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Questa è la versione Preprint (Submitted version) della seguente pubblicazione:

Original Citation:
Propagation of smallness and size estimate in the second order elliptic equation with discontinuous complex Lipschitz conductivity / Elisa Francini, Sergio Vessella, Jenn-Nan Wang. - In: JOURNAL OF DIFFERENTIAL EQUATIONS. - ISSN 0022-0396. - STAMPA. - 343:(2023), pp. 687-717. [10.1016/j.jde.2022.10.028]

Availability:
This version is available at: 2158/1286969 since: 2022-10-29T08:51:09Z

Published version:
DOI: 10.1016/j.jde.2022.10.028

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09 August 2024

arXiv:2007.01094v1 [math.AP] 2 Jul 2020

Propagation of smallness and size estimate in the second order elliptic equation with discontinuous complex Lipschitz conductivity

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July 3, 2020

Abstract

In this paper, we would like to derive three-ball inequalities and propagation of smallness for the complex second order elliptic equation with discontinuous Lipschitz coefficients. As an application of such estimates, we study the size estimate problem by one pair of Cauchy data on the boundary. The main ingredient in the derivation of three-ball inequalities and propagation of smallness is a local Carleman proved in our recent paper [FVW].

1 Introduction

In our recent paper [FVW], we derived a Carleman estimate for the second order elliptic equation with piecewise complex-valued Lipschitz coefficients. The theme of this paper is to prove some interesting results based on the Carleman estimate obtained in [FVW]. The ultimate goal is to give upper and lower bounds of the size of the inclusion embedded inside of a conductive body with discontinuous complex conductivity by only one pair of boundary measurements. A typical application of this study is to estimate the size of a cancerous tumor inside an organ by the electric impedance tomography (EIT).

For the conductivity equation with piecewise real Lipschitz coefficients, the same size estimate problem was considered in [FLVW]. We want to point out that, in many real world problems, the case of complex-valued coefficients arises naturally. The modeling of the current flows in biological tissues or the propagation of the electromagnetic waves in conductive media are typical examples. In these cases, the conductivities are complex-valued functions. On the other hand, in some situations,

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the conductivities are not continuous functions. In the human body, different organs have different conductivities. For instance, the conductivities of heart, liver, intestines are 0.70 (S/m), 0.10 (S/m), 0.03 (S/m), respectively. Therefore, to model the current flow in the human body, it is more reasonable to consider an anisotropic complex-valued conductivity with jump-type discontinuities [MPH].

In the size estimate problem studied in [FLVW], the essential tool is a three-region inequality which is obtained by applying the Carleman estimate for the second order elliptic equation with piecewise real Lipschitz coefficients derived in [DFLVW]. Since we have the similar Carleman estimate available for the case of piecewise complex-valued Lipschitz coefficients, we can proceed the method used in [FLVW] to prove the three-region inequality. In treating the size estimate problem, the three-region inequality is enough since one only needs to cross the interface once in propagating the information in the interior to the boundary. However, the three-region inequality is inconvenient in deriving the general propagation of smallness. Therefore, in this paper, we want to derive the usual three-ball inequality for the complex second order elliptic operator even when the coefficients are piecewise Lipschitz. We will follow the ideas outlined in [CW] where the three-ball inequality was proved for the real second order elliptic operator with piecewise Lipschitz coefficients. We then apply the three-ball inequality to derive a general propagation of smallness for the second order elliptic equation with piecewise complex Lipschitz coefficients.

We would like to raise an issue in the investigation of the size estimate problem when the background medium is complex valued. Following the method used in [ARS], an important step is to derive certain energy inequalities controlling the power gap. It was noted in [BFV] that when the current flows inside and outside of the inclusion obey the usual Ohm's law (the relation between current and voltage is linear) and the imaginary part of the conductivity outside of the inclusion is a nonzero variable function, energy inequalities (5.9), (5.10) are not likely to hold. Precisely, in this case, an example with $\delta W = 0$, but $|D| \neq 0$, is constructed in [BFV]. On the other hand, if both conductivities inside and outside of the inclusion are complex constants or the conductivity outside of the inclusion is real valued, then energy inequalities (5.9), (5.10) were obtained in [BFV].

A key observation found in [CNW] is that if the current flow inside of the inclusion obeys certain nonlinear Ohm's law, we can restore the energy inequalities (5.9), (5.10). In particular, our size estimate result applies to the case of a non-chiral medium with a chiral inclusion having real valued chirality.

The paper is organized as follows. In Section 2, we introduce some notations and state the Carleman estimate proved in [FVW]. In Section 3, we plan to prove a three-region inequality across the interface based on the Carleman estimate given in Section 2. We then combine the classical three-ball inequality and the three-region inequality to derive a three-ball inequality in Section 4. There, we also prove the propagation of smallness. Finally, we study the size estimate problem in Section 5.

2 Notations and Carleman estimate

In this section, we will state the Carleman estimate proved in [FVW] where the interface is assumed to be flat. Since our Carleman estimate is local near any point at the interface, for a general $C^{1,1}$ interface, it can be flatten by a suitable change of coordinates. Moreover, the transformed coefficients away from the interface remain Lipschitz. Define $H_\pm = \chi_{\mathbb{R}^n_\pm}$ where $\mathbb{R}^n_\pm = \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} | x_n \gtrless 0\}$ and $\chi_{\mathbb{R}^n_\pm}$ is the characteristic function of $\mathbb{R}^n_\pm$. In places we will use equivalently the symbols ∂, ∇ and $D = -i\nabla$ to denote the gradient of a function and we will add the index x' or x_n to denote gradient in $\mathbb{R}^{n-1}$ and the derivative with respect to x_n respectively. We further denote $\partial_\ell = \partial/\partial x_\ell$, $D_\ell = -i\partial_\ell$, and $\partial_{\xi_\ell} = \partial/\partial \xi_\ell$.

Let $u_\pm \in C^\infty(\mathbb{R}^n)$. We define

$$u = H_+ u_+ + H_- u_- = \sum_\pm H_\pm u_\pm,$$

hereafter, we denote $\sum_\pm a_\pm = a_+ + a_-$, and

$$\mathcal{L}(x, D)u := \sum_\pm H_\pm \mathrm{div}(A_\pm(x)\nabla u_\pm),$$

where

$$A_\pm(x) = \{a^\pm_{\ell j}(x)\}^n_{\ell,j=1} = \{a^\pm_{\ell j}(x', x_n)\}^n_{\ell,j=1}, \quad x' \in \mathbb{R}^{n-1}, x_n \in \mathbb{R} \tag{2.1}$$

is a Lipschitz symmetric matrix-valued function. Assume that

$$a^\pm_{\ell j}(x) = a^\pm_{j\ell}(x), \quad \forall \ \ell, j = 1, \cdots, n, \tag{2.2}$$

and furthermore

$$a^\pm_{\ell j}(x) = M^\pm_{\ell j}(x) + i\gamma N^\pm_{\ell j}(x), \tag{2.3}$$

where $(M^\pm_{\ell j})$ and $(N^\pm_{\ell j})$ are real-valued matrices and $\gamma > 0$. We further assume that there exists $\lambda_0 > 0$ such that for all $\xi \in \mathbb{R}^n$ and $x \in \mathbb{R}^n$ we have

$$\lambda_0 |\xi|^2 \le M^\pm(x)\xi \cdot \xi \le \lambda_0^{-1}|\xi|^2 \tag{2.4}$$

and

$$\lambda_0 |\xi|^2 \le N^\pm(x)\xi \cdot \xi \le \lambda_0^{-1}|\xi|^2. \tag{2.5}$$

In the paper, we consider Lipschitz coefficients $A_\pm$, i.e., there exists a constant $M_0 > 0$ such that

$$|A_\pm(x) - A_\pm(y)| \le M_0|x - y|. \tag{2.6}$$

To treat the transmission conditions, we write

$$h_0(x') := u_+(x', 0) - u_-(x', 0), \ \forall \ x' \in \mathbb{R}^{n-1}, \tag{2.7}$$

$$h_1(x') := A_+(x', 0)\nabla u_+(x', 0) \cdot \nu - A_-(x', 0)\nabla u_-(x', 0) \cdot \nu, \ \forall \ x' \in \mathbb{R}^{n-1}, \tag{2.8}$$

where $\nu = e_n$.

Let us now introduce the weight function. Let φ be

$$\varphi(x_n) = \begin{cases} \varphi_+(x_n) := \alpha_+ x_n + \beta x_n^2/2, & x_n \geq 0, \\ \varphi_-(x_n) := \alpha_- x_n + \beta x_n^2/2, & x_n < 0, \end{cases} \tag{2.9}$$

where α_+, α_- and β are positive numbers which will be determined later. In what follows we denote by φ_+ and φ_- the restriction of the weight function φ to $[0, +\infty)$ and to $(-\infty, 0)$ respectively. We use similar notation for any other weight functions. For any $\varepsilon > 0$ let

$$\psi_\varepsilon(x) := \varphi(x_n) - \frac{\varepsilon}{2}|x'|^2, \tag{2.10}$$

and let

$$\phi_\delta(x) := \psi_\delta(\delta^{-1}x), \quad \delta > 0. \tag{2.11}$$

For a function $h \in L^2(\mathbb{R}^n)$, we define

$$\hat{h}(\xi', x_n) = \int_{\mathbb{R}^{n-1}} h(x', x_n) e^{-ix'\cdot\xi}\, dx', \quad \xi' \in \mathbb{R}^{n-1}.$$

As usual we denote by $H^{1/2}(\mathbb{R}^{n-1})$ the space of the functions $f \in L^2(\mathbb{R}^{n-1})$ satisfying

$$\int_{\mathbb{R}^{n-1}} |\xi'||\hat{f}(\xi')|^2 d\xi' < \infty,$$

with the norm

$$\|f\|^2_{H^{1/2}(\mathbb{R}^{n-1})} = \int_{\mathbb{R}^{n-1}} (1 + |\xi'|^2)^{1/2} |\hat{f}(\xi')|^2 d\xi'. \tag{2.12}$$

Moreover we define

$$[f]_{1/2,\mathbb{R}^{n-1}} = \left[\int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \frac{|f(x) - f(y)|^2}{|x-y|^n} dy dx \right]^{1/2},$$

and recall that there is a positive constant C, depending only on n, such that

$$C^{-1} \int_{\mathbb{R}^{n-1}} |\xi'||\hat{f}(\xi')|^2 d\xi' \leq [f]^2_{1/2,\mathbb{R}^{n-1}} \leq C \int_{\mathbb{R}^{n-1}} |\xi'||\hat{f}(\xi')|^2 d\xi',$$

so that the norm (2.12) is equivalent to the norm $\|f\|_{L^2(\mathbb{R}^{n-1})} + [f]_{1/2,\mathbb{R}^{n-1}}$. We use the letters $C, C_0, C_1, \cdots$ to denote constants. The value of the constants may change from line to line, but it is always greater than 1.

We will denote by $B'_r(x')$ the $(n-1)$-ball centered at $x' \in \mathbb{R}^{n-1}$ with radius $r > 0$. Whenever $x' = 0$ we denote $B'_r = B'_r(0)$. Likewise, we denote $B_r(x)$ be the n-ball centered at $x \in \mathbb{R}^n$ with radius $r > 0$ and $B_r = B_r(0)$.

