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The price-leverage covariation as a measure of the response of the leverage effect to price and volatility changes

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Abstract

We study the sensitivity of the leverage effect to changes of the volatility and the price, showing the existence of an analytical link between the latter and the price-leverage covariation in settings with, respectively, stochastic and level-dependent volatility. From the financial standpoint, the results we obtain allow for the interpretation of the price-leverage covariation as a gauge of the responsiveness of the leverage effect to price and volatility changes. The empirical study of S&P500 high-frequency prices over the period March 2018–April 2018, carried out by means of nonparametric Fourier estimators, supports this interpretation of the role of the price-leverage covariation.

KEYWORDS

Fourier analysis, leverage effect, price-leverage covariation

1 | INTRODUCTION

Empirical evidence collected in the literature suggests that the leverage effect, that is, the (usually negative) correlation between the price and the volatility of a financial asset (see Reference 1), is time-varying. For instance, Kalnina and Xiu[2] point out that the intensity of the leverage effect gets stronger in turbulent periods, that is, in correspondence of volatility spikes or large returns, while Bandi and Renò[3] model the leverage process as a function of the stochastic volatility of the asset, based on empirical evidence. Thus, in order to get insight into the time-varying dynamics of the leverage process, it may be interesting to study its sensitivity to increments of the volatility or the price from an analytical perspective.

For the analytical study of the sensitivity of the leverage to the volatility, we assume that the price and variance processes are (continuous) semimartingales. This nonparametric framework is typically assumed in the literature on high-frequency econometrics (see Reference 4 and references therein). In fact, the semimartingale hypothesis for the price ensures the absence of arbitrage opportunities (see Reference 5), whereas the semimartingale hypothesis for the variance is convenient for estimation purposes (see Reference 4). In this setting, we show that if the volatility of volatility (hereinafter, vol-of-vol) is a sufficiently smooth function of the variance, then the derivative of the leverage with respect to the variance can be computed analytically and corresponds to the ratio of the price-leverage covariation and the leverage itself. Instead, for the analytical study of the sensitivity of the leverage to the log-price, we restrict our focus to the case when the variance is level-dependent, that is, a deterministic function of the price. In this framework, we show that the analytical derivative of the leverage with respect to the log-price equals the ratio of the price-leverage covariation and the variance, provided that the variance is a sufficiently smooth function of the log-price. It is worth stressing that these analytical results do not require a specific parametric assumption on the price and the volatility, but only some conditions on the functional form of the vol-of-vol and the variance, respectively. A number of popular parametric models used in the absence of jumps satisfy these conditions, as detailed in Section 2.

Overall, from the analytical study of the derivatives of the leverage it emerges that the price-leverage covariation represents a common driver of the response of the leverage to changes in either the variance or the price. This suggests that the price-leverage covariation can be interpreted, from the financial standpoint, as a model-free measure of the responsiveness of the leverage effect to the arrival of new information on the market that causes changes in the price or in the amount of risk perceived by market participants (i.e., in the volatility). The availability of this immediate nonparametric gauge of the responsiveness of the leverage could be useful for financial applications where leverage effects play a crucial role, such as, for instance, volatility forecasting (see, e.g., References 6-8) and portfolio risk management (see, e.g., References 9 and 10 and references therein). The price-leverage covariation has first been studied in Reference 11, where the authors derive a model-free indicator of financial instability whose analytical expression depends, other than on the volatility and the leverage, on the price-leverage covariation. However, only recently Sanfelici and Mancino[12] have provided a consistent nonparametric estimator of the price-leverage covariation, based on the Fourier method by Malliavin and Mancino.[13]

Additionally, as a by-product of our analysis, we find that for a number of widely used models in the absence of jumps (e.g., the model by Heston,[14] the 3/2 model by Platen,[15] the SABR model by Hagan et al.,[16] the continuous-time GARCH model by Nelson,[17] and the CEV model by Beckers[18]), the price-leverage covariation is a linear function of the vol-of-vol. Interpreting the vol-of-vol as the uncertainty about the actual level of risk perceived on the market, this finding suggests that the response of the leverage effect to changes in the price or the volatility is proportional to the intensity of this uncertainty: the larger the latter, the stronger the response of the leverage (and vice versa).

After uncovering the existence of the analytical links between the price-leverage covariation and the derivatives of the leverage, we test such links empirically, on a two-month sample of S&P500 high-frequency prices, ranging from March to April 2018. Specifically, first we reconstruct the paths of the volatility, the leverage and the price-leverage covariation from sample prices by means of consistent nonparametric Fourier estimators, which, as detailed in Section 3, are particularly well-suited for this task (see References 12, 13, 19, and 20). Then, we compare the analytical predictions of the derivatives, that is the ratio of the (estimated) price-leverage covariation and the (estimated) leverage or variance, with the corresponding "true" derivatives, obtained numerically via finite-differences from the reconstructed processes.

As a result of this empirical study, we uncover the existence of a statistically significant linear relationship between the price-leverage covariation scaled by the volatility or the leverage and the corresponding numerical derivative of the leverage. Remarkably, estimated regression coefficients are statistically significant and close to 1, with R^2 values around 0.9, thereby suggesting that analytical predictions in the level-dependent setting may provide an accurate proxy of the actual derivatives of the leverage for the sample object of study. Further, the empirical study also suggests that the variability of the price-leverage covariation may be partially explained by the vol-of-vol, whose path is also reconstructed nonparametrically by means of the Fourier methodology (see Reference 21).

Finally, it is worth noting that a limitation of the study presented in this article is that it does not consider the important contribution of price and volatility jumps to the leverage effect (see, e.g., References 2 and 22). The article is organized as follows. Section 2 contains the analytical study of the derivatives of the leverage. In Section 3, we give a brief description of Fourier estimators. Section 4 is devoted to a simulation exercise aimed at providing guidance for the empirical study of Section 5. Finally, Section 6 concludes.

2 | ANALYTICAL DERIVATIVES OF THE LEVERAGE PROCESS

In this section, we address the issue of the analytical computation of the derivatives of the leverage process w.r.t. the volatility and the log-price.

