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PAPER

Classical solutions for a van Roosbroeck–Helmholtz model for a semiconductor laser diode

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Abstract

We consider a coupled light-matter model for semiconductor lasers consisting of the transient van Roosbroeck system for charge transport and a Helmholtz eigenvalue problem for the transversal optical field. The coupling is realised through a stimulated recombination operator in the carrier continuity equations and a carrier-dependent dielectric function in the Helmholtz problem. In this paper, we establish, under physically relevant assumptions, local-in-time well-posedness of the coupled van Roosbroeck–Helmholtz system. The proof relies on the abstract framework of quasi-linear parabolic equations in Banach spaces developed by Kaiser, Neidhardt, and Rehberg which requires in particular a local Lipschitz continuity property of the nonlinear recombination operators. By deriving precise local Lipschitz bounds for the stimulated recombination operator, we verify the conditions needed to apply the abstract existence theorem. As a consequence, we obtain the existence and uniqueness of weak solutions to a drift-diffusion-Helmholtz model of semiconductor lasers that incorporates stimulated emission in a mathematically consistent way. To the best of our knowledge, this is the first rigorous existence and uniqueness result for the nonlinear coupling of the van Roosbroeck system with a Helmholtz eigenvalue problem under physically motivated assumptions.

1. Introduction

Semiconductor lasers are one of the most important opto-electronic applications. For such lasers, different models [1, 6, 7, 23, 33, 40], mathematical theory [25] and simulations [16, 34, 36, 39] exist. For the present paper, we consider a physically relevant mathematical model of a semiconductor laser which correctly couples electronic charge transport with optical stimulation, and then prove that it admits a weak, unique, local-in-time solution. The semiconductor laser model considered here consists of two different parts: a drift-diffusion system and a Helmholtz eigenvalue problem. We discuss the known mathematical theory for both models separately.

On the one hand, electron and hole charge transport in semiconductor material is mathematically described via a system of drift-diffusion equations, first introduced by van Roosbroeck [38]. A rigorous mathematical treatment of these equations is provided by Jerome in [24], while a comprehensive presentation, including numerical simulation aspects, can be found in the book of Selberherr [37] and in the book of Markowich, Ringhofer, and Schmeiser [30]. More recently, Gamba and collaborators have

contributed analytical and numerical results on drift-diffusion systems and their extensions, particularly in the context of coupled models and multiscale approaches (see, e.g. [17, 18]). For a review of the physical derivation and limitations of the drift-diffusion model, we refer to Baccarani *et al* [5]. In general, the van Roosbroeck system has been extensively analysed. Early contributions are due to Mock [32], who studied the stationary van Roosbroeck system and showed that, under reasonable assumptions on material parameters, boundary conditions, and recombination terms, the system of nonlinear equations describing electrons, holes, and electrostatic potential has at least one solution. This was one of the first rigorous mathematical frameworks for semiconductor models. Then, Gajewski proved existence, uniqueness, and asymptotic stability of solutions for the time-dependent drift-diffusion system under Boltzmann statistics [13]. This analysis was extended in joint work with Gröger, where they established global existence of weak solutions also under Fermi–Dirac statistics [14] and later treated stationary problems with variable mobilities [15] using monotone operator methods. They proved global existence of weak solutions to the transient drift-diffusion equations with Boltzmann or Fermi–Dirac statistics. A systematic presentation of the steady-state theory was then provided in the book of Markowich [29]. Since then, many related steady-state existence results have been published, for example coupling the van Roosbroeck system to electric circuits [3, 4] or for a laser-related stationary van Roosbroeck system, formulated in [11], which was perturbed by additional generation-recombination mechanisms [2].

On the other hand, to model electromagnetic waves in optoelectronic applications at the most general level, Maxwell's equations are typically used [9, 35]. However, it is often convenient to reduce the complexity of Maxwell's equations, which govern both the electric and magnetic fields. Therefore, in this paper, we consider a useful simplification of Maxwell's equations which results in a Helmholtz eigenvalue problem for the transversal electric field [6]. The existence theory for the Helmholtz eigenvalue problem has been studied in classical books [10, 19]. The perturbation theory for linear operators and their spectral properties can be found in [28].