Theorem 2.1 *Let $A_\pm(x)$ satisfy (2.1)-(2.6). There exist $\alpha_+, \alpha_-, \beta, \delta_0, r_0, \gamma_0$, τ_0, C depending on λ_0, M_0 such that if $\gamma \le \gamma_0$, $\delta \le \delta_0$ and $\tau \ge \tau_0$, then*

$$\begin{aligned}
&\sum_{\pm}\sum_{k=0}^{2}\tau^{3-2k}\int_{\mathbb{R}^n_\pm}|D^k u_\pm|^2 e^{2\tau\phi_{\delta,\pm}(x',x_n)}dx'dx_n+\sum_{\pm}\sum_{k=0}^{1}\tau^{3-2k}\int_{\mathbb{R}^{n-1}}|D^k u_\pm(x',0)|^2 e^{2\phi_\delta(x',0)}dx'\\
&+\sum_{\pm}\tau^2[e^{\tau\phi_\delta(\cdot,0)}u_\pm(\cdot,0)]^2_{1/2,\mathbb{R}^{n-1}}+\sum_{\pm}[D(e^{\tau\phi_{\delta,\pm}}u_\pm)(\cdot,0)]^2_{1/2,\mathbb{R}^{n-1}}\\
\le& C\left(\sum_{\pm}\int_{\mathbb{R}^n_\pm}|\mathcal{L}(x,D)(u_\pm)|^2\, e^{2\tau\phi_{\delta,\pm}(x',x_n)}dx'dx_n+[e^{\tau\phi_\delta(\cdot,0)}h_1]^2_{1/2,\mathbb{R}^{n-1}}\right.\\
&\left.+[D_{x'}(e^{\tau\phi_\delta}h_0)(\cdot,0)]^2_{1/2,\mathbb{R}^{n-1}}+\tau^3\int_{\mathbb{R}^{n-1}}|h_0|^2e^{2\tau\phi_\delta(x',0)}dx'+\tau\int_{\mathbb{R}^{n-1}}|h_1|^2e^{2\tau\phi_\delta(x',0)}dx'\right).
\end{aligned}\tag{2.13}$$

where $u = H_+u_+ + H_-u_-$, $u_\pm \in C^\infty(\mathbb{R}^n)$ and $\operatorname{supp} u \subset B'_{\delta r_0}\times[-\delta r_0, \delta r_0]$, and ϕ_δ is given by (2.11).

Remark 2.1 *In view of the proof of Theorem 2.1 in [FVW], the coefficients α_+, α_- are required to satisfy*

$$\frac{\alpha_+}{\alpha_-} \ge \kappa_0 > 1, \tag{2.14}$$

where κ_0 is general constant depending on the values of $A_\pm(0)$ at the interface.

Remark 2.2 *It is clear that (2.13) remains valid if can add lower order terms $\sum_\pm H_\pm(W\cdot\nabla u_\pm + Vu_\pm)$, where W, V are bounded functions, to the operator $\mathcal{L}$. That is, one can substitute*

$$\mathcal{L}(x,D)u = \sum_{\pm}H_\pm \mathrm{div}(A_\pm(x)\nabla u_\pm)+\sum_{\pm}H_\pm(W\cdot\nabla u_\pm + Vu_\pm) \tag{2.15}$$

in (2.13).

3 Three-region inequalities

Based on the Carleman estimate given in Theorem 2.1, we will derive three-region inequalities across the interface $x_n = 0$. Here we consider $u = H_+u_+ + H_-u_-$ satisfying

$$\mathcal{L}(x,D)u = 0 \quad \text{in} \quad \mathbb{R}^n, \tag{3.1}$$

where $\mathcal{L}$ is given in (2.15) and

$$\|W\|_{L^\infty(\mathbb{R}^n)} + \|V\|_{L^\infty(\mathbb{R}^n)} \le \lambda_0^{-1}.$$

Now all coefficients $\alpha_\pm$, β, δ_0, γ_0, r_0, τ_0 have been determined in Theorem 2.1.

Theorem 3.1 *Let u be a solution of (3.1) and $A_\pm(x)$ satisfy (2.1)-(2.6) with $h_0 = h_1 = 0$. Moreover, the constant γ in (2.3) satisfies $\gamma \le \gamma_0$ with γ_0 given in Theorem 2.1. Then there exist C and R, depending only on λ_0, M_0, n, such that if $0 < R_1, R_2 \le R$, then*

$$\int_{U_2} |u|^2 dx \le (e^{\tau_0 R_2} + C R_1^{-4}) \left(\int_{U_1} |u|^2 dx \right)^{\frac{R_2}{2R_1+3R_2}} \left(\int_{U_3} |u|^2 dx \right)^{\frac{2R_1+2R_2}{2R_1+3R_2}}, \tag{3.2}$$

where

$$U_1 = \left\{ z \ge -4R_2,\ \frac{R_1}{8a} < x_n < \frac{R_1}{a} \right\},$$
$$U_2 = \left\{ -R_2 \le z \le \frac{R_1}{2a},\ x_n < \frac{R_1}{8a} \right\},$$
$$U_3 = \left\{ z \ge -4R_2,\ x_n < \frac{R_1}{a} \right\},$$

$a = \alpha_+/\delta$,

$$z(x) = \frac{\alpha_- x_n}{\delta} + \frac{\beta x_n^2}{2\delta^2} - \frac{|x'|^2}{2\delta}, \tag{3.3}$$

and any $\delta \le \delta_0$.

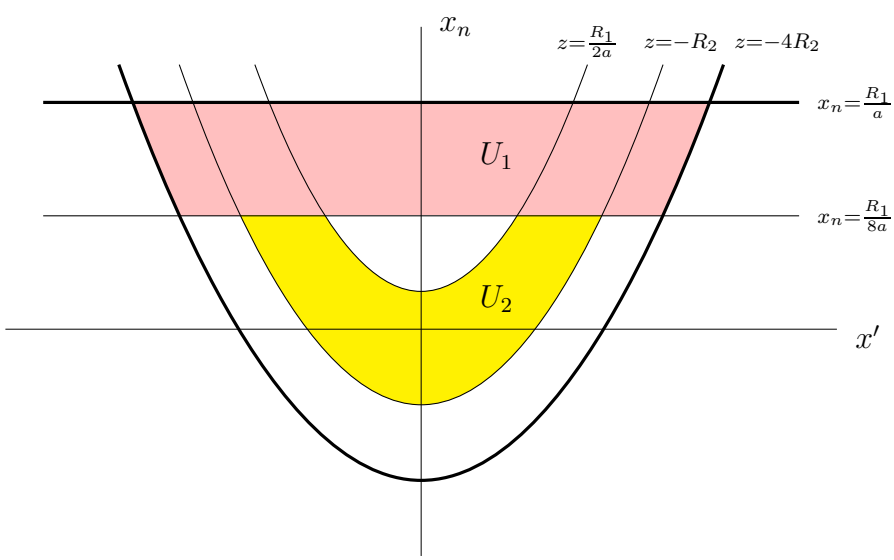

Figure 1: U_1 and U_2 are shown in pink and yellow, respectively. U_3 is the region enclosed by black boundaries. Note that since z is hyperbolic, there are parts similar to U_2 and U_3 lying below $x_n < -\alpha_-\delta/\beta$. Here we are only interested in the solution near $x_n = 0$. Thus we consider the cut-off function relative to U_2 and U_3 as in the figure.

Proof. Here we adopt the proof given in [FLVW]. To apply the estimate (2.13), we need to ensure that u satisfies the support condition. Let $r > 0$ be chosen satisfying

$$r \leq \min\left\{r_0^2, \frac{13\alpha_-}{8\beta}, \frac{2\delta r_0}{19\alpha_- + 8\beta}\right\}. \tag{3.4}$$

We then set

$$R = \frac{\alpha_- r}{16}.$$

It follows from (3.4) that

$$R \leq \frac{13\alpha_-^2}{128\beta}. \tag{3.5}$$

Given $0 < R_1 < R_2 \leq R$. Let $\vartheta_1(t) \in C_0^\infty(\mathbb{R})$ satisfy $0 \leq \vartheta_1(t) \leq 1$ and

$$\vartheta_1(t) = \begin{cases} 1, & t > -2R_2, \\ 0, & t \leq -3R_2. \end{cases}$$

Also, define $\vartheta_2(t) \in C_0^\infty(\mathbb{R})$ satisfying $0 \leq \vartheta_2(t) \leq 1$ and

$$\vartheta_2(t) = \begin{cases} 0, & t \geq \dfrac{R_1}{2a}, \\ 1, & t < \dfrac{R_1}{4a}. \end{cases}$$

Finally, we define $\vartheta(x) = \vartheta(x', x_n) = \vartheta_1(z(x))\vartheta_2(x_n)$, where z is defined by (3.3).

We now check the support condition for ϑ. From its definition, we can see that $\operatorname{supp}\vartheta$ is contained in

$$\begin{cases} z(x) = \dfrac{\alpha_- x_n}{\delta} + \dfrac{\beta x_n^2}{2\delta^2} - \dfrac{|x'|^2}{2\delta} > -3R_2, \\ x_n < \dfrac{R_1}{2a}. \end{cases} \tag{3.6}$$

In view of the relation

$$\alpha_+ > \alpha_- \quad (\text{see } (2.14)) \quad \text{and} \quad a = \frac{\alpha_+}{\delta},$$

we have that

$$\frac{R_1}{2a} < \frac{\delta}{2\alpha_-} \cdot R_1 < \frac{\delta}{\alpha_-} \cdot \frac{\alpha_- r}{16} < \delta r,$$

i.e., $x_n < \delta r \leq \delta r_0^2 \leq \delta r_0$. Next, we observe that

$$-3R_2 > -3R = -\frac{3\alpha_- r}{16} > \frac{\alpha_-}{\delta}(-\delta r) + \frac{\beta}{2\delta^2}(-\delta r)^2,$$

which gives $-\delta r_0 < -\delta r < x_n$ due to (3.4). Consequently, we verify that $|x_n| < \delta r < \delta r_0$. One the other hand, from the first condition of (3.6) and (3.4), we see that

$$\begin{aligned}\frac{|x'|^2}{2\delta} &< 3R_2 + \frac{\alpha_- x_n}{\delta} + \frac{\beta x_n^2}{2\delta^2} \le \frac{3\alpha_- r}{16} + \frac{\alpha_-}{\delta}\cdot \delta r + \frac{\beta}{2\delta^2}\cdot \delta^2 r^2 \\ &\le \frac{19\alpha_- + 8\beta}{16} r \le \frac{19\alpha_- + 8\beta}{16} r_0^2 \le \frac{\delta}{8} r_0^2,\end{aligned}$$

which gives $|x'| < \delta r_0/2$.

