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C-Finite Sequences and Riordan Arrays

Donatella Merlini

Dipartimento di Statistica, Informatica, Applicazioni, Università di Firenze, I-50134 Firenze, Italy;
donatella.merlini@unifi.it

Abstract: Many prominent combinatorial sequences, such as the Fibonacci, Lucas, Pell, Jacobsthal and Tribonacci sequences, are defined by homogeneous linear recurrence relations with constant coefficients. These sequences are often referred to as *C*-finite sequences, and a variety of representations have been employed throughout the literature, largely influenced by the author's background and the specific application under consideration. Beyond the representation through recurrence relations, other approaches include those based on generating functions, explicit formulas, matrix exponentiation, the method of undetermined coefficients and several others. Among these, the generating function approach is particularly prevalent in enumerative combinatorics due to its versatility and widespread use. The primary objective of this work is to introduce an alternative representation grounded in the theory of Riordan arrays. This representation provides a general formula expressed in terms of the vectors of constants and initial conditions associated with any recurrence relation of a given order, offering a new perspective on the structure of such sequences.

Keywords: Riordan arrays; combinatorial identities; generating functions; *C*-finite sequences

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1. Introduction

According to recent terminology (see, e.g., [1,2]), a *C-finite sequence* is any sequence defined by a homogeneous linear recurrence relation with constant coefficients. If $(u_k)_{k\in\mathbb{N}}$ is a *C*-finite sequence, the recurrence relation is written as follows:

$$u_{n+\omega} = c_1 u_{n+\omega-1} + c_2 u_{n+\omega-2} + \cdots + c_\omega u_n, \qquad (1)$$

where $c_1, c_2, \ldots, c_\omega \in \mathbb{Q}$ are constant with respect to n, $c_\omega \neq 0$ and ω is the *order* of the recurrence. The sequence is completely defined when we fix the *initial conditions*, that is, the value of the first ω elements (not all 0) of the sequence $u_0, u_1, \ldots u_{\omega-1}$. The defining recurrence relation (1) has the following obvious "generalization":

$$c_0 u_{n+\omega} = c_1 u_{n+\omega-1} + c_2 u_{n+\omega-2} + \cdots + c_\omega u_n,$$

with $c_0 \neq 0$, but obviously the two formulations are equivalent. Therefore, we will always suppose $c_0 = 1$.

Surely, *C*-finite sequences are the simplest and most used numeric sequences, for example, the well-known Fibonacci sequence, the properties of which have been widely studied and applied. Because of that, a number of different representations have been adopted in the literature, often depending on the background of the author and/or on the properties related to the specific application.

Among the various representations, we obviously have recurrence relations which, according to the definition of a *C*-finite sequence, have to be homogeneous, linear and with constant coefficients. For the relation (1), if $C = [c_1, c_2, \ldots, c_\omega]$ is the vector of coefficients and $U = [u_0,\ u_1,\ \dots,\ u_{\omega-1}]$ is the vector of initial conditions, we use the following notation:

$$(u_k)_{k\in\mathbb{N}} = \Re([c_1,\ c_2,\ \dots,\ c_\omega],\ [u_0,\ u_1,\ \dots,\ u_{\omega-1}]) = \Re^{(\omega)}(C, U), \tag{2}$$

where the order can be understood when no ambiguity arises. For the well-known Fibonacci numbers, for example, we have the following:

$$(F_k)_{k\in\mathbb{N}} = \Re([1,\ 1], [0,\ 1]) \qquad \text{corresponding to} \qquad F_{n+2} = F_{n+1} + F_n$$

valid for every $n \in \mathbb{N}$, and with $F_0 = 0,\ F_1 = 1$ as initial conditions.

Another representation is in terms of an explicit formula: a mathematical formula depending on a parameter n which returns u_n, the element at position n in the sequence. Especially important are *closed formulas*, that is, formulas the evaluation of which does not require a number of arithmetic operations depending on n. For Fibonacci numbers we have the well-known *Binet formula*:

$$F_n = \frac{\phi^n - \widehat{\phi}^n}{\sqrt{5}},$$

where $\phi = (\sqrt{5} + 1)/2 \sim 1.618033988$ is the golden ratio and $\widehat{\phi} = -1/\phi$.

