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Supersymmetric localization of (higher-spin) JT gravity: a bulk perspective

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Abstract: We study two-dimensional Jackiw-Teitelboim gravity on the disk topology by using a BF gauge theory in the presence of a boundary term. The system can be equivalently written in a supersymmetric way by introducing auxiliary gauginos and scalars with suitable boundary conditions on the hemisphere. We compute the exact partition function thanks to supersymmetric localization and we recover the result obtained from the Schwarzian theory by accurately identifying the physical scales. The calculation is then easily extended to the higher-spin generalization of Jackiw-Teitelboim gravity, finding perfect agreement with previous results. We argue that our procedure can also be applied to boundary-anchored Wilson lines correlators.

Keywords: 2D Gravity, Matrix Models, Models of Quantum Gravity, Supersymmetric Gauge Theory

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1 Introduction

The AdS/CFT correspondence [1–3] represents a promising framework to understand and, hopefully, to solve some subtle problems related to the quantization of gravity. Through the correspondence, the boundary theory can serve as a guide for understanding properties of the bulk physics. This is especially useful given the notorious difficulties in making sense of the functional integral of quantum gravity.

A powerful non-perturbative method to perform exact computations in certain quantum field theories is the localization technique [4], where the functional integral can be shown to "localize" over some solutions in field space, parametrizing a moduli space of suitable classical configurations. In simple cases, this finite-dimensional integral can be evaluated analytically, leading to a complete solution of the problem. When a system (or a particular set of observables) having a dual gravitational description in a bulk space, can be studied exactly through localization, we would expect to learn something about the structure of the related quantum gravity path integral. More ambitiously, we also hope that the bulk theory inherits some localization properties, opening to the possibility of obtaining exact results for integrations on fluctuating backgrounds.

The program of studying gravitational systems from localization techniques applied to the boundary theory has been successfully exploited to derive the Bekenstein-Hawking entropy of supersymmetric black holes in AdS_4 [5–7] and AdS_5 [8–10]. There have also been attempts to extend these localization methods directly to supergravity to evaluate the bulk quantum gravity partition function [11–13].

On general grounds, one expects that, in low dimensions, our understanding of quantum gravity should improve. In particular, 2d/1d holography provides probably the best frameworks to study fluctuating geometries beyond perturbation theory. Gravitons or gauge bosons in two dimensions have no dynamical degrees of freedom; therefore, the quantum path integral in general simplifies dramatically, even in non-supersymmetric settings. On the other hand, many of the open questions from higher dimensional holography, such as bulk reconstruction or the physics of black holes and wormholes, persist in the lowest dimensional case.

Two-dimensional Jackiw-Teitelboim (JT) theory [14, 15] involving, in the second-order formalism, a dilaton field Φ and the metric tensor $g_{\mu\nu}$, is a simple tractable example of holographic correspondence and has attracted much attention in the last few years (see [16] for a recent review). The dual holographic theory is a one-dimensional theory, the Schwarzian quantum mechanics [17], that effectively describes the bulk quantum gravity: a large class of correlation functions is precisely mapped to the boundary theory. Quite interestingly, the Schwarzian path integral can be exactly evaluated thanks to equivariant localization [18],[1] leading to a precise expression for the thermal-JT partition function at the basis of many recent developments [20–24]. In particular, higher-genus contributions to the path integral play a fundamental role in deriving a non-perturbative extension of the holographic duality in terms of an ensemble of theories described by a double-scaled matrix model [25].

This paper proposes that the same results can be obtained starting from the bulk theory and using a supersymmetric localization procedure [26]. The main idea[2] behind our computation is to use the well-known formulation of JT gravity as a BF gauge theory based on the algebra of $\mathrm{SL}(2,\mathbb{R})$ [29–31]. We map the theory into a supersymmetric $\mathcal{N}=(2,2)$ gauge theory on the hemisphere and apply supersymmetric localization to reproduce the known Schwarzian partition function.

A subtle point concerns the correct identification of the physical scales present in the gravitational theory with the geometrical scales appearing in the supersymmetric BF theory. We argue that our gauge model is actually obtained by reducing the three-dimensional Chern-Simons theory on a solid torus conformally equivalent to thermal AdS_3 and this provides us with the correct supersymmetric boundary terms and scales identification.

Interestingly our approach is easily extended for the higher-spin generalization of JT gravity [32, 33]. In this case, the relevant gauge theory is a $\mathrm{SL}(N,\mathbb{R})$ BF theory, and its partition function has been derived using equivariant localization on the boundary $\mathrm{SL}(N,\mathbb{R})$ Schwarzian quantum mechanics [33–35]. Our procedure reproduces precisely the result of [33, 35], bypassing the technical complications related to the derivation of the boundary quantum theory.

Obtaining the JT partition function through localization is fascinating as it suggests that we can use the same framework to compute general observables. In the gauge theory formulation, correlation functions of boundary-anchored Wilson lines [29] are the most

[1] See also [19] for previous results obtained by studying the SYK model.

[2] In three-dimensional gravity there have been similar attempts, both for the AdS [27] and the dS [28] cases.

natural candidates to be studied. Physically, they represent correlators of bi-local operators in the Schwarzian theory and contain essential information about the quantum structure of the bulk gravity. While bi-local correlators have been thoroughly studied in standard JT gravity [29, 31, 36] obtaining exact expressions from different methods, their higher-spin cousins have never been considered. Supersymmetric localization could provide a convenient framework for their calculation.

The structure of the paper is the following. We start section 2 by reviewing the gauge formulation of JT gravity, and then proceed to construct an equivalent supersymmetric BF theory. We pay particular attention to imposing supersymmetric boundary conditions on the hemisphere and to elucidate the identification between the physical and the geometrical scales. The actual localization of the path-integral is performed in section 3: we present the localizing term and compute the relevant functional determinants obtaining the well-known final result for the JT partition function. In section 4, we extend the computation to the higher-spin generalization of JT gravity recovering the disk partition function obtained in [33, 35]. Section 5 contains our conclusions and speculations about further uses of our procedure. The paper is completed with a couple of technical appendices.

2 JT gravity as a supersymmetric BF theory

2.1 JT gravity as a BF theory

Let us start by briefly reviewing how JT gravity can be formulated as a two-dimensional BF theory with $\mathrm{SL}(2,\mathbb{R})$ gauge group [31, 37–39]. In particular, our focus is on the theory defined on a two-dimensional manifold Σ with the topology of the disk. We start from the BF action

$$S_{\mathrm{BF}} = -i \int_{\Sigma} \mathrm{Tr}(\chi F)\,, \tag{2.1}$$

where $F = \mathrm{d}A - A \wedge A$ is the field strength associated with a gauge connection one form A, and χ is an auxiliary scalar field in the adjoint representation of the gauge group.

We consider a basis $\{\mathsf{P}_0, \mathsf{P}_1, \mathsf{P}_2\}$ for the generators of $\mathfrak{sl}(2,\mathbb{R})$ obeying the commutation relations

$$[\mathsf{P}_0, \mathsf{P}_1] = \mathsf{P}_2\,, \qquad [\mathsf{P}_0, \mathsf{P}_2] = -\mathsf{P}_1\,, \qquad [\mathsf{P}_1, \mathsf{P}_2] = -\mathsf{P}_0\,. \tag{2.2}$$

This algebra can be explicitly realized by choosing, for instance, a real two-dimensional representation in terms of Pauli matrices with

$$\mathsf{P}_0 = \frac{i\sigma_2}{2}\,, \qquad \mathsf{P}_1 = \frac{\sigma_1}{2}\,, \qquad \mathsf{P}_2 = \frac{\sigma_3}{2}\,. \tag{2.3}$$

The corresponding Killing form reads $\mathrm{Tr}(\mathsf{P}_i\mathsf{P}_j) = \frac{1}{2}\,\mathrm{diag}(-1,+1,+1)$.

We then expand the fields on such a basis as

$$A = \sqrt{\frac{\Lambda}{2}}\, e^a\, \mathsf{P}_a + \omega\, \mathsf{P}_0\,, \tag{2.4}$$

$$\chi = \chi^a\, \mathsf{P}_a + \chi^0\, \mathsf{P}_0\,. \tag{2.5}$$

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Here, we regard the index $a \in 1,2$ as an SO(2) frame index. In fact, the matching of gravitational and gauge degrees of freedom is obtained by identifying the one-forms e^a and ω with the zweibein and the spin connection, respectively. In the spirit of the first-order formulation of gravity, these are regarded as independent degrees of freedom.

Exploiting the expansion (2.4) and the commutation relations (2.2), we compute the non-abelian field strength

$$\begin{aligned} F &= F^a \, \mathsf{P}_a + F^0 \, \mathsf{P}_0 \\ &= \sqrt{\frac{\Lambda}{2}} \left(\mathrm{d}e^a + \epsilon^a{}_b \, \omega \wedge e^b \right) \mathsf{P}_a + \left(\mathrm{d}\omega + \frac{\Lambda}{4} \, \epsilon_{ab} \, e^a \wedge e^b \right) \mathsf{P}_0 \, . \end{aligned} \tag{2.6}$$

Upon plugging this expression into the BF action (2.1) one finds that the variation of S_{BF} with respect to the SO(2) vector χ^a yields the equations of motion

$$\mathrm{d}e^a + \epsilon^a{}_b \, \omega \wedge e^b = 0 \, . \tag{2.7}$$

This is precisely the zero-torsion condition, which, once solved, gives the spin connection ω in terms of the zweibein e^a. The action (2.1) evaluated on the solutions of (2.7) reduces to the second-order action

$$S_{\mathrm{BF}} = \frac{i}{2} \int_\Sigma \chi_0 \left(\mathrm{d}\omega(e) + \frac{\Lambda}{2} \, e^1 \wedge e^2 \right) \, . \tag{2.8}$$

In two dimensions, $\mathrm{d}\omega = R/2 \; e^1 \wedge e^2$, and we recognize in $e^1 \wedge e^2$ the two-dimensional volume form. We can then rewrite (2.8) in terms of the metric $g = \delta_{ab} \, e^a \otimes e^b$ as

$$S_{\mathrm{BF}} = \frac{i}{4} \int_\Sigma \mathrm{d}^2 x \, \sqrt{g} \, \chi_0 \, (R(g) + \Lambda) \, . \tag{2.9}$$

In (2.9), S_{BF} reproduces the bulk contribution of the JT action if we identify the dilaton field with $\Phi = -i \, \chi_0 / 4$.

At this point, within the gauge formulation of JT gravity, it is common practice to introduce a boundary term that, when combined with appropriate boundary conditions, replicates the dynamics of the Schwarzian theory on the boundary of the disk [29–31]. In the metric formulation, this consists in implementing the Gibbons-Hawking boundary term. In our approach, it will be the process of supersymmetrization that will naturally guide us towards incorporating a suitable boundary term. Such a term, upon identifying the correct physical scales, will allow us to obtain the correct Schwarzian partition function. A more detailed explanation of these steps will be provided in the following sections.