First, in Section 2.1, we provide a sufficient condition for the analytical computation of the derivative w.r.t. the variance in a fairly general nonparametric framework where the log-price and the volatility are continuous semimartingales. The sufficient condition we provide pertains to the functional form of the vol-of-vol process. Then, in Section 2.2, we restrict our focus to a level-dependent framework, that is, we assume that the volatility is a deterministic function of the log-price, and study the problem of the analytical computation of the derivative w.r.t. the log-price. In this regard, we provide a sufficient condition related to the smoothness of the volatility function. Finally, in Section 2.3, we discuss the financial meaning of the analytic results obtained.

The quantities appearing in the analytical expressions that we obtain for the derivatives can be consistently estimated nonparametrically by means of Fourier estimators from high-frequency prices, as detailed in Section 3. Thus, provided that the corresponding sufficient condition holds, one can reconstruct the analytical derivative of the leverage process w.r.t. the volatility (resp., the log-price) without assuming a specific parametric stochastic volatility model (resp., level-dependent model) as the data-generating process.

2.1 Analytical derivative w.r.t. the volatility

Let the processes x and v denote, respectively, the log-price of an asset and its instantaneous variance. We make the following assumption on their dynamics.

Assumption 1. The log-price x and the variance v are continuous semimartingales on the interval $[0, T]$, $T > 0$, that is,

$$\begin{cases} dx(t) = \sqrt{v(t)}dW(t) + a(t)dt, \\ dv(t) = \gamma(t)dZ(t) + b(t)dt, \end{cases}$$

where W and Z are two Brownian motions, with correlation ρ, on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, which satisfies the usual conditions; a, b, and γ are continuous, adapted, and bounded in absolute value; v and γ are strictly positive and bounded away from zero.

We define the leverage process η as the spot covariation between x and v, that is,

$$\eta(t) := \frac{d\langle x, v\rangle_t}{dt}. \tag{1}$$

Under Assumption 1, it is immediate to see that

$$\eta(t) = \rho\sqrt{v(t)}\gamma(t). \tag{2}$$

Based on Equation (2), the existence of an analytical derivative of η w.r.t. v under Assumption 1 depends on the functional form of the volatility of volatility γ. In this regard, the following holds.

Proposition 1. *Let Assumption 1 holds, with $\rho \neq 0$. Moreover, let $\gamma(t) = g(v(t))$, where $g : \mathbb{R}^+ \to \mathbb{R}^+$ is such that $g \in C^2(\mathbb{R}^+)$. Then*

$$\eta_v(t) = \frac{\chi(t)}{\eta(t)},$$

where $\chi(t) := \frac{d\langle x,\eta\rangle}{dt}$ denotes the price-leverage covariation.

Proof. If $g \in C^1(\mathbb{R}^+)$, the derivative of η w.r.t. v reads:

$$\eta_v(t) = \rho\sqrt{v(t)}\left[\gamma_v(t) + \frac{\gamma(t)}{2v(t)}\right]. \tag{3}$$

Moreover, if $g \in C^2(\mathbb{R}^+)$, Itô's lemma gives:

$$\begin{aligned} d\eta(t) = {} & \rho\gamma(t)\sqrt{v(t)}\left[\gamma_v(t) + \frac{\gamma(t)}{2v(t)}\right]dZ(t) \\ & + \frac{1}{2}\rho\gamma^2(t)\sqrt{v(t)}\left[\gamma_{vv}(t) - \frac{\gamma(t)}{4v^2(t)} + \frac{\gamma_v(t)}{v(t)}\right]dt. \end{aligned}$$

Therefore:

$$\chi(t) := \frac{d\langle x, \eta\rangle}{dt} = \rho^2 v(t)\gamma(t)\left[\gamma_v(t) + \frac{\gamma(t)}{2v(t)}\right] = \eta(t)\eta_v(t). \tag{4}$$

Finally, using (2) and (3), the latter equation can be rewritten as

$$\chi(t) = \eta(t)\eta_v(t).$$

■

Proposition 1 holds for a number of parametric models commonly used for financial applications in the absence of jumps. For instance, any stochastic volatility model where the variance process v is a Chan–Karolyi–Longstaff–Sanders (CKLS) process (see Reference 23) satisfies the hypotheses of Proposition 1. In this case, in fact, it holds that

$$\gamma(t) = \xi v^{\beta}(t), \quad \xi > 0, \quad \beta \neq 0. \tag{5}$$

In the absence of jumps, a model with a CKLS volatility may represent a parsimonious tool to reproduce a number of empirical stylized facts, including the presence of leverage effects, as pointed out, for example, by Christoffersen et al.[24] and Goard and Mazur.[25] Popular examples in this class are the Heston model (see Reference 14), the continuous-time GARCH model (see Reference 17), the SABR model (see Reference 16), and the 3/2 model (see Reference 15). Moreover, level-dependent models, that is, models where the variance is a deterministic function of the log-price, may also satisfy the hypotheses of Proposition 1 (see the next subsection).

Finally, Equation (4) reveals the existence of a link between the price leverage covariation and the vol-of-vol. Specifically, the process χ is a function of γ and γ_v, other than of v and the correlation parameter ρ. However, one may notice that, if

$$\gamma_v(t) = \kappa \frac{\gamma(t)}{v(t)}, \quad \kappa \neq 0, \tag{6}$$

the instantaneous price-leverage covariation χ reduces to a linear function of γ^2. Interestingly, the solution to the ODE in Equation (6) is exactly the function in Equation (5), for $\beta = \kappa$. Therefore, when the variance is a CKLS process, χ is a linear function of the variance-of-variance γ^2; precisely, one has that

$$\chi(t) = \rho^2 \left(\beta + \frac{1}{2}\right) \gamma^2(t). \tag{7}$$

2.2 | Analytical derivative w.r.t the log-price

In this subsection, we restrict our focus to the case when the variance process is level-dependent, that is, a deterministic function of the log-price process. Level-dependent models represent a parsimonious tool to reproduce some stylized facts of financial markets, for example, the implied volatility smile (see, e.g., References 26-28) and the leverage effect (see Reference 18). Specifically, we make the following assumption.

Assumption 2. The dynamics of the log-price x on the interval $[0, T]$, $T > 0$, satisfy

$$dx(t) = \sqrt{h(x(t))}dW(t) + a(t)dt,$$

where W is a Brownian motion on the filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$, which satisfies the usual conditions; a is continuous, adapted, and bounded in absolute value; the function $h : \mathbb{R} \to \mathbb{R}^+$ is such that $h \in C^3(\mathbb{R})$ and $h'(u) > 0$ $\forall u \in \mathbb{R}$.