Since $h_0 = 0$, we have that

$$\vartheta(x',0)u_+(x',0) - \vartheta(x',0)u_-(x',0) = 0,\ \forall\, x' \in \mathbb{R}^{n-1}. \tag{3.7}$$

Applying (2.13) to ϑu and using (3.7) yields

$$\begin{aligned}&\sum_{\pm}\sum_{|k|=0}^{2} \tau^{3-2|k|} \int_{\mathbb{R}^n_\pm} |D^k(\vartheta u_\pm)|^2 e^{2\tau\phi_{\delta,\pm}(x',x_n)} dx' dx_n \\ \le& C\sum_{\pm}\int_{\mathbb{R}^n_\pm} |\mathcal{L}(x,D)(\vartheta u_\pm)|^2\, e^{2\tau\phi_{\delta,\pm}(x',x_n)} dx' dx_n \\ &+ C\tau \int_{\mathbb{R}^{n-1}} |A_+(x',0)\nabla(\vartheta u_+(x',0))\cdot\nu - A_-(x',0)\nabla(\vartheta u_-)(x',0)\cdot\nu|^2 e^{2\tau\phi_\delta(x',0)} dx' \\ &+ C[e^{\tau\phi_\delta(x',0)}\big(A_+(x',0)\nabla(\vartheta u_+)(x',0)\cdot\nu - A_-(x',0)\nabla(\vartheta u_-)(x',0)\cdot\nu\big)]^2_{1/2,\mathbb{R}^{n-1}}.\end{aligned} \tag{3.8}$$

We now observe that $\nabla\vartheta_1(z) = \vartheta_1'(z)\nabla z = \vartheta_1'(z)(-\frac{x'}{\delta}, \frac{\alpha_-}{\delta} + \frac{\beta x_n}{\delta^2})$ and it is nonzero only when

$$-3R_2 < z < -2R_2.$$

Therefore, when $x_n = 0$, we have

$$2R_2 < \frac{|x'|^2}{2\delta} < 3R_2.$$

Thus, we can see that

$$|\nabla\vartheta(x',0)|^2 \le CR_2^{-2}\left(\frac{6R_2}{\delta} + \frac{\alpha_-^2}{\delta^2}\right) \le CR_2^{-2}. \tag{3.9}$$

By $h_0(x') = h_1(x') = 0$, (3.9), and the easy estimate of [DFLVW, Proposition 4.2],

we can estimate

$$
\begin{aligned}
&\tau \int_{\mathbb{R}^{n-1}} |A_+(x',0)\nabla(\vartheta u_+(x',0))\cdot\nu - A_-(x',0)\nabla(\vartheta u_-)(x',0)\cdot\nu|^2 e^{2\tau\phi_\delta(x',0)}dx' \\
&\quad + [e^{\tau\phi_\delta(x',0)}\big(A_+(x',0)\nabla(\vartheta u_+)(x',0)\cdot\nu - A_-(x',0)\nabla(\vartheta u_-)(x',0)\cdot\nu\big)]^2_{1/2,\mathbb{R}^{n-1}} \\
&\le CR_2^{-2}e^{-4\tau R_2}\left(\tau\int_{\{\sqrt{4\delta R_2}\le|x'|\le\sqrt{6\delta R_2}\}} |u_+(x',0)|^2dx' + [u_+(x',0)]^2_{1/2,\{\sqrt{4\delta R_2}\le|x'|\le\sqrt{6\delta R_2}\}}\right) \\
&\quad + C\tau^2R_2^{-3}e^{-4\tau R_2}\int_{\{\sqrt{4\delta R_2}\le|x'|\le\sqrt{6\delta R_2}\}} |u_+(x',0)|^2dx' \\
&\le C\tau^2R_2^{-3}e^{-4\tau R_2}E,
\end{aligned}
\tag{3.10}
$$

where

$$
E = \int_{\{\sqrt{4\delta R_2}\le|x'|\le\sqrt{6\delta R_2}\}} |u_+(x',0)|^2dx' + [u_+(x',0)]^2_{1/2,\{\sqrt{4\delta R_2}\le|x'|\le\sqrt{6\delta R_2}\}}.
$$

Writing out $\mathcal{L}(x,D)(\vartheta u_\pm)$ and considering the set where $\nabla\vartheta\neq 0$, it is not hard to estimate

$$
\begin{aligned}
&\sum_{\pm}\sum_{|k|=0}^{1}\tau^{3-2|k|}\int_{\{-2R_2\le z\le\frac{R_1}{2a},\, x_n<\frac{R_1}{4a}\}} |D^ku_\pm|^2e^{2\tau\phi_{\delta,\pm}(x',x_n)}dx'dx_n \\
&\le C\sum_{\pm}\sum_{|k|=0}^{1}R_2^{2(|k|-2)}\int_{\{-3R_2\le z\le -2R_2,\, x_n<\frac{R_1}{2a}\}} |D^ku_\pm|^2e^{2\tau\phi_{\delta,\pm}(x',x_n)}dx'dx_n \\
&\quad + C\sum_{|k|=0}^{1}R_1^{2(|k|-2)}\int_{\{-3R_2\le z,\, \frac{R_1}{4a}<x_n<\frac{R_1}{2a}\}} |D^ku_+|^2e^{2\tau\phi_{\delta,+}(x',x_n)}dx'dx_n \\
&\quad + C\tau^2R_2^{-3}e^{-4\tau R_2}E \\
&\le C\sum_{\pm}\sum_{|k|=0}^{1}R_2^{2(|k|-2)}e^{-4\tau R_2}e^{2\tau\frac{(\alpha_+-\alpha_-)}{\delta}\frac{R_1}{4a}}\int_{\{-3R_2\le z\le -2R_2,\, x_n<\frac{R_1}{4a}\}} |D^ku_\pm|^2dx'dx_n \\
&\quad + \sum_{|k|=0}^{1}R_1^{2(|k|-2)}e^{2\tau\frac{\alpha_+}{\delta}\frac{R_1}{2a}}e^{2\tau\frac{\beta}{2\delta^2}(\frac{R_1}{2a})^2}\int_{\{z\ge -3R_2,\, \frac{R_1}{4a}<x_n<\frac{R_1}{2a}\}} |D^ku_+|^2dx'dx_n \\
&\quad + C\tau^2R_2^{-3}e^{-4\tau R_2}E.
\end{aligned}
\tag{3.11}
$$

Let us recall $U_1=\{z\ge -4R_2,\ \frac{R_1}{8a}<x_n<\frac{R_1}{a}\}$, $U_2=\{-R_2\le z\le\frac{R_1}{2a},\ x_n<\frac{R_1}{8a}\}$. From (3.11) and interior estimates (Caccioppoli's type inequality), we can derive that

$$
\begin{aligned}
&\tau^3 e^{-2\tau R_2}\int_{U_2}|u|^2dx'dx_n = \tau^3 e^{-2\tau R_2}\int_{\{-R_2\le z\le \frac{R_1}{2a},\, x_n<\frac{R_1}{8a}\}}|u|^2dx'dx_n\\
\le& \sum_{\pm}\tau^3\int_{\{-2R_2\le z\le \frac{R_1}{2a},\, x_n<\frac{R_1}{4a}\}}|u_\pm|^2 e^{2\tau\phi_{\delta,\pm}(x',x_n)}dx'dx_n\\
\le& C\sum_{\pm}\sum_{|k|=0}^{1}R_2^{2(|k|-2)}e^{-4\tau R_2}e^{2\tau\frac{(\alpha_+-\alpha_-)}{\delta}\frac{R_1}{4a}}\int_{\{-3R_2\le z\le -2R_2,\, x_n<\frac{R_1}{4a}\}}|D^k u_\pm|^2dx'dx_n\\
&+\sum_{|k|=0}^{1}R_1^{2(|k|-2)}e^{2\tau\frac{\alpha_+}{\delta}\frac{R_1}{2a}}e^{2\tau\frac{\beta}{2\delta^2}(\frac{R_1}{2a})^2}\int_{\{z\ge -3R_2,\, \frac{R_1}{4a}<x_n<\frac{R_1}{2a}\}}|D^k u_+|^2dx'dx_n\\
&+C\tau^2R_2^{-3}e^{-4\tau R_2}E\\
\le& CR_1^{-4}e^{-3\tau R_2}\int_{\{-4R_2\le z\le -R_2,\, x_n<\frac{R_1}{a}\}}|u|^2dx'dx_n + C\tau^2R_2^{-3}e^{-4\tau R_2}E\\
&+CR_1^{-4}e^{(1+\frac{\beta R_1}{4\alpha_-^2})\tau R_1}\int_{\{z\ge -4R_2,\, \frac{R_1}{8a}<x_n<\frac{R_1}{a}\}}|u|^2dx'dx_n\\
\le& CR_1^{-4}\left(e^{2\tau R_1}\int_{U_1}|u|^2dx'dx_n+\tau^2e^{-3\tau R_2}F\right),
\end{aligned}
\tag{3.12}
$$

where

$$
F=\int_{U_3}|u|^2dx'dx_n
$$

and we used the inequality $\frac{\beta R_1}{4\alpha_-^2}<1$ in view of (3.5). Remark that we estimate E by F using the trace estimate and the interior estimate.

Dividing $\tau^3e^{-2\tau R_2}$ on both sides of (3.12) gives

$$
\int_{U_2}|u|^2dx\le CR_1^{-4}\left(e^{2\tau(R_1+R_2)}\int_{U_1}|u|^2dx'dx_n+e^{-\tau R_2}F\right). \tag{3.13}
$$

We now discuss two cases. If $\int_{U_1}|u|^2dx'dx_n\neq 0$ and

$$
e^{2\tau_0(R_1+R_2)}\int_{U_1}|u|^2dx'dx_n<e^{-\tau_0R_2}F,
$$

then we can choose a $\tau>\tau_0$ so that

$$
e^{2\tau(R_1+R_2)}\int_{U_1}|u|^2dx'dx_n=e^{-\tau R_2}F.
$$

With such τ, it follows from (3.13) that

$$
\begin{aligned}
\int_{U_2}|u|^2dx &\le CR_1^{-4}e^{2\tau(R_1+R_2)}\int_{U_1}|u|^2dx'dx_n\\
&=CR_1^{-4}\left(\int_{U_1}|u|^2dx'dx_n\right)^{\frac{R_2}{2R_1+3R_2}}(F)^{\frac{2R_1+2R_2}{2R_1+3R_2}}.
\end{aligned}
\tag{3.14}
$$

If $\int_{U_1}|u|^2dx'dx_n=0$, then letting $\tau\to\infty$ in (3.13) we have $\int_{U_2}|u|^2dx'dx_n=0$ as well. The three-regions inequality (3.2) obviously holds.

On the other hand, if

$$e^{-\tau_0R_2}F\le e^{2\tau_0(R_1+R_2)}\int_{U_1}|u|^2dx'dx_n,$$

then we have

$$\begin{aligned}\int_{U_2}|u|^2dx'dx_n&\le (F)^{\frac{R_2}{2R_1+3R_2}}\,(F)^{\frac{2R_1+2R_2}{2R_1+3R_2}}\\&\le \exp\left(\tau_0R_2\right)\left(\int_{U_1}|u|^2dx'dx_n\right)^{\frac{R_2}{2R_1+3R_2}}(F)^{\frac{2R_1+2R_2}{2R_1+3R_2}}.\end{aligned}\tag{3.15}$$

Putting together (3.14), (3.15) implies

$$\int_{U_2}|u|^2dx'dx_n\le\left(\exp\left(\tau_0R_2\right)+CR_1^{-4}\right)\left(\int_{U_1}|u|^2dx'dx_n\right)^{\frac{R_2}{2R_1+3R_2}}(F)^{\frac{2R_1+2R_2}{2R_1+3R_2}}.\tag{3.16}$$

□

4 Propagation of smallness

In this section, we will derive a general propagation of smallness for solutions satisfying (3.1), $\mathcal{L}(x,D)u=0$ in $\mathbb{R}^n$, using the ideas given in [CW]. For the region away from the interface, classical three-ball inequalities are shown to hold for the complex second order elliptic operators [CNW]. We will mainly focus on the inequalities across the interface. Let us first fix some notations. Assume that $\Omega\subset\mathbb{R}^n$ is an open bounded domain with Lipschitz boundary and $\Sigma\subset\Omega$ is a $C^{1,1}$ hypersurface. Furthermore, assume that $\Omega\setminus\Sigma$ only has two connected components, which we denote $\Omega_\pm$. Let $A_\pm(x)=(a_{\ell j}(x)^\pm)_{\ell,j=1}^n, W(x), V(x)$ be bounded measurable complex valued coefficients defined in Ω. We say that

$$\zeta^\pm:=(A_\pm,W,V)\in\mathscr{V}(\Omega_\pm,\lambda_0,M_0,K_1,K_2)$$

if $A_\pm$ satisfy (2.2)-(2.6) for $x\in\Omega_\pm$ and W,V satisfy

$$\|W\|_{L^\infty(\Omega)}\le K_1,\quad \|V\|_{L^\infty(\Omega)}\le K_2.$$

We will use the notation $\mathcal{L}_\zeta$ to denote

$$\mathcal{L}_\zeta(x,D)u=\sum_\pm H_\pm\mathrm{div}(A_\pm(x)\nabla u_\pm)+\sum_\pm H_\pm\left(W\cdot\nabla u_\pm+Vu_\pm\right)\quad\text{in}\quad\Omega.$$

Here, by an abuse of notation, we denote $H_\pm=\chi_{\Omega_\pm}$.