In order to now couple the charge transport model with the Helmholtz eigenvalue problem, we introduce a so-called stimulated recombination term in the van Roosbroeck system and take into account a dielectric function for the Helmholtz problem, which depends on the charge carriers. Related models have been discussed from a numerical [8] and physical [27] perspective. In this model, we neglect the photon balance equation and consider only the largest optical power component, which is justified since the remaining components are physically negligible.

The existence theory for such coupled van-Roosbroeck–Helmholtz light-matter models is still not well developed and has only been addressed to some extent in [25]. However, the existence result is stated for recombination terms, which are assumed to satisfy a local Lipschitz continuity condition. In the present paper, our goal is to rigorously prove the local Lipschitz continuity of the physically relevant stimulated recombination thereby enabling the subsequent novel proof of existence for the van Roosbroeck–Helmholtz model.

The key idea of the proof in [25], as well as of the present work, relies on the more abstract existence result for quasi-linear parabolic equations in [26]. In the present paper, we show that the assumptions of the aforementioned abstract existence result in [26] are satisfied.

The remainder of this paper is organised as follows: In section 2, we introduce the opto-electronically coupled model, in section 3, we introduce the mathematical notation and the assumptions needed in order to rewrite the problem in quasi-linear parabolic form. Then, in section 4, we state the quasi-linear parabolic system. In section 5 we show the main result before we conclude in the final section.

2. Presentation of the model

In this section, we present the opto-electronic model for a semiconductor laser. While the so-called van Roosbroeck model (section 2.1) will describe the flow of electrons and holes in a self-consistent electrical field due to drift and diffusion, a Helmholtz eigenvalue problem (section 2.2) will provide eigenmodes which describe the optical part of the laser. The electronic quantities of interest are the densities n and p of electrons and holes as well as the electrostatic potential φ of the self-consistent electrical field. Instead of electron and hole densities one may treat the electrochemical potentials φ_n and φ_p of electrons and holes as unknowns, which are in the physical literature typically referred to as quasi–Fermi potentials [12]. This is approach has advantages from a numerical and analytical point of view, since all the three potentials are on the same order of magnitude and quite naturally lead to a gradient structure [31]. These unknowns have to satisfy Poisson's equation and the continuity equations for the current

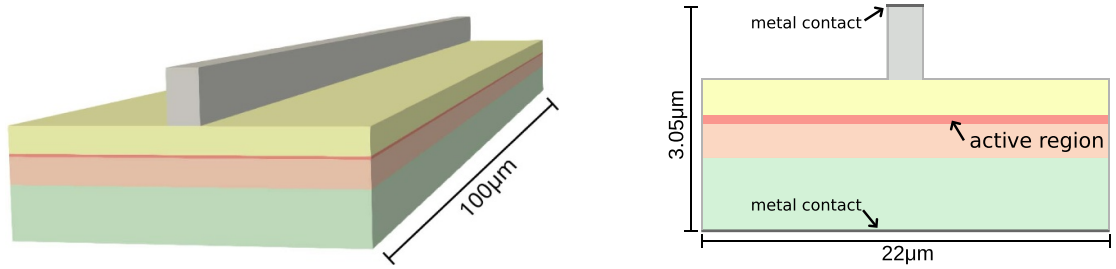

Figure 1. Schematic of the laser diode. Left is the 3-dimensional geometry, and right is a 2-dimensional cross section of the diode.

densities of electrons and holes with some side conditions. The latter are given by the relations between the potentials and the densities. The continuity equations for the current densities are excited by a stimulated generation term depending on the eigenvalues and eigenvectors of the Helmholtz eigenvalue problem for the electric potential which rules the optical behaviour of the laser. Important derived quantities are the electron and hole currents, $\mathbf{j}_n$ and $\mathbf{j}_p$.