Generating functions correspond to another important representation; it is well known that a sequence $(u_k)_{k\in\mathbb{N}}$ is C-finite if and only if its generating function is a rational function $u(t) = P(t)/Q(t)$, where $P(t),\ Q(t)$ are two polynomials having $\deg(P(t)) < \deg(Q(t))$. The function $u(t)$ contains the whole information relative to the sequence, so it is the ideal approach to most problems relative to C-finite sequences. A typical example is the generating function of Fibonacci numbers:

$$F(t) = \frac{t}{1 - t - t^2},$$

from which so many properties of Fibonacci numbers can be deduced, in particular the Binet formula.

Moreover, the evaluation of u_n through the recurrence relation can be seen as the computation of some power of a suitable *matrix*. In this case also, the appropriate example is given by the Fibonacci numbers. If we define

$$\mathbb{F} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

then by computing the power $\mathbb{F}^n$ we find a matrix containing some elements of the sequence:

$$\mathbb{F}^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}.$$

The method can be generalized to all C-finite sequences and allows many of their properties to be easily proven.

Another peculiar representation is by points: if we know the order ω of a C-finite sequence (or, at least, an upper bound thereof), we can represent it by a subsequence of 2ω consecutive elements. In fact, by substituting these values in (1), we obtain ω equations in the ω unknowns $c_1,\ c_2,\ \cdots,\ c_\omega$. If the points belong to the sequence, the system so obtained has one and only one solution. For example, the subsequence (5, 8, 13, 21) defines the Fibonacci numbers, the order of which is 2; from these points, we obtain the following system:

$$\begin{cases} 13 = 8c_1 + 5c_2 \\ 21 = 13c_1 + 8c_2 \end{cases}$$

with the obvious solution $c_1 = c_2 = 1$. This representation has been intensively used by Doron Zeilberger [2] and known as the method of *indeterminate coefficients*.

The previous and other representations can be found in [1–3]. In Table 1, we show the first values of some well-known C-finite sequences which will be used as examples in this paper; they are found, among others, in the *On-Line Encyclopedia of Integer Sequences* [4] and we attach a label Axxxxxx to a sequence if it appears in the OEIS with that identifier.

Table 1. Examples of C-finite sequences and their OEIS identifiers.

NAME	OEIS	ω	0	1	2	3	4	5	6	7	8	9	10
Fibonacci	A000045	2	0	1	1	2	3	5	8	13	21	34	55
Lucas	A000032	2	2	1	3	4	7	11	18	29	47	76	123
Pell	A000129	2	0	1	2	5	12	29	70	169	408	985	2378
Jacobsthal	A001045	2	0	1	1	3	5	11	21	43	85	171	341
Pell–Lucas	A002203	2	2	2	6	14	34	82	198	478	1154	2786	6726
$3^n - 2^n$	A001047	2	0	1	5	19	65	211	665	2059	6305	19,171	58,025
n	A001477	2	0	1	2	3	4	5	6	7	8	9	10
Tribonacci	A000073	3	0	1	1	2	4	7	13	24	44	81	149

In this work, we aim to introduce a novel representation for C-finite sequences, utilizing the framework of *Riordan arrays*, introduced in the literature in [5,6] as a generalization of the Pascal triangle.

A Riordan array $D = \mathcal{R}(d(t), h(t))$ is defined in terms of two generating functions $d(t)$ and $h(t)$ with $d(0) \neq 0$, $h(0) = 0$, $h'(0) \neq 0$ and corresponds to an infinite matrix $(d_{n,k})_{n,k\in\mathbb{N}}$, where the generic element can be found by extracting the coefficient of t^n from the generating function $d(t)h(t)^k$. Many properties of Riordan arrays have been studied in the literature; in particular, they are recognized as a powerful tool for proving combinatorial identities (see, e.g., [6,7]). Actually, if $(f_k)_{k\in\mathbb{N}}$ is any sequence having $f(t)$ as its generating function, it can be proven that the computation of the combinatorial sum $\sum_{k=0}^{n} d_{n,k} f_k$ can be reduced to the extraction of the coefficient from the generating function $d(t)f(h(t))$, obtained by transforming $f(t)$ in terms of the functions $d(t)$ and $h(t)$. In this paper, by using an approach based on Riordan arrays, we are able to present a new representation for any C-finite sequence, which in the case of Fibonacci numbers gives the following formula:

$$F_n = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-k-1}{n-2k-1}.$$

In Section 2, we provide the necessary theoretical foundation for understanding generating functions and Riordan arrays. Subsequently, in Section 3, we delve into the connection between C-finite sequences and Riordan arrays. A noteworthy feature of the representation discussed in this paper is that it can be generalized to entire classes of C-finite sequences of the same order, offering a unified approach to their study. A preliminary version of these results has been presented in [8].