For an open set $U \subset \mathbb{R}^n$ and a number $s > 0$, we define

$$U^s = \{x \in \mathbb{R}^n : \mathrm{dist}(x, U) < s\},$$

$$U_s = \{x \in U : \mathrm{dist}(x, \partial U) > s\},$$

and

$$sU = \{sx : x \in U\}.$$

Definition 4.1 *We say that $\Omega \in C^{k,1}$, $k \in \mathbb{N}$ with constants ρ_1, M_1 if for any point $P \in \partial\Omega$, after a rigid transformation, $P = 0$ and*

$$\Omega \cap \Gamma_{\rho_1, M_1}(0) = \{(x', x_n) : x' \in \mathbb{R}^{n-1}, |x'| < \rho_1, x_n \in \mathbb{R}, x_n > \Phi(x')\},$$

where Φ is a $C^{k,1}$ function such that $\Phi(0) = 0$, $\|\Phi\|_{C^{\alpha,1}(B_{\rho_1}(0))} \leq M_1$, and

$$\Gamma_{\rho_1, M_1}(0) = \{(x', x_n) : x' \in \mathbb{R}^{n-1}, |x'| < \rho_1, |x_n| \leq M_1\}.$$

Throughout this paper, when saying that a domain is $C^{k,1}$, we will mean that it is $C^{k,1}$ with constants ρ_1 and M_1.

Definition 4.2 *We say that Σ is $C^{1,1}$ with constants ρ_0, K_0 if for any point $P \in \Sigma$, after a rigid transformation, $P = 0$ and*

$$\Omega_\pm \cap C_{\rho_0, K_0}(0) = \{(x', x_n) : x' \in \mathbb{R}^{n-1}, |x'| < \rho_0, x_n \in \mathbb{R}, x_n \gtrless \psi(x)\},$$

where ψ is a $C^{1,1}$ function such that $\psi(0) = 0$, $\nabla_{x'}\psi(0) = 0$, $\|\psi\|_{C^{1,1}(B_{\rho_0}(0))} \leq K_0$, and

$$C_{\rho_0, K_0}(0) = \{(x', x_n) : x' \in \mathbb{R}^{n-1}, |x'| < \rho_0, |x_n| \leq \frac{1}{2} K_0 \rho_0^2\}.$$

If Σ is as above, then we may "flatten" the boundary around the point P (without loss of generality $P = 0$) via the local $C^{1,1}$-diffeomeorphism

$$\Psi_P(x, y) = (x, y - \psi(x)).$$

With these assumptions, we will prove a propagation of smallness result as follows.

Theorem 4.1 *Suppose $u \in H^1(\Omega)$ solves*

$$\mathcal{L}_\zeta u = 0 \quad \text{in} \quad \Omega.$$

Then there exist γ_0, depending on λ_0, M_0, and h_0, depending on λ_0, M_0, K_1, K_2, ρ_0, K_0, such that if $\gamma < \gamma_0$ and $0 < h < h_0$, $r/2 > h$, $D \subset \Omega$ is connected, open, and $D \setminus \Sigma$ has two connected components, denoted by $D_\pm$, such that $B_r(x_0) \subset D$, $\mathrm{dist}(D, \partial\Omega) \geq h$, then

$$\|u\|_{L^2(D)} \leq C \|u\|^\delta_{L^2(B_r(x_0))} \|u\|^{1-\delta}_{L^2(\Omega)},$$

where

$$C = C_1 \left(\frac{|\Omega|}{h^n}\right)^{\frac{1}{2}} e^{C_3 h^{-s}}, \quad \delta \geq \tau^{\frac{C_2|\Omega|}{h^n}},$$

with $s = s(\lambda_0, K_1, K_2)$, $C_1, C_2 > 0$, $\tau \in (0, 1)$ depending on λ_0, M_0, K_1, K_2, ρ_0, K_0.

We would like to remark that the propagation of smallness in Theorem 4.1 is valid regardless the locations of D and $B_{r_0}(x_0)$, which may intersect the interface Σ. The strategy of proving Theorem 4.1 consists two parts. When we are at one side of the interface, we can use the usual propagation of smallness for equations with Lipschitz complex coefficients based on [CNW]. When near the interface, we then use the three-region inequality derived above to propagate the smallness across the interface. The rest of this section is devoted to the proof of Theorem 4.1.

4.1 Propagation of smallness away from the interface

In this subsection, we want to derive a propagation of smallness for second order complex elliptic operators with Lipschitz leading coefficients. We consider $\zeta := (A, W, V) \in \mathscr{V}(U, \lambda_0, M_0, K_1, K_2)$, where U is an open bounded domain and the value of γ in (2.3) is irrelevant. Note that here A is Lipschitz without jumps in U. The following three-ball inequality was proved in [CNW, Theorem 3].

Proposition 4.1 *Assume that $\zeta := (A, W, V) \in \mathscr{V}(U, \lambda_0, M_0, K_1, K_2)$ and $u \in H^1_{loc}(\Omega)$ solves $\mathcal{L}_\zeta u = 0$ in U. Then there exist positive constants $R = R(n, \lambda_0, M_0)$ and $s = s(\lambda_0, K_1, K_2)$ such that if $0 < r_0 < r_1 < \lambda_0 r_2/2 < \sqrt{\lambda_0} R/2$ with $B_{r_2}(x_0) \subset U$, then*

$$\|u\|_{L^2(B_{r_1}(x_0))} \leq C \|u\|^{\tau}_{L^2(B_{r_0}(x_0))} \|u\|^{1-\tau}_{L^2(B_{r_2}(x_0))}, \tag{4.1}$$

where C is explicitly given by

$$C = e^{C_1(r_0^{-s} - r_2^{-s})}$$

with $C_1 >$ depending on λ_0, M_0, K_1, K_2, and

$$\tau = \frac{(2r_1/\lambda_0)^{-s} - r_2^{-s}}{r_0^{-s} - r_2^{-s}} = \frac{(2r_1/r_2\lambda_0)^{-s} - 1}{(r_0/r_2)^{-s} - 1}.$$

The proof of Proposition 4.1 relies on the Carleman estimate derived in [CGT].

Having established the three-ball inequality (4.1), we can prove the following propagation of smallness based on the chain of balls argument in [ARRV, Theorem 5.1]. We will not repeat the argument here.

Proposition 4.2 *Assume that $\zeta := (A, W, V) \in \mathscr{V}(U, \lambda_0, M_0, K_1, K_2)$ and $u \in H^1_{loc}(\Omega)$ solves $\mathcal{L}_\zeta u = 0$ in U. Let $0 < h < r/2$ with $r \leq \sqrt{\lambda_0} R/2$, $D \subset U$ connected, open, and such that $B_r(x_0) \subset D$, $\operatorname{dist}(D, \partial U) \geq h$. Then*

$$\|u\|_{L^2(D)} \leq C \|u\|^{\delta}_{L^2(B_r(x_0))} \|u\|^{1-\delta}_{L^2(U)},$$

where

$$C = C_2 \left(\frac{|U|}{h^n}\right)^{\frac{1}{2}} e^{C_3 h^{-s}}, \quad \delta \geq \tau^{\frac{C_4 |U|}{h^n}},$$

with $C_2, C_3, C_4 > 0$ depending on λ_0, M_0, K_1, K_2.

4.2 Propagation of smallness – an intermediate result

Here we would like to prove an intermediate propagation of smallness result in which the small ball lies entirely on one side of the interface. Assume that $D \subset\subset \Omega$ is open and connected. Recall that we have assumed that $\Omega \setminus \Sigma$ and $D \setminus \Sigma$ both have two connected components, denoted by $\Omega_\pm$ and $D_\pm$, respectively. Let ω_Σ be the surface measure induced on Σ by the Lebesgue measure on $\mathbb{R}^n$. We will consider coefficients

$$\zeta = (A_\pm, W, V) \in \mathscr{V}(\Omega_\pm, \lambda_0, M_0, K_1, K_2).$$

We can now prove the following propagation of smallness result.

Theorem 4.2 *Suppose $u \in H^1(\Omega)$ solves $\mathcal{L}_\zeta u = 0$ in Ω. Then there exist γ_0, depending on λ_0, M_0, and h_0, depending on λ_0, M_0, K_1, K_2, ρ_0, K_0, such that if $\gamma < \gamma_0$ and $0 < h \le h_0$, $h < r/2$, $B_r(x_0) \subset D_+$, and $\operatorname{dist}(D, \partial\Omega) \ge h$, then*

$$\|u\|_{L^2(D)} \le C \|u\|^{\delta}_{L^2(B_r(x_0))} \|u\|^{1-\delta}_{L^2(\Omega)},$$

where

$$C = C_1 \left(\frac{|\Omega|}{h^n}\right) \left[1 + \left(\frac{\omega_\Sigma(\Sigma \cap \Omega)}{h^{n-1}}\right)^{\frac{1}{2}}\right] e^{C_3 h^{-s}}, \quad \delta \ge \tau^{\frac{C_2|\Omega|}{h^n}},$$

with $C_1, C_2, C_3 > 0$, $\tau \in (0,1)$ depending on λ_0, M_0, K_1, K_2, ρ_0, K_0.

The difficult part of proving Theorem 4.2 is to obtain L^2 estimates of the solution in a neighborhood of Σ. We will use Theorem 3.1 to overcome this difficulty. However, we cannot apply Theorem 3.1 directly. The family of regions given in Theorem 3.1 has one serious drawback. If we choose the parameters $R_1 = \theta \bar{R}_1$, $R_2 = \theta \bar{R}_2$, $\theta \in (0,1)$, the vertical sizes of the regions would scale like θ, while their horizontal sizes would scale like $\theta^{\frac{1}{2}}$. Using just these two parameters in the proof would then lead to constants in the propagation of smallness inequality (i.e. the constants C and δ in Theorem 4.1) that depend on the geometry of Ω, D, and $B_r(x_0)$ in a way that is not invariant under a rescaling of these sets. Therefore, we will study how the three-region inequality (3.2) behaves under scaling.

Let us first introduce the scaled coefficients

$$\tilde{\zeta} = \left(\tilde{A}_\pm, \tilde{W}, \tilde{V}\right) \in \mathscr{V}(\mathbb{R}^n_\pm, \lambda_0, M_0, K_1, K_2).$$

For $0 < \theta \le 1$, let

$$\mathcal{L}^\theta_{\tilde{\zeta}}(\cdot, D)v = \sum_\pm H_\pm \operatorname{div}(\tilde{A}_\pm(\theta\cdot)\nabla v_\pm) + \sum_\pm H_\pm \left(\theta \tilde{W}(\theta\cdot) \cdot \nabla v_\pm + \theta^2 \tilde{V}(\theta\cdot) v_\pm\right).$$

Note that if $\tilde{\zeta} = \left(\tilde{A}_\pm, \tilde{W}, \tilde{V}\right) \in \mathscr{V}(\mathbb{R}^n_\pm, \lambda_0, M_0, K_1, K_2)$, then

$$\left(\tilde{A}_\pm(\theta\cdot), \theta\tilde{W}(\theta\cdot), \theta^2\tilde{V}(\theta\cdot)\right) \in \mathscr{V}(\mathbb{R}^n_\pm, \lambda_0, \theta M_0, \theta K_1, \theta^2 K_2).$$

It is also clear that if $\mathcal{L}_{\tilde{\zeta}}u = 0$ in Ω, then $u^\theta(x) = \theta^{-2}u(\theta x)$ solves

$$\mathcal{L}^\theta_{\tilde{\zeta}}u^\theta = 0 \quad \text{in} \quad \theta^{-1}\Omega.$$

Moreover, if $U \subset \theta^{-1}\Omega$, we have

$$\int_{\theta U} |u(x)|^2 dx = \theta^{n+4} \int_U |u^\theta(y)|^2 dy.$$

We therefore obtain, by scaling, the following corollary to Theorem 3.1.