Under Assumption 2, by applying Itô's lemma, one obtains that

$$dv(t) = \gamma(t)dW(t) + b(t)dt, \tag{8}$$

with

$$\gamma(t) = v_x(t)\sqrt{v(t)} \tag{9}$$

and

$$b(t) = v_x(t)a(t) + \frac{1}{2}v_{xx}(t)v(t). \tag{10}$$

Therefore, the leverage process reads

$$\eta(t) = \sqrt{v(t)}\gamma(t) = v(t)v_x(t). \tag{11}$$

Based on Equation (11), under Assumption 2 it is possible to view the leverage as a differentiable function of either v or x. The corresponding analytical derivatives are given in the following proposition.

Proposition 2. *Let Assumption 2 holds. Then*

$$\eta_v(t) = \frac{\chi(t)}{\eta(t)}, \tag{12}$$

$$\eta_x(t) = \frac{\chi(t)}{v(t)}. \tag{13}$$

Proof. The result in (12) follows from Proposition 1 by noting that:

- Assumption 2 is a special case of Assumption 1 with $Z \equiv W$ (see Equations 8–10).
- $\gamma(t) = g(v(t))$ with $g \in C^2(\mathbb{R}^+)$, since:

$$\gamma_v(t) = \sqrt{v(t)}\left[\frac{v_{xx}(t)}{v_x(t)} + \frac{v_x(t)}{2v(t)}\right],$$

$$\gamma_{vv}(t) = \sqrt{v(t)}\left[\frac{v_{xxx}(t)}{v_x^2(t)} + \frac{v_{xx}(t)}{v(t)v_x(t)} - \frac{v_{xx}^2(t)}{v_x^3(t)} - \frac{v_x(t)}{4v^2(t)}\right]. \tag{14}$$

For what concerns (13), it holds that

$$\eta_x(t) = \eta_v(t)v_x(t).$$

The latter expression can be rewritten as

$$\eta_x(t) = \eta_v(t)v_x(t) = \frac{\chi(t)}{\eta(t)}\frac{\eta(t)}{v(t)} = \frac{\chi(t)}{v(t)}, \tag{15}$$

using Equations (11) and (12). ■

The most popular example of a parametric model for which Proposition 2 holds is the CEV model by Beckers.[18] Under the CEV model, the dynamics of the asset price $S(t) := e^{x(t)}$ follow

$$dS(t) = \mu S(t)dt + \sigma S^{\delta}(t)dW(t), \tag{16}$$

where $\mu \in \mathbb{R}$, $\sigma > 0$ and $\delta > 0, \delta \neq 1$. The existence of a solution for the SDE in Equation (16) is discussed in Chapter 6 of Reference 29. Further, Itô's lemma gives:

$$dx(t) = \left(\mu - \frac{1}{2}\sigma^2 e^{2(\delta-1)x(t)}\right)dt + \sigma e^{(\delta-1)x(t)}dW(t). \tag{17}$$

Specifically, the CEV model is explicitly designed to capture leverage effects. In this regard, the role of the parameter δ is crucial: if $\delta < 1$, the price and the volatility are negatively correlated, as it commonly happens on equity markets;

instead, if $\delta > 1$, the price and the volatility move in the same direction, according to the so-called inverse leverage effect, a phenomenon usually observed on commodity markets.

Based on (17), under the CEV model the variance v is given by the function

$$h(x(t)) = \sigma^2 e^{2(\delta-1)x(t)}. \tag{18}$$

Bearing in mind (18), the following expressions are easily obtained from Equations (9), (11), and (15):

$$\eta(t) = 2(\delta - 1)v^2(t), \tag{19}$$

$$\gamma(t) = 2(\delta - 1)v^{\frac{3}{2}}(t), \tag{20}$$

$$\chi(t) = 8(\delta - 1)^2 v^3(t). \tag{21}$$

These expressions, with the exception of (20), can be also found in Section 2.1 of Reference 12. Interestingly, γ satisfies Equation (5) also in the case of the CEV model, with parameters $\xi = 2(\delta - 1)$ and $\beta = 3/2$. In fact, for the CEV model, χ is a linear function of γ^2, consistently with the prediction in (7); precisely, it holds that

$$\chi(t) = 2\gamma^2(t). \tag{22}$$

Remark 1. Assumption 2 is actually fully satisfied by the CEV model only when $\delta > 1$. In fact, for $\delta \in (0, 1)$, the function h in (18) is such that h' is negative. This implies that the process γ is negative (see Equations 9 and 20), which may be contrary to economic intuition. However, the results in Equations (12), (13), and (22)—relating, respectively, to the computation of the derivatives of the leverage and the existence of a linear link between the price-leverage covariation and the vol-of-vol—still hold, as one can easily verify from Equations (19)–(21).

2.3 | Financial interpretation of the price-leverage covariation

Based on the results on the analytical derivatives of the leverage given in Propositions 1 and 2, the instantaneous price-leverage covariation χ could be interpreted, from the financial viewpoint, as the process that captures the responsiveness of the leverage effect to the arrival of new information that causes changes in the price and the volatility. Thus, given the relevance of the leverage effect for, for example, volatility forecasting (see, for instance, References 6-8) or portfolio risk management (see, e.g., References 9 and 10 and references therein), the availability of reliable estimates of χ may be valuable for financial applications. As we will show in Section 3, the path of χ can be readily estimated from high-frequency prices in a nonparametric fashion using the Fourier methodology.

Moreover, the sign of χ plays a crucial role. In a level-dependent framework, it determines the sign of the derivative of the leverage w.r.t. the log-price. In such a framework, in fact, the latter is given by the ratio of χ and (the strictly positive) v, see Proposition 2. Thus, for instance, a positive χ implies that the leverage effect increases (i.e., the leverage process becomes more negative) in correspondence of a negative return, and vice versa. Instead, in the more general stochastic volatility framework of Proposition 1, the sign of the derivative of the leverage w.r.t. the volatility depends on the sign of both χ and η. Thus, assuming—for example—that at some point in time η is negative (as it usually is the case on equity markets) and χ is positive, then η becomes more (resp., less) negative in correspondence of an increment (resp., reduction) of the volatility.