2. Generating Functions and Riordan Arrays

One of the most important representations of C-finite sequences is through generating functions. Generating functions correspond to one of the most popular approaches to combinatorial problems and a clear exposition of this concept can be found in [9–11]. In fact, generating functions belong to the broader framework of the *method of coefficients* (see, e.g., [12,13]), which provides a unified approach to handling the algebraic properties of various types of sequences, particularly in the realm of enumerative combinatorics.

Let us consider a sequence of numbers $(f_0, f_1, f_2, \ldots) = (f_k)_{k\in\mathbb{N}}$; the *generating function* for the sequence $(f_k)_{k\in\mathbb{N}}$ is defined as the formal power series $f(t) = f_0 + f_1 t + f_2 t^2 + \cdots$, where the indeterminate t is arbitrary. Given the sequence $(f_k)_{k\in\mathbb{N}}$, we introduce the *generating function operator* $\mathcal{G}$, which applied to $(f_k)_{k\in\mathbb{N}}$ produces the generating function for the sequence, i.e., $\mathcal{G}(f_k)_{k\in\mathbb{N}} = f(t)$. The operator $\mathcal{G}$ is clearly linear:

$$\mathcal{G}(\alpha f_k + \beta g_k) = \alpha\mathcal{G}(f_k) + \beta\mathcal{G}(g_k) \tag{3}$$

and the function $f(t)$ can be shifted:

$$\mathcal{G}(f_{k+1}) = \frac{\mathcal{G}(f_k) - f_0}{t}. \tag{4}$$

For the Fibonacci sequence $(F_k)_{k\in\mathbb{N}}$, we have the following:

$$\mathcal{G}(F_{n+2}) = \mathcal{G}(F_{n+1}) + \mathcal{G}(F_n)$$

and by setting $F(t) = \mathcal{G}(F_n)$ we find the following:

$$\frac{F(t) - F_0 - F_1 t}{t^2} = \frac{F(t) - F_0}{t} + F(t).$$

Because we know that $F_0 = 0, F_1 = 1$, we have the following:

$$F(t) - t = tF(t) + t^2 F(t)$$

and by solving in $F(t)$ we have the explicit generating function:

$$F(t) = \frac{t}{1 - t - t^2}.$$

Moreover, the notation $[t^n]f(t)$ indicates the extraction of the coefficient of t^n from $f(t)$ and is known as the *coefficient operator*, $[t^n]f(t) = f_n$. The linearity and shifting properties also apply to the $[t^n]$ operator:

$$[t^n](\alpha f(t) + \beta g(t)) = \alpha[t^n]f(t) + \beta[t^n]g(t), \tag{5}$$

$$[t^n]t^k f(t) = [t^{n-k}]f(t). \tag{6}$$

An important property of this operator is *Newton's rule*:

$$[t^n](1 + \alpha t)^r = \binom{r}{n}\alpha^n, \tag{7}$$

which is one of the most frequently used results in coefficient extraction. Let us remark explicitly that when $r = -1$ we have the following:

$$[t^n]\frac{1}{1 + \alpha t} = \binom{-1}{n}\alpha^n = \binom{1 + n - 1}{n}(-1)^n\alpha^n = (-\alpha)^n.$$

For example, by using the $[t^n]$ operator, we can find an explicit expression for Fibonacci numbers. The denominator of $F(t)$ can be written as $1 - t - t^2 = (1 - \phi t)(1 - \widehat{\phi} t)$ where

$$\phi = \frac{1 + \sqrt{5}}{2}, \quad \widehat{\phi} = \frac{1 - \sqrt{5}}{2}.$$

By using partial fraction expansion we find the following:

$$F(t) = \frac{t}{(1 - \phi t)(1 - \widehat{\phi} t)} = \frac{A}{1 - \phi t} + \frac{B}{1 - \widehat{\phi} t} = \frac{A - A\widehat{\phi} t + B - B\phi t}{1 - t - t^2}.$$

We determine the two constants A and B by equating the coefficients in the first and last expression for $F(t)$:

$$\begin{cases} A + B = 0 \\ -A\widehat{\phi} - B\phi = 1 \end{cases} \qquad \begin{cases} A = 1/(\phi - \widehat{\phi}) = 1/\sqrt{5} \\ B = -A = -1/\sqrt{5} \end{cases}.$$

The value of F_n is now obtained by extracting the coefficient of t^n:

$$F_n = [t^n]F(t) = [t^n]\frac{1}{\sqrt{5}}\left(\frac{1}{1-\phi t} - \frac{1}{1-\widehat{\phi}t}\right) =$$

$$= \frac{1}{\sqrt{5}}\left([t^n]\frac{1}{1-\phi t} - [t^n]\frac{1}{1-\widehat{\phi}t}\right) = \frac{\phi^n - \widehat{\phi}^n}{\sqrt{5}}.$$

More details and other properties of the operators $\mathcal{G}$ and $[t^n]$ can be found in [13].

