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Exact spin polarization of massive and massless particles in relativistic fluids at global equilibrium

Andrea Palermo[1,2,a], **Francesco Becattini**[1,b]

[1] Università di Firenze and INFN Sezione di Firenze, Via G. Sansone 1, 50019 Sesto Fiorentino, Florence, Italy
[2] Institut für Theoretische Physik, Johann Wolfgang Goethe-Universität, Max-von-Laue-Straße 1, 60438 Frankfurt am Main, Germany

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Abstract We present the exact form of the spin polarization vector and the spin density matrix of massive and massless free particles of any spin and helicity at general global equilibrium in a relativistic fluid with non-vanishing thermal vorticity, thus extending the known expression at the linear order. The exact form is obtained by means of the analytic continuation of the relativistic density operator to imaginary thermal vorticity and the resummation of the obtained series. The phenomenological implications for the polarization of the Λ hyperon in relativistic heavy-ion collisions are addressed.

1 Introduction

Following the evidence of spin polarization of the Λ hyperon [1], spin physics in relativistic heavy ion collisions has become a very active research field both at the experimental [2–7] and theoretical level [8–16] (see [17] for a recent review).

At local thermodynamic equilibrium in a relativistic fluid, spin polarization turns out to be a function of the gradients of the thermo-hydrodynamic fields, particularly the gradient of the four-temperature vector β which is related to proper temperature and four-velocity of the fluid by $\beta^\mu = (1/T)u^\mu$. The gradients of β include an anti-symmetric part called thermal vorticity ϖ:

$$\varpi_{\mu\nu} = -\frac{1}{2}(\partial_\mu \beta_\nu - \partial_\nu \beta_\mu), \tag{1}$$

and a symmetric part called *thermal shear*. It has been recently found out that the thermal shear induces a significant polarization in relativistic nuclear collisions [18–22]. At global equilibrium, however, thermal shear vanishes because the field β must become a Killing vector [23, 24], and only a constant thermal vorticity survives.

For a spin-1/2 free Dirac field, the expression of the spin polarization vector at the leading order in thermal vorticity was derived in [25]:

$$S^\mu(p) = -\frac{1}{8m}\epsilon^{\mu\nu\rho\sigma} p_\sigma \frac{\int \mathrm{d}\Sigma \cdot p\, \varpi_{\nu\rho} n_F(1-n_F)}{\int \mathrm{d}\Sigma \cdot p\, n_F}, \tag{2}$$

where

$$n_F = \frac{1}{\exp(\beta \cdot p - \zeta) + 1}, \tag{3}$$

is the Fermi-Dirac distribution function and $\zeta = \mu/T$ is the ratio between the chemical potential and the temperature. The integrals in (2) must be performed over the freeze-out hypersurface in relativistic heavy ion physics. The equation (2) was confirmed in other derivations [20, 26, 27] and it is a good approximation for small values of ϖ. Yet, very little is known about higher-order terms, not even at global equilibrium where thermal vorticity is a constant.

In this work, we obtain the exact expression of the spin density matrix and spin polarization vector of massive and massless free fields of any spin at general global equilibrium with non-vanishing thermal vorticity, by means of the analytic continuation of the density operator proposed in refs. [28, 29]. We will first derive those expressions for free Dirac fermions by using the covariant Wigner function formalism, thereafter showing that they are a special case of a more general formula applying to any spin. Finally, we compare the newly found expressions to the linear approximation (2) which is commonly used in phenomenological studies, in order to assess the impact of higher-order corrections in thermal vorticity for the spin polarization measurements in relativistic heavy ion physics.

[a] e-mail: andrea.palermo@unifi.it (corresponding author)
[b] e-mail: becattini@fi.infn.it

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1.1 Notations

We use the natural units, with $\hbar = c = K = 1$. The Minkowskian metric tensor g is diag$(1, -1, -1, -1)$ and repeated indices are assumed to be saturated; for the Levi-Civita symbol, we use the convention $\epsilon^{0123} = 1$. Three vectors are denoted with bold symbols, for example $\boldsymbol{v}$. This notation corresponds to the contravariant space components of the corresponding four-vector, such that $v^\mu = (v^0, \boldsymbol{v})$. Unit vectors are denoted with a small upper hat, e.g. $\hat{p}$. The notation $a \cdot b$ is sometimes used for the scalar product of four-vectors and $X : Y$ for the double contraction of tensors, i.e. $X : Y = X^{\mu\nu} Y_{\mu\nu}$.

Operators in Hilbert space will be denoted by a wide upper hat, e.g. $\widehat{H}$, except the Dirac field operator which is denoted by a Ψ. The symbol Tr will stand for the trace over the Hilbert space of quantum states, while tr is the trace over a finite-dimensional vector space. We will use the notation $\langle \bullet \rangle = \mathrm{Tr}(\widehat{\rho}\, \bullet)$ for thermal expectation values, $\widehat{\rho}$ being the density operator.

2 Spin and helicity in quantum field theory

In a quantum relativistic framework, the spin polarization vector is defined as the expectation value of the Pauli-Lubanski (PL) operator [30]:

$$\widehat{\Pi}^\mu = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma} \widehat{J}_{\nu\rho} \widehat{P}_\sigma, \tag{4}$$

where $\widehat{J}^{\mu\nu}$ and $\widehat{P}^\mu$ are the angular momentum-boost and the four-momentum operators respectively. From the Lie algebra of the Poincaré group, it follows that the PL operator fulfills these relations:

$$[\widehat{\Pi}^\mu, \widehat{P}^\nu] = 0, \tag{5a}$$

$$[\widehat{\Pi}^\mu, \widehat{\Pi}^\nu] = -i\epsilon^{\mu\nu\rho\sigma} \widehat{\Pi}_\rho \widehat{P}_\sigma, \tag{5b}$$

$$\widehat{\Pi} \cdot \widehat{P} = 0. \tag{5c}$$

The restriction of the PL operator to the one-particle states with definite momentum $|p\rangle$ is defined as $\widehat{\Pi}(p)$:

$$\widehat{\Pi}^\mu(p) = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma} \widehat{J}_{\nu\rho} p_\sigma. \tag{6}$$

This operator generates the so-called little group of p, that is the group of Lorentz transformation leaving p^μ invariant, and plays a crucial role in the definition of the spin and helicity operators. We first notice that, due to Eq. (5c), $\widehat{\Pi}(p)$ can be decomposed along the directions perpendicular to p^μ. Such decomposition is different for massive and massless states, because in the latter case p^μ is orthogonal to itself.

In the massive case one can define a p-dependent spin operator:

$$\widehat{S}^\mu(p) = \frac{\widehat{\Pi}^\mu(p)}{m}. \tag{7}$$

Due to equation (5c), and since p^μ is time-like, the decomposition of the spin operator can be made along three orthogonal, normalized space-like vectors $n_i^\mu(p)$, such that $n_i(p) \cdot p = 0$ and $n_i \cdot n_j = -\delta_{ij}$. These vectors, along with p, make a momentum-dependent orthogonal basis of the Minkowski space-time. For a particle at rest, we have $p = (m, 0, 0, 0)$ and we define the n_i to coincide with the conventional basis vector e_i. In this special frame, vectors will be denoted by Gothic letters, i.e. $\mathfrak{p}^\mu = (m, 0, 0, 0)$ - which will be henceforth referred to as *standard vector* - and $\mathfrak{n}_i = \mathsf{e}_i$. Furthermore, a so-called *standard Lorentz transformation* $[p]$ is introduced, which transforms the conventional basis to the particle basis. Explicitly:

$$p^\mu = [p]^\mu_{\ \nu} \mathfrak{p}^\nu, \qquad n_i^\mu(p) = [p]^\mu_{\ \nu} \mathfrak{n}_i^\nu.$$

The decomposition of the spin operator along the tetrad $\{p, n_1, n_2, n_3\}$ reads:

$$\widehat{S}^\mu(p) = \sum_{i=1}^{3} \widehat{S}_i(p) n_i^\mu(p) \implies \widehat{S}_i(p) = -\widehat{S}(p) \cdot n_i(p). \tag{8}$$

It is well known that the components $\widehat{S}_i(p)$ fulfill a SO(3) Lie algebra and are related to the generators of rotations. Particularly:

$$\widehat{S}_i(\mathfrak{p}) = \widehat{\mathrm{J}}^i, \tag{9}$$

$\widehat{\mathrm{J}}^i$ being the i-th (contravariant) component of the angular momentum operator.

In a statistical system, the mean spin polarization vector of a massive particle with momentum p can be obtained with the formula [31]:

$$S^\mu(p) = \sum_{i=1}^{3} [p]^\mu_i \mathrm{tr}\Big(\Theta(p) D^S(\mathrm{J}^i)\Big), \tag{10}$$

where $D^S(\mathrm{J}^i)$ is the i-th generator of the rotation group in the spin S representation and tr denotes the trace on the $(2S+1)$-dimensional spin space. The matrix $\Theta(p)$ is the spin density matrix and in a quantum field theoretical framework reads:

$$\Theta_{sr}(p) = \frac{\mathrm{Tr}\Big(\widehat{\rho}\,\widehat{a}^\dagger_r(p)\widehat{a}_s(p)\Big)}{\sum_l \mathrm{Tr}\Big(\widehat{\rho}\,\widehat{a}^\dagger_l(p)\widehat{a}_l(p)\Big)} = \frac{\langle \widehat{a}^\dagger_r(p)\widehat{a}_s(p)\rangle}{\sum_l \langle \widehat{a}^\dagger_l(p)\widehat{a}_l(p)\rangle}, \tag{11}$$

where $\widehat{a}^\dagger_s(p)$ and $\widehat{a}_s(p)$ are the creation and annihilation operators for particles with momentum p and spin (helicity) s.