Based on these considerations, the presence of a positive χ is in principle consistent with the empirical findings related to time-varying leverage effects by Kalnina and Xiu[2] and Bandi and Renò,[3] who show that the leverage becomes stronger in periods of turbulence. In other words, given the results in Propositions 1 and 2, when χ is positive, the negative leverage becomes more negative after bad news hit the market, that is, after a price decrease or a volatility increase. In the empirical study of Section 5, we obtain strictly positive estimates of χ.

Further, from the modeling viewpoint, note that all parametric examples with a CKLS variance mentioned in the previous subsections imply a positive χ: indeed, based on Equation (7), for χ to be positive one only needs that $\beta > -\frac{1}{2}$. Specifically, it is worth recalling that for the Heston model β equals $1/2$. Instead, for the 3/2 model, β is indeed equal to $3/2$, and the same holds for the CEV model, as already shown in this section. Further, for both the SABR model and the continuous-time GARCH model, β equals 1. Besides, Equation (7) additionally tells that χ reduces to a linear function of the vol-of-vol when the variance is a CKLS process. From the financial standpoint, this linear link could be interpreted as follows. Taking the volatility as a measure of market risk and the vol-of-vol as a proxy of the uncertainty about the actual level of market risk perceived by market operators, the larger is the latter, the more intense is the response of the leverage to price and market risk changes, as captured by χ.

3 | FOURIER ESTIMATORS

The Fourier methodology, introduced by Malliavin and Mancino,[13] is particularly well-suited to build nonparametric estimators of second-order and third-order quantities. As a first step, one obtains estimates of the Fourier coefficients of the latent volatility v. Then, the knowledge of these coefficients allows iterating the procedure to compute the Fourier coefficients of the second-order quantities γ^2 and η. Finally, a third iteration yields estimates of the coefficients of the third-order quantity χ. In this regard, it is worth noting that these progressive iterations do not involve any differentiation procedure for the pre-estimation of the spot volatility (in order to estimate second-order quantities) or the spot leverage (in order to estimate the third-order quantity χ). Instead, they only require the pre-estimation of integrated quantities, namely the Fourier coefficients. Given the numerical instabilities which are typically linked to differentiation procedures, this feature represents a strength of the Fourier methodology, compared to the realized approach for the estimation of spot processes (see Chapter 8 in Reference 4 for a detailed description of realized spot estimators and their asymptotic properties).

Fourier estimators of v, η, γ^2, and χ are termed nonparametric in that, for their asymptotic properties to hold, they only require that the processes x, v, and η are continuous semimartingales. Specifically, as illustrated in the next subsections, Fourier estimators are consistent under the following assumption.

Assumption 3. The log-price x, the variance v, and the leverage η follow:

$$dx(t) = \sqrt{v(t)}dW(t) + a(t)dt,$$

$$dv(t) = \gamma(t)dZ(t) + b(t)dt,$$

$$d\eta(t) = \lambda(t)dY(t) + c(t)dt,$$

where W, Z, and Y are correlated Brownian motions on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, which satisfies the usual conditions; a, b, c, v, γ, and λ are continuous, adapted, and bounded in absolute value.

3.1 | Fourier estimator of the volatility

Assume that the log-price process x is observable on the equally spaced grid of mesh size $\rho(n) := 2\pi/n$ over the interval $[0, 2\pi]$. In applications, however, we can always assume that the price process x is observed on $[0, 2\pi]$ by rescaling the actual time interval. Moreover, it is worth noting that the Fourier method works also with unequally spaced samples (see, e.g., References 30 and 31).

For $|k| \leq N$, the kth (discrete) Fourier coefficient of the volatility is defined as

$$c_k\left(v_{n,N}\right) := \frac{2\pi}{2N+1} \sum_{|s|\leq N} c_s\left(dx_n\right) c_{k-s}\left(dx_n\right), \tag{23}$$

where for any integer k, $|k| \leq 2N$, $c_k\left(dx_n\right)$ is the kth (discrete) Fourier coefficient of the log-return process, namely

$$c_k\left(dx_n\right) := \frac{1}{2\pi} \sum_{j=0}^{n-1} e^{-\mathrm{i}kt_{j,n}} \delta_j^n(x), \tag{24}$$

where $\delta_j^n(x) := x_{t_{j+1},n} - x_{t_j,n}$, $t_{j,n} = j\frac{2\pi}{n}, j = 0, 1, \ldots, n$, while the symbol i denotes the imaginary unit, that is, $\mathrm{i} = \sqrt{-1}$.

Once the Fourier coefficients of the volatility (23) have been computed, the application of the Fourier–Fejér inversion formula allows reconstructing the volatility path. The definition of the Fourier spot volatility estimator is as follows.

Definition 1 (Fourier estimator of the spot volatility). The Fourier estimator of the spot volatility process is defined as the random function of time

$$\hat{v}_{n,N,S_v}(t) := \sum_{|k|<S_v} \left(1 - \frac{|k|}{S_v}\right) c_k\left(v_{n,N}\right) e^{ikt}, \tag{25}$$

where S_v is a positive integer smaller than N, while $c_k\left(v_{n,N}\right)$ is defined in (23).

The following theorem demonstrates the uniform consistency of the Fourier estimator of the spot volatility (25).

Theorem 1. *Let Assumption 3 holds. For any integer $|k| \le N$, as $n, N \to \infty$, if $N/n \to 1/2$, the following convergence in probability holds*

$$c_k\left(v_{n,N}\right) \xrightarrow{p} c_k(v),$$

where $c_k(v)$ is the kth Fourier coefficient of the volatility process v. Moreover, as n, $N, S_v \to \infty$, if $N/n \to 1/2$ and $S_v/n \to 0$, then

$$\sup_{t\in(0,2\pi)} \left|\hat{v}_{n,N,S_v}(t) - v(t)\right| \xrightarrow{p} 0.$$

Proof. See Reference 19. ∎

3.2 | Fourier estimator of the leverage

As mentioned, the knowledge of the Fourier coefficients of the latent instantaneous volatility v allows treating the latter as an observable process and iterate the procedure for computing the Fourier coefficients in order to reconstruct the leverage process η. In particular, to estimate the instantaneous leverage η we exploit the multivariate version of the Fourier method introduced in Reference 19. Accordingly, an estimator of the Fourier coefficients of the leverage is given by

$$c_k\left(\eta_{n,N,M}\right) := \frac{2\pi}{2M+1} \sum_{|j|\le M} c_j\left(dx_n\right) c_{k-j}\left(dv_{n,N}\right), \tag{26}$$

where M is a positive integer smaller than N, $c_j\left(dx_n\right)$ is given in (24) and we use the approximation $c_j\left(dv_{n,N}\right) \cong ijc_j\left(v_{n,N}\right)$ (see, e.g., Chapter 6 in Reference 31). Then the following theorem holds.