Proposition 4.3 *Assume that the assumptions in Theorem 3.1 hold. Let* $0 < R_1, R_2 \leq R$, $\theta \in (0, 1]$, *and*

$$\mathcal{L}_{\tilde{\gamma}}u = 0 \quad \textit{in} \quad \theta U_3,$$

then

$$\int_{\theta U_2} |u|^2 \leq (e^{\tau_0 R_2} + CR_1^{-4}) \left(\int_{\theta U_1} |u|^2 \right)^{\frac{R_2}{2R_1+3R_2}} \left(\int_{\theta U_3} |u|^2 \right)^{\frac{2R_1+2R_2}{2R_1+3R_2}}.$$

In order to adapt that result to the possibly curved surface Σ, we need to first consider how the three regions transform under a local boundary flattening diffeomorphism Ψ_P. Pick a point $P \in \Sigma$ and set $P = 0$ without loss of generality. Let $(x', x_n) \in C_{\rho_0, K_0}(0)$. We will try to determine when $(x', x_n) \in \Psi_P^{-1}(\theta U_2)$. To this end, we introduce the notation

$$y' = x', \quad y_n = x_n - \psi(x').$$

It is clear that $(x', x_n) \in \Psi_P^{-1}(\theta U_2)$ if and only if $\theta^{-1}(y', y_n) \in U_2$. We denote

$$\eta(x') = \frac{\psi(x')}{|x'|^2},$$

which is a bounded function due to the regularity assumption of Σ. It was proved in [CW, Lemma 3.1] that if

$$r < \theta \min \left\{ \frac{\delta R_1}{6a\alpha_-}, \frac{2\delta R_2}{3\alpha_-}, \frac{R_1}{12a}, \theta^{-1}\rho_0, \rho_1, \rho_2, \rho_3 \right\}, \tag{4.2}$$

then $\Psi_P(B_r(P)) \subset \theta U_2$, where

$$\begin{cases} \rho_1 = \dfrac{\alpha_-\delta}{\delta + \beta}, \\ \rho_2 \text{ is chosen such that } 2\|\eta\|\rho_2 + \|\eta\|^2\rho_2^2 < \dfrac{1}{2}, \ \|\eta\| = \|\eta\|_{L^\infty(B'_{\rho_0}(0))}, \\ \rho_3 = \dfrac{2\alpha_-\delta}{\beta}. \end{cases}$$

In [CW, Lemma 3.2], the following relation was established.

Lemma 4.1 *$\Psi_P^{-1}(\theta U_3)$ is contained in a ball of radius*

$$\theta\left[(1+2\|\eta\|^2)\left(\frac{2\alpha_-}{a}R_1+8\delta R_2\right)+\frac{1}{a^2}\left[2+(1+2\|\eta\|^2)\frac{\beta}{\delta}\right]R_1^2+\frac{128\delta^2R_2^2}{\left[\alpha_-+\sqrt{\alpha_-^2-8\beta R_2}\right]^2}\right]^{1/2}$$

centered at P.

Finally, we need to estimate the distance from $\Psi_P^{-1}(\theta U_1)$ to $\Sigma\cap C_{\rho_0,K_0}$. It was proved in [CW, Lemma 3.3] that

$$\operatorname{dist}(\Psi_P^{-1}(\theta U_1),\Sigma)>\theta\frac{R_1}{16a}. \tag{4.3}$$

We are now ready to prove Theorem 4.2. We will follow the arguments used in the proof of Theorem 3.1 in [CW]. Since we have slightly different constants here, we provide the proof for the sake of completeness.

Proof of Theorem 4.2. By the assumption, we may take D to be the set

$$D=\{x\in\Omega:\operatorname{dist}(x,\partial\Omega)>h\}.$$

We want to point out that even though the choice of $\alpha_\pm$ in Theorem 2.1 depends on $A_\pm(P)$ for $P\in\Sigma$, we can choose a pair of $\alpha_\pm$ such that Carleman estimate (2.13) holds near all $P\in\Sigma$ in view of the regularity assumptions of $A_\pm$ and Σ. Consequently, we can pick R_1, R_2 so that we can apply Proposition 4.3 at any point $P\in\Sigma\cap D$. By Lemma 4.1, there is a constant $d>0$, independent of P, such that $\Psi_P^{-1}(\theta U_3)\subset B_{\theta d}(P)$. We then choose θ such that $\theta d=\frac{h}{2}$, which implies $\Psi_P^{-1}(\theta U_3)\subset\Omega$ for any $P\in\Sigma\cap D$. Of course, this choice is not possible if h is too large. Therefore, we need to set h_0 small enough, depending on $\rho_0,K_0,\lambda_0,M_0,K_1,K_2$.

With this choice of parameters, by (4.3), there is a constant $0<\mu<1$, also independent on P, so that

$$\operatorname{dist}(\Psi_P^{-1}(\theta U_1),\Sigma)>\mu h.$$

Note that, depending on the geometry of Σ, we again need to set h_0 and R small enough so that $\Psi_P^{-1}(\theta U_1)\cap\Sigma^{\mu h}=\emptyset$, for any $P\in\Sigma\cap D$.

It follows from (4.2) that there exists a constant $\nu>0$, and without loss of generality $\nu<\mu<1$, such that $B_{5\nu h}(P)\subset\Psi_P^{-1}(\theta U_2)$. By Vitali's covering lemma, there exist finitely many $P_1,\dots,P_N\in\Sigma\cap D$ so that

$$\Sigma^{\nu h}\cap D\subset\bigcup_{j=1}^N\Psi_{P_j}^{-1}(\theta U_2), \tag{4.4}$$

and the balls $B_{\nu h}(P_j)$ are pairwise disjoint. By this last property, since for small h we have $\omega_\Sigma(\Sigma^{\nu h}\cap D)\sim\nu h\omega_\Sigma(\Sigma\cap D)$, it follows that there is a constant C such that

$$N\le C\frac{\omega_\Sigma(\Sigma\cap D)}{h^{n-1}}\le C\frac{\omega_\Sigma(\Sigma\cap\Omega)}{h^{n-1}}. \tag{4.5}$$

Let us denote $\tilde{D} = (D_+)^{h/2} \setminus (\Sigma^{\nu h} \cup \Omega_-)$, then by Proposition 4.2, we have that

$$\|u\|_{L^2(\tilde{D})} \leq C_+ \|u\|^{\delta_+}_{L^2(B_r(x_0))} \|u\|^{1-\delta_+}_{L^2(\Omega)}, \tag{4.6}$$

where

$$B_r(x_0) \subset \tilde{D}, \quad C_+ = C_2 \left(\frac{|\Omega|}{h^n}\right)^{\frac{1}{2}} e^{C_3 h^{-s}}, \quad \delta_+ \geq \tau^{\frac{C_4|\Omega|}{h^n}}.$$

The function $v = u \circ \Psi^{-1}_{P_j} \in H^1(\theta U_3)$ satisfies $\mathcal{L}_{\tilde{\zeta}} v = 0$ in θU_3 with $\tilde{\zeta}$ satisfying

$$\tilde{\zeta}^{\pm} = \left(\tilde{A}_{\pm}, \tilde{W}, \tilde{V}\right) \in \mathscr{V}(\mathbb{R}^n_{\pm}, \tilde{\lambda}_0, \tilde{M}_0, \tilde{K}_1, \tilde{K}_2),$$

with $C > 0$ and the parameters $\tilde{\lambda}_0, \tilde{M}_0, \tilde{K}_1, \tilde{K}_2$ depending on $\lambda_0, M_0, K_1, K_2, \rho_0, K_0$. We can then pull back the three regions inequality in Proposition 4.3 and apply it to u and the regions $\Psi^{-1}_{P_j}(\theta U_1)$, $\Psi^{-1}_{P_j}(\theta U_2)$, $\Psi^{-1}_{P_j}(\theta U_3)$.

Since

$$\Psi^{-1}_{P_j}(\theta U_1) \subset (D_+)^{h/2} \setminus (\Sigma^{\nu h} \cup \Omega_-),$$

we have that

$$\|u\|_{L^2(\Psi^{-1}_{P_j}(\theta U_2))} \leq C \|u\|^{\xi}_{L^2((D_+)^{h/2} \setminus (\Sigma^{\nu h} \cup \Omega_-))} \|u\|^{1-\xi}_{L^2(\Omega)},$$

where $\xi = \frac{R_2}{2R_1 + 3R_2}$. Combining this and (4.6), we obtain

$$\|u\|_{L^2(\Psi^{-1}_{P_j}(\theta U_2))} \leq C_1' \left(\frac{|\Omega|}{h^n}\right)^{\frac{\xi}{2}} e^{C_3 \xi h^{-s}} \|u\|^{\xi\delta_+}_{L^2(B_r(x_0))} \|u\|^{1-\xi\delta_+}_{L^2(\Omega)}.$$

Then it follows from (4.4) and (4.5) that

$$\|u\|_{L^2(\Sigma^{\nu h} \cap D)} \leq C_1'' \left(\frac{\omega_\Sigma(\Sigma \cap \Omega)}{h^{n-1}}\right)^{\frac{1}{2}} \left(\frac{|\Omega|}{h^n}\right)^{\frac{\xi}{2}} e^{C_3 \xi h^{-s}} \|u\|^{\xi\delta_+}_{L^2(B_r(x_0))} \|u\|^{1-\xi\delta_+}_{L^2(\Omega)}. \tag{4.7}$$

Applying Proposition 4.2 again (now with an appropriate small ball $\tilde{B}_{\tilde{r}} \subset \Sigma^{\nu h} \cap D_- \subset \Sigma^{\nu h} \cap D$), we have

$$\begin{aligned} \|u\|_{L^2(D_- \setminus \Sigma^{\nu h})} \leq & C_1''' \left(\frac{|\Omega|}{h^n}\right)^{\frac{1}{2}} \left(\frac{\omega_\Sigma(\Sigma \cap \Omega)}{h^{n-1}}\right)^{\frac{\delta_-}{2}} \left(\frac{|\Omega|}{h^n}\right)^{\frac{\delta_- \xi}{2}} e^{C_3(1+\delta_- \xi)h^{-s}} \\ & \times \|u\|^{\delta_- \xi \delta_+}_{L^2(B_r(x_0))} \|u\|^{1-\delta_- \xi \delta_+}_{L^2(\Omega)}, \end{aligned} \tag{4.8}$$

where

$$\delta_- \geq \tau^{\frac{C_2'|\Omega|}{h^n}}.$$

Combining estimates (4.6), (4.7), and (4.8), we obtain the conclusion of Theorem 4.2. □

4.3 Propagation of smallness – a general result

This section is devoted to the proof of Theorem 4.1. We will first prove a three-ball inequality, which is a direct consequence of Theorem 4.2. Such three balls inequality is a building block in the proof of Theorem 4.1.