2.2 Supersymmetrizing JT gravity

As a next step, we will introduce new auxiliary degrees of freedom in the BF action (2.1) with the aim of making it supersymmetric. In doing so, we first introduce a Riemannian structure on Σ.[3] We identify Σ with the hemisphere HS^2 endowed with the metric

$$\mathrm{d}s^2 = \ell^2 \left(\mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\varphi^2 \right) \, , \tag{2.10}$$

[3]The reader should not confuse the dynamical geometry associated with the degrees of freedom of the gauge theory with the background geometry introduced to construct the supersymmetry algebra.

written in terms of conventional spherical coordinates $\theta \in [0, \pi/2]$ and $\varphi \in [0, 2\pi)$. The boundary circle is located at $\theta = \pi/2$.

A simple scheme for localizing gauge theories on HS^2 was developed in [40–42] in the presence of $\mathcal{N} = (2,2)$ supersymmetry. In order to leverage such results, we need to embed the degrees of freedom of the BF theory into an $\mathcal{N} = (2,2)$ vector multiplet. This off-shell multiplet contains a two-dimensional gauge connection A, two scalars η and σ of dimension one, two Dirac fermions λ and $\bar{\lambda}$, and an auxiliary field D. The associated supersymmetry variations, parametrized by conformal Killing spinors ϵ and $\bar{\epsilon}$, are given in appendix B, where the geometry and the supersymmetry of the hemisphere are spelled out in detail.

While it is straightforward to identify the gauge field in the vector multiplet with the one appearing in the BF action, we have two options for the scalar field χ, namely σ and η. In choosing between them, we recall that the BF action can be constructed by dimensionally-reducing the Chern-Simons action. In this framework, the scalar χ can be identified with the third component of the gauge field in three dimensions. Similarly, the $\mathcal{N} = (2,2)$ vector multiplet can be obtained by performing a dimensional reduction of the $\mathcal{N} = 2$ vector multiplet in three dimensions, σ originates from the third component of the vector field. To address the disparity in dimensions between the dimensionless field χ and the dimensionful σ we set

$$\chi = L\sigma \,, \tag{2.11}$$

where L is a generic length scale. We will temporarily withhold any assumption about L, which will be determined in section 2.3.

With the identification (2.11), the BF action (2.1) reads

$$S_{\mathrm{BF}} = -iL \int_{\mathrm{HS}^2} \mathrm{d}^2x \, \sqrt{g} \, \mathrm{Tr}(\sigma f) \,, \tag{2.12}$$

where the scalar $f = \star F$ is the Hodge dual of the field strength two-form.

The above has its supersymmetric completion in the bulk action

$$S_{\mathrm{bulk}} = -iL \int_{\mathrm{HS}^2} \mathrm{d}^2x \, \sqrt{g} \, \mathrm{Tr}\left(\sigma f - \frac{1}{2}\bar{\lambda}\lambda + D\eta\right) \,, \tag{2.13}$$

which is still equivalent to (2.12) since the additional degrees of freedom are non-dynamical.

For the hemisphere, however, the supersymmetric variation of (2.13) produces a boundary term, namely

$$\delta_{\epsilon,\bar{\epsilon}} S_{\mathrm{bulk}} = -iL \int_{\partial \mathrm{HS}^2} \mathrm{d}\varphi \;\, \mathrm{Tr}\left(\frac{i}{2}\ell^2 \eta (\bar{\lambda}\gamma^\theta \epsilon - \bar{\epsilon}\gamma^\theta \lambda) + \sigma \delta_{\epsilon,\bar{\epsilon}} A_\varphi\right) \,, \tag{2.14}$$

originating from the integration of a total divergence. In order to obtain a supersymmetric action we should then complement S_{bulk} with the boundary term

$$S_{\mathrm{bdry}} = L\ell \int_{\partial \mathrm{HS}^2} \mathrm{d}\varphi \;\, \mathrm{Tr}\left(\sigma^2\right) . \tag{2.15}$$

In fact, for half of the supersymmetry[4] on $\partial \mathrm{HS}^2$, the second term in (2.14) is exactly canceled by the supersymmetric variation of (2.15), since they nicely combine into

$$\delta_{\epsilon,\bar{\epsilon}}(S_{\mathrm{bulk}} + S_{\mathrm{bdry}}) = -iL \int_{\partial \mathrm{HS}^2} \mathrm{d}\varphi \;\, \mathrm{Tr}(\sigma \, \delta_{\epsilon,\bar{\epsilon}}(A_\varphi + i\ell\sigma)) + \dots \,. \tag{2.16}$$

[4] Specifically, the preserved supercharges are those generated by (B.8).

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The combination $A_\varphi + i\ell\sigma$ can be regarded as the putative connection of a 1/2-BPS Wilson loop running along the boundary. The dots stand for a remaining term proportional to η coming from (2.14) that can be eliminated by imposing the boundary condition[5]

$$\eta\big|_{\partial \mathrm{HS}^2} = 0\,. \tag{2.17}$$

In summary, we managed to build a supersymmetric version of the BF action,

$$S_{\rm tot} = L\left[-i\int_{\mathrm{HS}^2} \mathrm{d}^2x\,\sqrt{g}\;\mathrm{Tr}\left(\sigma f - \frac{1}{2}\bar{\lambda}\lambda + D\eta\right) + \frac{\ell}{2}\int_{\partial\mathrm{HS}^2} \mathrm{d}\varphi\;\mathrm{Tr}\!\left(\sigma^2\right)\right], \tag{2.18}$$

which preserves half of the off-shell supersymmetry enjoyed by the bulk model on the sphere. We emphasize that, up to this point, both L and ℓ are general length scales. We will provide the correct identification for them in the following subsection by comparison with the relevant scales in JT gravity.

Finally, let us briefly turn our attention to the variational principle associated with the action (2.18). Upon variation of the fields, we obtain, as before, a boundary term that contains the variation $\delta(A_\varphi + i\ell\sigma)$. For the action (2.18) to be extremized by the bulk Euler-Lagrange equations, this term must vanish. This can be obtained by imposing for the combination $A_\varphi + i\ell\sigma$ to equal some constant on the boundary. We will come back to this point at the end of the next section.

2.3 Gauge and gravitational scales

At this point, we need to identify the relevant parameters in the BF theory with their gravitational counterparts in order to ensure a precise match between the partition function of JT gravity on the disk topology with that of our supersymmetric theory.

We start by reviewing some well-known facts about three-dimensional gravity with negative cosmological constant. The spectrum of 3d gravity includes global thermal AdS_3 and a collection of Euclidean BTZ solutions [43, 44], separated from the AdS_3 vacuum by a mass gap. All these Euclidean saddles are characterized by the topology of a solid torus. Notably, the modular invariance of 3d gravity naturally acts on the boundary torus of complex structure τ, allowing for their mapping to one another through modular transformations. In particular, a Euclidean geometry with torus boundary is specified once one chooses which cycle of the boundary torus is contractible in the bulk. In the case where the time cycle is contractible, we obtain a Euclidean BTZ solution. On the other hand, when the spatial cycle is contractible, we have thermal AdS_3 as the solution. Specifically, the non-rotating BTZ solution is related to thermal AdS_3 by a modular S-transformation $\tau \to -1/\tau$, which acts by swapping the two cycles.

Considering 3d coordinates $(t_{\rm E}, r, \phi)$ playing the role of time, radial coordinate, and angular coordinate respectively, it is known that the spherically symmetric $(t_{\rm E}, r)$ sector of 3d gravity is directly governed by JT gravity [16, 30]. For instance, the JT black hole can be obtained as the dimensional reduction of the BTZ three-dimensional one, by reducing

[5]The term proportional to η in (2.14) could also be canceled by the variation of an additional boundary contribution proportional to $-i\int \eta^2$.

along the circle parametrized by ϕ. In particular, the inverse temperature $\beta_{2\mathrm{d}}$ of the JT black hole is given by

$$\beta_{2\mathrm{d}} = \frac{4G_3 C}{\ell_{\mathrm{AdS}}^2}\beta_{3\mathrm{d}}\,, \tag{2.19}$$

in terms of the inverse temperature $\beta_{3\mathrm{d}}$ of the BTZ. In (2.19) ℓ_{AdS} represents the AdS radius, $G_{3\mathrm{d}}$ is the 3d Newton's constant, while C is the usual coupling of JT gravity[6] [16, 18]. This relation, which will turn useful later, can be proven by equating the corresponding entropies of the BTZ and JT black holes.[7] However, as noticed in [45], in most of the literature the JT gravity action is found to be supported in the (r, ϕ) section [17, 46, 47]. For instance, in [48] the Schwarzian model, which describes the boundary degree of freedom of JT gravity, emerges on the spatial angular direction, after compactifying the time circle. This second identification turns out to be the correct one for our purposes, where the coordinate φ, which we introduced in (2.10) to parametrize the hemisphere, is to be interpreted as a spatial direction.

Based on the digression above, we now present a brief argument to identify the correct values of the unspecified scales ℓ and L appearing in (2.18). Our starting point is the gauge formulation of 3d gravity with negative cosmological constant, which can be rephrased as a double Chern-Simons theory [49, 50] with action

$$S_{3\mathrm{d}} = i(S_{\mathrm{CS}}[A] - S_{\mathrm{CS}}[\bar{A}])\,, \tag{2.20}$$

where

$$S_{\mathrm{CS}}[A] = \frac{k}{4\pi}\int \mathrm{Tr}\left(A\wedge \mathrm{d}A + \frac{2}{3}A\wedge A\wedge A\right)\,. \tag{2.21}$$

Here, A and $\bar{A}$ are independent $\mathfrak{sl}(2)$ gauge fields, and k is the Chern-Simons level, related to the gravitational parameters by $k = \frac{\ell_{\mathrm{AdS}}}{4G_3}$. For our derivation, we focus on the holomorphic sector of the theory, which is described by the connection A. By virtue of what was argued before, we consider the Chern-Simons theory to be supported on a solid torus $D\times S^1$ which has the same topology of thermal AdS_3, with the Euclidean time coordinate running along the non-contractible cycle S^1. We equip it with the metric

$$\mathrm{d}s^2_{\mathrm{EAdS}_3} = \mathrm{d}t_{\mathrm{E}}^2 + C^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right)\,, \tag{2.22}$$

where the time variable is identified as $t_{\mathrm{E}} \sim t_{\mathrm{E}} + \beta_{2\mathrm{d}}k$.[8] We then dimensionally reduce

[6]In the JT literature, $C = \frac{\phi_r}{8\pi G_2}$, where ϕ_r is the renormalized value of the dilaton on the boundary and G_2 the 2d Newton's constant.

[7]The entropy of the BTZ black hole is given by the Hawking formula $S_{\mathrm{BTZ}} = \frac{2\pi r_{\mathrm{h}}}{4G_3}$, where the shorthand $r_{\mathrm{h}} = 2\pi\ell_{\mathrm{AdS}}^2/\beta_{3\mathrm{d}}$ is used to denote the radius of the event horizon [43]. The entropy of the JT black hole is instead given by $S_{\mathrm{JT}} = 4\pi^2 C/\beta_{2\mathrm{d}}$ [16]. We expect $S_{\mathrm{BTZ}} = S_{\mathrm{JT}}$ in the strict s-wave reduction, connecting the three dimensional and the two-dimensional theory.

