

$$||u_\nu - R||_{2,\Gamma}^2 \le R\left\{d_\Omega + \frac{M(M+R)}{r_i}\left(1 + \frac{N}{r_i\,\mu(\Omega,z)}\right)\right\} \times \|H_0 - H\|_{2,\Gamma}\,\|u_\nu - R\|_{2,\Gamma},$$

from which (4.3) follows at once. ∎

Proof of Theorem 1.2. Discarding the second summand on the lefthand side of (2.6) and applying Hölder's inequality on its righthand side, as in the previous proof, gives that

$$\begin{aligned}\frac{1}{N-1}\int_\Omega |\nabla^2 h|^2\,dx &\le \\ &\le R\left\{d_\Omega + \frac{M(M+R)}{r_i}\left(1 + \frac{N}{r_i\,\mu(\Omega,z)}\right)\right\}\|H_0 - H\|_{2,\Gamma}\,\|u_\nu - R\|_{2,\Gamma}, \\ &\le R^2\left\{d_\Omega + \frac{M(M+R)}{r_i}\left(1 + \frac{N}{r_i\,\mu(\Omega,z)}\right)\right\}^2 ||H_0 - H||_{2,\Gamma}^2,\end{aligned}$$

where the second inequality follows from Lemma 4.2.

Inequalities (1.16) and (1.17) then result from (4.1) and (4.2). ∎

Remark 4.3 (On the constants** C **and** ε**). Needless to say, the proof of Theorem 1.2 tells us that, by proceeding as in Remarks 3.7, 3.8, and 4.1, we can reduce the dependence of C and ε to the parameters N, r_i, r_e, d_Ω, and $\delta_\Gamma(z)$, if Γ is of class C^2, and to N, r_i, and d_Ω, if Ω is also convex or chosen as described in Remark 3.9.

Remark 4.4. (i). Assertion (ii) of Theorem 1.2 can also be proved as a direct corollary of Theorem 1.1, by noting that (4.3) together with (1.13) gives (1.17).

(ii). Analogs of Theorem 3.6 and Corollary 3.10 can be easily derived by following the steps of their proofs.

Remark 4.5. The assumption of smallness of the relevant deviation η required in Theorems 1.1 and 3.6, Corollary 3.10, and (ii) of Theorem 1.2 is only apparent because, if $\eta \geq \varepsilon$, then it is a trivial matter to obtain an upper bound for $\rho_e - \rho_i$ in terms of η. Thus, all the stability estimates that we presented are global.

Another consequence of Lemma 4.2 is the following inequality that shows an optimal stability exponent for any $N \geq 2$. The number $\mathcal{A}(\Omega)$, defined in (1.14), is some sort of asymmetry similar to the so-called Fraenkel asymmetry (see [Fr]).

Theorem 4.6 (Stability by asymmetry). *Let $N \geq 2$ and let Γ be the connected boundary of class C^2 of a bounded domain $\Omega \subset \mathbb{R}^N$. Denote by H the mean curvature function for Γ and let H_0 be the constant defined in* (1.8).

Then, it holds that

$$\mathcal{A}(\Omega) \leq C \, \|H_0 - H\|_{2,\Gamma}, \tag{4.4}$$

for some positive constant C.

Proof. We use [Fe, inequality (2.14)]: we have that

$$\frac{|\Omega \Delta B_R^z|}{|B_R^z|} \leq C \, \|u_\nu - R\|_{2,\Gamma},$$

where B_R^z is a ball of radius R (as defined in (1.8)) and centered at the point z described in item (ii) of Remark 3.9. Hence, we obtain that

$$\mathcal{A}(\Omega) \leq C \, \|u_\nu - R\|_{2,\Gamma},$$

by the definition (1.14). Thus, thanks to (4.3), we obtain (4.4).

Here, if we estimate L_0 as done in item (ii) of Remark 3.9, we can see that C depends on N, d_Ω, r_i, and $|\Gamma|/|\Omega|$. ◻

Remark 4.7 (On the asymmetry $\mathcal{A}(\Omega)$). Notice that, for any $x \in \Omega$, we have that

$$\frac{|\Omega \Delta B_R^x|}{|B_R^x|} \leq \frac{|B_{\rho_e}^x \setminus B_{\rho_i}^x|}{|B_R^x|} = \frac{\rho_e^N - \rho_i^N}{R^N} \leq \frac{N \rho_e^{N-1}}{R^N} (\rho_e - \rho_i),$$

and $\rho_e \leq d_\Omega$. Thus, if d_Ω/R remains bounded and $(\rho_e - \rho_i)/R$ tends to 0, then the ratio $|\Omega \Delta B_R^x|/|B_R^x|$ does the same.