4.3.1 Three-ball inequality

Recall that we assume $\Omega \subset \mathbb{R}^n$ is an open Lipschtiz domain, Σ is a $C^{1,1}$ hypersurface with constants ρ_0, K_0, and $\Omega \setminus \Sigma$ has two connected components, $\Omega_\pm$. We consider coefficients

$$\zeta = (A_\pm, W, V) \in \mathscr{V}(\Omega_\pm, \lambda_0, M_0, K_1, K_2)$$

with $\gamma < \gamma_0$. Also, let $u \in H^1(\Omega)$ be a solution of

$$\mathcal{L}_\zeta u = 0 \quad \text{in} \quad \Omega.$$

Proposition 4.4 *There exists $\bar{r} > 0$, depending on ρ_0, K_0, such that if $0 < r_1 < r_2 < r_3 < \bar{r}$, $Q \in \Omega$, $\operatorname{dist}(Q, \partial\Omega) > r_3$, then there exist $C > 0$, $0 < \delta < 1$ such that*

$$\|u\|_{L^2(B_{r_2}(Q))} \leq C \|u\|^{\delta}_{L^2(B_{r_1}(Q))} \|u\|^{1-\delta}_{L^2(B_{r_3}(Q))} \tag{4.9}$$

with

$$C = C_1 e^{C_2 r_3^{-s}},$$

where C_1, C_2, and δ depend on λ_0, M_0, ρ_0, K_0, K_1, K_2, $\frac{r_1}{r_3}$, $\frac{r_2}{r_3}$, $\operatorname{diam}(\Omega)$.

Proof. We would like to use the propagation of smallness result in Theorem 4.2 with $r = \frac{r_1}{10}$, $D = B_{r_2}(Q)$, and replacing Ω there by $B_{r_3}(Q)$. We can choose the constant $\bar{r}$ so that $B_{r_j}(Q) \setminus \Sigma$ can all only have at most two connected components. This would be the case for example if $\bar{r} \leq \min\{\rho_0, \frac{1}{2}K_0\rho_0^2\}$. Fix an $\bar{r}$ as described. Then we can always find $Q' \in B_{r_1}(Q)$ so that $B_{r_1/10}(Q') \subset B_{r_1}(Q) \cap \Omega_+$ or $B_{r_1/10}(Q') \subset B_{r_1}(Q) \cap \Omega_-$. Without loss of generality we may assume that $B_{r_1/10}(Q') \subset B_{r_1}(Q) \cap \Omega_+$.

Let g^Σ be the metric induced on Σ by the Euclidean metric of $\mathbb{R}^n$. Around a point $P \in \Sigma$ at which we have chosen coordinates as in Definition 4.2, we can use the coordinates $(x_1, \ldots, x_{n-1})$ as a local map for Σ. In these coordinates, we have

$$g^\Sigma_{jk} = \delta_{jk} + \partial_j\psi\partial_k\psi.$$

This observation implies that there exists a constant κ so that

$$\omega_\Sigma(\Sigma \cap B_{r_3}(Q)) < \kappa r_3^{n-1}.$$

Here we can also observe that we can cover $\Sigma \cap \Omega$ with balls $B_{\rho_0}(P)$, $P \in \Omega$, and on each of those the same estimate for $\omega_\Sigma(\Sigma \cap B_{\rho_0}(P))$ holds. Using again Vitali's covering lemma, it is not hard to see that we can make sure that the number N of these balls satisfies

$$N < C \operatorname{diam}(\Omega)^n,$$

with C depending only on the dimension n. It follows that $\omega_\Sigma(\Sigma\cap\Omega)$ can be bound from above by a quantity depending on ρ_0, K_0, and $\mathrm{diam}(\Omega)$.

We will treat several cases separately. The first case is when $r_3-r_2<\min\{\frac{r_1}{20},h_0\}$. Remind that the constant h_0 is given in Theorem 4.2. We now apply Theorem 4.2, with $h=r_3-r_2$, to obtain

$$\|u\|_{L^2(B_{r_2}(Q))}\le C\|u\|^{\delta}_{L^2(B_{\frac{r_1}{10}}(Q'))}\|u\|^{1-\delta}_{L^2(B_{r_3}(Q))},$$

where

$$C=C_1\left(\frac{r_3^n}{(r_3-r_2)^n}\right)\left[1+\left(\frac{\kappa r_3^{n-1}}{(r_3-r_2)^{n-1}}\right)^{\frac12}\right]e^{C_3(r_3-r_2)^{-s}}, \tag{4.10}$$

$$\delta\ge\tau^{C_2\frac{r_3^n}{(r_3-r_2)^n}}. \tag{4.11}$$

The second case is when $\frac{r_1}{10}<2h_0$, $r_3-r_2\ge\frac{r_1}{20}$. Let $r_3'=r_2+\frac{r_1}{21}$ (note $r_3'<r_3$), $h=\frac{r_1}{21}$ and again apply Theorem 4.2 to obtain

$$\|u\|_{L^2(B_{r_2}(Q))}\le C\|u\|^{\delta}_{L^2(B_{\frac{r_1}{10}}(Q'))}\|u\|^{1-\delta}_{L^2(B_{r_3}(Q))},$$

where

$$C=C_1\left(\frac{(r_2+r_1/21)^n}{(r_1/21)^n}\right)\left[1+\left(\frac{\kappa(r_2+r_1/21)^{n-1}}{(r_1/21)^{n-1}}\right)^{\frac12}\right]e^{C_3(r_1/21)^{-s}}, \tag{4.12}$$

$$\delta\ge\tau^{C_2\frac{(r_2+r_1/21)^n}{(r_1/21)^n}}. \tag{4.13}$$

The third and final case is when $\frac{r_1}{10}\ge 2h_0$, $r_3-r_2\ge h_0$. In this case we take $h=h_0$, and use the estimates

$$|B_{r_3}(Q)|\le(\mathrm{diam}(\Omega))^n,\quad \omega_\Sigma(B_{r_3}(Q)\cap\Sigma)\le\omega_\Sigma(\Sigma\cap\Omega).$$

We then have

$$||u||_{L^2(B_{r_2}(Q))}\le C\|u\|^{\delta}_{L^2(B_{\frac{r_1}{10}}(Q'))}\|u\|^{1-\delta}_{L^2(B_{r_3}(Q))},$$

where

$$C=C_1\frac{(\mathrm{diam}(\Omega))^n}{h_0^n}\left[1+\left(\frac{\omega_\Sigma(\Sigma\cap\Omega)}{h_0^{n-1}}\right)^{\frac12}\right]e^{C_3h_0^{-s}}, \tag{4.14}$$

$$\delta\ge\tau^{C_2\frac{(\mathrm{diam}(\Omega))^n}{h_0^n}}. \tag{4.15}$$

It follows that, in all cases, we have our three ball inequality with the constant C being the maximum of the ones in (4.10), (4.12), and (4.14), and the exponent δ being the minimum of the ones in (4.11), (4.13), and (4.15).

4.3.2 Proof of Theorem 4.1

Once we have established the three balls inequality in Theorem 4.4, the proof of Theorem 4.1 is standard. We include it here for the benefit of the reader. Let

$$r_3 = \frac{h}{2}, \quad r_2 = \frac{1}{5}r_3 = \frac{1}{10}h, \quad r_1 = \frac{1}{3}r_3 = \frac{1}{30}h,$$

and

$$\tilde{D} = \{x \in \Omega : \operatorname{dist}(x, D) < r_1\},$$

which is an open connected subset of Ω, such that $D \subset \tilde{D}$, $\operatorname{dist}(\tilde{D}, \partial\Omega) > h/2$. Let $y \in \tilde{D}$ and $c(t) \in C([0,1]; \tilde{D})$ be a continuous curve such that $c(0) = x_0$, and $c(1) = y$. Define

$$0 = t_0 < t_1 < \cdots < t_N = 1$$

so that

$$t_{k+1} = \max\{t : |c(t) - c(t_k)| = 2r_1\}, \text{ as long as } |y - c(t_k)| > 2r_1,$$
$$\text{otherwise } N = k+1, t_N = 1.$$

Then $B_{r_1}(c(t_k)) \cap B_{r_1}(c(t_{k-1})) = \emptyset$, and $B_{r_1}(c(t_{k+1})) \subset B_{r_2}(c(t_k))$, $k = 1, \ldots, N-1$. By Theorem 4.4 we have

$$\|u\|_{L^2(B_{r_1}(c(t_{k+1})))} \le C\|u\|^{\tau}_{L^2(B_{r_1}(c(t_k)))}\|u\|^{1-\tau}_{L^2(\Omega)},$$

where $k = 0, \ldots, N-1$. Note that $C = C_1 e^{C_2 h^{-s}}$ with C_1, C_2 depending on λ_0, M_0, ρ_0, K_0, K_1, K_2, $\operatorname{diam}(\Omega)$.

Let

$$m_k = \frac{\|u\|_{L^2(B_{r_1}(c(t_k))}}{\|u\|_{L^2(\Omega)}},$$

then $m_{k+1} \le C m_k^{\tau}$, $k = 0, \ldots, N-1$, and so

$$m_N \le C^{1+\tau+\cdots+\tau^{N-1}} m_0^{\tau^N}.$$

Since the balls $B_{r_1}(c(t_k))$ are pairwise disjoint,

$$N \le \frac{|\Omega|}{\omega_n r_1^n} \le \frac{C_2|\Omega|}{h^n}.$$

Then it is easy to see that

$$\tau^N \ge \tau^{\frac{C_2|\Omega|}{h^n}}, \quad C^{1+\tau+\cdots+\tau^{N-1}} \le C^{\frac{1}{1-\tau}}.$$

From a family of disjoint open cubes of side $2r_1/\sqrt{n}$ whose closures cover $\mathbb{R}^n$, extract the finite number of cubes which intersect D non-trivially: Q_j, $j = 1, \ldots, J$. The number of these cubes satisfies $J \le \frac{n^{n/2}|\Omega|}{2^n r_1^n}$. For each j there exists $w_j \in \tilde{D}$ such that $Q_j \subset B_{r_1}(w_j)$. Then

$$\int_D |u|^2 \le \sum_{j=1}^{J} \int_{Q_j} |u|^2 \le \sum_{j=1}^{J} \int_{B_{r_1}(w_j)} |u|^2 \le J C^{2/(1-\tau)} \|u\|^{2\delta}_{L^2(B_{r_1}(x_0))} \|u\|^{2(1-\delta)}_{L^2(\Omega)}.$$

□

5 Size estimate

In this section, we will study the size estimate problem using the uniqueness estimate derived above. Precisely speaking, we will extend the result obtained for the Lipschitz complex coefficients in [CNW] to the piecewise Lipschitz complex coefficients. So we follow closely the arguments used in [CNW].

Let Ω be a conducting body with an anisotropic background with current-voltage relation (or Ohm's law) given by

$$I_0(\nabla u) = (\sigma_0(x) + i\varepsilon_0(x))\nabla u(x), \tag{5.1}$$

where σ_0, ε_0 are symmetric matrix valued functions. As above, we assume that $\Omega \subset \mathbb{R}^n$ is an open bounded domain with Lipschitz boundary with constants ρ_1, M_1 and $\Sigma \subset \Omega$ is a $C^{1,1}$ hypersurface with constants ρ_0, K_0 such that $\Omega \setminus \Sigma$ only has two connected components, which are denoted by $\Omega_\pm$. Here Ω_- represents the inner component, i.e., $\partial\Omega_- = \Sigma$. Assume that $\mathrm{dist}(\Sigma, \partial\Omega) > 0$.

Let $\sigma_0(x) + i\varepsilon_0(x) = A^0_\pm(x)$ be bounded measurable complex valued and

$$A^0_\pm \in \mathscr{V}(\Omega_\pm, \lambda_0, M_0),$$

i.e., $A^0_\pm$ satisfy (2.1)-(2.6) for $x \in \Omega_\pm$. Here γ in (2.3) has been chosen so that Theorem 4.1 holds.