The generating functions of C-finite sequences are particularly simple and can be derived directly from their recurrence relation, as previously carried out for Fibonacci numbers.

Theorem 1. *A sequence $(u_k)_{k\in\mathbb{N}}$ is C-finite if and only if its generating function $u(t)$ is a rational function:*

$$u(t) = \frac{P(t)}{Q(t)} = \frac{a_0 + a_1 t + a_2 t^2 + \cdots + a_k t^k}{1 - c_1 t - c_2 t^2 - \cdots - c_\omega t^\omega},$$

where $k = \deg(P(t)) < \omega = \deg(Q(t))$.

Proof. The theorem is well known and is sometimes traced back to A. de Moivre [14], so we limit ourselves to a simple sketch. If $(u_k)_{k\in\mathbb{N}}$ is C-finite, relation (1) holds for every $n \in \mathbb{N}$ and we can apply the rules of the generating function operator linearity and shifting, this latter in the general form

$$\mathcal{G}(u_{n+j}) = \frac{u(t) - u_0 - u_1 t - \cdots - u_{j-1}t^{j-1}}{t^j}$$

for $j = 0, 1, \ldots, \omega$. Multiplying everything by t^ω and isolating $u(t)$, we obtain $Q(t)u(t) = P(t)$ where $\deg(P(t))$ is less than $\deg(Q(t))$ by construction. For the converse, let $u(t) = P(t)/Q(t)$, write this identity as $Q(t)u(t) = P(t)$ and extract the coefficient of $t^{n+\omega}$:

$$\begin{aligned}[t^{n+\omega}]Q(t)u(t) &= [t^{n+\omega}](1 - c_1 t - c_2 t^2 - \cdots - c_\omega t^\omega)u(t) \\ &= [t^{n+\omega}]u(t) - c_1[t^{n+\omega-1}]u(t) - c_2[t^{n+\omega-2}]u(t) - \cdots - c_\omega[t^n]u(t) \\ &= u_{n+\omega} - c_1 u_{n+\omega-1} - c_2 u_{n+\omega-2} - \cdots - c_\omega u_n.\end{aligned}$$

On the other hand, we have $[t^{n+\omega}]P(t) = 0$, since $\deg(P(t)) < \deg Q(t) = \omega$, and this completes the proof. □

Some important observations are to be emphasized relative to the conditions of this theorem. First of all, the last recurrence constants c_ω and at least one among the initial conditions must be different from 0. In fact, in the former case, $c_\omega = 0$ would imply an order less than ω; in the latter case, we would obtain the 0 sequence in correspondence of every other specification. Finally, let us consider an example of a generating function for which the condition $\deg(P(t)) < \deg(Q(t))$ is not true:

$$G(t) = \frac{3 + t - 3\,t^2 - 4\,t^3 - t^4}{1 - t - t^2} = 3 + 4\,t + 4\,t^2 + 4\,t^3 + 7\,t^4 + 11\,t^5 + 18\,t^6 + 29\,t^7 + 47\,t^8 + \cdots.$$

Since the denominator is the same as that of Fibonacci numbers, the relation $G_{n+2} = G_{n+1} + G_n$ should be valid. However, by a simple inspection, we see that the relation holds true only from the fifth element on, with initial conditions $G_0 = 4$ and $G_1 = 7$. If we perform the division, we obtain $1 + 3t + t^2$ as quotient and $2 - t$ as remainder; therefore,

$$G(t) = 1 + 3t + t^2 + \frac{2 - t}{1 - t - t^2};$$

now, everything is clear and we can go on ignoring the first elements, not conforming to the recurrence relation.