For light-like states, the decomposition of the PL operator is different and the formula (10) no longer holds. The four-momentum p can be included in a non-orthonormal basis of Minkowski space-time defined by the tetrad $\{p, q, n_1, n_2\}$, where q is a light-like vector *not* orthogonal to p and $n_{1,2}$ are two normalized space-like vectors orthogonal to p, q and to each other (i.e., $n_i(p)\cdot p = n_i(p)\cdot q = 0$, $n_i(p)\cdot n_j(p) = -\delta_{ij}$). Similarly to the massive case, these vectors can be written as Lorentz-transformed of a basis of standard vectors. They are, for some $\kappa > 0$:

$$\mathfrak{p}^\mu = (\kappa, 0, 0, \kappa), \qquad \mathfrak{q}^\mu = (\kappa, 0, 0, -\kappa), \qquad \mathfrak{n}^\mu_i = \delta^\mu_i.$$

Likewise, a standard Lorentz transformation $[p]$ is introduced (a typical choice being a boost along the z direction followed by a rotation around the $\hat{\mathbf{k}} \times \hat{\mathbf{p}}$ axis), turning the standard basis into the particle basis:

$$p^\mu = [p]^\mu_{\ \nu}\mathfrak{p}^\nu, \qquad q^\mu = [p]^\mu_{\ \nu}\mathfrak{q}^\nu, \qquad n^\mu_i(p) = [p]^\mu_{\ \nu}\mathfrak{n}^\nu_i.$$

Taking into account $\widehat{\Pi}(p)\cdot p = 0$, the component of the PL vector operator restricted to single particle states along q^μ vanishes and one has:

$$\widehat{\Pi}^\mu(p) = \widehat{h}(p)p^\mu + \widehat{\Pi}_1(p)n^\mu_1(p) + \widehat{\Pi}_2(p)n^\mu_2(p). \tag{12}$$

Using equations (5), it is possible to show that the components of the above decomposition obey the following commutation rules:

$$[\widehat{h}(p), \widehat{\Pi}_1(p)] = i\widehat{\Pi}_2(p), \tag{13a}$$

$$[\widehat{h}(p), \widehat{\Pi}_2(p)] = -i\widehat{\Pi}_1(p), \tag{13b}$$

$$[\widehat{\Pi}_1(p), \widehat{\Pi}_2(p)] = 0. \tag{13c}$$

This is the algebra of the euclidean group in two dimensions, ISO(2). The algebra is semi-simple, including an abelian sub-algebra generated by $\widehat{\Pi}_{1,2}(p)$. It is well known that actual physical states are such that:

$$\widehat{\Pi}_1(p)|p\rangle = \widehat{\Pi}_2(p)|p\rangle = 0. \tag{14}$$

A straightforward consequence of (14) is that the only physically relevant component of the PL vector is $\widehat{h}(p)$ and a basis of the Hilbert space can be chosen using the common eigenvectors of $\widehat{h}(p)$ and $\widehat{P}^\mu$:

$$\widehat{P}^\mu|p, h\rangle = p^\mu|p, h\rangle, \tag{15a}$$

$$\widehat{h}(p)|p, h\rangle = h|p, h\rangle. \tag{15b}$$

The eigenvalue of the $\widehat{h}(p)$ operator is referred to as the *helicity* of the particle. Due to the topology of the Lorentz group, helicity can only be integer or half-integer, and it is also known that helicity-S massless states only exhibit the two extremal helicity states, S and $-S$ [32]. Contracting the Eq. (12) with q, it can be realized that the helicity operator can be written as:

$$\widehat{h}(p) = \frac{\widehat{\Pi}(p)\cdot q}{p\cdot q} = -\frac{1}{2p\cdot q}\epsilon^{\mu\nu\rho\sigma}\widehat{J}_{\nu\rho}p_\sigma q_\mu.$$

For a single relativistic particle we thus have, if $\widehat{\rho}$ is the single-particle density operator:

$$\Pi^\mu(p) = \sum_h \langle p, h|\widehat{\Pi}^\mu(p)\widehat{\rho}|p, h\rangle = p^\mu \sum_h h\langle p, h|\widehat{\rho}|p, h\rangle = p^\mu \sum_h h\Theta(p)_{hh}, \tag{16}$$

where we used the decomposition (12) and the Eq. (14). The sum in Eq. (16), as it is known, has just two terms, i.e. $h = -S$ and $h = S$. In a statistical system, the mean polarization vector of a particle with momentum p is obtained from the Eq. (12) with the spin density matrix:

$$\Theta(p)_{hh} = \frac{\langle \widehat{a}_h^\dagger(p)\widehat{a}_h(p)\rangle}{\sum_{l=\pm S}\langle \widehat{a}_l^\dagger(p)\widehat{a}_l(p)\rangle}, \tag{17}$$

which can be interpreted as the fraction of particles with helicity h. Altogether, the mean PL vector reads:

$$\Pi^\mu(p) = p^\mu \frac{\sum_{h=\pm S} h\langle \widehat{a}_h^\dagger(p)\widehat{a}_h(p)\rangle}{\sum_{l=\pm S}\langle \widehat{a}_l^\dagger(p)\widehat{a}_l(p)\rangle}. \tag{18}$$

3 Spin polarization of Dirac fermions and the Wigner function

The Wigner function is a useful tool in spin polarization studies. For a non-interacting Dirac field, either massive or massless, the Wigner function is defined as [23]:

$$W(x,k) = -\frac{1}{(2\pi)^4}\int \mathrm{d}^4y\, \mathrm{e}^{-ik\cdot y}\,\mathrm{Tr}\Big(\widehat{\rho} : \Psi\Big(x - \frac{y}{2}\Big)\overline{\Psi}\Big(x + \frac{y}{2}\Big) :\Big),$$

where the colons imply normal ordering, Ψ is the Dirac field operator (the Dirac adjoint being $\overline{\Psi} = \Psi^\dagger\gamma^0$) and $\widehat{\rho}$ is the density operator. Notice that the pseudo-momentum k is not in general on-shell. In fact, depending on k, it is possible to decompose the Wigner function into particle, antiparticle, and spacelike components, denoted as W_+, W_- and W_S respectively:

$$W(x,k) = W_+(x,k)\theta(k^2)\theta(k^0) + W_-(x,k)\theta(k^2)\theta(-k^0) + W_s(x,k)\theta(-k^2).$$

Since the definition of the Wigner function, as well as the plane wave expansion of the non-interacting Dirac field, are the same for massive and massless particles, some properties of the Wigner function are common to the two cases. For instance, it can be shown that:

$$k^\mu\partial_\mu W_{\pm s}(x,k) = 0. \tag{19}$$

The above equation implies that, provided that some boundary conditions are fulfilled, $k^\mu W_{\pm s}(x,k)$ can be integrated over any hypersurface, and the result is independent thereof. This property makes it possible to define an on-shell Wigner function, denoted as $w(p)$ [31]. Focusing on the particle component, one has:

$$\frac{1}{2\varepsilon}\delta(k^0 - \varepsilon)w_+(k) = \int \mathrm{d}\Sigma_\mu k^\mu W_+(x,k), \tag{20}$$

and, after using the plane wave expansion of the Dirac field, explicit integration leads to:

$$w_+(p) = \frac{1}{2}\sum_{h,l}\langle \widehat{a}_l^\dagger(p)\widehat{a}_h(p)\rangle u_h(p)\bar{u}_l(p). \tag{21}$$

A derivation of the formula connecting the mean spin polarization vector to the Wigner function in the massive case was presented in ref. [31]. Using the normalization of the spinors, $\bar{u}_s(p)u_r(p) = 2m\delta_{rs}$, one gets:

$$S^\mu(p) = \frac{1}{2}\frac{\int \mathrm{d}\Sigma\cdot p\ \mathrm{tr}(\gamma^\mu\gamma_5 W_+(x,p))}{\int \mathrm{d}\Sigma\cdot p\ \mathrm{tr}(W_+(x,p))}, \tag{22}$$

where in heavy-ions applications the integral is computed over the freeze-out hypersurface. In fact, the derivation of a formula such as (22) does not trivially extend to the massless case, the reason being that $\bar{u}_s(p)u_r(p) = 0$. In this case, using the relation $\bar{u}_s(p)\gamma^\mu u_r(p) = 2p^\mu\delta_{rs}$, it is easy to see:

$$\mathrm{tr}(w_+(p)\gamma^\mu) = \frac{1}{2}\sum_h\langle \widehat{a}_l^\dagger(p)\widehat{a}_h(p)\rangle\mathrm{tr}(u_h(p)\bar{u}_l(p)\gamma^\mu) = \frac{1}{2}\sum_{hl}\langle \widehat{a}_l^\dagger(p)\widehat{a}_h(p)\rangle\bar{u}_l(p)\gamma^\mu u_h(p) = p^\mu\sum_h\langle \widehat{a}_h^\dagger(p)\widehat{a}_h(p)\rangle, \tag{23}$$

where the cyclicity of the trace has been used. Notice how this trace is just the denominator of the formula (17) multiplied by p^μ. Therefore, the denominator of Eq. (17) can be obtained by contracting the (23) with *any* vector v, provided that it is not orthogonal to p. However, any four-vector v can be decomposed along the basis $\{p, q, n_1, n_2\}$, as we have seen in Sect. 2:

$$v^\mu = v_p p^\mu + v_q q^\mu + v_i n_i^\mu\ .$$

Equation (23) implies that only the component of v^μ along q^μ is relevant to invert Eq. (23), as it is the only one contributing to the product $v \cdot p$. Therefore, we can choose conveniently $v = q$ to find:

$$\sum_h \langle \widehat{a}_h^\dagger(p)\widehat{a}_h(p)\rangle = \frac{\mathrm{tr}(w_+(k)\not{q})}{p \cdot q}. \tag{24}$$

With the same steps of the derivation of the Eq. (23), it can be shown that:

$$\mathrm{tr}(w_+(p)\gamma^\mu\gamma_5) = 2p^\mu \sum_h h\langle \widehat{a}_h^\dagger(p)\widehat{a}_h(p)\rangle,$$

where we used the equation $\gamma_5 u_h(p) = 2hu_h(p)$, notably applying to massless fermions. By contracting with q the above equation and using the (24), the equation (18) can be rewritten as:

$$\Pi^\mu(p) = \frac{p^\mu}{2}\frac{\mathrm{tr}(\not{q}\gamma_5 w_+(p))}{\mathrm{tr}(w_+(p)\not{q})},$$

and, by using the Eq. (20):

$$\Pi^\mu(p) = \frac{p^\mu}{2}\frac{\int \mathrm{d}\Sigma \cdot p\ \mathrm{tr}(\not{q}\gamma^5 W_+(x,p))}{\int \mathrm{d}\Sigma \cdot p\ \mathrm{tr}(W_+(x,p)\not{q})}. \tag{25}$$

This formula is the corresponding of (22) for massless particles and it is the final result of this section. It should be emphasized that, in spite of its appearance, the Eq. (25) does not depend on the particular vector q chosen; this dependence cancels out, though not manifestly, if the same q is used in the numerator and the denominator. We point out that the result (25) differs from others in literature [33] in that the polarization vector is manifestly parallel to the four-momentum.

4 Exact spin polarization of Dirac fermions

In a previous paper of ours [29] we derived the exact form of the Wigner function of free Dirac fermions in global equilibrium with non-vanishing thermal vorticity. We are now in a position to use that result to calculate the exact expression of the spin polarization vector of Dirac fermions under those conditions.