Let $D \subset \Omega$ represent the region of anomaly hidden in Ω. In [FLVW], D is assumed to be contained entirely in Ω_-, i.e., $\bar{D} \subset \Omega_-$. It should be emphasized that here the location of D is arbitrary as long as it stays away from $\partial\Omega$ at a fixed distance. It can intersect the interface Σ. The underlying reason is that here we have more powerful propagation of smallness result, Theorem 4.1. Assume that the current-voltage relation in D is given by

$$I_1(\nabla u) = (\sigma_1(x) + i\varepsilon_1(x))\nabla u(x) + \zeta_1(x)\overline{\nabla u}(x), \tag{5.2}$$

where σ_1, ε_1, ζ_1 are real symmetric matrix-valued functions such that the supports of $\sigma_1 - \sigma_0$, $\varepsilon_1 - \varepsilon_0$, ζ_1 are contained in D. We want to remark that (5.2) can be derived from the Maxwell equations describing the propagation of electromagnetic waves traveling in a chiral medium. We refer the reader to [CNW, Section 1] for detailed explanation. Here ζ_1 denotes the chirality of the medium.

Given a Neumann boundary data $g \in L^2(\partial\Omega)$ satisfying $\int_{\partial\Omega} g = 0$, let u_0 be the unique solution of the unperturbed equation

$$\begin{cases} \nabla \cdot I_0(\nabla u_0) = 0 \quad \text{in} \quad \Omega, \\ I_0(\nabla u_0) \cdot \nu = g \quad \text{on} \quad \partial\Omega, \\ \displaystyle\int_\Omega u_0 = 0, \end{cases} \tag{5.3}$$

and u_1 be the solution of the perturbed equation with the same boundary data, i.e.

$$\begin{cases} \nabla \cdot I_1(\nabla u_1) = 0 \quad \text{in} \quad \Omega, \\ I_1(\nabla u_1) \cdot \nu = g \quad \text{on} \quad \partial\Omega, \\ \displaystyle\int_\Omega u_1 = 0. \end{cases} \tag{5.4}$$

The inverse problem studied here is to estimate the size of the inclusion D by one pair of boundary measurement $\{u|_{\partial\Omega}, g\}$. More precisely, one tries to estimate $|D|$ using the power gap $\delta W = W_0 - W_1$, where

$$W_1 = \int_{\partial\Omega} u_1 g, \quad W_0 = \int_{\partial\Omega} u_0 g.$$

Before stating the main result of the size estimate, we list several conditions imposed on $\sigma_1, \varepsilon_1, \zeta_1$. We denote Id the n-dimensional identity matrix.

- (Boundedness and Ellipticity) for some $\lambda_1 \leq 1$,

$$\begin{aligned} &\lambda_1 Id \leq \sigma_1 + \zeta_1 \leq \lambda_1^{-1} Id, \quad \lambda_1 Id \leq \sigma_1 - \zeta_1 \leq \lambda_1^{-1} Id, \\ &\|\varepsilon_0\|_{L^\infty(\Omega)} \leq \lambda_1^{-1}, \quad \|\varepsilon_1\|_{L^\infty(\Omega)} \leq \lambda_1^{-1}. \end{aligned} \tag{5.5}$$

- (Jump condition) There exists $\varrho > 0$ such that either

$$\begin{aligned} &\text{(i)} \qquad \zeta_1 \leq (\sigma_1 - \sigma_0) - \varrho Id, \;\; \zeta_1 \leq (\sigma_0 - \sigma_1) - \varrho Id \quad \text{a.e. } x \in D, \\ &\text{or} \\ &\text{(ii)} \qquad \zeta_1 \geq (\sigma_1 - \sigma_0) + \varrho Id, \;\; \zeta_1 \geq (\sigma_0 - \sigma_1) + \varrho Id \quad \text{a.e. } x \in D. \end{aligned} \tag{5.6}$$

  Morever, for $\delta = \delta(\alpha, \varrho) > 0$ sufficiently small, we have

$$\|\varepsilon_1 - \varepsilon_0\|_{L^\infty(D)} \leq \delta. \tag{5.7}$$

Note that $\|\varepsilon_0\|_{L^\infty(\Omega)} \leq \lambda_1^{-1}$ provided $\gamma\lambda_0^{-1} \leq \lambda_1^{-1}$. Let us remark that these assumptions are rather standard in this area, see e.g. [ARS] and [BFV]. The main difference between ours and those in previous work is the jump condition which is adapted to our physical model (5.2). It plays an important role in the derivation of the power gap (see Proposition 5.1).

We are ready to state the main theorem about the size estimates. Here we define the free energy

$$W_0' = \int_{\partial\Omega} \bar{u}_0 g = \int_\Omega \sigma_0 \nabla u_0 \cdot \nabla \bar{u}_0 + i \int_\Omega \varepsilon_0 \nabla u_0 \cdot \nabla \bar{u}_0.$$

Since u_0 is completely determined, W_0' is known.

Theorem 5.1 *Let Ω be a bounded domain in $\mathbb{R}^n$ with $\partial\Omega \in C^{3,1}$. Assume that for some $d_0, d_1 > 0$, the inclusion $D \subset \Omega$ satisfies* $\operatorname{dist}(D, \partial\Omega) \geq d_0$ *and*

$$|D_{d_1}| \geq \frac{1}{2}|D| \quad \text{(fatness condition)}. \tag{5.8}$$

Suppose that (5.5)-(5.7) are satisfied. Then there exist two positive constants C_1, and C_2 such that

$$C_1 \frac{|\Re \delta W|}{\Re W_0'} \leq |D| \leq C_2 \frac{|\Re \delta W|}{\Re W_0'},$$

where C_1 depends on $d_0, \lambda_0, M_0, \lambda_1, \rho_0, K_0$ and C_2 depends on λ_0, M_0, λ_1, ρ_0, K_0, ρ_1, K_1, ϱ, $|\Omega|$, d_0, d_1, and $\|g\|_{L^2(\partial\Omega)}/\|g\|_{H^{-1/2}(\partial\Omega)}$.

We remark that the fatness condition (5.8) is only used to obtain an upper bound of $|D|$. In [BFV], an estimate was obtained for general inclusions without the fatness condition (5.8). This was possible because under their conditions, u_0 satisfies a doubling inequality, which is not known in the case considered here. The proof of Theorem 5.1 relies on the following energy inequalities proved in [CNW]. We include the proof here for readers' convenience.

Proposition 5.1 [CNW, Proposition 5] *Assume that (5.5) and (5.7) hold. Then there exist positive constants $\kappa_1 = \kappa_1(\lambda_0, \lambda_1, \varrho)$ and $\kappa_2 = \kappa_2(\lambda_0, \lambda_1)$ such that*

- *If (5.6)(i) holds then*

$$\kappa_1 \int_D |\nabla u_0|^2 \leq \Re \delta W \leq \kappa_2 \int_D |\nabla u_0|^2. \tag{5.9}$$

- *If (5.6)(ii) holds then*

$$\kappa_1 \int_D |\nabla u_0|^2 \leq -\Re \delta W \leq \kappa_2 \int_D |\nabla u_0|^2. \tag{5.10}$$

Proof. In this proof, C denotes a positive constant whose value may change from line to line. We have, for $j = 0, 1$,

$$\begin{pmatrix} \Re I_j(\nabla u_j) \\ \Im I_j(\nabla u_j) \end{pmatrix} = \begin{pmatrix} \sigma_j + \zeta_j & -\varepsilon_j \\ \varepsilon_j & \sigma_j - \zeta_j \end{pmatrix} \begin{pmatrix} \Re \nabla u_j \\ \Im \nabla u_j \end{pmatrix}, \tag{5.11}$$

where we set $\zeta_0 = 0$. Following Cherkaev and Gibiansky [CG], (5.11) can be expressed as

$$\begin{pmatrix} \Re \nabla u_j \\ \Im I_j(\nabla u_j) \end{pmatrix} = \begin{pmatrix} (\sigma_j + \zeta_j)^{-1} & (\sigma_j + \zeta_j)^{-1}\varepsilon_j \\ \varepsilon_j(\sigma_j + \zeta_j)^{-1} & \sigma_j - \zeta_j + \varepsilon_j(\sigma_j + \zeta_j)^{-1}\varepsilon_j \end{pmatrix} \begin{pmatrix} \Re I_j(\nabla u_j) \\ \Im \nabla u_j \end{pmatrix}. \tag{5.12}$$

We denote the square matrix on the right hand side B_j and $v_j = (\Re I_j(\nabla u_j), \Im \nabla u_j)^t$. From our assumptions, it can be easily checked that B_j is positive-definite. Using (5.12), we have, for $j, k \in \{0, 1\}$,

$$\begin{aligned}\int_\Omega B_j v_j \cdot v_k &= \int_\Omega \Re \nabla u_j \cdot \Re I_k(\nabla u_k) + \Im I_j(\nabla u_j) \cdot \Im \nabla u_k \\ &= \Re \int_\Omega \nabla \cdot (I_k(\nabla u_k) \Re u_j) + \Im \int_\Omega \nabla \cdot (I_j(\nabla u_j) \Im u_k) \\ &= \int_{\partial\Omega} (\Re u_j \Re h + \Im u_k \Im h).\end{aligned} \tag{5.13}$$

From this and $B_1 = B_1^T$, we can deduce that

$$\Re \delta W = \Re \int_{\partial\Omega} (\bar{u}_0 - \bar{u}_1) h = \int_\Omega (B_0 - B_1) v_0 \cdot v_1 = \int_D (B_0 - B_1) v_0 \cdot v_1. \tag{5.14}$$

Using (5.13) and $B_0 = B_0^T$, we obtain

$$\begin{aligned}\int_\Omega B_0(v_0 - v_1) \cdot (v_0 - v_1) &= \int_\Omega (B_0 v_0 \cdot v_0 - 2B_0 v_0 \cdot v_1 + B_1 v_1 \cdot v_1) \\ &\quad + \int_\Omega (B_0 - B_1) v_1 \cdot v_1 \\ &= \Re \int_{\partial\Omega} (u_1 - u_0) h + \int_D (B_0 - B_1) v_1 \cdot v_1.\end{aligned} \tag{5.15}$$

Thus,

$$\Re \delta W = \int_\Omega B_0(v_0 - v_1) \cdot (v_0 - v_1) + \int_D (B_1 - B_0) v_1 \cdot v_1. \tag{5.16}$$

Swapping the indices 0 and 1, we obtain a similar identity

$$\Re \delta W = -\int_\Omega B_1(v_0 - v_1) \cdot (v_0 - v_1) + \int_D (B_1 - B_0) v_0 \cdot v_0. \tag{5.17}$$

Here we have used the fact that $\operatorname{supp}(B_1 - B_0) \subset \bar{D}$.

Case 1: (5.6)(i) is satisfied.

The positivity of B_1 and (5.17) give the second inequality of (5.9). Using (5.16) and the triangle inequality, we see that the first inequality of (5.9) follows if $B_1 - B_0$ is positive-definite. We have $B_1 - B_0 = P + Q$, where

$$P = \begin{pmatrix} M & M\varepsilon_1 \\ \varepsilon_1 M & N + \varepsilon_1 M \varepsilon_1 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & \sigma_0^{-1}(\varepsilon_1 - \varepsilon_0) \\ (\varepsilon_1 - \varepsilon_0)\sigma_0^{-1} & (\varepsilon_1 - \varepsilon_0)\sigma_0^{-1}\varepsilon_1 + \varepsilon_0 \sigma_0^{-1}(\varepsilon_1 - \varepsilon_0) \end{pmatrix},$$

and $M = (\sigma_1 + \zeta_1)^{-1} - \sigma_0^{-1}$, $N = (\sigma_1 - \zeta_1) - \sigma_0$. From (5.6)(i), it follows that M and N are positive-definite.