The spatial domain of the semiconductor laser diode is three-dimensional, with the longitudinal dimension much greater than the transversal dimensions, see figure 1. In particular, an active layer is present along the longitudinal direction between two thicker semiconductor layers with higher doping. The active layer will be excited by an electric potential with appropriate resonance conditions.

It is possible to reduce the dimensionality of the model from three to two space dimensions by considering a transversal section of the space domain. The semiconductor equations keep the same structure, but the Helmholtz eigenvalue problem decouples into two subsystems, one for the longitudinal variable and one for the transversal variables [9, Chapter 7], [35, Chapter 4], [41]. We consider the laser model for a fundamental excitation frequency ω_0 which will be taken as part of the model data.

The transversal Helmholtz eigenvalue problem is coupled with the semiconductor equations by means of the dielectric function ε_{opt}, which is a function of the charge carrier densities.

Let Ω be the two-dimensional (bounded) domain representative of a transverse cut of the diode domain. The metal contacts are in $D \subset \partial\Omega$ and the rest of the boundary of Ω is denoted by Γ.

2.1. The van Roosbroeck system

The van Roosbroeck system [38] describes how charge carriers, electrons and holes, flow through a semiconductor device. It couples the Poisson equation for the electrostatic potential φ self-consistently with the continuity equations for the electron and hole densities n and p.

The Poisson problem, with mixed boundary conditions, is given by:

$$-\nabla\cdot(\varepsilon\nabla\varphi) = q_{\text{e}}(C+p-n) \qquad \text{in } \Omega, \tag{2.1a}$$

$$\varphi = \varphi_{\text{D}} \qquad \text{on } D, \tag{2.1b}$$

$$\boldsymbol{\nu}\cdot(\varepsilon\nabla\varphi) = 0 \qquad \text{on } \Gamma, \tag{2.1c}$$

where ε and C are bounded, measurable functions on Ω with values in the set of real, and correspond to the spatially varying dielectric permittivity and doping profile, respectively, and q_{e} is the modulus of the electron charge. With φ_{D} we indicate the voltage applied at the contacts of the device.

The continuity equations for the electron and hole current densities are

$$-q_{\text{e}}\frac{\partial n}{\partial t} + \nabla\cdot\mathbf{j}_n = q_{\text{e}}(R+R_{\text{s}}) \qquad \text{in } \Omega\times(0,T), \tag{2.2a}$$

$$q_{\text{e}}\frac{\partial p}{\partial t} + \nabla\cdot\mathbf{j}_p = -q_{\text{e}}(R+R_{\text{s}}) \qquad \text{in } \Omega\times(0,T), \tag{2.2b}$$

with constitutive relations for the current densities $\mathbf{j}_n$, $\mathbf{j}_p$,

$$\mathbf{j}_n = -q_{\text{e}}n\mu_n\nabla\varphi_n, \tag{2.2c}$$

$$\mathbf{j}_p = -q_{\text{e}}p\mu_p\nabla\varphi_p, \tag{2.2d}$$

supplemented with mixed boundary conditions

$$\varphi_n = \varphi_{n,\text{D}}, \qquad \varphi_p = \varphi_{p,\text{D}} \qquad \text{on } D\times(0,T), \tag{2.2e}$$

$$\boldsymbol{\nu}\cdot\mathbf{j}_n = 0, \qquad \boldsymbol{\nu}\cdot\mathbf{j}_p = 0 \qquad \text{on } \Gamma\times(0,T), \tag{2.2f}$$

and initial values

$$\varphi_n(0) = \varphi_n^{\mathrm{I}}, \qquad \varphi_p(0) = \varphi_p^{\mathrm{I}} \qquad \text{in } \Omega. \tag{2.2g}$$

In (2.2*a*) and (2.2*b*), μ_n and μ_p are the mobilities for electrons and holes, respectively, which are measurable, bounded real functions on Ω (in particular independent of the electric field), R is the semiconductor generation-recombination term and R_{s} is the so-called *stimulated* generation term due the Helmholtz problem, which will be specified later.