A (proper) Riordan array is defined by a pair of generating functions $D = \mathcal{R}(d(t), h(t))$ with $d(0) \neq 0$, $h(0) = 0$, $h'(0) \neq 0$ and the usual way to represent the Riordan array $\mathcal{R}(d(t), h(t))$ is by means of an infinite matrix $(d_{n,k})_{n,k \in \mathbb{N}}$, its generic element being as follows:

$$d_{n,k} = [t^n]d(t)h(t)^k. \tag{8}$$

If $h'(0) = 0$, the Riordan array is said to be *non-proper*; moreover, we point out that (8) is also well defined in the case $d(0) = 0$. The Pascal triangle is simply the following case:

$$P = \mathcal{R}\left(\frac{1}{1-t}, \frac{t}{1-t}\right),$$

while

$$C = \mathcal{R}\left(\frac{1 - \sqrt{1-4t}}{2t}, \frac{1 - \sqrt{1-4t}}{2}\right)$$

represents the Catalan triangle. The first six rows of the involved matrices are given below:

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 & 0 \\ 1 & 4 & 6 & 4 & 1 & 0 \\ 1 & 5 & 10 & 10 & 5 & 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 & 0 & 0 \\ 5 & 5 & 3 & 1 & 0 & 0 \\ 14 & 14 & 9 & 4 & 1 & 0 \\ 42 & 42 & 28 & 14 & 5 & 1 \end{bmatrix}.$$

Both triangles are very well known and studied in the literature.

Many properties of Riordan arrays, in particular their connection with combinatorial sums, have been studied in the literature and are collected in the recent book [15]. Actually, if $(f_k)_{k \in \mathbb{N}}$ is any sequence having $f(t) = \sum_{k=0}^{\infty} f_k t^k$ as its generating function, it is possible to prove the following:

$$\sum_{k=0}^{n} d_{n,k} f_k = [t^n]d(t)f(h(t)) \tag{9}$$

thus reducing the sum to the extraction of a coefficient from a formal power series by using the operator $[t^n]$ (see, e.g., [6,7]). Equation (9) is known as the *Fundamental Rule of Riordan Arrays* (FRRA) and reduces to the partial sum theorem

$$\sum_{k=0}^{n} f_k = [t^n]\frac{f(t)}{1-t},$$

when

$$D = \mathcal{R}\left(\frac{1}{1-t}, t\right),$$

and to the Euler transformation in the case of Pascal triangle:

$$\sum_{k=0}^{n} \binom{n}{k} f_k = [t^n]\frac{1}{1-t} f\left(\frac{t}{1-t}\right).$$

Let us see an application related to Harmonic Numbers $H_n = \sum_{k=1}^{n} \frac{1}{k}$, starting from the generating function of the sequence $(0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots)$:

$$\mathcal{G}\left(\frac{1}{n}\right) = \ln\frac{1}{1-t}$$

and the alternate sign version of it:

$$\mathcal{G}\left(\frac{(-1)^n}{n}\right) = \ln\frac{1}{1+t}.$$

By using the FRRA, we have

$$\mathcal{G}\left(\sum_{k=1}^{n}\frac{1}{k}\right) = \mathcal{G}(H_n) = \frac{1}{1-t}\ln\frac{1}{1-t}$$

and

$$\sum_{k=1}^{n}\binom{n}{k}\frac{(-1)^{k-1}}{k} = [t^n]\frac{1}{1-t}\ln\frac{1}{1-t} = H_n.$$

Equation (9) can be expressed in a different way, by observing that it corresponds to the product between the matrix $(d_{n,k})_{n,k\in\mathbb{N}}$ and the vector of coefficients $(f_k)_{k\in\mathbb{N}}$. For example, the first few rows corresponding to the Euler transformation $\sum_{k=0}^{n}\binom{n}{k} = 2^n$ go as follows:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 & 0 \\ 1 & 4 & 6 & 4 & 1 & 0 \\ 1 & 5 & 10 & 10 & 5 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \\ 8 \\ 16 \\ 32 \end{bmatrix}.$$

This reasoning leads to the following formula, which expresses the FRRA in terms of generating functions:

$$\mathcal{R}(d(t),\, h(t)) * f(t) = d(t)f(h(t)), \tag{10}$$

where $*$ denotes the action of Riordan arrays on generating functions.

More generally, the product can be performed between two Riordan arrays $\mathcal{R}(d(t),\, h(t))$ and $\mathcal{R}(a(t),\, b(t))$, as follows:

$$\mathcal{R}(d(t),\, h(t)) * \mathcal{R}(a(t),\, b(t)) = \mathcal{R}(d(t)a(h(t)),\, b(h(t))). \tag{11}$$

A key characteristic of proper Riordan arrays is their group structure under the specified product operation. For an in-depth discussion on the theory and practical applications of Riordan arrays, readers can refer to [15].