Theorem 2. *Let Assumption 3 holds. As $n, N, M \to \infty$, if $N/n \to 1/2$ and $M^2/n \to 0$, then*

$$c_k\left(\eta_{n,N,M}\right) \xrightarrow{p} c_k(\eta),$$

where $c_k(\eta)$ is the kth Fourier coefficient of the leverage process η.

Proof. See Reference 20. ∎

Finally, a consistent estimator of the instantaneous leverage η is obtained as

$$\hat{\eta}_{n,N,M,S_\eta}(t) := \sum_{|k|<S_\eta} \left(1 - \frac{|k|}{S_\eta}\right) c_k\left(\eta_{n,N,M}\right) e^{ikt}, \tag{27}$$

where S_η is a positive integer smaller than M, while $c_k\left(\eta_{n,N,M}\right)$ is defined in (26).

3.3 | Fourier estimator of the vol-of-vol

The knowledge of the coefficients of the volatility process ν also allows building an estimator of its quadratic variation, the vol-of-vol γ^2. In particular, an estimator of the coefficients of γ^2 is given by

$$c_k\left(\xi_{n,N,M}\right) := \frac{2\pi}{2M+1}\sum_{|j|\leq M} c_j\left(d\nu_{n,N}\right) c_{k-j}\left(d\nu_{n,N}\right), \tag{28}$$

where, again, $c_j\left(d\nu_{n,N}\right)$ is approximated with $\mathrm{i}jc_j\left(\nu_{n,N}\right)$. Then the following theorem holds.

Theorem 3. *Let Assumption 3 holds. As $n, N, M \to \infty$, if $N/n \to 0$ and $M^4/N \to 0$, then*

$$c_k\left(\xi_{n,N,M}\right) \xrightarrow{p} c_k(\xi),$$

where $c_k(\xi)$ is the kth Fourier coefficient of the vol-of-vol process γ^2.

Proof. See Reference 21. ■

Finally, a consistent estimator of the spot vol-of-vol γ^2 can be obtained as

$$\hat{\xi}_{n,N,M,S_\xi}(t) := \sum_{|k|<S_\xi}\left(1-\frac{|k|}{S_\xi}\right) c_k\left(\xi_{n,N,M}\right) e^{\mathrm{i}kt}, \tag{29}$$

where S_ξ is a positive integer smaller than M, while $c_k\left(\xi_{n,N,M}\right)$ is defined in (28).

3.4 | Fourier estimator of the price-leverage covariation

Similarly to what we have done for the volatility process ν, once its Fourier coefficients have been estimated, we can treat the second-order quantity η as an observable process and exploit the multivariate Fourier method to estimate the third-order quantity χ. The following asymptotic result is obtained.

Theorem 4. *Let Assumption 3 holds. If $N/n \to 1/2$ and $L^2M^2/N \to 0$ when $n, N, M, L \to \infty$, then the following convergence in probability holds*

$$c_k\left(\chi_{n,N,M,L}\right) \xrightarrow{p} c_k(\chi),$$

where, for L positive integer and smaller than M, we define

$$c_k\left(\chi_{n,N,M,L}\right) := \frac{2\pi}{2L+1}\sum_{|j|\leq L} c_j\left(dx_n\right) c_{k-j}\left(d\eta_{n,N,M}\right).$$

Proof. See Reference 12. ■

Accordingly, a consistent spot estimator of the process χ is defined as

$$\hat{\chi}_{n,N,M,L,S_\chi}(t) := \sum_{|k|<S_\chi}\left(1-\frac{|k|}{S_\chi}\right) c_k\left(\chi_{n,N,M,L}\right) e^{\mathrm{i}kt}, \tag{30}$$

where S_χ is a positive integer smaller than L.

Remark 2. The assumptions on M in Theorem 4 imply that $M^2/n \to 0$ as $M, n \to \infty$, thereby satisfying the hypothesis on M in Theorem 2.

4 | SIMULATION STUDY

The Fourier methodology may allow estimating the trajectories of the volatility, the leverage and the price-leverage covariation with accuracy and, thereby, treating these quantities as observable. Hence, after reconstructing the sample paths of these processes using the Fourier estimators described in the previous section, a simple test to check with empirical data if the true derivatives of the leverage match the corresponding analytical predictions given in Propositions 1 and 2 would entail performing a linear regression between the actual numerical derivatives, obtained via finite differences, and the corresponding analytical predictions.

The computation of the numerical derivative of a function contaminated by noise (such as the noise related to estimation error) is discussed, for example, in Reference 32 and references therein. For instance, a simple method to reduce the variance arising from the numerical differentiation of noisy functions is as follows: first, one performs a preliminary smoothing of the noisy signal via local-polynomials fitting; then, finite differences are applied to the de-noised signal. In our simulation study, we follow this approach, using the "lowess" smoothing method by Cleveland.[33] The smoothed version of the noisy process $\hat{y}$ is denoted by y^*.