To begin with, we make a brief recap of the key concepts of global thermodynamic equilibrium in quantum relativistic statistical mechanics. The density operator corresponding to the most general global equilibrium allowed by special relativity reads:

$$\widehat{\rho} = \frac{1}{Z}\exp\left[-b \cdot \widehat{P} + \frac{\varpi : \widehat{J}}{2} + \zeta\widehat{Q}\right], \tag{26}$$

where the operators $\widehat{P}$, $\widehat{J}$ and $\widehat{Q}$ are the four-momentum, the angular momentum-boost, and the charge operators. The Lagrange multipliers b^μ and $\varpi^{\mu\nu}$ are a constant vector and a constant anti-symmetric tensor respectively. Together, they define the four-temperature β^μ as the Killing vector:

$$\beta^\mu = b^\mu + \varpi^{\mu\nu}x_\nu. \tag{27}$$

The four-temperature naturally identifies a hydrodynamic frame (defining a four-velocity $u^\mu = T\beta^\mu$, being $T = 1/\sqrt{\beta^2}$ the proper temperature). The additional Lagrange multiplier ζ is the ratio of the chemical potential and the proper temperature, $\zeta = \mu/T$, and it is constant at global equilibrium. The tensor ϖ is the thermal vorticity, since from the Eq. (27) one readily obtains the (1). The thermal vorticity can be decomposed in two space-like vectors by using the four-velocity of the fluid:

$$\varpi^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma}w_\rho u_\sigma + \alpha^\mu u^\nu - \alpha^\nu u^\mu,$$

where:

$$w^\mu = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\varpi_{\nu\rho}u_\sigma, \qquad \alpha^\mu = \varpi^{\mu\nu}u_\nu. \tag{28}$$

The general expressions of w^μ and α^μ can be obtained from the above definition and the (1); at global equilibrium they reduce to the ratio of the angular velocity ω^μ and the four-acceleration A^μ with the proper temperature T. Explicitly, one has:

$$w^\mu = \frac{\omega^\mu}{T}, \qquad \alpha^\mu = \frac{A^\mu}{T}. \tag{29}$$

A method to calculate the expectation values with the density operator (26) was proposed in refs. [28, 29]. The idea is to analytically continue the density operator (26) to imaginary vorticity, i.e. setting $\varpi \mapsto -i\phi$, and to make an analytic continuation back to real thermal vorticity *after* the analytic results in ϕ are obtained. With this technique, expectation values are expressed as series of functions and the analytic continuation to real thermal vorticity generally requires an intermediate operation that we dubbed as *analytic distillation*. Yet, in the case of the spin density matrix and the spin polarization vector, we will see that analytic distillation is not necessary and the analytic continuation can be done straightforwardly.

The expectation value of the quadratic combination of creation and annihilation operators with the analytically continued density operator reads [29]:

$$\langle \widehat{a}_s^\dagger(p)\widehat{a}_t(p')\rangle = 2\varepsilon' \sum_{n=1}^{\infty}(-1)^{2S(n+1)}\delta^3(\Lambda^n\mathbf{p}-\mathbf{p}')D(W(\Lambda^n,\, p))_{ts}\mathrm{e}^{-\tilde{b}\cdot\sum_{k=1}^n \Lambda^k p}\mathrm{e}^{n\zeta}. \tag{30}$$

In the above series, we have tacitly introduced Λ as a notation for the Lorentz transformation $\Lambda = \exp[-i\phi : J/2]$, while $\tilde{b}(\phi)$ is given by:

$$\tilde{b}^\mu(\phi) = \sum_{k=0}^{\infty}\frac{1}{(k+1)!}\phi^\mu_{\ \nu_1}\phi^{\nu_1}_{\ \nu_2}\dots\phi^{\nu_{k-1}}_{\ \nu_k}b^{\nu_k}. \tag{31}$$

The Eq. (30), which applies to both massive and massless particles, involves the representation of the transformation $W(\Lambda,\, p) = [\Lambda p]^{-1}\Lambda[p]$, where $[p]$ is the standard Lorentz transformation mapping the conventional basis to particle basis, that is $p^\mu = [p]^\mu_{\ \nu}\mathfrak{p}^\nu$ as defined in Sect. 2. The transformation $W(\Lambda,\, p)$ belongs to the little group of the standard vector $\mathfrak{p}$, that is, it leaves $\mathfrak{p}$ invariant. For massive spin-S states, $D(W(\Lambda,\, p))$ is an element of the S-irreducible representation of the rotation group SO(3), $D(W(\Lambda,\, p)) = D^S(W(\Lambda,\, p))$, and $W(\Lambda,\, p)$ is thus commonly known as *Wigner rotation*. In fact, as a consequence of (14), in the massless case the transformation $W(\Lambda,\, p)$ is a composition of a rotation with a Lorentz transformation, but its representation reduces to a phase factor [32]:

$$D(W(\Lambda, p))_{rs} = \exp[i\, s\vartheta(\Lambda, p)]\delta_{rs}, \tag{32}$$

where r and s can only be $\pm S$, being S the helicity of the particle. The sum over n in Eq. (30) can be interpreted as the quantum statistics expansion, whose first term $n = 1$ is the Boltzmann statistics contribution [28].

Using Eq. (30), the particle component of the analytically continued Wigner function for Dirac fermions appearing in Eqs. (22) and (25) can be obtained [29]:

$$W_+(x,k) = \frac{1}{(2\pi)^3}\int\frac{\mathrm{d}^3p}{2\varepsilon}\sum_{n=1}^{\infty}(-1)^{n+1}\mathrm{e}^{-\tilde{b}(\phi)\cdot\sum_{k=1}^n\Lambda^k p}\mathrm{e}^{n\zeta}\mathrm{e}^{-ix\cdot(\Lambda^n p-p)}\mathrm{e}^{n\zeta}\exp\left(-in\frac{\phi:\Sigma}{2}\right)(m+\not{p})\delta^4\left(k-\frac{\Lambda^n p+p}{2}\right), \tag{33}$$

where $\not{p} = p_\mu\gamma^\mu$, γ^μ being the gamma matrices and $\Sigma^{\mu\nu} = (i/4)[\gamma^\mu,\, \gamma^\nu]$ is the generator of Lorentz transformations in the Dirac representation. The Wigner function for massless fermions is simply obtained by setting $m = 0$ in the Eq. (33).

Plugging the Eq. (33) in the (22) one can obtain the spin polarization vector of massive Dirac fermions. In view of the relation (19), it is possible to compute the integral in Eq. (22) over a constant-time hypersurface $t = t_0$, obtaining:

$$S^\mu(p) = \frac{1}{2m}\frac{\sum_{n=1}^{\infty}(-1)^{n+1}\mathrm{e}^{-\tilde{b}(\phi)\cdot\sum_{k=1}^n\Lambda^k p}\mathrm{e}^{n\zeta}\mathrm{tr}(\gamma^\mu\gamma_5\exp[-in\phi:\Sigma/2]\not{p})\delta^3(\Lambda^n p-p)}{\sum_{n=1}^{\infty}(-1)^{n+1}\mathrm{e}^{-\tilde{b}(\phi)\cdot\sum_{k=1}^n\Lambda^k p}\mathrm{e}^{n\zeta}\mathrm{tr}(\exp[-in\phi:\Sigma/2])\delta^3(\Lambda^n p-p)}. \tag{34}$$

The above expression is the ratio of two series of δ-functions, which appears daunting. This is not surprising, though, as the Eq. (22), as well as (25), was originally derived from the spin density matrix (11), whose definition is based on $\langle\widehat{a}_s^\dagger(p)\widehat{a}_t(p)\rangle$. These expectation values read, according to the Eq. (30):

$$\langle \widehat{a}_s^\dagger(p)\widehat{a}_t(p)\rangle = 2\varepsilon \sum_{n=1}^{\infty}(-1)^{2S(n+1)}\delta^3(\Lambda^n\mathbf{p}-\mathbf{p})D^S(W(\Lambda^n,\, p))_{rs}\mathrm{e}^{-\tilde{b}\cdot\sum_{k=1}^n \Lambda^k p}\mathrm{e}^{n\zeta},$$

and they vanish unless $\Lambda^n p = p$. This relation is fulfilled for specific $n \neq 1$ or if $\Lambda p = p$. In the latter case, the equation holds $\forall n$, whereas if the equation is solved for $n \neq 1$ and given p, Λ would be a discrete transformation, which would make the analytic continuation impossible; therefore, we will focus on the case $\Lambda p = p$. This constraint requires Λ to belong to the little group of p, i.e. the group of transformations leaving p invariant. By using the exponential parametrization of the Lorentz group, $\Lambda = \exp[-i\phi : J/2]$, and expanding for infinitesimal ϕ:

$$\exp[-i\phi : J/2]p = p \implies \phi^{\mu\nu}p_\nu = 0, \tag{35}$$

being $(J_{\mu\nu})^\rho_{\ \sigma} = i(\delta^\rho_\mu g_{\nu\sigma} - \delta^\rho_\nu g_{\mu\sigma})$. The general solution of Eq. (35) can be expressed in terms of an auxiliary vector ξ^μ:

$$\xi^\rho = -\frac{1}{2m}\epsilon^{\rho\mu\nu\sigma}\phi_{\mu\nu}p_\sigma. \tag{36}$$

In this case, it can be shown that:

$$\phi^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma}\xi_\rho\frac{p_\sigma}{m}, \qquad \phi:\phi = -2\xi^2, \qquad \phi:\widetilde{\phi} = 0, \tag{37}$$

where the tensor $\tilde{\phi}$ is the dual of ϕ:
$$\tilde{\phi}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\phi_{\rho\sigma}.$$

Notice that, since $\xi \cdot p = 0$ and p is time-like, ξ is a space-like vector and $-\xi^2 > 0$. If $\Lambda p = p$, it turns out that all the δ functions in the Eq. (34) reduce to $\delta^3(0)$ that simplifies in the ratio, leaving:

$$\begin{aligned} S^\mu(p) =& \frac{1}{2m}\frac{\sum_{n=1}^{\infty}(-1)^{n+1}\mathrm{e}^{-\tilde{b}(\phi)\cdot\sum_{k=1}^{n}\Lambda^k p}\mathrm{e}^{n\zeta}\,\mathrm{tr}(\gamma^\mu\gamma_5\exp[-in\phi:\Sigma/2]\not{p})\delta^3(0)}{\sum_{n=1}^{\infty}(-1)^{n+1}\mathrm{e}^{-\tilde{b}(\phi)\cdot\sum_{k=1}^{n}\Lambda^k p}\mathrm{e}^{n\zeta}\,\mathrm{tr}(\exp[-in\phi:\Sigma/2])\delta^3(0)} \\ =& \frac{1}{2m}\frac{\sum_{n=1}^{\infty}(-1)^{n+1}\mathrm{e}^{-\tilde{b}(\phi)\cdot\sum_{k=1}^{n}\Lambda^k p}\mathrm{e}^{n\zeta}\,p_\nu\mathrm{tr}(\gamma^\mu\gamma_5\exp[-in\phi:\Sigma/2]\gamma^\nu)}{\sum_{n=1}^{\infty}(-1)^{n+1}\mathrm{e}^{-\tilde{b}(\phi)\cdot\sum_{k=1}^{n}\Lambda^k p}\mathrm{e}^{n\zeta}\,\mathrm{tr}(\exp[-in\phi:\Sigma/2])}. \end{aligned} \tag{38}$$