The fundamental variables of the continuity equations are the quasi–Fermi potentials φ_n, φ_p, and the carrier densities are related the the quasi–Fermi potentials and the electrostatic potential by the state equations

$$n(\varphi,\varphi_n) = N_{\mathrm{c}}\mathcal{F}(\chi_n), \qquad p(\varphi,\varphi_p) = N_{\mathrm{v}}\mathcal{F}(\chi_p), \tag{2.3}$$

where N_{c} and N_{v} are the conduction and valence band densities of states, and χ_p and χ_n are the chemical potentials, and the function $\mathcal{F}$ represents the statistical distribution of the electrons and holes on the energy band. We consider the Fermi–Dirac statistics

$$\mathcal{F}(s) = \frac{2}{\sqrt{\pi}}\int_0^\infty \frac{\sqrt{\xi}}{1+\mathrm{e}^{\xi-s}}\,\mathrm{d}\xi, \quad s\in\mathbb{R}. \tag{2.4}$$

The chemical potentials χ_n, χ_p are related to the quasi–Fermi potentials φ_n, φ_p by

$$\chi_n = \frac{q_{\mathrm{e}}(\varphi-\varphi_n) - E_{\mathrm{c}}}{k_{\mathrm{B}}T_{\mathrm{L}}}, \qquad \chi_p = \frac{q_{\mathrm{e}}(\varphi_p-\varphi) + E_{\mathrm{v}}}{k_{\mathrm{B}}T_{\mathrm{L}}}, \tag{2.5}$$

where E_{c} and E_{v} are the conduction and valence band-edge energies, respectively, k_{B} is the Boltzmann constant and T_{L} is the lattice temperature which we consider to be constant.

Finally, the semiconductor generation-recombination term R describes the production of electrons and holes and is the sum of different mechanisms like the Shockley–Read–Hall recombination and Auger recombination. It is given by

$$R = s(n,p)\left(np - n_{\mathrm{i}}(\chi_n,\chi_p)^2\right), \tag{2.6}$$

where the non-equilibrium intrinsic concentration n_{i} is given by:

$$n_{\mathrm{i}}(\chi_n,\chi_p)^2 = N_{\mathrm{c}}N_{\mathrm{v}}\exp\left(-\frac{E_{\mathrm{c}}-E_{\mathrm{v}}}{k_{\mathrm{B}}T_{\mathrm{L}}}\right)\frac{\mathcal{F}(\chi_n)\mathcal{F}(\chi_p)}{\exp(\chi_n)\exp(\chi_p)},$$

and $s(n,p)$ is a model-dependent generation-recombination rate. We will consider three recombination processes, namely, Shockley–Read–Hall, Auger and spontaneous recombination. This implies that the function s in (2.6) consists of the following three parts:

$$s := s_{\mathrm{SRH}} + s_{\mathrm{Auger}} + s_{\mathrm{spont}} := \frac{1}{\tau_p(n+n_*)+\tau_n(p+p_*)} + (C_n n + C_p p) + s_{\mathrm{spont}}, \tag{2.7}$$

where the carrier life times τ_n, τ_p, the reference carrier densities n_*, p_* and the coefficients s_{spont}, C_n, C_p are material-dependent parameters.