The converse is not true in general. For example, consider a lollipop made by a ball and a stick with fixed length L and vanishing width; as that width vanishes, the ratio d_Ω/R remains bounded, while $|\Omega \Delta B_R^x|/|B_R^x|$ tends to zero and $\rho_e - \rho_i \geq L > 0$.

If we fix r_i and r_e, we have the following result.

Theorem 4.8. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain satisfying the uniform interior and exterior sphere conditions with radii r_i and r_e.*

Then, we have that

$$\rho_e - \rho_i \leq 4R\mathcal{A}(\Omega)^{1/N} \quad \text{if } \mathcal{A}(\Omega) \leq \left(\frac{r_i}{R}\right)^N.$$

Proof. Let x be any point in Ω. It is clear that

$$\max(\rho_e - R, R - \rho_i) \geq \frac{\rho_e - \rho_i}{2}. \tag{4.5}$$

If that maximum is $\rho_e - R$, at a point y where the ball centered at x with radius ρ_e touches Γ, we consider the interior touching ball B_{r_i}.

If $2r_i < (\rho_e - \rho_i)/2$, then $B_{r_i} \subset \Omega \setminus B_R^x$, and hence

$$\frac{|\Omega \Delta B_R^x|}{|B_R^x|} \geq \left(\frac{r_i}{R}\right)^N.$$

If, otherwise, $2r_i \geq (\rho_e - \rho_i)/2$, then B_{r_i} contains a ball of radius $(\rho_e - \rho_i)/4$ still touching Γ at y. Such a ball is contained in $\Omega \setminus B_R^x$, and hence

$$\frac{|\Omega \Delta B_R^x|}{|B_R^x|} \geq \left(\frac{\rho_e - \rho_i}{4R}\right)^N.$$

Thus, we have proved that

$$\frac{|\Omega \Delta B_R^x|}{|B_R^x|} \geq \min\left\{\left(\frac{r_i}{R}\right)^N, \left(\frac{\rho_e - \rho_i}{4R}\right)^N\right\},$$

and the conclusion easily follows, since x was arbitrarily chosen in Ω.

If, otherwise, the maximum in (4.5) is $R - \rho_i$, we proceed similarly, by reasoning on the exterior ball B_{r_e} and $\mathbb{R}^N \setminus \bar{\Omega}$, instead. ◻

Remark 4.9. Theorem 4.8 and (4.4) give the inequality

$$\rho_e - \rho_i \leq C\|H_0 - H\|_{2,\Gamma}^{1/N}$$

that, for any $N \geq 2$, is poorer than that obtained in Theorem 1.2.

ACKNOWLEDGEMENTS.

The paper was partially supported by the Gruppo Nazionale Analisi Matematica Probabilità e Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

REFERENCES

[ABR] A. AFTALION, J. BUSCA, AND W. REICHEL, *Approximate radial symmetry for overdetermined boundary value problems*, Adv. Differential Equations **4** (1999), no. 6, 907–932. MR1729395.

[Al1] A. D. ALEKSANDROV, *Uniqueness theorems for surfaces in the large. V*, Vestnik Leningrad. Univ. **13** (1958), no. 19, 5–8 (Russian, with English summary); English transl., Amer. Math. Soc. Transl. (2) **21** (1962), 412–416. MR0150710. MR0102114.

[Al2] ———, *A characteristic property of spheres*, Ann. Mat. Pura Appl. (4) **58** (1962), 303–315. http://dx.doi.org/10.1007/BF02413056. MR0143162.

[BNST] B. BRANDOLINI, C. NITSCH, P. SALANI, AND C. TROMBETTI, *On the stability of the Serrin problem*, J. Differential Equations **245** (2008), no. 6, 1566–1583. http://dx.doi.org/10.1016/j.jde.2008.06.010. MR2436453.

[BMS] L. BRASCO, R. MAGNANINI, AND P. SALANI, *The location of the hot spot in a grounded convex conductor*, Indiana Univ. Math. J. **60** (2011), no. 2, 633–659. http://dx.doi.org/10.1512/iumj.2011.60.4578. MR2963787.

[BS] H. P. BOAS AND E. J. STRAUBE, *Integral inequalities of Hardy and Poincaré type*, Proc. Amer. Math. Soc. **103** (1988), no. 1, 172–176. http://dx.doi.org/10.2307/2047547. MR938664.

[CM] G. CIRAOLO AND F. MAGGI, *On the shape of compact hypersurfaces with almost-constant mean curvature*, Comm. Pure Appl. Math. **70** (2017), no. 4, 665–716. http://dx.doi.org/10.1002/cpa.21683. MR3628882.