Ideally, one would then test the analytical results in Propositions 1 and 2 by estimating the linear models

$$\frac{\eta^*{}_{n,N,M,S_\eta}(t+h) - \eta^*{}_{n,N,M,S_\eta}(t)}{\nu^*{}_{n,N,S_\nu}(t+h) - \nu^*{}_{n,N,S_\nu}(t)} = \alpha_1 \frac{\hat{\chi}_{n,N,M,L,S_\chi}(t)}{\hat{\eta}_{n,N,M,S_\eta}(t)} \tag{31}$$

and

$$\frac{\eta^*{}_{n,N,M,S_\eta}(t+h) - \eta^*{}_{n,N,M,S_\eta}(t)}{x(t+h) - x(t)} = \alpha_2 \frac{\hat{\chi}_{n,N,M,L,S_\chi}(t)}{\hat{\nu}_{n,N,S_\nu}(t)}, \tag{32}$$

where the numerical derivatives of the leverage and the corresponding analytical predictions appear, respectively, on the l.h.s. and the r.h.s. However, ratios of estimated quantities, as the ones appearing on the r.h.s. of both Equations (31) and (32), are typically biased estimators of the corresponding true ratios. To overcome this additional issue, we then rewrite the two equations as, respectively,

$$\begin{aligned}&\left[\eta^*{}_{n,N,M,S_\eta}(t+h) - \eta^*{}_{n,N,M,S_\eta}(t)\right] \hat{\eta}_{n,N,M,S_\eta}(t)\\ &= \alpha_1 \left[\nu^*{}_{n,N,S_\nu}(t+h) - \nu^*{}_{n,N,S_\nu}(t)\right] \hat{\chi}_{n,N,M,L,S_\chi}(t),\end{aligned} \tag{33}$$

and

$$\begin{aligned}&\left[\eta^*{}_{n,N,M,S_\eta}(t+h) - \eta^*{}_{n,N,M,S_\eta}(t)\right] \hat{\nu}_{n,N,S_\nu}(t)\\ &= \alpha_2 \left[x(t+h) - x(t)\right] \hat{\chi}_{n,N,M,L,S_\chi}(t).\end{aligned} \tag{34}$$

Finally, to obtain reliable results from the tests (33) and (34), it is crucial that the step h is carefully selected. Accordingly, the aim of the simulation study of this section is to provide guidance for such selection. The simulation study is organized as follows. First, we illustrate the setup of the simulation exercise. Then, we focus on the optimal selection of h.

4.1 | Simulation setup

For simulations, we consider three different models: the Heston and 3/2 models, which satisfy the hypotheses of Proposition 1 (but not Proposition 2), and the CEV model, which satisfies the hypotheses of Proposition 2 (bearing in mind Remark 1 when $\delta < 1$). The models read, respectively, as follows:

$$\begin{cases} dx(t) = \left(\mu - \frac{1}{2}\nu(t)\right) dt + \sqrt{\nu(t)} dW(t), \\ d\nu(t) = \theta\left(\alpha - \nu(t)\right) dt + \xi\sqrt{\nu(t)} dZ(t), \\ \rho dt = d\langle W, Z\rangle_t, \end{cases} \tag{35}$$

$$\begin{cases} dx(t) = \left(\mu - \frac{1}{2}v(t)\right) dt + \sqrt{v(t)}dW(t), \\ dv(t) = \theta v_t\,(\alpha - v(t))\,dt + \xi v^{3/2}(t)dZ(t), \\ \rho dt = d\langle W, Z\rangle_t, \end{cases} \tag{36}$$

$$dx(t) = \left(\mu - \frac{1}{2}\sigma^2 e^{2(\delta-1)x(t)}\right) dt + \sigma e^{(\delta-1)x(t)} dW(t). \tag{37}$$

Note that the CEV model in (37) is exactly the same as in (17), but was illustrated again here for convenience of exposition. The parameter vectors used for simulations are as follows (see, e.g., References 12, 34, and 35 for similar parameter choices):

- $(\mu, \theta, \alpha, \xi, \rho, x(0), v(0)) = (0.05, 5, 0.2, 0.5, -0.8, 1, 0.2)$ for model (35).
- $(\mu, \theta, \alpha, \xi, \rho, x(0), v(0)) = (0.05, 10, 0.2, 0.5, -0.8, 1, 0.2)$ for model (36).
- $(\mu, \sigma, \delta, x(0)) = (0.01, 0.3, 0.5, 1)$ for model (37).

Notice that ρ and δ are selected such that returns and volatility changes are negatively correlated, as it is usually the case on equity markets. For each model, we simulate a 250-day long trajectory of 1-second observations. Each simulated day is 6.5-h long. Specifically, we simulate the realistic scenario where one can only observe the 1-s path of the noisy price $\tilde{x} := x + \varepsilon$, that is, the efficient price x contaminated by the presence of an i.i.d. zero-mean noise component ε that reflects the presence of microstructure phenomena, such as bid-ask bounces (see, e.g., References 4 or 36, and references therein). For the simulation of the process ε, we use a Gaussian distribution, with standard deviation parameter equal to 10^{-4}.

For the estimation of v, η, and χ, we use all available prices, that is we select n corresponding to the 1-s sampling frequency. The efficiency of the Fourier methodology with noisy high-frequency data is illustrated in Reference 37. For the selection of the cutting frequencies, we use as guidance the extensive numerical study by Sanfelici and Mancino.[12] At the first level, we select N based on the noise-robust procedure proposed by Mancino and Sanfelici;[37] then we select $S_v = n^{0.5}$. At the second level, we select $M = n^{0.5}$ and $S_\eta = 2n^{0.25}$. Finally, at the third level, we select $L = 2n^{0.25}$ and $S_\chi = \sqrt{2}n^{0.125}$.

4.2 | Simulation results

The estimation of α_1 and α_2 in (33) and (34) is performed using the robust regression method with a bisquare weighting scheme to penalize outliers (see Reference 38). To account for auto-correlations in the residuals, we compute Newey–West standard errors, see Reference 39.

The estimates of α_1 and α_2 for the CEV model appear to be quite satisfactory for h between 10 and 20 min. In fact, such estimates are in the range 0.9–1.1 and show p-values equal to 0. The corresponding R^2 is greater than 0.8. The analogous situation is experienced for α_1 in the case of the Heston and 3/2 models. In particular, the distance of coefficient estimates from the true value of 1 is optimized for h equal to 15 min, for all the three models considered. The estimates of α_1 and α_2 for such value of h, along with other relevant outputs of the estimation, are reported in Table 1.