2.2. The optical model

The optical part is described by a complex amplitude Φ, of a monochromatic electromagnetic wave, which satisfies the Helmholtz equation [16]

$$-\left(\nabla^2 + \varepsilon_{\mathrm{opt}}\right)\Phi = \lambda\Phi \quad \text{in } \Omega \tag{2.8a}$$

$$\Phi = 0 \quad \text{on } \partial\Omega. \tag{2.8b}$$

This is an eigenvalue problem for the eigenpair (λ,Φ), where Φ is the eigenfunction associated to the eigenvalue λ. In (2.8), the function $\varepsilon_{\mathrm{opt}}$ is defined as

$$\varepsilon_{\mathrm{opt}} := \frac{\omega_0^2}{c^2}\left(\tilde{n} + i\frac{c(g-\alpha)}{2\omega_0}\right)^2, \tag{2.9}$$

where c the speed of light, and $\tilde{n}$ is the refractive index, g the gain, and α the absorption, which are functions of the space variable $\mathbf{x}$ and the carrier densities n, p (see, for instance [16]).

The Helmholtz problem (2.8) admits a sequence of eigenpairs (λ_m, Φ_m), $m \in \mathbb{N}$. The resulting monochromatic electromagnetic wave affects the semiconductor diode by means of the stimulated recombination term R_s (resulting in the stimulated emission of photons), which couples the electronic model to the optical one and has the form

$$R_s = \frac{\tilde{n}g}{\hbar c} \sum_{m=1}^{\infty} \frac{P_m}{\mathrm{Re}\,(\beta_m)} \frac{|\Phi_m|^2}{\int_\Omega |\Phi_m|^2 \,\mathrm{d}\mathbf{x}}. \tag{2.10}$$

The value β_m is related to the eigenvalue λ_m by $\beta_m^2 = -\lambda_m$, with $\mathrm{Re}(\beta_m) \geqslant 0$. The coefficients P_m, $m = 1, 2, \ldots$, are the optical powers, which are solutions of the photon balance equation:

$$\frac{\partial}{\partial t} P_m = v_{g,m} \left(G_m - \hat{\alpha}_m - \gamma_m\right) P_m + \dot{P}_{\mathrm{sp},m}. \tag{2.11}$$

Here, $v_{g,m}$ is the modal group velocity, $G_m = 2\mathrm{Im}(\beta_m)$ the net gain of mode travelling along the cavity, $\hat{\alpha}_m$ the additional losses due to longitudinal scattering, $\gamma_m = \frac{1}{2L} \log \frac{1}{R_m(0) R_m(L)}$ the outcoupling losses of a Fabry–Perot laser, $R_m(0), R_m(L)$ the reflectivities at the facets at $z = 0, z = L$, and $\dot{P}_{\mathrm{sp},m}(\mathbf{x}, n, p)$ is the rate of spontaneous emission into the mode [16].

2.3. The van Roosbroeck–Helmholtz model

We consider a simplified model by making the following assumptions:

(1) The imaginary part of the dielectric function $\varepsilon_{\mathrm{opt}}$ defined in (2.9) is small compared to the real part, so that $\varepsilon_{\mathrm{opt}} \approx \varepsilon_* := \frac{\omega_0^2}{c^2} \tilde{n}^2$.

(2) The main optical power is associated with the Dirichlet principal eigenvalue λ of the Helmholtz operator $-(\nabla^2 + \varepsilon_{\mathrm{opt}})$.

(3) The photon balance equation (2.11) relaxes to a stationary solution faster than the semiconductor equations.

The above three assumptions are based on empirical observations in [16, 41]. In realistic simulations, two or three modes are usually considered, but one mode is already sufficient to provide a good approximation. Assumption (2) allows us to focus on the dominant effect, leaving open the discussion of the corrections introduced by additional modes. Assumption (1) then justifies a perturbative approach, which leads to the following approximation. The Dirichlet principal eigenvalue of the Helmholtz operator $-(\nabla^2 + \varepsilon_{\mathrm{opt}})$ is approximated by the Dirichlet principal eigenvalue of the Helmholtz operator $-(\nabla^2 + \varepsilon_*)$, so that (2.8) reduces to a self-adjoint eigenvalue problem:

$$-\left(\nabla^2 + \varepsilon_*\right) \Phi = \lambda \Phi \qquad \text{in } \Omega, \tag{2.12a}$$