[8]One can easily prove that the metric (2.22), by using the identification $\beta_{2\mathrm{d}} = \frac{C}{k\ell_{\mathrm{AdS}}}\beta_{3\mathrm{d}}$ given by (2.19) and by performing the change of variables $\theta = \arctan(r/\ell_{\mathrm{AdS}})$, is Weyl equivalent to the metric

$$\mathrm{d}s^2_{\mathrm{EAdS}_3} = \left(\frac{r^2}{\ell_{\mathrm{AdS}}^2}+1\right)\mathrm{d}t_{\mathrm{E}}^2 + \left(\frac{r^2}{\ell_{\mathrm{AdS}}^2}+1\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\phi^2$$

of thermal AdS_3.

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Chern-Simons theory (2.21) along $t_{\rm E}$ and by setting $A_{t_{\rm E}} = \sigma/k^2$ we obtain

$$S_{\rm 2d} = -i\frac{\beta_{\rm 2d}}{2\pi}\left(\int_\Sigma \mathrm{Tr}\,(\sigma F) - \frac{1}{2}\int_{\partial\Sigma} \mathrm{Tr}(\sigma A)\right) . \tag{2.23}$$

By enforcing the boundary condition

$$A_\varphi = -iC\sigma\,, \tag{2.24}$$

we finally arrive at the action

$$S_{\rm 2d} = \frac{\beta_{\rm 2d}}{2\pi}\left(-i\int_\Sigma \mathrm{Tr}\,(\sigma F) + \frac{C}{2}\int_{\partial\Sigma} \mathrm{d}\varphi\; \mathrm{Tr}\big(\sigma^2\big)\right) . \tag{2.25}$$

After integrating in the auxiliary fields of the $\mathcal{N} = (2,2)$ vector multiplet, we note that the action (2.25) precisely corresponds to our supersymmetric theory (2.18). A comparison between the two leads to the identification of the physical scales with

$$L = \frac{\beta_{\rm 2d}}{2\pi}\,, \qquad\qquad \ell = C\,. \tag{2.26}$$

We also note that the latter is consistent with the dimensional reduction of the metric (2.22) to the metric (2.10) of a hemisphere with a radius of C. Finally, the boundary condition (2.24) is consistent with requiring a well-defined variational principle for the action (2.18), as discussed at the end of section 2.2.

3 Localization of the supersymmetric BF model

The partition function of certain supersymmetric gauge theories can be evaluated by using a technique known as supersymmetric localization. The method relies on the fact that if a theory enjoys a fermionic symmetry δ_Q, one can deform the action by adding a δ_Q-exact term $t\,\delta_Q\mathcal{V}$ without altering the result of the path integral.

The proof of this property is trivial and goes as follows. One first introduces an auxiliary quantity

$$Z(t) = \int [D\Phi]\; e^{-S[\Phi]-t\,\delta_Q V[\Phi]}\,, \tag{3.1}$$

where we denote with Φ the full set of dynamical fields in the theory. By construction, $Z(0)$ gives the undeformed partition function that we are set out to compute. If we assume that $\delta_Q^2 V = 0$,[9] we can easily show that

$$\frac{\mathrm{d}}{\mathrm{d}t} Z(t) = \int [D\Phi]\; \delta_Q\Big(V\, e^{-S[\Phi]-t\,\delta_Q V[\Phi]}\Big)\,, \tag{3.2}$$

which, in turn, vanishes since it is the integral of δ_Q-exact expression.[10] This means that (3.1) computes the undeformed partition function for any value of t. In particular, we

[9] In general, δ_Q^2 does not vanish, but it yields a bosonic symmetry of the theory. Thus, requiring $\delta_Q^2 V = 0$ is equivalent to the invariance of V under this bosonic symmetry.

[10] The validity of (3.2) also assumes that the convergence of the path integral does not depend on t and the measure of integration is invariant under δ_Q.

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can take the limit $t \to \infty$. If the bosonic part of $\delta_Q V$ is positive-definite, the path integral is *exactly* captured by the semiclassical expansion of $Z(t)$ around the saddle points of $\delta_Q V$, rather than around those of the classical action.

In the following, we will apply this technique to the supersymmetric BF theory defined by the action in (2.18). By virtue of the considerations of section 2.3, this procedure gives an alternative way to calculate the partition function of JT gravity.

3.1 Localizing term

Following [41], we choose the localizing supercharge $\delta_Q = \delta_{\epsilon,\bar{\epsilon}}$, where the specific form of the Killing spinors ϵ and $\bar{\epsilon}$ is given in (B.8). The localizing term reads

$$V \equiv \delta_{0,\bar{\epsilon}} \int_{\mathrm{HS}^2} \mathrm{d}^2x\, \sqrt{g}\, \mathrm{Tr}\left(\frac{1}{2}\bar{\lambda}\gamma_3\lambda - 2iD\sigma + i\eta^2\right) . \tag{3.3}$$

The variation of (3.3) can be split as

$$\delta_Q V = S_{\mathrm{SYM}} + W_{\mathrm{bos}} + W_{\mathrm{ferm}} \tag{3.4}$$

in terms of a bulk action

$$\begin{aligned} S_{\mathrm{SYM}} = \int_{\mathrm{HS}^2} \mathrm{d}^2x\, \sqrt{g}\, \frac{\bar{\epsilon}\epsilon}{2}\, \mathrm{Tr}\Big[\left(f + \frac{\eta}{\ell}\right)^2 + \mathrm{D}^\mu\eta \mathrm{D}_\mu\eta + \mathrm{D}^\mu\sigma\mathrm{D}_\mu\sigma - [\eta,\sigma]^2 + D^2 \\ - \frac{i}{2}\left(\mathrm{D}_\mu\bar{\lambda}\gamma^\mu\lambda - \bar{\lambda}\gamma^\mu\mathrm{D}_\mu\lambda\right) + i\bar{\lambda}[\eta,\lambda] + \bar{\lambda}\gamma^3[\sigma,\lambda]\Big] , \end{aligned} \tag{3.5}$$

which is the super-Yang-Mills action for the $\mathcal{N} = (2,2)$ multiplet, and a boundary term that we further split into its bosonic and fermionic parts

$$W_{\mathrm{bos}} = \int_{\partial\mathrm{HS}^2} \mathrm{d}\varphi\;\, \mathrm{Tr}\left(-\sigma\mathrm{D}_\theta\sigma + i\ell\sigma f + i\eta\sigma\right) , \tag{3.6}$$

$$W_{\mathrm{ferm}} = \int_{\partial\mathrm{HS}^2} \mathrm{d}\varphi\;\, \mathrm{Tr}\left(-\frac{i}{4}\ell\bar{\lambda}\lambda\right) . \tag{3.7}$$

A set of sufficient conditions to make the fermionic boundary term (3.7) vanish is given, in our conventions, by $\lambda_1 = \lambda_2$ and $\bar{\lambda}_1 = \bar{\lambda}_2$ on $\partial\mathrm{HS}^2$.

In section 2.2, we observed that the action (2.18) preserves the relevant supersymmetry if one imposes (2.17). To preserve supersymmetry for the full deformed action at a generic t, i.e. of the supersymmetric action (2.18) plus the localizing term $t\,\delta_Q V$, one needs to impose additional boundary conditions on the bosonic fields. A set of conditions preserving full gauge invariance on the boundary is given by

$$\eta|_{\partial\mathrm{HS}^2} = 0\,, \qquad F|_{\partial\mathrm{HS}^2} = 0\,, \qquad \mathrm{D}_\theta\sigma|_{\partial\mathrm{HS}^2} = 0\,. \tag{3.8}$$

From the point of view of the original gravitational theory, these conditions carry an interesting physical interpretation, as they correspond to imposing for the asymptotic geometry to satisfy the classical equation of motion of JT gravity. In particular the equations of motion of the BF action (2.1)

$$F = 0\,, \qquad \mathrm{D}_\mu\chi = 0\,, \tag{3.9}$$

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have their equivalent as equations for the metric and the dilaton field in

$$R = -\Lambda\,, \qquad\qquad \nabla_\mu \nabla_\nu \Phi = \frac{\Lambda}{2} g_{\mu\nu} \Phi\,. \tag{3.10}$$

Upon identifying χ with σ, we note that the equation $\mathrm{D}_\varphi \sigma = 0$ is the usual *Gauss-law constraint*, while the equation for the component normal to the boundary, namely $\mathrm{D}_\theta \sigma = 0$, gives rise to a Neumann boundary condition for the scalar σ as in (3.8).

Notice that the last two identities in (3.8) here are presented as alternative to (2.24). In this regard, we observe that another choice of boundary conditions for the t-deformed action can be generated by considering its variational principle. In fact, the variation of the bosonic part of the full action produces the boundary term

$$\begin{aligned} \delta S\Big|_{\text{bos}} = L \int_{\partial \mathrm{HS}^2} \mathrm{d}\varphi \; \mathrm{Tr}\Big[-i\sigma\,\delta\left(A_\varphi + i\ell\sigma - \frac{t}{L}(\ell f + \eta + i\mathrm{D}_\theta \sigma)\right) \\ + \frac{t}{L}\mathrm{D}_\theta \eta\,\delta\eta + \frac{t}{L}\left(f + \frac{\eta}{\ell}\right)\delta(A_\varphi + i\ell\sigma)\Big]\,. \end{aligned} \tag{3.11}$$

The above can be made to vanish by imposing, for any $t > 0$,

$$\eta\Big|_{\partial \mathrm{HS}^2} = 0\,, \qquad F\Big|_{\partial \mathrm{HS}^2} = 0\,, \qquad A_\varphi + i\ell\sigma - i\frac{t}{L}\mathrm{D}_\theta \sigma\,\Big|_{\partial \mathrm{HS}^2} = c\,, \tag{3.12}$$

where c is a constant. Upon setting $c = 0$, the last condition effectively interpolates between (2.24) at $t \to 0$ and the Neumann boundary condition for σ introduced in (3.8) for $t \to \infty$.

When applied to the path-integral (3.1), an interpolating boundary conditions such as (3.12) does not necessarily lead to a partition function that is independent of t. In fact, its derivative in (3.2) should contain an additional term associated with a variation of the domain of integration. Yet, because the boundary condition (2.24), as shown in previous literature [29, 31], is responsible for generating the correct Schwarzian dynamics on the boundary, one is led to speculate that the alternative boundary conditions considered earlier should give rise to the same partition function that the localization procedure will compute, given that the latter will be shown to match the known result for JT gravity.