3. Identities for *C*-Finite Sequences

Typically, the FRRA, in both Formulations (9) and (10), is applied from left to right, as demonstrated in the examples of the previous section. In this section, however, we introduce a new approach for representing C-finite sequences, which involves applying the formula from right to left. Specifically, given the generating function of a C-finite sequence of a particular order, the goal is to derive a combinatorial expression for its coefficients through repeated applications of the FRRA. To do so, we consider generating functions of the form $t/(1 + c_1 t + \cdots + c_\omega t^\omega)$ with a simple t at the numerator.

Let $\mathbf{F}$ be the set of all C-finite sequences and $\mathbf{F}_\omega$ denotes the class of C-finite sequences of order ω. Every class is, in its turn, subdivided into families, joining all the C-finite sequences having their generating functions with the same denominator. For example, Fibonacci and Lucas numbers belong to the same family, as do Jacobsthal and Jacobsthal–Lucas numbers. Finally, as the representative of the family $P(t)/(1+c_1t+\cdots+c_\omega t^\omega)$, we choose the sequence whose numerator is t; this sequence will be called the *canonical sequence* of its family. The following theorem assures that by studying the canonical sequence we actually study the representation of all the sequences in the family.

Theorem 2. *Let*

$$u(t)=\frac{P(t)}{Q(t)}=\frac{a_0+a_1t+a_2t^2+\cdots+a_{\omega-1}t^{\omega-1}}{1+c_1t+c_2t^2+\cdots+c_\omega t^\omega}$$

be the generating function of a C-finite sequence of order ω and let $f(t)=t/Q(t)$ be the generating function of the canonical sequence of the relative family; then, u_n can be expressed as a linear combination of at most $\omega-1$ elements of the canonical sequence.

Proof. We make use of the method of coefficients by applying linearity and the shifting property $[t^n]t^kf(t)=[t^{n-k}]f(t)$ whenever $k\le n$. Arranging the powers of t, we obtain

$$\begin{aligned}u_n&=[t^n]\frac{a_0+a_1t+a_2t^2+\cdots+a_{\omega-1}t^{\omega-1}}{1+c_1t+c_2t^2+\cdots+c_\omega t^\omega}\\&=a_0[t^{n+1}]\frac{t}{Q(t)}+a_1[t^n]\frac{t}{Q(t)}+a_2[t^{n-1}]\frac{t}{Q(t)}+\cdots+a_{\omega-1}[t^{n-\omega+2}]\frac{t}{Q(t)}\end{aligned}$$

and $t/Q(t)$ is the generating function of the canonical sequence. □

We can now concentrate on canonical sequences by introducing some new concepts. Let us begin with the following generating functions:

$$\begin{aligned}G^{[k]}(t)&=\frac{1}{1+c_kt}=1-c_kt+c_k^2t^2-c_k^3t^3+c_k^4t^4-c_k^5t^5+\cdots,\\Z^{[k]}(t)&=\frac{t}{1+c_1t+c_2t^2+\cdots+c_kt^k},\end{aligned}$$

and then go on with two families of (non-proper) Riordan arrays:

$$\begin{aligned}S^{[k]}&=\mathcal{R}\left(\frac{t}{1+c_1t+c_2t^2+\cdots+c_kt^k},\ \frac{t^{k+1}}{1+c_1t+c_2t^2+\cdots+c_kt^k}\right),\\N^{[k]}&=\mathcal{R}\left(\frac{1}{1+c_kt},\ \frac{t}{1+c_kt}\right).\end{aligned}$$

Moreover, we need the following transformation α:

$$\alpha(\mathcal{R}(d(t),\ h(t))=\mathcal{R}(d(t),\ th(t)),$$

whose effect consists in pushing down by k positions the k-th column of the Riordan array. In fact, we have the following lemma:

Lemma 1. *Let $d_{n,k}$ be any element of the Riordan array $D=\mathcal{R}(d(t),\ h(t))$; then, for any element $\widehat{d}_{n,k}$ of $\alpha(D)=\mathcal{R}(d(t),\ th(t))$, we have $\widehat{d}_{n,k}=d_{n-k,k}$.*

Proof. Clearly, we obtain the following:

$$\widehat{d}_{n,k}=[t^n]d(t)(th(t))^k=[t^{n-k}]d(t)h(t)^k=d_{n-k,k}.$$

□

The theory of Riordan arrays, with formula (10), allows us to prove the following identity.