For $X = (p, q)^T \in \mathbb{R}^{2n}$, we have

$$PX \cdot X = M(p + \varepsilon_1 q) \cdot (p + \varepsilon_1 q) + Nq \cdot q$$

and

$$|QX \cdot X| \leq C\|\varepsilon_1 - \varepsilon_0\|_{L^\infty(D)}|X|.$$

The required positivity then follows from condition (5.7) for δ small depending on α, ϱ.

Case 2: (5.6)(ii) is satisfied.

Similar argument as in the previous case shows that $B_0 - B_1$ is positive-definite. Thus, the first inequality of (5.10) follows from (5.17). To prove the second inequality of (5.10), we first deduce from (5.16) and (5.17) that

$$\begin{aligned}\int_D (B_0 - B_1)v_1 \cdot v_1 &= -\Re\delta W + \int_\Omega B_0(v_0 - v_1) \cdot (v_0 - v_1) \\ &\leq -\Re\delta W + C\int_\Omega B_1(v_0 - v_1) \cdot (v_0 - v_1) \\ &= -(C+1)\Re\delta W + C\int_D (B_1 - B_0)v_0 \cdot v_0. \end{aligned} \tag{5.18}$$

Using (5.14) and Cauchy-Schwarz we have

$$-\Re\delta W \leq 2(C+1)\int_D (B_0 - B_1)v_0 \cdot v_0 + \frac{1}{2(C+1)}\int_D (B_0 - B_1)v_1 \cdot v_1. \tag{5.19}$$

Eestimating the last term of (5.19) by (5.18), we obtain

$$-\Re\delta W \leq 4(C+1)\int_D (B_0 - B_1)v_0 \cdot v_0. \tag{5.20}$$

The proof is complete. □

Another tool we need is the following Lipschitz propagation of smallness which is a consequence of Theorem 4.1.

Proposition 5.2 *Assume that $\partial\Omega \in C^{3,1}$. Let u_0 be the solution of (5.3). Then for any $a > 0$ and every $x \in \Omega_{4a}$, we have*

$$\int_{B_a(x)} |\nabla u_0|^2 \geq C_a \int_\Omega |\nabla u_0|^2, \tag{5.21}$$

where C_a depends on λ_0, M_0, ρ_1, M_1, ρ_0, K_0, $|\Omega|$, a, and $\|g\|_{L^2(\partial\Omega)}/\|g\|_{H^{-1/2}(\partial\Omega)}$.

Before proving this proposition, we first establish a technical lemma.

Lemma 5.1 *Let the assumptions described in Proposition 5.2 hold. Then for $a > 0$ sufficiently small, we have*

$$\int_{\Omega\setminus\Omega_{a/4}} |\nabla u_0|^2 \leq C a^{1/n} \|g\|^2_{L^2(\partial\Omega)}. \tag{5.22}$$

Proof. Let c_0 be a positive number such that $4c_0 < \mathrm{dist}(\partial\Omega, \Sigma)$. Denote $\mathcal{U}_{c_0} = \Omega\setminus\Omega_{c_0}$, that is, $\mathcal{U}_{c_0} = \{x \in \Omega \,:\, \mathrm{dist}(x, \partial\Omega) \leq c_0\}$. Let η_0 be a cut-off function, such that $\eta_0 \in H^1(\Omega)$ and

$$\eta_0 = \begin{cases} 1 & \text{in } \mathcal{U}_{c_0} \\ 0 & \text{in } \Omega \setminus \mathcal{U}_{2c_0} = \overline{\Omega}_{2c_0} \end{cases}$$

Now let $u = u_0\eta_0$, since $\nabla \cdot I_0(\nabla u_0) = 0$, we have

$$\nabla \cdot I_0(\nabla u) = \left[2I_0(\nabla\eta_0) \cdot \nabla u_0 + u_0 \nabla \cdot (I_0(\nabla\eta_0)\right] \chi_{\mathcal{U}_{2c_0}}$$

Moreover, since $u \equiv 0$ in $\overline{\Omega}_{2c_0}$, we have actually that $\nabla \cdot I_0(\nabla u) = \nabla \cdot A^0_+(\nabla u)$, hence

$$\nabla \cdot (A^0_+ \nabla u) = \left[2I_0(\nabla\eta_0) \cdot \nabla u_0 + u_0 \nabla \cdot (I_0(\nabla\eta_0)\right] \chi_{\mathcal{U}_{2c_0}}$$

We now apply Theorem 2.2, Ch. 4, Sect. 4.2.7, in [N] and get that

$$\|u\|_{H^2(\Omega)} \leq C \left(\|f\|_{L^2(\Omega)} + \|g\|_{H^{1/2}(\partial\Omega)}\right) \tag{5.23}$$

where

$$f = \left[2I_0(\nabla\eta_0) \cdot \nabla u_0 + u_0 \nabla \cdot (I_0(\nabla\eta_0)\right] \chi_{\mathcal{U}_{2c_0}}$$

and C depends on $\lambda_0, M_0, \rho_1, M_1$. We can estimate

$$\begin{aligned} \|f\|_{L^2(\Omega)} = \|f\|_{L^2(\mathcal{U}_{2c_0})} &\leq C \left(\|\nabla\eta \cdot \nabla u_0\|_{L^2(\mathcal{U}_{2c_0}} + \|u_0 \nabla \cdot (I_0(\nabla\eta_0)\|_{L^2(\mathcal{U}_{2c_0}}\right) \\ &\leq C\|u_0\|_{H^1(\Omega)} \leq C\|g\|_{H^{-1/2}(\partial\Omega)}. \end{aligned} \tag{5.24}$$

Here C depends on $\lambda_0, M_0, \rho_1, M_1$, and c_0. Combining (5.23), (5.24), and $\eta_0 = 1$ in $\mathcal{U}_{c_0}$, we obtain

$$\|u_0\|_{H^2(\mathcal{U}_{c_0})} \leq C\|g\|_{H^{1/2}(\partial\Omega)}.$$

This estimate, together with the standard energy estimate

$$\|u_0\|_{H^1(\mathcal{U}_{c_0})} \leq C\|g\|_{H^{-1/2}(\partial\Omega)}$$

gives, by interpolation,

$$\|u_0\|_{H^{3/2}(\mathcal{U}_{c_0})} \leq C\|g\|_{L^2(\partial\Omega)}. \tag{5.25}$$

Hölder inequality implies

$$\|\nabla u_0\|^2_{L^2\left(\Omega\setminus\Omega_{a/4}\right)} \leq \left|\Omega \setminus \Omega_{a/4}\right|^{1/n} \|\nabla u_0\|^2_{L^{2n/(n-1)}(\Omega\setminus\Omega_{a/4})}. \tag{5.26}$$

By Sobolev inequality and if $a/4 \leq c_0$, we have

$$\|\nabla u_0\|^2_{L^{2n/(n-1)}(\Omega\setminus\Omega_{a/4})} \leq C\|\nabla u_0\|^2_{H^{1/2}(\Omega\setminus\Omega_{a/4})} \leq C\|u_0\|_{H^{3/2}(\mathcal{U}_{c_0})}. \tag{5.27}$$

From (5.26), (5.27) and (5.25), and in view of the estimate

$$|\Omega \setminus \Omega_{a/4}| \leq Ca$$

(see [AR, (A.3)]), we finally prove (5.22). □

We now prove Proposition 5.2.

Proof. It suffices to consider a sufficiently small. Assume $a < h_0$, where h_0 is the constant given in Theorem 4.1. To apply Theorem 4.1, we choose $h = a/3$ and obtain

$$\|u_0\|_{L^2(\Omega_{a/3})} \leq C\|u_0\|^{\delta}_{L^2(B_a(x))}\|u_0\|^{1-\delta}_{L^2(\Omega)}. \tag{5.28}$$

Using Caccioppoli's inequality and a generalized Poincaré inequality [AMR, (3.8)], we deduce from (5.28)

$$\|\nabla u_0\|_{L^2(\Omega_{a/4})} \leq C\|\nabla u_0\|^{\delta}_{L^2(B_a(x))}\|\nabla u_0\|^{1-\delta}_{L^2(\Omega)},$$

i.e.,

$$\frac{\|\nabla u_0\|_{L^2(\Omega_{a/4})}}{\|\nabla u_0\|_{L^2(\Omega)}} \leq C\left(\frac{\|\nabla u_0\|_{L^2(B_a(x))}}{\|\nabla u_0\|_{L^2(\Omega)}}\right)^{\delta}. \tag{5.29}$$

With the help of (5.22) and the estimate

$$\|g\|^2_{H^{-1/2}(\partial\Omega)} \leq C\|\nabla u_0\|^2,$$

we can see that

$$\frac{\|\nabla u_0\|^2_{L^2(\Omega_{a/4})}}{\|\nabla u_0\|^2_{L^2(\Omega)}} = 1 - \frac{\|\nabla u_0\|^2_{L^2(\Omega\setminus\Omega_{a/4})}}{\|\nabla u_0\|^2_{L^2(\Omega)}} \geq 1 - Ca^{1/n}\frac{\|g\|^2_{L^2(\partial\Omega)}}{\|g\|^2_{H^{-1/2}(\partial\Omega)}} \geq \frac{1}{2} \tag{5.30}$$

by taking a sufficiently small. Estimate (5.21) follows immediately from (5.29) and (5.30). □

Finally, we provide the proof of the size estimate result.

Proof of Theorem 5.1. We follow the proof of Theorem 2.1 in [ARS, page 61]. By the estimate for elliptic system with discontinuous coefficients [LN, Theorem 1.1] and the Poincaré inequality, we have the interior estimate

$$\sup_D |\nabla u_0| \leq C\|\nabla u_0\|_{L^2(\Omega)} \leq C\left(\Re W_0'\right)^{1/2},$$

where C depends on $\lambda_0, M_0, \rho_0, K_0, d_0$. We want to point out that the equation $\nabla \cdot I_0(\nabla u_0) = 0$ can be transformed to the elliptic system with required conditions

considered in [LN, Theorem 1.1]. Applying the second inequality of (5.9) or (5.10), we obtain

$$\Re\delta W \leq \kappa_2|D|\left(\sup_D|\nabla u_0|\right)^2 \leq C_1^{-1}|D|\Re W_0',$$

which gives a lower bound of $|D|$.

Let $\ell = \min(d_0/2, d_1/2)$ and cover D_{d_1} with squares $\{Q_k\}_{k=1,\dots,N}$ of side length ℓ and disjoint interiors. It is clear that $N \geq \ell^{-2}|D_{d_1}| \geq \frac{1}{2\ell^2}|D|$. Applying Proposition 5.2 with $a = \ell/2$ we see that $\int_{Q_k}|\nabla u_0|^2 \geq C\Re W_0'$, hence

$$\int_D |\nabla u_0|^2 \geq \sum_{k=1}^N \int_{Q_k} |\nabla u_0|^2 \geq NC\Re W_0' \geq C|D|\Re W_0'.$$

Combining this with the first inequality of (5.9) or (5.10) provides an upper bound of $|D|$. □

Acknowledgements

The research of Francini and Vessella was partly funded by: (a) Research Project 201758MTR2 of the Italian Ministry of Education, University and Research (MIUR) Prin 2017 Direct and inverse problems for partial differential equations: theoretical aspects and applications; (b) GNAMPA of the Italian INdAM National Institute of High Mathematics (grant number not available). Wang was partially supported by MOST 108-2115-M-002-002-MY3 & 109-2115-M-002-001-MY3.

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