In both cases, however, in the limit of interest, namely the $t \to \infty$ limit where the path integral is dominated by the supersymmetric saddles, the boundary conditions are those in (3.8). Using a gauge-fixed form of (3.8) proves to be convenient. In particular, we use our gauge freedom to set $A_\theta = 0$ on the boundary. With this choice, we can exploit some results already present in the literature [41]. The bosonic boundary conditions now reduce to the following standard form:

$$\partial_\theta A_\varphi|_{\partial \mathrm{HS}^2} = 0\,, \qquad \partial_\theta \sigma|_{\partial \mathrm{HS}^2} = 0\,, \qquad \eta|_{\partial \mathrm{HS}^2} = 0\,, \qquad A_\theta|_{\partial \mathrm{HS}^2} = 0\,, \tag{3.13}$$

which amounts to considering Dirichlet boundary conditions for η and A_θ, and Neumann boundary conditions for A_φ and σ.

Before moving to the actual localization computation, a few comments are in order. Performing a path integral over bosonic fields involves a choice of a half-dimensional

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integration contour in the space of complex fields. Specifically, the integration contour for bosonic fields in Euclidean gauge theories must be chosen in such a way that the resulting gauge group is some compact subgroup of the complexification $G_{\mathbb{C}}$ of the original gauge group G, and the bosonic action is positive definite. In supersymmetric localization this applies both to the original and the localizing actions separately.

In order to give meaning to the path integral of the supersymmetric BF theory at hand, defined at the classical level over the gauge group $\mathrm{SL}(2,\mathbb{R})$, we find that the correct choice to reproduce the Schwarzian result is to pick a contour in such way that all fields are real and the resulting gauge group is $\mathrm{SU}(2)$, a compact subgroup of $\mathrm{SL}(2,\mathbb{C})$.

For a detailed discussion on the choice of integration contour for supersymmetric $\mathcal{N}=(2,2)$ gauge theories on the hemisphere, we refer the reader to [42].

3.2 Matrix model

The minimum of the bosonic gauge sector in (3.5) is realized when the following set of conditions holds

$$f=-\eta\,,\qquad \mathrm{D}_\mu\sigma=\mathrm{D}_\mu\eta=0\,,\qquad D=0\,,\qquad [\sigma,\eta]=0\,. \tag{3.14}$$

The localization locus defined by (3.14) is easy to characterize. Consider first the scalar field η. Since it vanishes at the boundary because of (3.13) and is covariantly constant ($\mathrm{D}_\mu\eta=0$), it must vanish everywhere. This implies $f=0$. In other words, the gauge field A must be a flat connection. However, since the hemisphere is contractible, every flat connection is gauge equivalent to $A_\mu=0$. The only non-vanishing field is the scalar field σ, fixed to be an arbitrary constant σ_0. In summary, the set of field configurations that satisfies (3.14), compatible with the boundary conditions (3.13), is

$$\sigma=\frac{\sigma_0}{\ell}\,,\qquad A_\mu=0\,,\qquad \eta=0\,,\qquad D=0\,. \tag{3.15}$$

Similarly to the case of the Chern-Simons reformulation of three-dimensional gravity, the BF theory is (classically) equivalent to JT gravity only when the zweibein is invertible. At the perturbative level, this requirement is implemented by expanding the path integral around the geometrical (semiclassical) saddle point $e^a_\mu=\delta^a_\mu$ and $\omega=0$, which is quite far from the non-geometrical saddle obtained in (3.15). When we turn on the localizing parameter t, we are instead allowing for the emergence of new saddle points that may compete and eventually replace the semiclassical one. We implicitly assume that complexification permits to deform the original semiclassical contour into a new one, picking the dominant contribution from the non-geometrical saddle $A_\mu=0$ (see (3.15)).

We now evaluate the total classical action (2.18) on the locus (3.15). The bulk term vanishes, and we are left only with the boundary term

$$S_{\mathrm{tot}}\big|_{\mathrm{locus}}=\frac{L}{2\ell}\int_{\partial\mathrm{HS}^2}\mathrm{d}\varphi\ \mathrm{Tr}\left(\sigma_0^2\right). \tag{3.16}$$

Therefore, the infinite-dimensional path integral, which evaluates the partition function of JT gravity, localizes to a matrix model with the following structure:

$$\mathrm{Z}_{\mathrm{JT}}=\frac{1}{\mathrm{vol}(\mathfrak{g})}\int_{\mathfrak{g}}\mathrm{d}\sigma_0\ \exp\left(-\frac{L}{2\ell}\int_{\partial\mathrm{HS}^2}\mathrm{d}\varphi\ \ \mathrm{Tr}\left(\sigma_0^2\right)\right)\mathcal{Z}_{\text{1-loop}}\left[\sigma_0\right]. \tag{3.17}$$

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In (3.17) $\mathcal{Z}_{\text{1-loop}}[\sigma_0]$ encodes the contributions of the one-loop determinants arising from the Gaussian integrals originating from the localizing term $\delta_Q V$ when we expand around the locus (3.15). The subscript $\mathfrak{g}$ on the integral means we are integrating over the Lie algebra of the gauge group; in this case, $\mathfrak{g} \simeq \mathfrak{su}(2)$.

Since the initial action is gauge invariant, the integrand in (3.17) turns out to be invariant under the adjoint action of the gauge group. We can use this freedom to diagonalize the matrix σ_0 through a gauge transformation and reduce the integral over the entire Lie algebra $\mathfrak{g}$ to an integral over a chosen realization of the Cartan subalgebra $\mathfrak{t}$. The Jacobian of this transformation will produce the usual Vandermonde determinant at the level of the integration measure. These steps are summarized by the following general identity that holds for any integral of an adjoint invariant function $\mathfrak{f}(\sigma)$

$$\frac{1}{\text{vol}(\mathfrak{g})}\int_{\mathfrak{g}} \mathrm{d}^{d_g}\sigma\ \mathfrak{f}(\sigma) = \frac{1}{|W|}\int_{\mathfrak{t}} \mathrm{d}^{l_{\mathfrak{g}}}\sigma\ \mathfrak{f}(\sigma) \prod_{\alpha\in\Delta_+} \alpha(\sigma)^2 \,. \tag{3.18}$$

Above $d_{\mathfrak{g}}$ and $l_{\mathfrak{g}}$ are the dimension and the rank of $\mathfrak{g}$, while Δ_+ is the set of its positive roots, denoted with α. In (3.18), we normalize the l.h.s. by the order of the Weyl group, $|W|$, to account for the residual gauge symmetry. Then we are left with

$$\mathrm{Z}_{\mathrm{JT}} = \frac{1}{2!}\int_{\mathfrak{t}} \mathrm{d}\sigma_0\ \alpha(\sigma_0)^2\ \exp\left(-\frac{L}{2\ell}\int_{\partial \mathrm{HS}^2} \mathrm{d}\varphi\ \mathrm{Tr}\left(\sigma_0^2\right)\right)\ \mathcal{Z}_{\text{1-loop}}[\sigma_0] \tag{3.19}$$

where σ_0 is now assumed to be in the Cartan of $\mathfrak{su}(2)$, i.e., the component along the diagonal generator γ_3.

The expression for $\mathcal{Z}_{\text{1-loop}}[\sigma_0]$ is obtained by combining the integration of the Gaussian fluctuation for both the bosonic and fermionic degrees of freedom around the localization locus. The steps necessary to obtain such quantities are presented in appendix C. Specifically, by taking the product of (C.21) and (C.29) and simplifying the common factors between the numerator and denominator, we arrive at

$$\begin{aligned}\mathcal{Z}_{\text{1-loop}} &= \mathcal{Z}^{\text{bos}}_{\text{1-loop}} \times \mathcal{Z}^{\text{fer}}_{\text{1-loop}} \\ &\sim \prod_{j=1}^{\infty}\left(j^2+\alpha(\sigma_0)^2\right) \\ &\sim \left(\prod_{j=1}^{\infty} j^2\right)\prod_{j=1}^{\infty}\left(1+\frac{\alpha(\sigma_0)^2}{j^2}\right).\end{aligned} \tag{3.20}$$

The first of the two infinite products can be regularized by using the zeta function regularization, giving

$$\prod_{j=1}^{\infty} j^2 = e^{-2\zeta'(0)} = 2\pi\,, \tag{3.21}$$

where we used $\zeta'(0) = -\frac{1}{2}\ln 2\pi$. In the second one, we recognize the representation of the hyperbolic sine as an infinite product. Then

$$\mathcal{Z}_{\text{1-loop}} = \frac{2\sinh(\pi\alpha(\sigma_0))}{\alpha(\sigma_0)}\,. \tag{3.22}$$

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By substituting the one-loop determinant (3.22) into the localization formula (3.19), we find that the partition function is given by

$$\mathrm{Z_{JT}} = \int_{\mathfrak{t}} \mathrm{d}\sigma_0\, \alpha(\sigma_0)\, \sinh(\pi\alpha(\sigma_0))\, \exp\left(-\frac{\pi L}{\ell}\,\mathrm{Tr}\left(\sigma_0^2\right)\right). \tag{3.23}$$

Since σ_0 lies in the Cartan subalgebra $\mathfrak{t}$, we can parametrize it as $\sigma_0 = s\gamma_3$ with $s \in \mathbb{R}$. For $\mathfrak{su}(2)$, the only positive root is 1, so we have $\alpha(s\gamma_3) = 2s$ and $\mathrm{Tr}(\gamma_3^2) = 2$. Consequently, (3.23) becomes

$$\mathrm{Z_{JT}} = 4\int_0^\infty \mathrm{d}s\; s\, \sinh(2\pi s)\, \exp\left(-\frac{\beta}{C}s^2\right), \tag{3.24}$$

where we have reinstated the gravitational scales $C = \ell$ and $\beta \equiv \beta_{2\mathrm{d}} = 2\pi L$, and utilized the parity of the integrand function to limit the integral to the range $[0, +\infty)$. The result (3.24) coincides with the one obtained in [31] through $\mathrm{SL}(2,\mathbb{R})$ Hamiltonian quantization.

We can now evaluate the integral over s and obtain

$$\mathrm{Z_{JT}} \propto \left(\frac{C\pi}{\beta}\right)^{3/2} e^{\frac{C\pi^2}{\beta}}, \tag{3.25}$$

which reproduces the result obtained through equivariant localization of the Schwarzian theory [18] and via conformal bootstrap from Liouville CFT [36].

4 Higher-spin JT gravity

Our results can be easily generalized to compute the partition function of the higher-spin version of JT gravity on the disk. Here, we are interested in higher-spin theories living on an AdS background. The first remarkable constructions were developed in four dimensions [51–55], and later extended to a generic number of spacetime dimensions [56]; higher-spin theories also play an important role in holography [57, 58].

One may hope that in lower dimensions, some simplifications should happen. That is the case for three-dimensional higher-spin gravity, as there are no propagating local degrees of freedom. Moreover, higher-spin theories admit a Chern-Simons formulation [59] that generalizes the pure gravitational construction. Unlike the higher dimensional cases, there is no need to consider an infinite number of higher-spin fields for having consistent interactions [60–62]. A natural and simple example is the higher-spin theory corresponding to the Chern-Simons theory $\mathrm{SL}(N,\mathbb{R}) \times \mathrm{SL}(N,\mathbb{R})$, containing fields with spin up to N.