Theorem 3. *For every positive k, we have* $S^{[k]} * G^{[k+1]}(t) = Z^{[k+1]}(t)$.

Proof. The proof consists in a straightforward computation:

$$\begin{aligned}
S^{[k]} * G^{[k+1]}(t) &= \mathcal{R}\left(\frac{t}{1+c_1t+c_2t^2+\cdots+c_kt^k}, \frac{t^{k+1}}{1+c_1t+c_2t^2+\cdots+c_kt^k}\right) * \frac{1}{1+c_{k+1}t} \\
&= \frac{t}{1+c_1t+c_2t^2+\cdots+c_kt^k} \cdot \frac{1}{1+c_{k+1}t^{k+1}/(1+c_1t+c_2t^2+\cdots+c_kt^k)} \\
&= \frac{t}{1+c_1t+c_2t^2+\cdots+c_kt^k} \cdot \frac{1+c_1t+c_2t^2+\cdots+c_kt^k}{1+c_1t+c_2t^2+\cdots+c_{k+1}t^{k+1}} \\
&= \frac{t}{1+c_1t+c_2t^2+\cdots+c_{k+1}t^{k+1}}
\end{aligned}$$

and this is simply $Z^{[k+1]}(t)$. We explicitly observe that the computation implies the recursive relation $Z^{[k+1]}(t) = Z^{[k]}(t)/(1+c_{k+1}t^kZ^{[k]}(t))$. □

This result has an immediate application. In fact, for $k = 1$, we have

$$S^{[1]} * G^{[2]}(t) = \mathcal{R}\left(\frac{t}{1+c_1t}, \frac{t^2}{1+c_1t}\right) * \frac{1}{1+c_2t} = Z^{[2]}(t) = \frac{t}{1+c_1t+c_2t^2},$$

the generating function of the generic canonical C-finite sequence of order 2. Therefore, we have found a universal formula which can be extended to all C-finite sequences of order 2 by means of Theorem 2. The problem is now to find an explicit formula for the elements of the Riordan array $S^{[1]}$. Happily, the theory supplies a simple method to find $S^{[1]}_{n,k}$:

$$\begin{aligned}
S^{[1]}_{n,k} &= [t^n]\frac{t}{1+c_1t}\left(\frac{t^2}{1+c_1t}\right)^k = [t^{n-2k-1}]\left(\frac{1}{1+c_1t}\right)^{k+1} \\
&= \binom{-k-1}{n-2k-1}c_1^{n-2k-1} = \binom{n-k-1}{n-2k-1}(-c_1)^{n-2k-1}.
\end{aligned}$$

Finally, we compute the following:

$$u_n = \sum_{k=0}^{n} S^{[1]}_{n,k}(-c_2)^k = \sum_{k=0}^{n}\binom{n-k-1}{n-2k-1}(-c_1)^{n-2k-1}(-c_2)^k,$$

and conclude the following:

Theorem 4. *For the canonical sequence of class* $\mathbf{F}_2$, $u_n = \Re([-c_1, -c_2,],\ [0,1])$, *we have the following general formula:*

$$u_n = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor}\binom{n-k-1}{n-2k-1}(-c_1)^{n-2k-1}(-c_2)^k.$$

Proof. At this point, the only thing that remains to be proven concerns the limits of the sum. The binomial coefficient is different from 0 when $n-2k-1 \geq 0$ and $n-k-1-(n-2k-1) \geq 0$. This happens if and only if $0 \leq k \leq \lfloor (n-1)/2 \rfloor$. □

Corollary 1. *The following identities hold true. For Fibonacci numbers* A000045*, defined by* $(F_n) = \Re([1,1],[0,1])$*,*

$$F_n = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-k-1}{n-2k-1}.$$

For Pell numbers A000129*, defined by* $(P_n) = \Re([2,1],[0,1])$*,*

$$P_n = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-k-1}{n-2k-1} 2^{n-2k-1}.$$

For Jacobsthal numbers A001045*, defined by* $(J_n) = \Re([1,2],[0,1])$*,*

$$J_n = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-k-1}{n-2k-1} 2^{k}.$$

For $(3^n - 2^n)$ *numbers* A001047*, defined by* $(D_n) = \Re([5,-6],[0,1])$*,*

$$D_n = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-k-1}{n-2k-1} (-6)^k 5^{n-2k-1}.$$

For n numbers A001477*, defined by* $(N_n) = \Re([2,-1],[0,1])$*,*

$$N_n = n = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-k-1}{n-2k-1} (-1)^k 2^{n-2k-1}.$$

In order to find a general formula for class $\mathbf{F}_\omega$, we proceed in an analogous way.