This formula coincides with Eq. (9.6) reported in [29], although its derivation was not carried out in detail. The constraint $\Lambda p = p$ implies more simplifications, particularly:

$$\tilde{b}(\phi)\cdot\sum_{k=1}^{n}\Lambda^k p = n\tilde{b}(\phi)\cdot p = n\sum_{k=0}^{\infty}\frac{1}{(k+1)!}p_\mu\underbrace{(\phi^\mu_{\ \alpha_1}\phi^{\alpha_1}_{\ \alpha_2}\ldots\phi^{\alpha_{k-1}}_{\ \alpha_k})}_{k\text{ times}}b^{\alpha_k} = nb\cdot p, \tag{39}$$

where the definition (31) and Eq. (35) have been used. Thanks to Eqs. (37), especially $\phi : \tilde{\phi} = 0$, also the traces in the equation (38) take a simpler form (calculations are reported in appendix A):

$$\mathrm{tr}(\exp[-in\phi:\Sigma/2]) = 4\cos\left(\frac{n}{2}\sqrt{\frac{\phi:\phi}{2}}\right), \tag{40a}$$

$$\mathrm{tr}(\gamma^\nu\gamma^\mu\exp[-in\phi:\Sigma/2]) = 4g^{\mu\nu}\cos\left(\frac{n}{2}\sqrt{\frac{\phi:\phi}{2}}\right) + 4\phi^{\mu\nu}\frac{\sin\left(\frac{n}{2}\sqrt{\frac{\phi:\phi}{2}}\right)}{\sqrt{\frac{\phi:\phi}{2}}}, \tag{40b}$$

$$\mathrm{tr}(\gamma^\nu\gamma^\mu\gamma_5\exp[-in\phi:\Sigma/2]) = 4i\tilde{\phi}^{\mu\nu}\frac{\sin\left(\frac{n}{2}\sqrt{\frac{\phi:\phi}{2}}\right)}{\sqrt{\frac{\phi:\phi}{2}}}. \tag{40c}$$

By plugging the above traces and the Eq. (39) into the (38) we obtain:

$$S^\mu(p) = \frac{i\tilde{\phi}^{\mu\nu}p_\nu}{2m\sqrt{-\xi^2}}\frac{\sum_{n=1}^{\infty}(-1)^{n+1}\mathrm{e}^{-nb\cdot p+n\zeta}\sin\left(n\sqrt{-\xi^2}/2\right)}{\sum_{n=1}^{\infty}(-1)^{n+1}\mathrm{e}^{-nb\cdot p+n\zeta}\cos\left(n\sqrt{-\xi^2}/2\right)}.$$

Both series in the above equation are convergent $\forall\phi \in \mathbb{R}$ and $b \cdot p > \zeta$. Expanding $\tilde{\phi}$, the summation yields:

$$S^\mu(p) = \frac{i\epsilon^{\mu\nu\rho\sigma}\phi_{\rho\sigma}p_\nu}{4m\sqrt{-\xi^2}}\frac{\sin\left(\sqrt{-\xi^2}/2\right)}{\cos\left(\sqrt{-\xi^2}/2\right) + \mathrm{e}^{-b\cdot p+\zeta}}, \tag{41}$$

which is an analytic function in ϕ.

The Eq. (41) can be analytically continued to real thermal vorticity. Introducing the vector:

$$\theta^\mu = -\frac{1}{2m}\epsilon^{\mu\nu\rho\sigma}\varpi_{\nu\rho}p_\sigma, \tag{42}$$

the mapping $\phi \mapsto i\varpi$ implies $\xi \mapsto i\theta$. Notice that, the vector θ^μ is just the ratio between the local angular velocity seen by the particle (p/m replacing u in Eq. (28)) and the temperature. Therefore, the continuation of Eq. (41) reads:

$$S^\mu(p) = -\frac{1}{4m}\epsilon^{\mu\nu\rho\sigma}\varpi_{\nu\rho}p_\sigma\frac{1}{\sqrt{-\theta^2}}\frac{\sinh\left(\sqrt{-\theta^2}/2\right)}{\cosh\left(\sqrt{-\theta^2}/2\right) + \mathrm{e}^{-b\cdot p+\zeta}}. \tag{43}$$

The polarization vector can also be expressed solely in terms of θ^μ, yielding a more suggestive expression:

$$S^\mu(p) = \frac{1}{2}\frac{\theta^\mu}{\sqrt{-\theta^2}}\frac{\sinh\left(\sqrt{-\theta^2}/2\right)}{\cosh\left(\sqrt{-\theta^2}/2\right) + \mathrm{e}^{-b\cdot p+\zeta}} = \frac{1}{2}\hat{\theta}^\mu P_{1/2}\left(\sqrt{-\theta^2}, b\cdot p - \zeta\right). \tag{44}$$

The above equation has been written such that the factor 1/2 is the spin of the particle, the vector $\hat{\theta}^\mu = \theta^\mu/\sqrt{-\theta^2}$ provides the direction of $S^\mu(p)$, while its magnitude is determined by the weight function:

$$P_{1/2}(x, y) = \frac{\sinh(x/2)}{\cosh(x/2) + \mathrm{e}^{-y}}. \tag{45}$$

where $x = \sqrt{-\theta^2}$ and $y = b \cdot p - \zeta$. Note that, since the polarization vector of a spin-S particle is defined as:

$$P^\mu = \frac{S^\mu}{S},$$

the function $P_{1/2}$ is in fact the polarization itself. The function $P_{1/2}$ monotonically increases in both arguments for $x \geq 0$, and it is bounded in the interval $0 \leq P_{1/2} \leq 1$ at it should. Its limiting values are:

$$\lim_{x\to\infty} P_{1/2}(x, y) = 1, \qquad P_{1/2}(x, y) \simeq \frac{x}{2}(1 - n_F(y)) \quad \text{for} \quad x \ll 1 \tag{46}$$

with $n_F(y) = [\exp(y) + 1]^{-1}$ as in (3). Therefore:

$$\lim_{\sqrt{-\theta^2}\to\infty} S^\mu(p) = \frac{1}{2}\frac{\theta^\mu}{\sqrt{-\theta^2}}, \tag{47a}$$

$$S^\mu(p) \simeq \frac{\theta^\mu}{4}(1 - n_F(b \cdot p - \zeta)), \qquad \text{if} \quad \sqrt{-\theta^2} \ll 1. \tag{47b}$$

Recalling the definition of θ^μ, Eq. (42), it can be seen that the second limit agrees with the Eq. (2) in the case of global equilibrium. Besides, Eq. (47a) shows that, for an infinitely large vorticity, particles become fully polarized in the direction of θ.

It is worth discussing in more detail the obtained results. A crucial role in the determination of the exact formula of the spin polarization vector has been played by the constraint $\Lambda p = p$, equivalent to $\phi_{\mu\nu} p^\nu = 0$, which has been used to determine the form of the imaginary vorticity $\phi_{\mu\nu}$ in the Eq. (37). Nevertheless, it should be emphasized that, after the analytic continuation $\phi = i\varpi$ of the Eq. (41) to the final (43), such constraint does not extend to ϖ. Otherwise stated, the continuation of ϕ to real thermal vorticity does not bring the constraint along.

It should also be emphasized that our obtained expression of the spin polarization vector has the correct bound of polarization, given by the Eq. (47a). This is an important point, as a violation of the unitarity bound of polarization was noted in ref. [9] where the spin polarization vector was calculated at all orders in thermal vorticity in the Boltzmann limit. The violation was attributed to a problem in the *ansatz* of the Wigner function used in the derivation. Indeed, it can be shown that the violation also appears in the single quantum relativistic particle framework [31] if the constraint $\phi_{\mu\nu} p^\nu = 0$ is neglected before the analytic continuation to real thermal vorticity.

4.1 Massless Dirac fermions

We can now move to the case of the massless Dirac field. Using the Eq. (25) and the exact Wigner function of Dirac fermions (33) with $m = 0$ and $\zeta = 0$, and calculating the integrals in Eq. (25) over a constant-time hypersurface, one finds:

$$\Pi^\mu = \frac{p^\mu}{2}\frac{\sum_{n=1}^\infty (-1)^{n+1} \mathrm{e}^{-\tilde{b}(\phi)\cdot\sum_{k=1}^n \Lambda^k p} \mathrm{tr}(\not{q}\gamma_5 \exp[-in\phi : \Sigma/2]\not{p})\delta^3(\Lambda^n p - p)}{\sum_{n=1}^\infty (-1)^{n+1} \mathrm{e}^{-\tilde{b}(\phi)\cdot\sum_{k=1}^n \Lambda^k p} \mathrm{tr}(\not{q} \exp[-in\phi : \Sigma/2]\not{p})\delta^3(\Lambda^n p - p)}. \tag{48}$$

The constraint $\Lambda p = p$ implies $\phi^{\mu\nu} p_\nu = 0$, just like in the massive case. Indeed, as it is shown in appendix B, the most general decomposition of an anti-symmetric tensor solving Eq. (35) for a light-like p is:

$$\phi^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma}\frac{h_\rho p_\sigma}{p \cdot q}, \qquad h^\mu = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\phi_{\nu\rho}q_\sigma. \tag{49}$$

with $h \cdot q = 0$. All the traces in the Eq. (48) can be simplified much like in the massive case. Indeed, it can be shown that:

$$\phi : \phi = 2\eta^2, \qquad \phi : \tilde{\phi} = 0, \tag{50}$$

where we have defined:

$$\eta = \frac{h \cdot p}{q \cdot p} = \frac{\tilde{\phi}^{\mu\nu} q_\mu p_\nu}{p \cdot q} = \frac{1}{2(p \cdot q)}\epsilon^{\mu\nu\alpha\beta}\phi_{\alpha\beta}p_\nu q_\mu. \tag{51}$$

Since $\phi : \tilde{\phi} = 0$, the identities (40) hold, and taking into account the Eq. (39) we have:

$$\Pi^\mu(p) = \frac{p^\mu}{2|\eta|}\frac{i\tilde{\phi}^{\alpha\beta}q_\alpha p_\beta}{p \cdot q}\frac{\sum_{n=1}^\infty (-1)^{n+1}\mathrm{e}^{-nb\cdot p}\sin(n|\eta|/2)}{\sum_{n=1}^\infty (-1)^{n+1}\mathrm{e}^{-nb\cdot p}\cos(n|\eta|/2)}.$$

The above series converges for $b \cdot p > 0$, and the summation yields:

$$\Pi^\mu(p) = \frac{p^\mu}{2}\frac{i\eta}{|\eta|}\frac{\sin(|\eta|/2)}{\cos(|\eta|/2) + \mathrm{e}^{-b\cdot p}} = \frac{p^\mu}{2}\frac{i\sin(\eta/2)}{\cos(\eta/2) + \mathrm{e}^{-b\cdot p}},$$

where in the last step we have used the parity of the trigonometric functions.