A somewhat analogous situation also occurs in higher-spin extensions of JT gravity. They can be constructed from the gauge theory formulation and allowing the gauge group to be $\mathrm{SL}(N,\mathbb{R})$ [32, 63, 64].[11] While all these formulations require some relevant modifications of the considerations worked out in the standard JT gravity, see for instance [33–35, 65], the technology developed in the previous section can be readily extended to the higher-spin case $\mathrm{SL}(N,\mathbb{R})$.

[11] Here we are not precise on the global structure of the gauge group, as it will not be relevant.

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4.1 Supersymmetric higher-spin theory

To begin with, we shall briefly review how an $\mathrm{SL}(N,\mathbb{R})$ version of the BF theory realizes a higher-spin theory [63]. As done for standard JT gravity in section 2.1, we work with the first-order formalism and organize fields into a connection and dilaton field. Let us now be slightly more general and work with a gauge group G with the property that it contains an $\mathrm{SL}(2,\mathbb{R})$ sector generated by the P_i of section 2.1. This factor corresponds to the AdS_2 isometry group. Then we demand that all the other generators in the adjoint of G can be decomposed into a totally symmetric irreducible representation of the $\mathrm{SL}(2,\mathbb{R})$ factor. Let us denote these generators as $\mathsf{T}_{a_1\ldots a_s}$ for some integer s. They are totally symmetric and traceless, namely $\eta^{a_1a_2}\mathsf{T}_{a_1a_2\ldots a_s} = 0$.

Any Lie-algebra valued field Φ in our BF theory will have an expansion

$$\Phi = \Phi^a\,\mathsf{P}_a + \sum_s \Phi^{a_1\ldots a_s}\,\mathsf{T}_{a_1\ldots a_s}\,. \tag{4.1}$$

Because of the specific properties of $\Phi^{a_1\ldots a_s}$, it is natural to interpret it as a higher-spin s field [61, 62].

It turns out that $\mathrm{SL}(N,\mathbb{R})$ satisfies all the above conditions [66]. Therefore, from now on, we will focus on this specific case and construct the corresponding generalization of JT gravity. To do so, we mimic the BF construction of section 2.1. That is, we introduce an $\mathrm{SL}(N,\mathbb{R})$ dilaton field χ expanded as in (4.1) and a $\mathrm{SL}(N,\mathbb{R})$ connection

$$A = \sum_s A_\mu^{a_1\ldots a_s}\,\mathsf{T}_{a_1a_2\ldots a_s}\,\mathrm{d}x^\mu\,. \tag{4.2}$$

The action is just the BF one

$$S_{\mathrm{BF}} = -i\int_\Sigma \mathrm{Tr}\,(\chi F) \tag{4.3}$$

The equations of motion are

$$F_{\mu\nu}^{a_1\ldots a_s} = 0\,, \qquad\qquad D_\mu\chi^{a_1\ldots a_s} = 0\,, \tag{4.4}$$

where $\mathrm{D}_\mu = \nabla_\mu + A_\mu$ and $F_{\mu\nu}^{a_1\ldots a_s}$ is the field strength related to $A_\mu^{a_1\ldots a_s}$. One can study them in the metric formulation around the AdS_2 background and show that indeed reproduce a 2d higher-spin gravity theory, identifying the spectrum [63].

The next step would be studying the asymptotic boundary conditions that reproduce a consistent generalization of the Schwarzian dynamics [33, 35]. This requires the addition of the familiar boundary term

$$S_{\mathrm{bdy}} \propto \int_{\partial\Sigma} \mathrm{d}\varphi\;\mathrm{Tr}\Big(\chi^2\Big)\,. \tag{4.5}$$

Not so surprisingly, one must also consider asymptotic boundary conditions preserving the W_N-algebra (a non-linear extension of Virasoro).

However, we do not follow this approaches here. We rather adopt an extension of the method of the previous section to define and quantize the higher-spin version of JT gravity. Indeed, one advantage of our method is that there are no further technical difficulties in moving from SU(2) to an arbitrary gauge group.

The quantization scheme is always the same. We start with an $\mathrm{SL}(N,\mathbb{R})$ BF theory with the given boundary term and make the implicit assumption, based on [34, 67], that analogous steps to those outlined in section 2.3 can be performed. We can make the theory supersymmetric on the hemisphere (topologically equivalent to the disk) as explained in section 2.2. At the end of the day, our action reads again

$$S_{\rm tot} = \frac{\beta}{2\pi}\left[-i\int_{{\rm HS}^2}{\rm d}^2x\,\sqrt{g}\,{\rm Tr}\left(\sigma f - \frac{1}{2}\bar\lambda\lambda + D\eta\right) + \frac{C}{2}\int_{\partial{\rm HS}^2}{\rm d}\varphi\;{\rm Tr}\left(\sigma^2\right)\right]. \tag{4.6}$$

As discussed in the JT gravity case in section 3, we need to specify an integration contour in the field space to make sense of the path integral with action (4.6). Again, we can directly generalize the procedure successfully introduced for the standard JT case. Namely, we choose the half-dimensional integration contour in the space of complex fields such that all the fields are real with gauge group SU(N), the compact real section of the complexification of the original gauge group SL($N,\mathbb{R}$). We conjecture that this prescription reproduces the higher-spin version of JT gravity.

In the following, we shall recover from a supersymmetric localization perspective all the results in the literature for the partition function on the disk [33–35] and explicitly show the relations among the different expressions.

4.2 Exact partition function

It is fairly easy to generalize the results of section 3.2 to more general choices of gauge group, as such a choice enters only when one has to make explicit the roots or weights appearing in the classical contribution and in the 1-loop determinant. Therefore, we can start directly from the extension of (3.19) to a generic group, with L and ℓ replaced by their physical values β and C respectively, i.e.

$$\mathrm{Z}_{\rm JT} = \frac{1}{N!}\int_{\mathfrak{t}}{\rm d}\sigma_0\,\exp\left(-\frac{\beta}{2C}\int{\rm d}\varphi\;{\rm Tr}\left(\sigma_0^2\right)\right)\prod_{\alpha>0}\left[2\alpha(\sigma_0)\,\sinh(\pi\alpha(\sigma_0))\right], \tag{4.7}$$

where α labels the roots of SU(N). If we use their explicit expression, we find

$$\mathrm{Z}_{\rm JT} = \frac{1}{N!}\int\prod_i{\rm d}\sigma_i\,\delta\left(\sum_i\sigma_i\right)\prod_{i<j}\left[2(\sigma_i-\sigma_j)\,\sinh(\pi(\sigma_i-\sigma_j))\right]\,\exp\left(-\frac{\beta}{2C}\sum_i\sigma_i^2\right), \tag{4.8}$$

where σ_i, $i=1,\dots,N$ are the eigenvalues of the constant matrix σ_0, and the delta function set to zero the trace of σ_0.

We aim to evaluate the integral, which computes the partition function of the SL($N,\mathbb{R}$) higher-spin JT gravity on the disk. Using the integral representation of the Dirac and the Weyl denominator formula,[12] we arrive at the expression

$$\mathrm{Z}_{\rm JT} = \frac{(-1)^{\frac{N(N-1)}{2}}}{N!}\int\frac{{\rm d}k}{2\pi}\sum_{\eta,\lambda\in S_N}(-1)^{\lambda+\eta}\prod_i\frac{1}{(2\pi)^{\lambda(i)-1}} \\ \times\frac{\partial}{\partial u_i^{\lambda(i)-1}}\int{\rm d}\sigma_i\,e^{2\pi\left(\frac{N+1}{2}-\eta(i)+\frac{ik}{2\pi}+u_i\right)\sigma_i-\frac{\beta}{2C}\sigma_i^2}\Big|_{u_i=0}, \tag{4.9}$$

where we introduced the sources u_i to perform the Gaussian integrals.

[12] We use the two following variants of the formula

$$\prod_{1<i<j<N}2\sinh(\pi(\sigma_i-\sigma_j)) = \sum_{\eta\in S_N}(-1)^\eta\prod_i e^{2\pi\left(\frac{N+1}{2}-\eta(i)\right)\sigma_i},$$

$$\prod_{1<i<j<N}\sigma_j-\sigma_i = \sum_{\lambda\in S_N}(-1)^\lambda\sigma_1^{\lambda(1)-1}\dots\sigma_N^{\lambda(N)-1} = \sum_{\lambda\in S_N}(-1)^\lambda\prod_i\sigma_i^{\lambda(i)-1},$$

where S_N denotes the set of permutations of N elements.

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We can now perform both integrals and obtain

$$Z_{\rm JT} = \frac{(-1)^{\frac{N(N-1)}{2}}}{N!\sqrt{N}} \left(\frac{2\pi C}{\beta}\right)^{\frac{N-1}{2}} e^{\frac{\pi^2 CN(N^2-1)}{6\beta}} \sum_{\lambda,\eta\in S_N} (-1)^{\eta+\lambda} \prod_i \frac{1}{(2\pi)^{\lambda(i)-1}} \frac{\partial e^{\mathcal{P}}}{\partial u_i^{\lambda(i)-1}}\Big|_{u_i=0}, \tag{4.10}$$

where

$$\mathcal{P} = \frac{2\pi^2 C}{\beta}\left(\sum_j u_j^2 - \frac{1}{N}\sum_{j,k} u_j u_k + (N+1)\sum_j u_j - 2\sum_j \eta(j) u_j\right). \tag{4.11}$$

We argue that, in $\mathcal{P}$, the quadratic part in u_i does not contribute to the final result. To show this, we use the Weyl denominator formula in the opposite direction for the sum over η and restore a product of hyperbolic sines:

$$\sum_{\eta\in S_N} (-1)^\eta e^{\mathcal{P}} = \exp\left(\frac{2\pi^2 C}{\beta}\left(\sum_j u_j^2 - \frac{1}{N}\sum_{j,k} u_j u_k\right)\right) \prod_{i<j} 2\sinh\left(\frac{2\pi^2 C}{\beta}(u_i - u_j)\right). \tag{4.12}$$

The only non-vanishing terms arise when all the derivatives act an all the hyperbolic sines. But this can only occur when no derivative acts on the exponential term. As a consequence, we can safely drop the latter from (4.10).