Finally, note that we replicated the simulation exercise using the Heston model with $\rho = 0$, all other parameters left unchanged. We did this to check the robustness of tests (33) and (34) to "false positives." If $\rho = 0$, in fact, the leverage

TABLE 1 Estimation results for h equal to 15 min

	Coeff. est.	Std. err.	p-value	R^2
$\hat{\alpha}_1$, CEV	1.003	0.001	0	0.966
$\hat{\alpha}_2$, CEV	1.002	0.001	0	0.951
$\hat{\alpha}_1$, Heston	0.994	0.002	0	0.935
$\hat{\alpha}_1$, 3/2	0.996	0.002	0	0.941

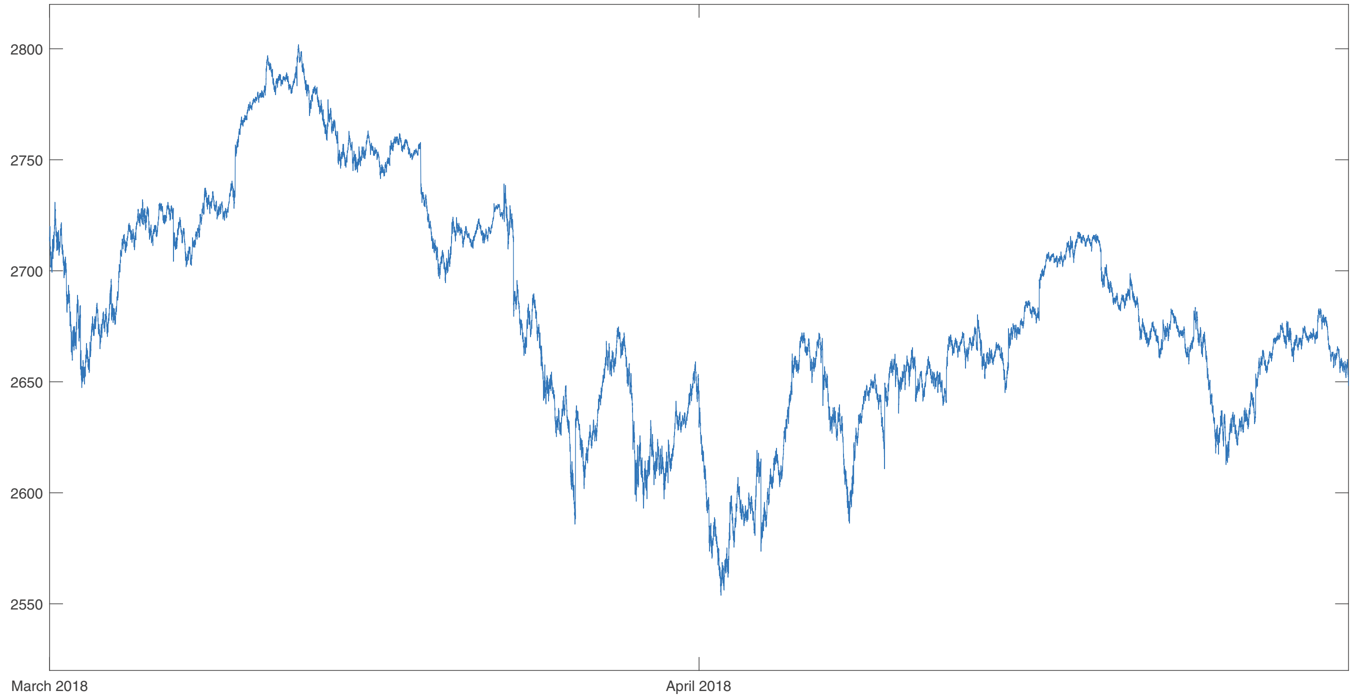

FIGURE 1 S&P 500 (March 2018–April 2018), 1-s prices

process is identically zero. In this case, estimation results suggested not to reject the hypothesis that α_1 (resp., α_2) is equal to zero, yielding a p-value larger than 0.5 in both cases.

5 | EMPIRICAL STUDY

In this section, we perform tests (33) and (34) with empirical data. Additionally, we also empirically investigate the existence of a linear link between χ and γ^2, as predicted by (7). For this study, we use the series of 1-s S&P500 price observations over the period March 2018–April 2018 (see Figure 1).

To obtain Fourier estimates of the paths of ν, η, χ, and γ^2, we use all data in the sample, that is, we select n corresponding to the 1-s sampling frequency. Further, to reconstruct ν, η, and χ, we select the same noise-robust cutting frequencies as in the simulation exercise of Section 4, which, as mentioned, are taken from Reference 12. As for γ^2, we select N via the noise-robust approach by Mancino and Sanfelici,[37] then we choose $M = 2n^{0.25}$ and $S_{\xi} = n^{0.25}$, based on the optimization of the mean squared error with the simulated data of Section 4. The estimated trajectories of the processes are plotted in Figure 2. Note that, before performing the estimation, we have removed days with price jumps from the 2-month sample, using the jump detection test by Corsi et al.[40] In particular, the test at 99.9% confidence level detects only two days with jumps, namely March 20th and March 23rd. These two days are associated with market turbulence related to the so-called "trade war" between China and the US.

Once the reconstructed processes are available, we can then perform tests (33) and (34). However, to avoid performing a spurious regression (see Reference 41), we test in advance for the null hypothesis of the presence of a unit root in all the series of regressors and regressands involved, using the augmented Dickey–Fuller test (see Reference 42). For all series, test results at the 99.9% confidence level reject the null hypothesis.

The results of tests (33) and (34) for the selection h equal to 15 min, which was optimal in the simulated setting, are summarized in Table 2. Note that the estimated values of α_1 and α_2 are close to 1 and statistically significant, with p-values equal to 0; moreover, note that the resulting R^2 values are quite large. Note that we have also attempted to fit a linear model with intercept, but the resulting estimate of the latter was not statistically significant for both tests, with p-values larger than 0.45.