The latter result can be readily continued to real thermal vorticity. Introducing:

$$H = \frac{\tilde{\varpi}^{\mu\nu} q_\mu p_\nu}{p\cdot q} = \frac{1}{2(p\cdot q)}\epsilon^{\mu\nu\alpha\beta}\varpi_{\alpha\beta} p_\nu q_\mu, \tag{52}$$

and realizing that $\phi \mapsto i\varpi$ implies $\eta \mapsto iH$, one finds:

$$\Pi^\mu(p) = -\frac{p^\mu}{2}\frac{\sinh(H/2)}{\cosh(H/2) + \mathrm{e}^{-b\cdot p}}. \tag{53}$$

The Lorentz scalar H can be written in a way which is independent of q by breaking manifest covariance. Since $\mathfrak{q} = (\kappa, 0, 0, -\kappa)$ is the parity conjugate of $\mathfrak{p}$ and $[p]^\mu_{\ \nu}\mathfrak{p}^\nu = p^\mu = (\varepsilon, \mathbf{p})$, where $\varepsilon = \|\mathbf{p}\|$ is the energy, it can be readily shown that:

$$q^\mu = [p]^\mu_{\ \nu}\mathfrak{q}^\nu = \kappa^2(1/\varepsilon, -\mathbf{p}/\varepsilon^2).$$

Therefore:

$$H = \frac{1}{2(p\cdot q)}\epsilon^{\mu\nu\alpha\beta}\varpi_{\alpha\beta} p_\nu q_\mu = \frac{1}{4\kappa^2}\epsilon^{0n\alpha\beta}\varpi_{\alpha\beta} p_n q_0 + \frac{1}{4\kappa^2}\epsilon^{m0\alpha\beta}\varpi_{\alpha\beta} p_0 q_m = \frac{1}{2\varepsilon}\epsilon^{0n\alpha\beta}\varpi_{\alpha\beta} p_n,$$

for the combinations where p and q both have a space index vanish due to $\epsilon^{mn\alpha\beta} p_n q_m = -\epsilon^{mn\alpha\beta} p_n p_m = 0$. Defining:

$$\varphi^n = -\frac{1}{2}\epsilon^{n\alpha\beta 0}\varpi_{\alpha\beta}, \qquad \hat{\mathbf{p}} = \frac{\mathbf{p}}{\varepsilon}.$$

we obtain:

$$H = -\frac{1}{2\varepsilon}\epsilon^{0n\alpha\beta}\varpi_{\alpha\beta} p_n = \frac{1}{\varepsilon}\varphi^n p_n = -\boldsymbol{\varphi}\cdot\hat{\mathbf{p}}. \tag{54}$$

We can finally rewrite the PL vector for massless particles using φ as:

$$\Pi^\mu(p) = \frac{p^\mu}{2} P_{1/2}\big(\boldsymbol{\varphi}\cdot\hat{\mathbf{p}}, b\cdot p\big),$$

where we made use of the function $P_{1/2}$ defined in Eq. (45). The limits of very large and very small thermal vorticity, are easily obtained from (46):

$$\lim_{|\boldsymbol{\varphi}\cdot\hat{\mathbf{p}}|\to\infty} \Pi^\mu(p) \simeq \frac{p^\mu}{2}\mathrm{sgn}(\boldsymbol{\varphi}\cdot\hat{\mathbf{p}}),$$
$$\Pi^\mu(p) \simeq \frac{p^\mu}{4}\boldsymbol{\varphi}\cdot\hat{\mathbf{p}}(1 - n_F(b\cdot p)), \qquad \text{if } \left|\boldsymbol{\varphi}\cdot\hat{\mathbf{p}}\right| \ll 1.$$

5 Particles with any spin

The previous results can be extended to particles of any spin S. For this purpose, since we are working at global equilibrium, the spin density matrix can be used in the first place without introducing the covariant Wigner function. Still, the calculation requires the analytic continuation of the density operator to imaginary thermal vorticity. Using the definition (11) and the exact expression of $\langle \widehat{a}^\dagger_s(p)\widehat{a}_r(p')\rangle$ at general global equilibrium, equation (30), one finds:

$$\Theta_{rs}(p) = \frac{\langle \widehat{a}^\dagger_s(p)\widehat{a}_r(p)\rangle}{\sum_t \langle \widehat{a}^\dagger_t(p)\widehat{a}_t(p)\rangle} = \frac{\sum_{n=1}^\infty (-1)^{2S(n+1)}\mathrm{e}^{-\widetilde{b}\cdot\sum_{k=1}^n \Lambda^k p}\mathrm{e}^{n\zeta} D_{rs}(W(\Lambda^n, p))\delta^3(\Lambda^n p - p)}{\sum_{n=1}^\infty (-1)^{2S(n+1)}\mathrm{e}^{-\widetilde{b}\cdot\sum_{k=1}^n \Lambda^k p}\mathrm{e}^{n\zeta}\mathrm{tr}[D(W(\Lambda^n, p))]\delta^3(\Lambda^n p - p)}.$$

According to the discussion in Sect.4, the Dirac δ-function constrains the tensor ϕ such that the equation $\Lambda p = p$ is fulfilled, hence $\phi_{\mu\nu}p^\nu = 0$. Therefore all the Dirac δ-functions boil down to a common divergent factor $\delta^3(0)$ in the numerator and the denominator, similarly to Eq. (38), and so:

$$\Theta_{rs}(p) = \frac{\sum_{n=1}^\infty (-1)^{2S(n+1)}\mathrm{e}^{-nb\cdot p}\mathrm{e}^{n\zeta} D_{rs}(W(\Lambda^n, p))}{\sum_{n=1}^\infty (-1)^{2S(n+1)}\mathrm{e}^{-nb\cdot p}\mathrm{e}^{n\zeta}\mathrm{tr}[D(W(\Lambda^n, p))]}, \tag{55}$$

where we have set $-\widetilde{b}\cdot\sum_{k=1}^n \Lambda^k p = nb\cdot p$ according to the Eq. (39). Using the above equation, one can determine the exact form at global equilibrium of spin related observables, such as the polarization vector or the spin alignment parameter, for both massive and massless particles of any spin. In what follows, we will confine ourselves with the spin vector, tackling the massive and massless case separately.

5.1 Massive particles

The first step to derive the spin density matrix and the spin polarization vector, both for massive and massless particles, is to determine the Wigner rotation for Lorentz transformations Λ in the little group of p; this calculation is carried out in the appendix C.

For massive particles, this Wigner rotation turns out to be:

$$D^S(W(\Lambda, p)) = \exp\Big[-i\boldsymbol{\xi}_0 \cdot D^S(\mathbf{J})\Big]\,, \tag{56}$$

where $\xi_0^\mu = [p]^{-1\mu}_{\nu}\xi^\nu = (0, \boldsymbol{\xi}_0)$ and $\mathbf{J}$ is the three-vector of the generators of SO(3). The time-component of ξ_0 vanishes because $\xi_0 \cdot \mathfrak{p} = \xi \cdot p = 0$. The Eq. (56) shows that the Wigner rotation corresponds to a rotation of an angle $\sqrt{\boldsymbol{\xi}_0 \cdot \boldsymbol{\xi}_0} = \sqrt{-\xi \cdot \xi}$ around the axis $\hat{\boldsymbol{\xi}}_\mathbf{0} = \boldsymbol{\xi}_0/\sqrt{\boldsymbol{\xi}_0 \cdot \boldsymbol{\xi}_0}$. Using this result, the Eq. (55) reduces to:

$$\Theta(p) = \frac{\sum_{n=1}^\infty (-1)^{2S(n+1)} \mathrm{e}^{-nb\cdot p} \mathrm{e}^{n\zeta}\, \mathrm{e}^{-in\boldsymbol{\xi}_0\cdot D^S(\mathbf{J})}}{\sum_{n=1}^\infty (-1)^{2S(n+1)} \mathrm{e}^{-nb\cdot p} \mathrm{e}^{n\zeta}\, \mathrm{tr}\Big(\mathrm{e}^{-in\boldsymbol{\xi}_0\cdot D^S(\mathbf{J})}\Big)}.$$

The spin polarization vector can be readily found from the above equation using the Eq. (10):

$$S^\mu(p) = \sum_{i=1}^3 [p]^\mu_i \mathrm{tr}\Big(\Theta(p) D^S(\mathrm{J}^i)\Big) = \sum_{i=1}^3 [p]^\mu_i \frac{\sum_{n=1}^\infty (-1)^{2S(n+1)} \mathrm{e}^{-nb\cdot p} \mathrm{e}^{n\zeta}\, \mathrm{tr}\Big(\mathrm{e}^{-in\boldsymbol{\xi}_0\cdot D^S(\mathbf{J})} D^S(\mathrm{J}^i)\Big)}{\sum_{n=1}^\infty (-1)^{2S(n+1)} \mathrm{e}^{-nb\cdot p} \mathrm{e}^{n\zeta}\, \mathrm{tr}\Big(\mathrm{e}^{-in\boldsymbol{\xi}_0\cdot D^S(\mathbf{J})}\Big)}. \tag{57}$$

Developing the expression (57) requires some intermediate steps. We can calculate the trace in the denominator by choosing the z axis along $\boldsymbol{\xi}_0$ so that $D^S(\mathrm{J}^3)$ is diagonal:

$$\mathrm{tr}\Big(\mathrm{e}^{-in\boldsymbol{\xi}_0\cdot D^S(\mathbf{J})}\Big) = \sum_{k=-S}^{S} \mathrm{e}^{-ikn\sqrt{-\xi^2}} \equiv \chi_n^S\Big(\sqrt{-\xi^2}\Big). \tag{58}$$