The β-dependence follows from a straightforward scaling argument. Upon renaming

$$x_i = \frac{2\pi^2 C}{\beta} u_i\,, \tag{4.13}$$

we find

$$Z_{\rm JT} = \frac{(-1)^{\frac{N(N-1)}{2}}}{N!\sqrt{N}} \left(\frac{2\pi C}{\beta}\right)^{\frac{N^2-1}{2}} e^{\frac{\pi^2 CN(N^2-1)}{6\beta}} \mathcal{K}\,, \tag{4.14}$$

where $\mathcal{K}$ is an overall normalization given by

$$\begin{aligned} \mathcal{K} &= 2^{-\frac{N(N-1)}{2}} \sum_{\eta,\lambda\in S_N} (-1)^{\eta+\lambda} \prod_i \frac{\partial}{\partial x_i^{\lambda(i)-1}} e^{(N+1)\sum_j x_j - 2\sum_j \eta(j) x_j}\Big|_{x_i=0} \\ &= 2^{-\frac{N(N-1)}{2}} \sum_{\eta,\lambda\in S_N} (-1)^{\eta+\lambda} \prod_i [N+1-2\eta(i)]^{\lambda(i)-1}\,. \end{aligned} \tag{4.15}$$

We can get rid of one sum over the permutations by renaming $i \to \eta^{-1}(i)$ and $\pi = \lambda \circ \eta^{-1}$. This gives

$$\begin{aligned} \mathcal{K} &= 2^{-\frac{N(N-1)}{2}} \sum_{\pi\in S_N} (-1)^\pi \prod_i [N+1-2\eta(i)]^{\pi(i)-1} \\ &= N! \prod_{i<j} (i-j) \\ &= (-1)^{\frac{N^2-N}{2}} G(N+2)\,, \end{aligned} \tag{4.16}$$

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where $G(x)$ is the Barnes function. In summary, we find

$$Z_{\mathrm{JT}} = \frac{G(N+2)}{N!\sqrt{N}} \left(\frac{2\pi C}{\beta}\right)^{\frac{N^2-1}{2}} \exp\left(\pi^2 C \frac{N(N^2-1)}{6\beta}\right). \tag{4.17}$$

The β-dependent part agrees nicely with the results obtained by a different localization scheme in [35], up to the identification $C = 2\gamma$. This computation provides additional and non-trivial evidence that our quantization procedure is, not only self-consistent but also suggests an alternative and novel computational method in the context of JT gravity and its generalizations.

5 Conclusions and outlook

This paper proposes a localization procedure for JT gravity and its higher-spin generalization on the disk topology. We have used a supersymmetric completion of the related gauge theory, involving only auxiliary fields, and complexified the path integration to reduce the computation to a "standard" BF theory on the hemisphere.

The correct $s\sinh(2\pi s)$ measure for JT gravity, which in the previous formulations was obtained as the Plancherel measure associated either with the positive semigroup $\mathrm{SL}^+(2,\mathbb{R})$ [68] or with the analytic continuation of the universal cover of $\mathrm{SL}(2,\mathbb{R})$ [31], in our framework is directly provided by gaussian integral over quadratic fluctuations around the dominant saddle point as $t \to +\infty$. The supersymmetric boundary conditions play a crucial role in annihilating any discrete sum within the moduli space of the localized theory, with the constant configurations of the field σ being the only locus to integrate over.

Furthermore, supersymmetry provided us with a crucial boundary potential, quadratic in σ, which carries the information of the gravitational Gibbons-Hawking term: by carefully establishing the identification between physical and geometrical scales, we have recovered the well-known partition function of JT gravity and confirmed the results [33, 35] for the higher-spin theory.

The natural follow-up of this work is to generalize the supersymmetric localization of boundary-anchored Wilson lines correlation functions: they admit a simple representation at the level of gauge theory and correspond to correlators of bi-local operators in the boundary Schwarzian quantum mechanics [29, 31]. The explicit expression for the two-point and the four-point functions has appeared in [29, 31, 36] and was checked to be consistent with direct Schwarzian calculations [69]. It would be nice to reproduce the result of [29, 31] from the localization perspective: we expect that representing the Wilson lines would need an extension of the field content of the original BF, probably involving the presence of a chiral multiplet. If working in the JT gravity case, the procedure could be easily extended to the higher-spin generalization. In this case, exact results are more difficult to extract compared to the purely gravitational case. For instance, if one insists on computing them from the generalized BF approach of [29, 31], the complication comes from the lack of explicit expressions for the representation matrices of $\mathrm{SL}(N;\mathbb{R})$.[13] Perhaps our localization approach

[13] See nonetheless [29] for some progress with a focus on the spin-3 case where one can rely on the results of [70].

can be used to sidestep such difficulties. Another issue would be to apply our machinery to super-JT gravity, where supersymmetric localization should work along similar lines.

Moreover, JT gravity is known to emerge in the near-horizon limit of four-dimensional extremal black-holes [71] and recently supersymmetric localization has been applied to the computation of their entropy [72, 73]. In light of these advances, it would be interesting to perform the localization of JT gravity in the metric variables in the spirit of [74] analogous higher-dimensional cases [75].

As a final comment, although our computation has been performed on a disk topology, JT gravity is known to admit a celebrated non-perturbative completion as a sum over different topologies [25]. It would be tempting to extend our BF gauge-theoretic approach to higher genus/multi-boundary surfaces. Usually, however, one is faced with the issue that the mapping class group is not taken into account in the BF formulation.[14] An easier case where to perform a bulk localization should be given by the singular disk geometry (i.e. the "trumpet"), realized at the gauge-theory level by the insertion of vortex configuration [76].

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A Conventions

We recall the conventions for spinors. A spinor ψ_α is a two components column vector. Indices are raised and lowered according to $\psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta$, $\psi_\alpha = \epsilon_{\alpha\beta}\psi^\beta$, where

$$\epsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\,, \qquad \epsilon_{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\,. \tag{A.1}$$

Standard bilinears are

$$\psi\chi \equiv \psi^\alpha\chi_\alpha\,, \qquad \psi\gamma_\mu\chi \equiv \psi^\alpha(\gamma_\mu)_\alpha{}^\beta\chi_\beta\,. \tag{A.2}$$

Notice that

$$\psi\chi = (-1)^{h+1}\,\chi\psi\,, \qquad \psi\gamma_\mu\chi = (-1)^h\,\chi\gamma_\mu\psi\,, \tag{A.3}$$

where $h = 1$ if both ψ and χ are odd, otherwise $h = 0$.

[14] We thank T. G. Mertens for pointing out this aspect to us.

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The (flat) gamma matrices satisfy the relations

$$\gamma_a\gamma_b = \delta_{ab} + i\epsilon_{ab}\gamma_3\,, \qquad \gamma_3\gamma_a = i\epsilon_{ab}\gamma^b \tag{A.4}$$

Usefull Fierz identities can be derived just by reducing those for 3d spinors. The basic Fierz identity is

$$\chi_\alpha\psi^\beta = \frac{(-1)^h}{2}\left[\delta_\alpha^\beta(\psi\chi) + (\psi\gamma_a\chi)(\gamma^a)_\alpha{}^\beta + (\psi\gamma_3\chi)(\gamma_3)_\alpha{}^\beta\right]. \tag{A.5}$$

For instance, for any spinors χ, ψ, and λ

$$\chi_\alpha(\psi\lambda) + \psi_\alpha(\lambda\chi) + \lambda_\alpha(\chi\psi) = 0\,. \tag{A.6}$$

B Supersymmetry on the hemisphere

The metric of the hemisphere reads

$$\mathrm{d}s^2 = \ell^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right), \tag{B.1}$$

where $\theta \in [0, \frac{\pi}{2}]$ and $\varphi \in [0, 2\pi)$, while as vielbein we choose

$$e^1 = \ell\,\mathrm{d}\theta\,, \qquad\qquad e^2 = \ell\,\sin\theta\,\mathrm{d}\varphi\,. \tag{B.2}$$

We also choose $\gamma_1 = \sigma_1$, $\gamma_2 = \sigma_2$ and $\gamma_3 = \sigma_3$. The spin connection is $\omega_{12} = -\cos\theta\mathrm{d}\varphi$.

We need to describe only the $\mathcal{N} = (2,2)$ vector multiplet. Its components are the gauge field A_μ, two dimension 1 scalars η and σ, two Dirac fermions λ, $\bar\lambda$, and the auxiliary field D. The corresponding supersymmetry variations are[15]

$$\delta_{\epsilon,\bar\epsilon}\,A_\mu = -\frac{i}{2}\left(\bar\epsilon\gamma_\mu\lambda + \bar\lambda\gamma_\mu\epsilon\right), \tag{B.3a}$$

$$\delta_{\epsilon,\bar\epsilon}\,\eta = \frac{1}{2}\left(\bar\epsilon\lambda + \bar\lambda\epsilon\right), \tag{B.3b}$$

$$\delta_{\epsilon,\bar\epsilon}\,\sigma = -\frac{i}{2}\left(\bar\epsilon\gamma_3\lambda + \bar\lambda\gamma_3\epsilon\right), \tag{B.3c}$$

$$\delta_{\epsilon,\bar\epsilon}\,\lambda = \left(+i\not{D}\eta + if\gamma_3 - [\eta,\sigma]\gamma_3 + \frac{i}{\ell}\,\eta\gamma_3 - D + i\epsilon_{\mu\nu}\gamma^\mu\mathrm{D}^\nu\sigma\right)\epsilon\,, \tag{B.3d}$$

$$\delta_{\epsilon,\bar\epsilon}\,\bar\lambda = \left(-i\not{D}\eta + if\gamma_3 + [\eta,\sigma]\gamma_3 + \frac{i}{\ell}\,\eta\gamma_3 + D + i\epsilon_{\mu\nu}\gamma^\mu\mathrm{D}^\nu\sigma\right)\bar\epsilon\,, \tag{B.3e}$$

$$\delta_{\epsilon,\bar\epsilon}\,D = -\frac{i}{2}\,\bar\epsilon\not{D}\lambda - \frac{i}{2}\,[\eta,\bar\epsilon\lambda] - \frac{1}{2}\,[\sigma,\bar\epsilon\gamma_3\lambda] + \frac{i}{2}\,\epsilon\not{D}\bar\lambda + \frac{i}{2}\,[\eta,\bar\lambda\epsilon] + \frac{1}{2}\,[\sigma,\bar\lambda\gamma_3\epsilon]\,. \tag{B.3f}$$

ϵ and $\bar\epsilon$ are bosonic spinors satisfying the conformal Killing spinor equation

$$\nabla_\mu\epsilon = \gamma_\mu\tilde\epsilon\,, \tag{B.4}$$

for some spinor $\tilde\epsilon$. It is solved by

$$\epsilon = e^{-s\frac{i}{2}\theta\gamma^2}\begin{pmatrix} e^{\frac{i}{2}\varphi} \\ 0 \end{pmatrix} \qquad \text{and} \qquad \epsilon = e^{-s\frac{i}{2}\theta\gamma^2}\begin{pmatrix} 0 \\ e^{-\frac{i}{2}\varphi} \end{pmatrix}, \tag{B.5}$$

where $s = \pm 1$. The same solutions hold for $\bar\epsilon$.

[15] To match the notation of [41], one needs to identify $\sigma_1 = \eta$ and $\sigma_2 = \sigma$.