After the estimation, we also perform the test by Breusch and Pagan[43] on the residuals, whose result suggests that residuals are not heteroskedastic at the 99% confidence level in both cases. Finally, it is worth mentioning that

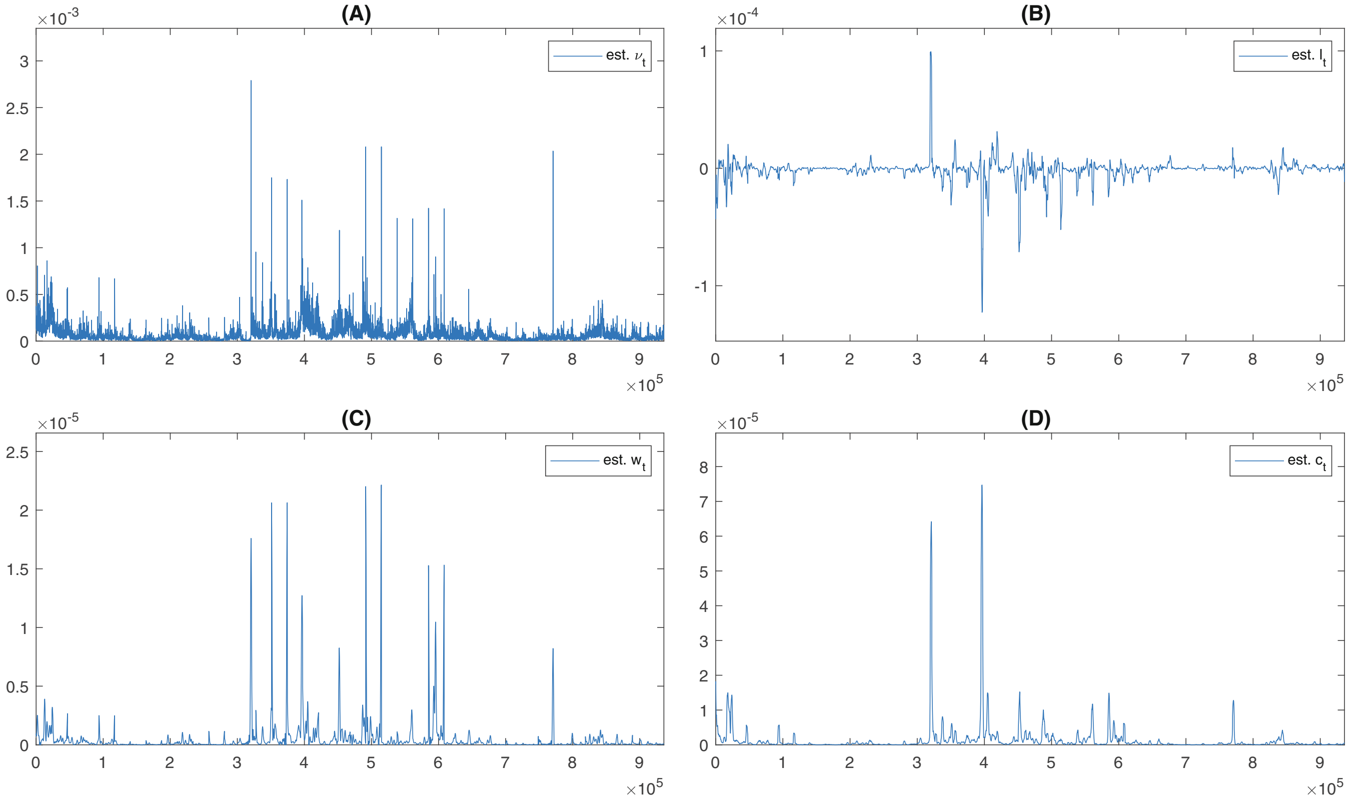

FIGURE 2 S&P 500 (March 2018–April 2018), reconstructed 1-s trajectories of: (A) ν; (B) η; (C) γ^2; (D) χ

TABLE 2 S&P 500 (March 2018-April 2018), estimation results for h equal to 15 min

	Coeff. est.	**Std. err.**	***p*-value**	R^2
$\hat{\alpha}_1$	1.034	0.005	0	0.898
$\hat{\alpha}_2$	0.972	0.005	0	0.976

estimation results were quite stable for h in the range 5–20 min and the range 10–25 min for, respectively, test (33) and test (34).

Overall, this empirical investigation supports the interpretation of the price-leverage covariation χ as a process that jointly captures the instantaneous response of the leverage to the arrival of new information that causes changes in the price and the volatility.

We conclude the empirical exercise with the investigation of the existence of a linear link between the price-leverage covariation and the vol-of-vol, as predicted by (7) under the assumption of a CKLS variance process. Obtaining insight on the dependence between χ and γ^2 could provide additional financial intuition on the role of χ, as discussed in Section 2.3. Specifically, sampling $\hat{\chi}_{n,N,M,L,S_\chi}$ and $\hat{\xi}_{n,N,M,S_\xi}$ at the 5-min frequency (which appears to be a suitable choice based on unreported simulations), we perform the following linear regression:

$$\hat{\chi}_{n,N,M,L,S_\chi}(t) = \alpha_3 \hat{\xi}_{n,N,M,S_\xi}(t). \tag{38}$$

As a result, we obtain that the estimate $\hat{\alpha}_3$ is equal to 1.886, with p-value equal to 0.003. The resulting R^2 is equal to 0.426, indicating that the variability of the price-leverage covariation may be partially explained by the vol-of-vol. This empirical result, if considered jointly with the results of tests (33) and (34), suggests that the sensitivity of the leverage to changes of the volatility or the price tends to be larger when the uncertainty about the actual level of risk perceived by market operators (i.e., the vol-of-vol) is larger. In this regard, it is worth noting that an exploratory study recently proposed by Livieri et al.[44] suggests the existence of an auto-regressive structure in the vol-of-vol of the S&P500 and EUROSTOXX indices. A possible direction for future research is thus to investigate if a statistically significant auto-regressive structure appears also in the price-leverage covariation series, thereby possibly explaining the variability not explained by the simultaneous vol-of-vol.

6 | CONCLUSIONS

The main finding of this article is uncovering the analytical relationship between the price-leverage covariation and the derivatives of the leverage process w.r.t. the volatility and the price in nonparametric settings with, respectively, stochastic and level-dependent volatility. The price-leverage covariation could then be understood by market operators as a gauge of the responsiveness of the leverage effect to the arrival of new information causing a change in the price and/or the volatility. Given the role of the leverage effect in applications such as volatility forecasting or portfolio risk management, the relevance of this finding is immediate.

The analytical results we obtained were tested on S&P500 sample prices, after reconstructing the paths of the variance, the leverage and the price-leverage covariation via the nonparametric Fourier methodology, which is particularly well-suited for the task. Empirical evidence, although limited to a single index and a 2-month sample, appears to support those analytical results.

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Research data are not shared.

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