Besides:

$$\mathrm{tr}\Big(\mathrm{e}^{-in\boldsymbol{\xi}_0\cdot D^S(\mathbf{J})} D^S(\mathrm{J}^i)\Big) = \frac{i}{n}\frac{\partial}{\partial \xi_0^i}\mathrm{tr}\Big(\mathrm{e}^{-in\boldsymbol{\xi}_0\cdot D^S(\mathbf{J})}\Big) = \frac{i}{n}\frac{{\xi_0}^i}{\sqrt{-\xi^2}}\frac{\partial \chi_n^S}{\partial\sqrt{-\xi^2}},$$

where we used $\boldsymbol{\xi}_0^2 = -\xi^2$, that follows from Lorentz invariance. The standard boost in Eq. (10) is such that $\sum_{i=1}^3 [p]^\mu_i {\xi_0}^i = \xi^\mu$, hence the polarization vector (57) becomes:

$$S^\mu(p) = \frac{i\xi^\mu}{\sqrt{-\xi^2}} \frac{\sum_{n=1}^\infty (-1)^{2S(n+1)} (1/n) \mathrm{e}^{-nb\cdot p} \mathrm{e}^{n\zeta}\, \frac{\partial \chi_n^S}{\partial\sqrt{-\xi^2}}}{\sum_{n=1}^\infty (-1)^{2S(n+1)} \mathrm{e}^{-nb\cdot p} \mathrm{e}^{n\zeta}\, \chi_n^S\Big(\sqrt{-\xi^2}\Big)}. \tag{59}$$

The series can be rewritten using the Eq. (58):

$$\sum_{n=1}^\infty (-1)^{2S(n+1)} \mathrm{e}^{-nb\cdot p} \mathrm{e}^{n\zeta}\, \chi_n^S\Big(\sqrt{-\xi^2}\Big) = \sum_{n=1}^\infty \sum_{k=-S}^{S} (-1)^{2S(n+1)} \mathrm{e}^{-nb\cdot p + n\zeta - ikn\sqrt{-\xi^2}} = \sum_{k=-S}^{S} \frac{1}{\mathrm{e}^{b\cdot p - \zeta + ik\sqrt{-\xi^2}} - (-1)^{2S}},$$

$$\sum_{n=1}^\infty (-1)^{2S(n+1)} \mathrm{e}^{-nb\cdot p} \mathrm{e}^{n\zeta}\, \frac{1}{n}\frac{\partial \chi_n^S}{\partial\sqrt{-\xi^2}} = \sum_{n=1}^\infty (-1)^{2S(n+1)} \mathrm{e}^{-nb\cdot p + n\zeta} \sum_{k=-S}^{S} -ik\mathrm{e}^{-ikn\sqrt{-\xi^2}} = \sum_{k=-S}^{S} \frac{-ik}{\mathrm{e}^{b\cdot p - \zeta + ik\sqrt{-\xi^2}} - (-1)^{2S}},$$

where, in both cases, the finite sum over k was exchanged with the series in n.

After having resummed the series in n, the analytic continuation is readily done by just mapping $\phi \mapsto i\varpi$ and $\xi \mapsto i\theta$. Therefore, the spin polarization vector for a massive spin-S particle and for real thermal vorticity reads:

$$S^\mu(p) = \frac{\theta^\mu}{\sqrt{-\theta^2}} \frac{\sum_{k=-S}^{S} k\Big[\mathrm{e}^{b\cdot p - \zeta - k\sqrt{-\theta^2}} - (-1)^{2S}\Big]^{-1}}{\sum_{k=-S}^{S} \Big[\mathrm{e}^{b\cdot p - \zeta - k\sqrt{-\theta^2}} - (-1)^{2S}\Big]^{-1}}. \tag{60}$$

The above formula shows that the spin polarization vector at global equilibrium can be expressed as a finite sum of Fermi-Dirac or Bose-Einstein distributions functions, depending on the spin, where $\sqrt{-\theta^2}$ acts as a sort of chemical potential. It can be checked that this expression reduces to (34) for $S = 1/2$. We can also write the spin polarization vector extending Eq. (44) and introducing the polarization function P_S for spin-S fields:

$$S^\mu(p) = S\frac{\theta^\mu}{\sqrt{-\theta^2}} P_S\Big(\sqrt{-\theta^2}, b\cdot p - \zeta\Big), \qquad P_S\Big(\sqrt{-\theta^2}, b\cdot p - \zeta\Big) = \frac{1}{S}\frac{\sum_{k=-S}^{S} k\Big[\mathrm{e}^{b\cdot p - \zeta - k\sqrt{-\theta^2}} - (-1)^{2S}\Big]^{-1}}{\sum_{k=-S}^{S} \Big[\mathrm{e}^{b\cdot p - \zeta - k\sqrt{-\theta^2}} - (-1)^{2S}\Big]^{-1}}. \tag{61}$$

In the limit of small $\sqrt{-\theta^2}$, the Eq. (60) can be approximated by:

$$S^\mu(p) = \theta^\mu(1 + (-1)^{2S} n_{F/B}(b\cdot p - \zeta)) \frac{\sum_{k=-S}^{S} k^2}{\sum_{k=-S}^{S} 1} = \theta^\mu \frac{S(S+1)}{3}(1 + (-1)^{2S} n_{F/B}(b\cdot p - \zeta)),$$

with n_F being the Fermi-Dirac distribution, Eq. (3), and $n_B(b\cdot p - \zeta) = [\exp(b\cdot p - \zeta) - 1]^{-1}$ is the Bose-Einstein distribution. The above result is in agreement with the known linear approximations in thermal vorticity [34].

We can also check the limit of Boltzmann statistics. Since the sum over n corresponds to the quantum statistics expansion, the Boltzmann case is obtained retaining the $n = 1$ term in the series. It is straightforward to see that, after mapping $\xi \mapsto i\theta$, Eq. (59) yields:

$$S^\mu(p) = \frac{\theta^\mu}{\sqrt{-\theta^2}} \frac{\chi_1'\left(\sqrt{-\theta^2}\right)}{\chi_1\left(\sqrt{-\theta^2}\right)},$$

where with χ' we denote the derivative of χ with respect to $\sqrt{-\theta^2}$. This expression is formally the same as that in refs. [35, 36] obtained for a rotating fluid.

5.2 Massless particles

In the massless case, due to Eq. (14), the "Wigner rotation" appearing in (55) is just the Eq. (32):

$$D(W(\Lambda, p))_{hk} = \mathrm{e}^{i\vartheta(\Lambda,p)k}\delta_{hk},$$

where h, k can only be $+S$ or $-S$. The derivation of the angle ϑ associated to a transformation Λ such that $\Lambda p = p$ is reported in appendix C; the result is:

$$\vartheta(\Lambda, p) = \frac{h\cdot p}{q\cdot p} = \eta$$

where η was defined in Eq. (51).

Now we can obtain the spin density matrix and the spin polarization vector. From Eq. (55) one has:

$$\Theta_{hk}(p) = \frac{\sum_{n=1}^{\infty}(-1)^{2S(n+1)}\mathrm{e}^{-nb\cdot p}\mathrm{e}^{in\eta h}\delta_{hk}}{\sum_{n=1}^{\infty}(-1)^{2S(n+1)}\mathrm{e}^{-nb\cdot p} 2\cos n\eta}, \tag{62}$$

whence, by using the Eq. (16), the spin polarization vector is obtained:

$$\Pi^\mu(p) = p^\mu \sum_{h=\pm S} h\Theta_{hh} = ip^\mu S \frac{\sum_{n=1}^{\infty}(-1)^{2S(n+1)}\mathrm{e}^{-nb\cdot p}\sin(n\eta S)}{\sum_{n=1}^{\infty}(-1)^{2S(n+1)}\mathrm{e}^{-nb\cdot p}\cos(n\eta S)},$$

where it has been taken into account that h, k can only take on values $\pm S$, with $S = 1/2, 1, 3/2\ldots$ denoting the magnitude of the helicity of the particle. The above series can be straightforwardly resummed, yielding:

$$\Pi^\mu(p) = ip^\mu S \frac{\sin(\eta S)}{\cos(\eta S) - (-1)^{2S}\mathrm{e}^{-b\cdot p}}.$$

This result can be continued to real thermal vorticity. Mapping $\phi \mapsto i\varpi$ and $\eta \mapsto iH$ we have:

$$\Pi^\mu(p) = -p^\mu S \frac{\sinh(SH)}{\cosh(SH) - (-1)^{2S}\mathrm{e}^{-b\cdot p}} = p^\mu S \frac{\sinh\left(S\boldsymbol{\varphi}\cdot\hat{\mathbf{p}}\right)}{\cosh\left(S\boldsymbol{\varphi}\cdot\hat{\mathbf{p}}\right) - (-1)^{2S}\mathrm{e}^{-b\cdot p}}, \tag{63}$$

where in the last step we have used Eq. (54). The Eq. (63) reproduces Eq. (53) for $S = 1/2$.

6 Application to Λ polarization in heavy ion collisions

In relativistic heavy ion collisions, the theoretical estimates of the spin polarization vector of spin 1/2 hyperons are obtained at linear order in thermal vorticity, see Eq. (2). Even though it is known that thermal vorticity is generally $\ll 1$ throughout the freeze-out hypersurface [21] at high energy, it would be important to have a quantitative assessment the accuracy of this approximation. Indeed, we are in a position to provide such an assessment by comparing the exact formula of the spin polarization vector at global equilibrium - that is with constant thermal vorticity - with its linear approximation.

We can focus on the relative difference between the exact vorticity-induced spin vector Eq. (44) and its linear approximation (47b) in global equilibrium. As the direction of the spin vector is given by θ^μ in both formulae, the relative difference is the same for all components and reads:

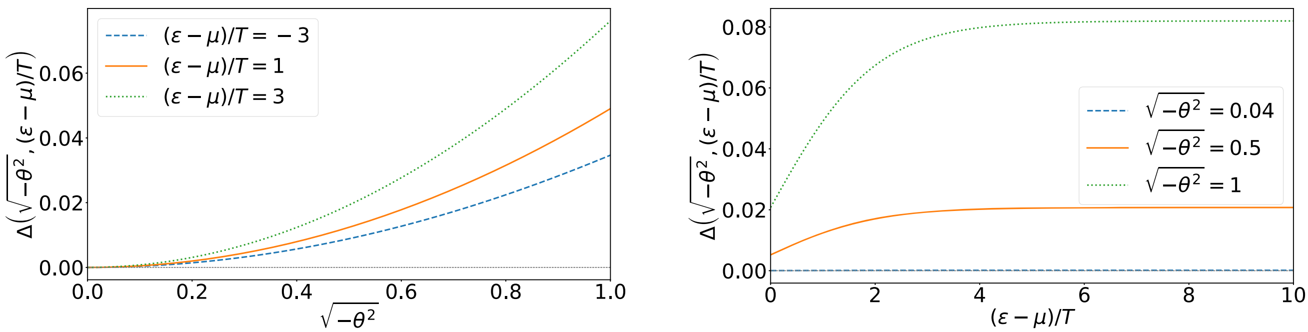

Fig. 1 The behaviour of the relative difference between the exact polarization vector and the linear approximation as a function of vorticity, temperature and energy and chemical potential for $b \cdot p = \varepsilon/T$