Lemma 2. *With the notations above, we have*

$$\alpha(S^{[k]} * N^{[k+1]}) = S^{[k+1]}.$$

Proof. By applying the rule for the product of two Riordan arrays, we obtain

$$\begin{aligned} S^{[k]} * N^{[k+1]} &= \mathcal{R}\left(\frac{t}{1 + c_1 t + c_2 t^2 + \cdots + c_k t^k}, \frac{t^{k+1}}{1 + c_1 t + c_2 t^2 + \cdots + c_k t^k} \right) * \\ &* \mathcal{R}\left(\frac{1}{1 + c_{k+1} t}, \frac{t}{1 + c_{k+1} t} \right) \\ &= \mathcal{R}\left(\frac{t}{1 + c_1 t + c_2 t^2 + \cdots + c_{k+1} t^{k+1}}, \frac{t^{k+1}}{1 + c_1 t + c_2 t^2 + \cdots + c_{k+1} t^{k+1}} \right). \end{aligned}$$

Finally, α pushes down by k positions the elements of column k, for every $k \in \mathbb{N}$. □

From Lemma 2 and Theorem 3, we find the following:

Theorem 5. *For every positive k, we have* $\alpha(S^{[k]} * N^{[k+1]}) * G^{[k+2]}(t) = Z^{[k+2]}(t)$*.*

Varying k in Theorem 5, we have the following:

$$Z^{[3]}(t) = \alpha(S^{[1]} * N^{[2]}(t)) * G^{[3]}(t),$$

$$Z^{[4]}(t) = \alpha(\alpha(S^{[1]} * N^{[2]}) * N^{[3]}) * G^{[4]}(t),$$

$$Z^{[5]}(t) = \alpha(\alpha(\alpha(S^{[1]} * N^{[2]}) * N^{[3]}) * N^{[4]}) * G^{[5]}(t)$$

and so on, thus obtaining decompositions in terms of products of Riordan arrays and generating functions, which easily translate into combinatorial identities.

In particular, let us consider the case $\omega = 3$, that is, recurrence relations of the third order. For the canonical sequences we have the following:

Theorem 6. *For the canonical sequence of class* $\mathbf{F}_3$, $u_n = \Re([-c_1, -c_2, -c_3],\ [0, 1, -c_1])$, *we have the following general formula:*

$$u_n = \sum_{k=0}^{n} \sum_{j=0}^{\lfloor (n-k-1)/2 \rfloor} \binom{n-k-j-1}{n-k-2j-1} \binom{j}{k} (-c_1)^{n-k-2j-1} (-c_2)^{j-k} (-c_3)^k.$$

Proof. We begin by computing the product $S^{[1]} * N^{[2]}$, the generic element of which is

$$\sum_{j=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-j-1}{n-2j-1} \binom{j}{k} (-c_1)^{n-2j-1} (-c_2)^{j-k}.$$

The next step consists in applying the transformation α, but, as we have seen, it reduces to the change in variable $n \mapsto n-k$. The last step is to perform the product by $G^{[3]}(t)$, which is carried out by applying the fundamental rule of Riordan arrays. □

As a simple but meaningful example, we consider Tribonacci numbers A000073

$$(T_n) = \Re([1, 1, 1],\ [0, 1, 1]),$$

corresponding to the following canonical generating function:

$$T(t) = \frac{t}{1 - t - t^2 - t^3}.$$

The previous theorem gives the following identity:

$$T_n = \sum_{k=0}^{n} \sum_{j=0}^{\lfloor (n-k-1)/2 \rfloor} \binom{n-k-j-1}{n-k-2j-1} \binom{j}{k}.$$

4. Conclusions

In conclusion, we have explored various well-known combinatorial sequences, such as the Fibonacci, Lucas, Pell, Jacobsthal and Tribonacci sequences, all of which are defined by homogeneous linear recurrence relations with constant coefficients and belong to the class of C-finite sequences. Throughout the literature, several representations for these sequences have been adopted, each influenced by the author's approach or the specific context of the application.

In this work, we focused on introducing a new representation based on the theory of Riordan arrays. Unlike traditional methods, such as generating functions, explicit formulas, or matrix exponentiation, our approach provides a unified formula that encapsulates both the constants and initial conditions of the involved recurrence relation. This offers a new perspective on how C-finite sequences can be studied.

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