[CMV] G. CIRAOLO, R. MAGNANINI, AND V. VESPRI, *Hölder stability for Serrin's overdetermined problem*, Ann. Mat. Pura Appl. (4) **195** (2016), no. 4, 1333–1345. http://dx.doi.org/10.1007/s10231-015-0518-7. MR3522349.

[CV] G. CIRAOLO AND L. VEZZONI, *A sharp quantitative version of Alexandrov's theorem via the method of moving planes*, J. Eur. Math. Soc. (JEMS) **20** (2018), no. 2, 261–299, available at http://arxiv.org/abs/arXiv:1501.07845v3. http://dx.doi.org/10.4171/JEMS/766. MR3760295.

[Fe] W. M. FELDMAN, *Stability of Serrin's problem and dynamic stability of a model for contact angle motion*, SIAM J. Math. Anal. **50** (2018), no. 3, 3303–3326, available at http://arxiv.org/abs/arXiv:1707.06949. http://dx.doi.org/10.1137/17M1143009. MR3817760.

[Fr] L. E. FRAENKEL, *An Introduction to Maximum Principles and Symmetry in Elliptic Problems*, Cambridge Tracts in Mathematics, vol. 128, Cambridge University Press, Cambridge, 2000. http://dx.doi.org/10.1017/CBO9780511569203. MR1751289.

[HS] R. HURRI-SYRJÄNEN, *An improved Poincaré inequality*, Proc. Amer. Math. Soc. **120** (1994), no. 1, 213–222. http://dx.doi.org/10.2307/2160188. MR1169032.

[Ko] N. J. KOREVAAR, *Convex solutions to nonlinear elliptic and parabolic boundary value problems*, Indiana Univ. Math. J. **32** (1983), no. 4, 603–614. http://dx.doi.org/10.1512/iumj.1983.32.32042. MR703287.

[KM] B. KRUMMEL AND F. MAGGI, *Isoperimetry with upper mean curvature bounds and sharp stability estimates*, Calc. Var. Partial Differential Equations **56** (2017), no. 2, Art. 53, 43, available at http://arxiv.org/abs/arXiv:1606.00490. http://dx.doi.org/10.1007/s00526-017-1139-3. MR3627438.

[MP] R. MAGNANINI AND G. POGGESI, *On the stability for Alexandrov's Soap Bubble theorem*, J. Anal. Math. (2016), to appear, available at http://arxiv.org/abs/arXiv:1610.07036.

[MS] R. MAGNANINI AND S. SAKAGUCHI, *Polygonal heat conductors with a stationary hot spot*, J. Anal. Math. **105** (2008), 1–18. http://dx.doi.org/10.1007/s11854-008-0029-1. MR2438419.

[Po] S. I. POHOŽAEV, *On the eigenfunctions of the equation* $\Delta u + \lambda f(u) = 0$, Dokl. Akad. Nauk SSSR **165** (1965), 36–39 (Russian). MR0192184.

[PS] L. E. PAYNE AND PH. W. SCHAEFER, *Duality theorems in some overdetermined boundary value problems*, Math. Methods Appl. Sci. **11** (1989), no. 6, 805–819. http://dx.doi.org/10.1002/mma.1670110606. MR1021402.

[Re] R. C. REILLY, *Mean curvature, the Laplacian, and soap bubbles*, Amer. Math. Monthly **89** (1982), no. 3, 180–188, 197–198. http://dx.doi.org/10.2307/2320201. MR645791.

[Se] J. SERRIN, *A symmetry problem in potential theory*, Arch. Rational Mech. Anal. **43** (1971), 304–318. http://dx.doi.org/10.1007/BF00250468. MR0333220.

[We] H. F. WEINBERGER, *Remark on the preceding paper of Serrin*, Arch. Rational Mech. Anal. **43** (1971), 319–320. http://dx.doi.org/10.1007/BF00250469. MR0333221.

[Zi] W. P. ZIEMER, *A Poincaré-type inequality for solutions of elliptic differential equations*, Proc. Amer. Math. Soc. **97** (1986), no. 2, 286–290. http://dx.doi.org/10.2307/2046515. MR835882.

Dipartimento di Matematica ed Informatica "U. Dini"
Università di Firenze
viale Morgagni 67/A
50134 Firenze, Italy
E-MAIL: magnanin@math.unifi.it; giorgio.poggesi@unifi.it
URL: http://web.math.unifi.it/users/magnanin

KEY WORDS AND PHRASES: Serrin's overdetermined problem, Alexandrov soap bubble theorem, torsional rigidity, constant mean curvature, integral identities, stability, quantitative estimates.

2010 MATHEMATICS SUBJECT CLASSIFICATION: 35N25, 53A10, 35B35 (35A23).

Received: November 22, 2017.