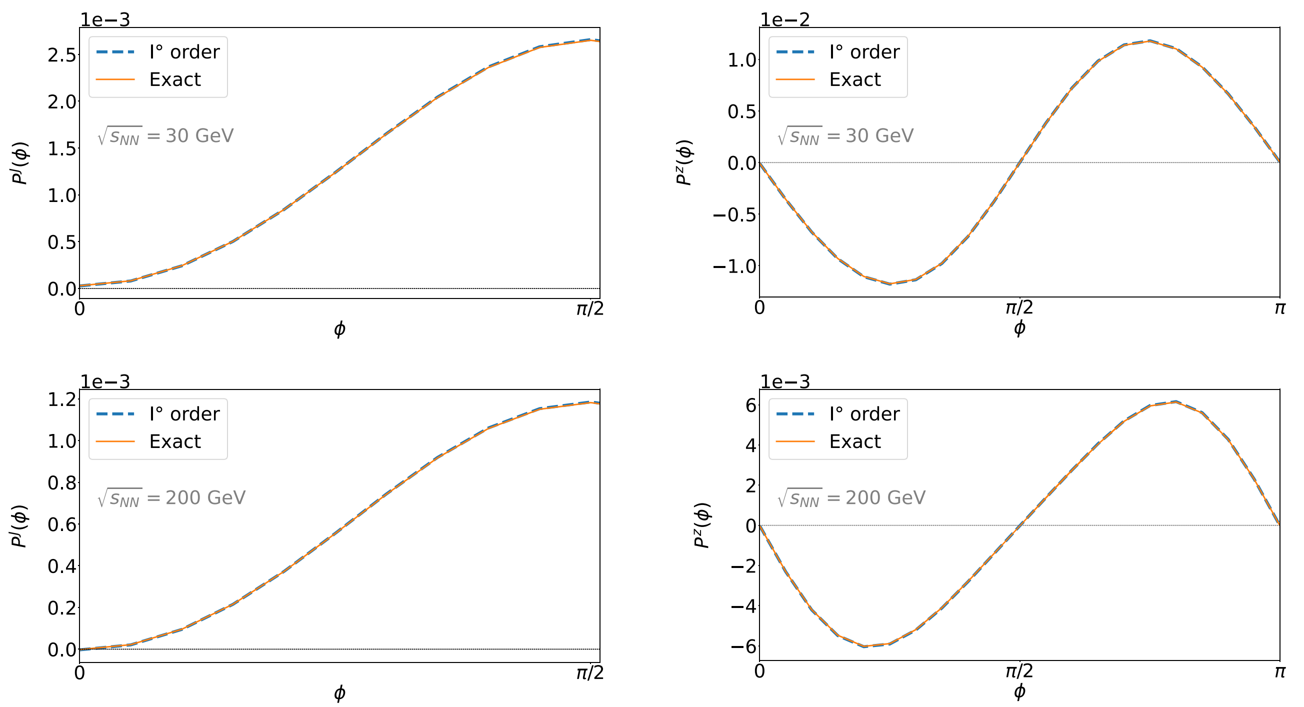

Fig. 2 The components of the polarization vector along the angular momentum, P^J, and the beam axis, P^z, are shown as functions of the azimuthal angle ϕ in the left and right panels respectively. They are calculated at freeze-out in Au-Au collisions at $\sqrt{s_{NN}} = 30$ (upper panels) and $\sqrt{s_{NN}} = 200$ GeV (lower panels)

$$\Delta\left(\sqrt{-\theta^2}, b \cdot p - \zeta\right) = \frac{1}{P_{1/2}\left(\sqrt{-\theta^2}, b \cdot p - \zeta\right)} \left(P_{1/2}\left(\sqrt{-\theta^2}, b \cdot p - \zeta\right) - \frac{\sqrt{-\theta^2}}{2}\left(1 - n_F(b \cdot p - \zeta)\right)\right).$$

In Fig. 1 we show Δ in terms of its arguments, having set $b \cdot p = \varepsilon/T$. It can be seen that, even for $\sqrt{-\theta^2} \sim 1$, the relative difference is less than 10%.

At local equilibrium, thermal vorticity ϖ can be promoted to a local variable, i.e. $\varpi = \varpi(x)$. Extending the formula (2) by using the Eq. (44), the spin polarization vector of the Λ induced by thermal vorticity at freeze-out turns out to be:

$$S^\mu(p) = -\frac{1}{4m}\epsilon^{\mu\nu\rho\sigma} p_\sigma \frac{\int \mathrm{d}\Sigma \cdot p\, n_F \varpi_{\nu\rho}\, P_{1/2}\left(\sqrt{-\theta^2}, \frac{\varepsilon-\mu}{T}\right)/\sqrt{-\theta^2}}{\int \mathrm{d}\Sigma \cdot p\, n_F}. \tag{64}$$

It should be emphasized that the above expression is not the exact spin vector at local equilibrium, as the contributions from thermal shear as well as from dissipative corrections and higher-order derivatives of the thermodynamic fields are not included. Nevertheless, the formula (64) resums all the terms involving thermal vorticity and it is certainly a better approximation than (2).

We have evaluated the Eq. (64) by performing $3 + 1D$ hydrodynamic simulation of Au-Au collisions at $\sqrt{s_{NN}} = 30$ and $\sqrt{s_{NN}} = 200$ GeV with centrality $10 - 60\%$ by using the code vHLLE [37] for the hydrodynamic evolution and the integration over the freeze-out hypersurface. An averaged entropy density profile, generated by GLISSANDO v.2.702 [38], is used as initial state. Only particles with zero rapidity have been taken into account.

Figure 2 shows the comparison between (64) and the Eq. (2), the former being labelled as "Exact" and the latter as "I° order". In particular, as it is customary, we study the azimuthal dependence P^z and P^J, which are the projections of the polarization vector along the beam and the angular momentum directions respectively. We remind the reader that the polarization vector of Dirac fermions is twice the spin vector, $P^\mu = 2S^\mu$.

Figure 2 confirms the expectations from Fig. 1, as the difference between the exact polarization and the linear approximation is tiny for the physical value of thermal vorticity in relativistic heavy ion collisions at $30 < \sqrt{s_{\rm NN}} < 200$ GeV. It is possible that at lower energy, where thermal vorticity is larger [39–43], such corrections may play a more significant role.

7 Summary and conclusions

To summarize, we have derived the analytic formulae of the exact spin polarization vector and spin density matrix for massive and massless free fields at general global equilibrium with non-vanishing thermal vorticity. Our formulae are effectively a resummation of all higher-order corrections in thermal vorticity to the spin polarization vector and the spin density matrix. Furthermore, the unitary polarization bound is fulfilled.

We have developed the basic tools to study the polarization of massless particles, expressing the mean Pauli-Lubanski vector in terms of the spin density matrix and, for spin-1/2 particles, of the Wigner function. In agreement with the expectation, the mean Pauli-Lubanki vector is parallel to the four-momentum of the particle.

We have studied the phenomenological implications of the improved formulae showing that the higher-order corrections to the spin polarization vector in thermal vorticity contribute marginally to the local polarization for $30 < \sqrt{s_{NN}} < 200$ GeV. For collisions with $\sqrt{s_{NN}} \sim 3 - 7$ GeV, where the vorticity is larger, they might be more significant.

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Data Availability Statement the authors declare that the data supporting the findings of this study are available within the paper and its supplementary material.

Appendix A: Traces

To compute the traces appearing in Eqs. (34) and (48), we resort to the techniques used in ref. [44]. First, we define the auxiliary variables:

$$z = \frac{\phi : \phi}{2} + i\frac{\phi : \tilde{\phi}}{2}, \qquad \bar{z} = \frac{\phi : \phi}{2} - i\frac{\phi : \tilde{\phi}}{2}, \tag{A1}$$

where $\widetilde{\phi}$ is the dual of ϕ:

$$\tilde{\phi}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\phi_{\rho\sigma}.$$

It is possible to show that the following identities hold [44]:

$$\begin{aligned}
\mathrm{tr}(\Sigma^{\mu\nu}(\phi : \Sigma)^{2k+1}) &= (\phi^{\mu\nu} + i\tilde{\phi}^{\mu\nu})z^k + (\phi^{\mu\nu} - i\tilde{\phi}^{\mu\nu})\bar{z}^k,\\
\mathrm{tr}(\gamma_5\Sigma^{\mu\nu}(\phi : \Sigma)^{2k+1}) &= (\phi^{\mu\nu} + i\tilde{\phi}^{\mu\nu})z^k - (\phi^{\mu\nu} - i\tilde{\phi}^{\mu\nu})\bar{z}^k,\\
\mathrm{tr}(\Sigma^{\mu\nu}(\phi : \Sigma)^{2k}) &= 0,
\end{aligned}$$

$$\mathrm{tr}(\gamma_5 \Sigma^{\mu\nu}(\phi : \Sigma)^{2k}) = 0.$$

The above equations allow us to determine the traces involved in the calculations of Sect. 4. For example one has:

$$\begin{aligned}
\mathrm{tr}(\exp[-in\phi : \Sigma/2])) &= \sum_{k=0}^{\infty} \frac{n^k(-i)^k}{2^k k!} \phi_{\mu\nu} \mathrm{tr}(\Sigma^{\mu\nu}(\phi : \Sigma)^{k-1}) \\
&= \sum_{k=-1}^{\infty} \frac{n^{2k+2}(-i)^{2k+2}}{2^{2k+2}(2k+2)!} \phi_{\mu\nu} \mathrm{tr}(\Sigma^{\mu\nu}(\phi : \Sigma)^{2k+1}) \\
&= \sum_{k=-1}^{\infty} \frac{n^{2k+2}(-i)^{2k+2}}{2^{2k+1}(2k+2)!} (z^{k+1} + \bar{z}^{k+1}) \\
&= 2\cos\left(\frac{n\sqrt{z}}{2}\right) + 2\cos\left(\frac{n\sqrt{\bar{z}}}{2}\right).
\end{aligned}$$

Similarly:

$$\begin{aligned}
\mathrm{tr}(\gamma_5 \exp[-in\phi : \Sigma/2])) &= 2\cos\left(\frac{n\sqrt{z}}{2}\right) - 2\cos\left(\frac{n\sqrt{\bar{z}}}{2}\right), \\
\mathrm{tr}(\Sigma^{\mu\nu} \exp[-in\phi : \Sigma/2]) &= -i(\phi^{\mu\nu} + i\tilde{\phi}^{\mu\nu}) \frac{\sin(\sqrt{z}/2)}{\sqrt{z}} - i(\phi^{\mu\nu} - i\tilde{\phi}^{\mu\nu}) \frac{\sin(\sqrt{\bar{z}}/2)}{\sqrt{\bar{z}}}, \\
\mathrm{tr}(\gamma_5 \Sigma^{\mu\nu} \exp[-in\phi : \Sigma/2]) &= -i(\phi^{\mu\nu} + i\tilde{\phi}^{\mu\nu}) \frac{\sin(\sqrt{z}/2)}{\sqrt{z}} + i(\phi^{\mu\nu} - i\tilde{\phi}^{\mu\nu}) \frac{\sin(\sqrt{\bar{z}}/2)}{\sqrt{\bar{z}}}.
\end{aligned}$$

These formulae hold for any ϕ, and in the general z and $\bar{z}$ are complex numbers. However, the cases of interest are always such that $\phi : \widetilde{\phi} = 0$ and $z = \bar{z} = \phi : \phi/2$, see for instance Eqs. (37) and (50).