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Together, ϵ and $\bar{\epsilon}$ generate the entire 2d superconformal algebra. To apply localization on S^2, we can restrict to the $\mathfrak{su}(2|1)$ Poincaré subalgebra, generated by the only four Killing spinors

$$\epsilon = e^{-\frac{i}{2}\theta\gamma^2}\begin{pmatrix} e^{\frac{i}{2}\varphi} \\ 0 \end{pmatrix}, \qquad \epsilon = e^{+\frac{i}{2}\theta\gamma^2}\begin{pmatrix} 0 \\ e^{-\frac{i}{2}\varphi} \end{pmatrix},$$

$$\bar{\epsilon} = e^{+\frac{i}{2}\theta\gamma^2}\begin{pmatrix} e^{\frac{i}{2}\varphi} \\ 0 \end{pmatrix}, \qquad \bar{\epsilon} = e^{-\frac{i}{2}\theta\gamma^2}\begin{pmatrix} 0 \\ e^{-\frac{i}{2}\varphi} \end{pmatrix}. \tag{B.6}$$

They satisfy the equations

$$\nabla_\mu \epsilon = \frac{1}{2\ell}\gamma_\mu\gamma_3\epsilon\,, \qquad \nabla_\mu\bar{\epsilon} = -\frac{1}{2\ell}\gamma_\mu\gamma_3\bar{\epsilon}\,. \tag{B.7}$$

We have four solutions. In restricting to the hemisphere, the SU(2) isometry group breaks down to the U(1) group of azimuthal rotations. Therefore, we expect that supersymmetry will close on an $\mathfrak{su}(1|1)$ algebra, with spacetime symmetry reduced to rotations along φ. We choose the first couple in (B.6)

$$\epsilon = e^{\frac{i\varphi}{2}}\begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix}, \qquad \bar{\epsilon} = e^{-\frac{i\varphi}{2}}\begin{pmatrix} \sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix}. \tag{B.8}$$

Some useful Killing spinor bilinears are

$$\bar{\epsilon}\gamma^3\epsilon = 1\,, \qquad \bar{\epsilon}\epsilon = \cos\theta\,, \qquad \bar{\epsilon}\gamma^\mu\epsilon = \left(0, -\frac{i}{\ell}\right). \tag{B.9}$$

C One-loop determinants

In this appendix, we present the computation of the one-loop determinants contributing to $\mathcal{Z}_{\text{1-loop}}$. The analysis of the possible contributions for $\mathcal{N} = (2,2)$ theories on the hemisphere was done in detail in [40–42]. Below, we shall simply review the essential steps of the calculation and collect the relevant results.

To begin with, we expand each field of our supersymmetric model around the background value given by the localization locus (3.15). Schematically, $\Phi \mapsto \Phi_0 + \Phi/\sqrt{t}$.

Plugging this into the localizing (bulk) term (3.5) and subsequently expanding in t, we can easily single out quadratic part of $\delta_Q V$. Since this quantity vanishes on the locus (3.15), we do not have any "classical contribution", and we can write

$$\begin{aligned}
\delta_Q V^{(2)} &= \lim_{t\to\infty} tS_{\text{SYM}} \\
&= \int_{\text{HS}^2} \mathrm{d}^2x\,\sqrt{g}\,\text{Tr}\Bigg[-A_\mu\nabla_\nu\nabla^\nu A^\mu + A_\mu\nabla_\nu\nabla^\mu A^\nu + \frac{2}{\ell}\eta\,\epsilon^{\mu\nu}\nabla_\mu A_\nu + \frac{\eta^2}{\ell^2} \\
&\quad -\frac{1}{\ell^2}[\sigma_0, A^\mu][\sigma_0, A_\mu] - \sigma\nabla_\mu\nabla^\mu\sigma - \frac{2i}{\ell}[\sigma_0, A^\mu]\nabla_\mu\sigma - \eta\nabla_\mu\nabla^\mu\eta \\
&\quad +D^2 - \frac{1}{\ell^2}\,[\sigma_0, \eta]^2 + \frac{i}{2}\bar{\lambda}\gamma^\mu\nabla_\mu\lambda - \frac{i}{2}\nabla_\mu\bar{\lambda}\gamma^\mu\lambda + \frac{1}{\ell}\bar{\lambda}\gamma_3[\sigma_0, \lambda]\Bigg]\,,
\end{aligned} \tag{C.1}$$

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where we have integrated by parts some terms taking advantage of the boundary conditions for the fluctuation fields. The quadratic integral over D is trivial and, as such, can be neglected.

Gauge fixing. Before moving on, we must gauge-fix the theory [77]. In particular, we choose to impose Lorenz gauge and set

$$\nabla_\mu A^\mu = 0\,. \tag{C.2}$$

To do so, we exploit the standard BRST construction by introducing two ghost fields c, $\bar{c}$, and a Lagrange multiplier b, all living in the adjoint representation of the gauge algebra. Next, we add the following term[16] to the localizing action (3.5)

$$\delta_Q V_{\rm BRST} = \int_{\rm HS^2} {\rm d}^2x\,\sqrt{g}\;{\rm Tr}\left(\bar{c}\,\nabla_\mu \mathcal{D}^\mu c + b\,\nabla_\mu A^\mu\right)\,. \tag{C.3}$$

Expanding it as before around the combined locus (3.15) and $c = \bar{c} = b = 0$, we get

$$\delta_Q V^{(2)}_{\rm BRST} = \int_{\rm HS^2} {\rm d}^2x\,\sqrt{g}\;{\rm Tr}\left(\frac{1}{2}\bar{c}\,\nabla_\mu \nabla^\mu c + b\,\nabla_\mu A^\mu\right)\,. \tag{C.4}$$

The integration over the bosonic Lagrange multiplier b gives $\delta(\nabla_\mu A^\mu)$, which enforces the gauge-fixing condition in the path integral.

Following [78], we then separate the gauge field into a divergenceless and pure divergence part

$$A_\mu = \partial_\mu u + A'_\mu\,, \tag{C.5}$$

where A'_μ is the divergenceless part of A_μ, i.e. $\nabla_\mu A'^\mu = 0$. Exploiting this decomposition, the delta function imposing the gauge-fixing becomes $\delta(-\nabla^2 u)$, with $\nabla^2 \equiv \nabla^\mu\nabla_\mu$. Thus, the integration measure for the gauge field can be rewritten as follows

$$\begin{aligned}[DA_\mu]\;\delta(\nabla^\mu A_\mu) &= [DA'_\mu][Du]\;\delta(-\nabla^2 u)\\ &= [DA'_\mu][Du]\;\delta(u)\,{\rm det}\Big(-\nabla^2\Big)^{-\frac{1}{2}}\,.\end{aligned} \tag{C.6}$$

The scalar u can then be integrated out, leaving only the Jacobian factor ${\rm det}(-\nabla^2)^{-1/2}$. Subsequently, we can perform the functional integrations over σ and the ghosts. The former gives an additional factor ${\rm det}(-\nabla^2)^{-1/2}$, while the latter provides a factor of ${\rm det}(-\nabla^2)$, so that the above three contributions exactly cancel.

The gauge fixed quadratic localizing term now reads

$$\begin{aligned}\delta_Q V^{(2)}_{\rm gf} = \int_{\rm HS^2} {\rm d}^2x\,\sqrt{g}\;{\rm Tr}\Big[&-A'_\mu\nabla_\nu\nabla^\nu A'_\mu + \frac{1}{\ell^2}A'_\mu A'^\mu + \frac{2}{\ell}\eta\;\epsilon^{\mu\nu}\nabla_\mu A'_\nu - \frac{1}{\ell^2}[\sigma_0, A'^\mu][\sigma_0, A'_\mu]\\ &-\eta\nabla_\mu\nabla^\mu\eta + \eta^2 - \frac{1}{\ell^2}[\sigma_0,\eta]^2 + \frac{i}{2}\lambda\gamma^\mu\nabla_\mu\bar{\lambda} + \frac{i}{2}\bar{\lambda}\gamma^\mu\nabla_\mu\lambda + \frac{1}{\ell}\bar{\lambda}\gamma_3[\sigma_0,\lambda]\Big]\,.\end{aligned} \tag{C.7}$$

[16] The total supersymmetric transformation would now become $\delta_Q \mapsto \delta_Q + \delta_{\rm BRST}$.

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The next step consists in using the Cartan decomposition, that is, to expand the adjoint field X as

$$X = \sum_i X^i \mathsf{H}_i + \sum_{\alpha\in\Delta_+} \left(X^\alpha \mathsf{E}_\alpha + X^{-\alpha}\mathsf{E}_{-\alpha}\right), \tag{C.8}$$

where H_i are the Cartan generators, E^α is the generator corresponding to the root α and Δ_+ is the set of positive roots. They satisfy the following relations

$$[\mathsf{H}_i, \mathsf{E}_\alpha] = \alpha(\mathsf{H}_i)\,\mathsf{E}_\alpha\,, \qquad \mathsf{E}_\alpha^\dagger = \mathsf{E}_{-\alpha}\,, \qquad \mathrm{Tr}\left(\mathsf{E}_\alpha \mathsf{E}_\beta\right) = \delta_{\alpha+\beta}\,, \qquad \mathrm{Tr}\left(\mathsf{E}_\alpha \mathsf{H}_i\right) = 0\,. \tag{C.9}$$

For $\mathfrak{sl}(2,\mathbb{R})$, we only have one Cartan generator γ_3 and one positive root α. Therefore, we shall drop the sum over the positive roots in the following.

Bosonic determinants. Using the commutation and trace relations (C.9) one can find the bosonic part of (C.7) is proportional to

$$\begin{aligned}\delta_Q V^{(2)}_{\text{bos}} = \int_{\text{HS}^2} \mathrm{d}^2x\,\sqrt{g}\,\Big[&-A^{-\alpha}_\mu \nabla_\nu \nabla^\nu A^\alpha_\mu + \frac{1}{\ell^2} A^{-\alpha}_\mu A^{\alpha,\mu} + \frac{1}{\ell}\eta^{-\alpha}\,\epsilon^{\mu\nu}\nabla_\mu A^\alpha_\nu + \frac{1}{\ell}\eta^\alpha\,\epsilon^{\mu\nu}\nabla_\mu A^{-\alpha}_\nu \\ &+ \frac{1}{\ell^2}\alpha(\sigma_0)^2 A^{-\alpha}_\mu A^{\alpha\mu} - \eta^{-\alpha}\nabla_\mu\nabla^\mu \eta^\alpha + \frac{1}{\ell^2}\eta^{-\alpha}\eta^\alpha + \frac{1}{\ell^2}\alpha(\sigma_0)^2 \eta^{-\alpha}\eta^\alpha\Big]\,,\end{aligned} \tag{C.10}$$

where we omitted the prime and implicitly considered only the divergenceless part of A_μ. We find it convenient to expand the gauge field in terms of the vector spherical harmonics $\mathcal{C}^\lambda_{jm,\mu}$ and to write:

$$A^\alpha_\mu = \sum_{\lambda=1,2}\sum_{j=1}^{\infty}\sum_{m=-j}^{j} A^{\alpha,\lambda}_{jm}\,\mathcal{C}^\lambda_{jm,\mu}(\vartheta,\varphi) \tag{C.11}$$

These special functions enjoy the following two properties

$$\nabla^\mu \mathcal{C}^1_{jm,\mu} = -\frac{\sqrt{j(j+1)}}{\ell^2}\,\mathcal{Y}_{jm}\,, \qquad \nabla^\mu \mathcal{C}^2_{jm,\mu} = 0\,. \tag{C.12}$$

We indicated the usual scalar spherical harmonics with $\mathcal{Y}_{jm}$. Since A_μ is divergenceless, only the component with helicity $\lambda = 2$ can appear in the above expansion. Therefore, we can drop the sum over λ in (C.11) and write

$$A^\alpha_\mu = \sum_{j=1}^{\infty}\sum_{m=-j}^{j} A^{\alpha,2}_{jm}\,\mathcal{C}^2_{jm,\mu}(\vartheta,\varphi)\,, \qquad A^{-\alpha}_\mu = (A^\alpha_\mu)^*\,. \tag{C.13}$$

The boundary conditions satisfied by A_μ further restrict this sum, and the coefficients $A^{\alpha,2}_{jm}$ are different from zero only when $j - m$ is an odd integer (see [41]). Similarly, we can expand the scalar field η in terms of the usual spherical harmonics:

$$\eta^\alpha = \sum_{j=0}^{\infty}\sum_{m=-j}^{j} \eta^\alpha_{jm}\,\mathcal{Y}_{jm}(\vartheta,\varphi)\,, \qquad \eta^{-\alpha} = (\eta^\alpha)^*\,. \tag{C.14}$$

The vanishing of this field at the boundary again imposes that the expansion coefficients differ from zero only when $j - m = \text{odd}$ [41].