For $z = \bar{z}$ the above traces simplify, and one has:

$$\begin{aligned}
\mathrm{tr}(\exp[-in\phi : \Sigma/2])) &= 4\cos\left(\frac{n\sqrt{z}}{2}\right), \\
\mathrm{tr}(\gamma_5 \exp[-in\phi : \Sigma/2])) &= 0, \\
\mathrm{tr}(\Sigma^{\mu\nu} \exp[-in\phi : \Sigma/2]) &= -2i\phi^{\mu\nu} \frac{\sin(\sqrt{z}/2)}{\sqrt{z}}, \\
\mathrm{tr}(\gamma_5 \Sigma^{\mu\nu} \exp[-in\phi : \Sigma/2]) &= 2\tilde{\phi}^{\mu\nu} \frac{\sin(\sqrt{z}/2)}{\sqrt{z}}.
\end{aligned}$$

Writing the product of two gamma matrices in terms of their commutator and anticommutator, $[\gamma^\mu, \gamma^\nu] = -4i\Sigma^{\mu\nu}$ and $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ respectively, we have:

$$\begin{aligned}
\mathrm{tr}(\gamma^\nu\gamma^\mu \exp[-in\phi : \Sigma/2]) &= g^{\mu\nu}\mathrm{tr}(\exp[-in\phi : \Sigma/2]) + 2i\,\mathrm{tr}(\Sigma^{\mu\nu} \exp[-in\phi : \Sigma/2]), \\
\mathrm{tr}(\gamma^\nu\gamma^\mu\gamma_5 \exp[-in\phi : \Sigma/2]) &= g^{\mu\nu}\mathrm{tr}(\gamma_5 \exp[-in\phi : \Sigma/2]) + 2i\,\mathrm{tr}(\Sigma^{\mu\nu}\gamma_5 \exp[-in\phi : \Sigma/2]),
\end{aligned}$$

and finally we find:

$$\mathrm{tr}(\exp[-in\phi : \Sigma/2])) = 4\cos\left(\frac{n\sqrt{z}}{2}\right), \tag{A3a}$$

$$\mathrm{tr}(\gamma^\nu\gamma^\mu \exp[-in\phi : \Sigma/2]) = 4g^{\mu\nu}\cos\left(\frac{n\sqrt{z}}{2}\right) + 4\phi^{\mu\nu}\frac{\sin(\sqrt{z}/2)}{\sqrt{z}}, \tag{A3b}$$

$$\mathrm{tr}(\gamma^\nu\gamma^\mu\gamma_5 \exp[-in\phi : \Sigma/2]) = 4i\tilde{\phi}^{\mu\nu}\frac{\sin(\sqrt{z}/2)}{\sqrt{z}}, \tag{A3c}$$

which are precisely the identities (40).

Appendix B: Little group for massless particles

In this section, the form of the tensor ϕ fulfilling $\phi^{\mu\nu}p_\nu = 0$ for a light-like momentum p is obtained. To begin with, we find the most general decomposition of an anti-symmetric tensor using the the basis $\{p, q, n_1, n_2\}$, with $p^2 = q^2 = 0$, which has been introduced in Sect. 2. It can be shown that two vectors h and y exist, with $h \cdot q = 0$ and $y \cdot p = 0$, such that:

$$\phi^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma}\frac{h_\rho p_\sigma}{p\cdot q} + y^\mu q^\nu - y^\nu q^\mu.$$

Their existence can be proved by inverting the above relation. Indeed, contracting $\phi^{\mu\nu}$ with p_ν and taking into account that $y \cdot p = 0$:

$$\phi^{\mu\nu} p_\nu = y^\mu p\cdot q \Rightarrow y^\mu = \frac{\phi^{\mu\nu} p_\nu}{p\cdot q},$$

what is consistent with the requirement $y \cdot p = 0$. Furthermore, if $h \cdot q = 0$:

$$\epsilon_{\mu\nu\alpha\beta}\phi^{\mu\nu} q^\alpha = \epsilon_{\mu\nu\alpha\beta}\epsilon^{\mu\nu\rho\sigma}\frac{h_\rho p_\sigma}{p\cdot q} q^\alpha = -2(\delta^\rho_\alpha\delta^\sigma_\beta - \delta^\rho_\beta\delta^\sigma_\alpha)h_\rho\frac{p_\sigma q^\alpha}{p\cdot q} = 2h_\beta,$$

which leads to:

$$h^\mu = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\phi_{\nu\rho}q_\sigma,$$

which is again consistent with the requirement $h \cdot q = 0$.

Using the above decompostion, the solution of $\phi^{\mu\nu}p_\nu = 0$ simply yields $y = 0$, which implies that the little group of massless particles is generated by tensors ϕ parametrized as:

$$\phi^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma}\frac{h_\rho p_\sigma}{p\cdot q},$$

with $h \cdot q = 0$, that is the Eq. (49).

Appendix C: The little-group transformations with $\Lambda p = p$

Our goal is to calculate the Wigner rotation, for massive and massless particles, for Lorentz transformations such that $\Lambda p = p$. For this set of transformations, the Wigner rotation can be written as follows:

$$W(\Lambda, p) = [\Lambda p]^{-1}\Lambda[p] = [p]^{-1}\Lambda[p] = \exp\left[-i\frac{\phi_{\mu\nu}}{2}[p]^{-1}J^{\mu\nu}[p]\right] = \exp\left[-i\frac{\phi_{0\mu\nu}}{2}J^{\mu\nu}\right], \tag{C1}$$

where we have used the condition $\Lambda p = p$, the transformation rules of the generators $J^{\mu\nu}$, and denoted

$$\phi_{0\mu\nu} = \phi_{\rho\sigma}[p]^\rho{}_\mu[p]^\sigma{}_\nu \quad .$$

For massive particles, using the parametrization (37) and the properties of the Levi-Civita symbol, one has:

$$\phi_{0\mu\nu} = \phi_{\rho\sigma}[p]^\rho{}_\mu[p]^\sigma{}_\nu = \frac{1}{m}\underbrace{\epsilon_{\rho\sigma\lambda\tau}[p]^\rho{}_\mu[p]^\sigma{}_\nu[p]^\lambda{}_{\lambda'}[p]^\tau{}_{\tau'}}_{=\epsilon_{\mu\nu\lambda'\tau'}}[p]^{-1\lambda'}{}_\alpha[p]^{-1\tau'}{}_\gamma\xi^\alpha p^\gamma = \frac{1}{m}\epsilon_{\mu\nu\rho\sigma}\xi_0{}^\rho\mathfrak{p}^\sigma, \tag{C2}$$

where $\xi_0^\mu = [p]^{-1\mu}{}_\nu\xi^\nu$, the vector ξ being defined by (36), and $\mathfrak{p}^\mu = [p]^{-1\mu}{}_\nu p^\nu$. Plugging this expression in (C1) and going to the representation D^S of the rotation group, one obtains:

$$D^S(W(\Lambda,\ p)) = \exp\left[-i\frac{\phi_{0\mu\nu}}{2}D^S(J^{\mu\nu})\right] = \exp[-i\boldsymbol{\xi}_\mathbf{0}\cdot D^S(\mathbf{J})],$$

which gives Eq. (56).

We now move to the massless case. Similarly as in (C2) and using the parametrization (49), the tensor ϕ_0 reads:

$$\phi_0^{\mu\nu} = \frac{1}{\mathfrak{p}\cdot\mathfrak{q}}\epsilon^{\mu\nu\rho\sigma}\mathfrak{h}_\rho\mathfrak{q}_\sigma,$$

where $\mathfrak{h}^\mu = [p]^{-1\mu}{}_\nu h^\nu$, the vector h being defined in (49). Using (C1) and the decomposition (49):

$$W(\Lambda,\ p) = \exp\left[-i\frac{\phi_0 : J}{2}\right] = \exp\left[-i\frac{\mathfrak{h}\cdot\Pi(\mathfrak{p})}{\mathfrak{q}\cdot\mathfrak{p}}\right] = \exp\left[i\frac{\mathfrak{h}\cdot\mathfrak{p}}{\mathfrak{q}\cdot\mathfrak{p}}h(\mathfrak{p}) + i\frac{\mathfrak{h}\cdot\mathfrak{n}_1}{\mathfrak{q}\cdot\mathfrak{p}}\Pi_1(\mathfrak{p}) + i\frac{\mathfrak{h}\cdot\mathfrak{n}_2}{\mathfrak{q}\cdot\mathfrak{p}}\Pi_2(\mathfrak{p})\right].$$

The generator $h(\mathfrak{p})$ coincides with J^3 (see Sect. 2). Since, according to the algebra (13), the commutator between h and $\Pi_{1,2}$ can be written in terms of $\Pi_{1,2}$, it can be realized that the Baker-Cambpell-Haussdorf formula for the factorization of exponentials of operators implies the existence of a_1, a_2 such that:

$$\exp\left[i\frac{\mathfrak{h}\cdot\mathfrak{p}}{\mathfrak{q}\cdot\mathfrak{p}}h(\mathfrak{p}) + i\frac{\mathfrak{h}\cdot\mathfrak{n}_1}{\mathfrak{q}\cdot\mathfrak{p}}\Pi_1(\mathfrak{p}) + i\frac{\mathfrak{h}\cdot\mathfrak{n}_2}{\mathfrak{q}\cdot\mathfrak{p}}\Pi_2(\mathfrak{p})\right] = \exp[ia_1\Pi_1(\mathfrak{p}) + ia_2\Pi_2(\mathfrak{p})]\exp\left[i\frac{\mathfrak{h}\cdot\mathfrak{p}}{\mathfrak{q}\cdot\mathfrak{p}}h(\mathfrak{p})\right],$$

The representation of such transformation onto the Hilbert space of on-shell states (see Sect. 2) is such that the first exponential is the identity due to (14), and only the rightmost exponential contributes. We thus have Eq. (32) with:

$$\vartheta(\Lambda,\ p) = \frac{\mathfrak{h}\cdot\mathfrak{p}}{\mathfrak{q}\cdot\mathfrak{p}} = \frac{h\cdot p}{q\cdot p} = \eta$$

where we used the definition (51).

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