Both scalar and vector spherical harmonics are eigenvectors of the corresponding Laplacian, though with different eigenvalues, i.e.

$$-\nabla^{\mu}\nabla_{\mu}\,\mathcal{Y}_{jm}=\frac{j(j+1)}{\ell^2}\,\mathcal{Y}_{jm}\,,\qquad -\nabla^{\mu}\nabla_{\mu}\,\mathcal{C}^{\lambda}_{jm}=\frac{j(j+1)-1}{\ell^2}\,\mathcal{C}^{\lambda}_{jm}\,. \tag{C.15}$$

Moreover, the vector harmonics satisfy these further set of relations

$$\epsilon^{\mu\nu}\nabla_{\mu}\left(\mathcal{C}^{\lambda}_{jm}\right)_{\nu}=-\delta^{\lambda}_{2}\frac{\sqrt{j(j+1)}}{\ell^2}\mathcal{Y}_{jm}. \tag{C.16}$$

The above properties will allow us to deal with the mixed terms present in the bosonic sector. Then, by taking advantage of (C.15) and (C.16) as well as the orthogonality relations on the hemisphere

$$\int_0^{2\pi}\mathrm{d}\varphi\int_0^{\frac{\pi}{2}}\mathrm{d}\vartheta\;\sin\vartheta\;\mathcal{Y}_{jm}(\vartheta,\varphi)^*\;\mathcal{Y}_{j'm'}(\vartheta,\varphi)=\frac{1}{2}\delta_{jj'}\delta_{mm'}\,, \tag{C.17}$$

$$\int_0^{2\pi}\mathrm{d}\varphi\int_0^{\frac{\pi}{2}}\mathrm{d}\vartheta\;\sin\vartheta\;\mathcal{C}^{\lambda}_{jm,\mu}(\vartheta,\varphi)^*\;\mathcal{C}^{\lambda'}_{j'm',\nu}(\vartheta,\varphi)\;g^{\mu\nu}=\frac{1}{2}\delta_{jj'}\delta_{mm'}\delta^{\lambda\lambda'}\,, \tag{C.18}$$

we can easily show that (C.10) reduces to

$$\delta_Q V^{(2)}_{\text{bos}}=\sum_{j,m}\left(\mathbb{Q}^{\alpha}_{jm}\right)^{\dagger}\Delta^{\text{bos}}_{j}\left(\mathbb{Q}^{\alpha}_{jm}\right) \tag{C.19}$$

where $\mathbb{Q}^{\alpha}_{jm}=\left(A^{\alpha,2}\quad\eta^{\alpha}\right)^{\mathrm{T}}_{jm}$. The explicit form of the matrix Δ^{bos}_{j} is

$$\Delta^{\text{bos}}_{j}=\frac{1}{\ell^2}\begin{pmatrix} j\,(j+1)+\alpha(\sigma_0)^2 & \sqrt{j\,(j+1)} \\ \sqrt{j\,(j+1)} & j\,(j+1)+\alpha(\sigma_0)^2+1\end{pmatrix}. \tag{C.20}$$

Recall now that $\mathbb{Q}^{\alpha}_{jm}$ vanishes when $j-m$ is even because of the boundary conditions. This fact reduces the usual degeneracy in m of a spherical symmetric problem from $2j+1$ to j. Thus, the total bosonic contribution to $\mathcal{Z}_{\text{1-loop}}$ will read (taking into account that integration variable $\mathbb{Q}^{\alpha}_{jm}$ is complex) [41]

$$\begin{aligned}\mathcal{Z}^{\text{bos}}_{\text{1-loop}}&=\prod_{j}\left(\det\Delta^{\text{bos}}_{j}\right)^{-j}\\ &=\prod_{j=1}^{\infty}\frac{1}{\ell^{4j}\,[j^2+\alpha(\sigma_0)^2]^j\,[(j+1)^2+\alpha(\sigma_0)^2]^j}\,.\end{aligned} \tag{C.21}$$

Fermionic determinants. The fermionic part of (C.7) after Cartan decomposition and some integration by parts becomes

$$\delta_Q V^{(2)}_{\text{fer}}=\int\mathrm{d}^2x\sqrt{g}\left(\lambda^{\alpha}\;\bar{\lambda}^{\alpha}\right)^{\dagger}\begin{pmatrix}0 & i\gamma_3\gamma^{\mu}\nabla_{\mu}-\frac{1}{\ell}\alpha(\sigma_0)\\ i\gamma_3\gamma^{\mu}\nabla_{\mu}+\frac{1}{\ell}\alpha(\sigma_0) & 0\end{pmatrix}\begin{pmatrix}\lambda^{\alpha}\\ \bar{\lambda}^{\alpha}\end{pmatrix}. \tag{C.22}$$

tThe symbol † denotes a Dirac-like conjugation containing also a factor γ_3, that is

$$\left(\lambda^\alpha \ \bar{\lambda}^\alpha\right)^\dagger = \left(\lambda^{-\alpha}\gamma_3\,,\ \bar{\lambda}^{-\alpha}\gamma_3\right). \tag{C.23}$$

The gauginos λ, $\bar{\lambda}$ are fields of spin 1/2, and we denote the two different helicities with $s = \pm(\pm\frac{1}{2})$. We can expand both of them into spin spherical harmonics $\mathcal{Y}^s_{jm}$:

$$\lambda^\alpha = \sum_{s=\pm}\sum_{j=\frac{1}{2}}^{\infty}\sum_{m=-j}^{j}{}' \lambda^{\alpha,s}_{jm}\,\mathcal{Y}^s_{jm}(\vartheta,\varphi)\,, \qquad \bar{\lambda}^\alpha = \sum_{s=\pm}\sum_{j=\frac{1}{2}}^{\infty}\sum_{m=-j}^{j}{}' \bar{\lambda}^{\alpha,s}_{jm}\,\mathcal{Y}^s_{jm}(\vartheta,\varphi)\,. \tag{C.24}$$

The spin spherical harmonics are eigenvectors of the Dirac operator, i.e.

$$i\gamma_3\gamma^\mu\nabla_\mu\mathcal{Y}^\pm_{jm} = \pm\frac{i}{\ell}\left(j+\frac{1}{2}\right)\mathcal{Y}^\pm_{jm} \tag{C.25}$$

with $j = \frac{1}{2}, \frac{3}{2}, \cdots$, $m = -j, -j+1, \cdots, j$, and they are normalized on the hemisphere with

$$\int_0^{2\pi}\mathrm{d}\varphi\int_0^{\frac{\pi}{2}}\mathrm{d}\vartheta\ \sin(\vartheta)\,\mathcal{Y}^s_{jm}(\vartheta,\varphi)^*\,\mathcal{Y}^{s'}_{j'm'}(\vartheta,\varphi) = \frac{1}{2}\,\delta_{jj'}\delta_{mm'}\delta^{ss'}\,. \tag{C.26}$$

In (C.27), the prime in the internal sum over m means that we must restrict the values of m to even $j-m$ if $s=+$ and to even $j-m$ if $s=-$. This constraint stems from the boundary conditions imposed on the fermions [41].

Plugging the expansion (C.27) into (C.22) and exploiting the orthogonality relations to perform the angular integrations, we find the fermionic term (C.22) can be reorganized into the sum of two series

$$\begin{aligned}\delta_Q V^{(2)}_{\text{fer}} &= \delta_Q V^{(2)}_{\text{fer},+} + \delta_Q V^{(2)}_{\text{fer},-}\\ &= \sum_{j=\frac{1}{2}}^{\infty}\sum_{m=-j}^{j}{}' (\Lambda^{\alpha,+}_{jm})^\dagger\ \Delta^{\text{fer},+}_j\ (\Lambda^{\alpha,+}_{jm}) + \sum_{j=\frac{1}{2}}^{\infty}\sum_{m=-j}^{j}{}' (\Lambda^{\alpha,-}_{jm})^\dagger\ \Delta^{\text{fer},-}_j\ (\Lambda^{\alpha,-}_{jm})\,,\end{aligned} \tag{C.27}$$

where $\Lambda^{\alpha,\pm}_{jm} = \left(\lambda^{\alpha,\pm}\ \ \bar{\lambda}^{\alpha,\pm}\right)^{\mathrm{T}}_{jm}$ with

$$\Delta^{\text{fer},\pm}_j = \frac{1}{\ell}\begin{pmatrix} 0 & \pm j \pm \frac{1}{2} + i\alpha(\sigma_0) \\ \pm j \pm \frac{1}{2} - i\alpha(\sigma_0) & 0\end{pmatrix}. \tag{C.28}$$

Since this matrix depends only on j, we have the usual degeneracy in m for each eigenvalue. This degeneracy is reduced from $2j+1$ to $j+1/2$ by the constraint on the sum over m.

Since the matrices $\Delta^{\text{fer},+}_j$ and $\Delta^{\text{fer},-}_j$ possess the same determinant, the total fermionic contribution to $\mathcal{Z}^{\text{fer}}_{\text{1-loop}}$ can be written, up to a phase, as follows [26]:

$$\begin{aligned}\mathcal{Z}^{\text{fer}}_{\text{1-loop}} &= \prod_{j=\frac{1}{2}}^{\infty}\left(\det\Delta^{\text{fer},+}_j\right)^{j+\frac{1}{2}}\left(\det\Delta^{\text{fer},-}_j\right)^{j+\frac{1}{2}}\\ &= \prod_{j=\frac{1}{2}}^{\infty}\ell^{4j+2}\left(j+\frac{1}{2}+i\alpha(\sigma_0)\right)^{2j+1}\left(j+\frac{1}{2}-i\alpha(\sigma_0)\right)^{2j+1}\,.\end{aligned} \tag{C.29}$$

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