$$\Phi = 0 \qquad \text{on } \partial\Omega. \tag{2.12b}$$

In proposition 3.3, we introduce a condition that ensures $\lambda < 0$. Therefore, equations (2.10) and (2.11) simplify to

$$R_s = \alpha_s(n,p) \frac{|\Phi|^2}{\beta}, \tag{2.13}$$

$$\alpha_s(n,p) = \frac{\tilde{n}(n,p)\; g(n,p)}{\hbar c}\; P(n,p), \qquad P(n,p) = -\frac{\dot{P}_{\mathrm{sp}}(n,p)}{v_g\left(G - \hat{\alpha} - \gamma\right)}, \tag{2.14}$$

where $\beta := \sqrt{-\lambda}$, with $\lambda < 0$ the principal eigenvalue of (2.12), and Φ is the principal eigenfunction, normalised so that $\int_\Omega |\Phi|^2 \,\mathrm{d}\mathbf{x} = 1$, $\Phi > 0$ in Ω. For the justification of the last statement, see Proposition 3.3.

2.3.1. Nondimensional van Roosbroeck–Helmholtz model

In the following, we rewrite the van Roosbroeck–Helmholtz model (also referred to as drift-diffusion-Helmholtz model) into a nondimensional one. For any physical quantity F we write $F = \bar{F}\hat{F}$ where $\bar{F}$ is

a reference value and $\hat{F}$ is the nondimensional quantity. For the independent and dependent variables, we choose

$$\bar{x} := \operatorname{diam}(\Omega), \qquad \bar{t} := \frac{\bar{x}^2}{\bar{\mu} U_{\mathrm{th}}}, \qquad \bar{\varphi} = \bar{\varphi}_n = \bar{\varphi}_p = U_{\mathrm{th}} := \frac{k_{\mathrm{B}} T_{\mathrm{L}}}{q_{\mathrm{e}}},$$

and for the other physical quantities we choose the scaling factors

$$\bar{\varepsilon} := \frac{q \bar{C} \bar{x}^2}{U_{\mathrm{th}}}, \quad \bar{C} := \sup_{\mathbf{x}\in\Omega} C(\mathbf{x}), \quad \bar{\mu}_n = \bar{\mu}_p = \bar{\mu} := \max\left\{ \sup_{\mathbf{x}\in\Omega} \mu_n(\mathbf{x}), \sup_{\mathbf{x}\in\Omega} \mu_p(\mathbf{x}) \right\}.$$

We scale n, p with the reference doping $\bar{C}$ so that the scaled state equations are

$$\hat{n}(\hat{\varphi}, \hat{\varphi}_n) = \hat{N}_{\mathrm{c}} \mathcal{F}(\hat{\chi}_n), \qquad \hat{p}(\hat{\varphi}, \hat{\varphi}_p) = \hat{N}_{\mathrm{v}} \mathcal{F}(\hat{\chi}_p), \tag{2.15}$$

with $\bar{N}_{\mathrm{c}} = \bar{N}_{\mathrm{v}} = \bar{C}$, and

$$\hat{\chi}_n = \hat{\varphi} - \hat{\varphi}_n - \hat{E}_{\mathrm{c}}, \qquad \hat{\chi}_p = \hat{\varphi}_p - \hat{\varphi} + \hat{E}_{\mathrm{v}}, \tag{2.16}$$

with $\bar{E}_{\mathrm{c}} = \bar{E}_{\mathrm{v}} = k_{\mathrm{B}} T_{\mathrm{L}}$. For the recombination terms we have

$$\hat{R}(\hat{n}, \hat{p}) = \frac{\bar{t}}{\bar{C}} R(\bar{C}\hat{n}, \bar{C}\hat{p}), \tag{2.17}$$

$$\hat{R}_{\mathrm{s}}(\hat{n}, \hat{p}) = \hat{\alpha}_s(\hat{n}, \hat{p}) \frac{|\hat{\Phi}|^2}{\hat{\beta}}, \quad \hat{\alpha}_s(\hat{n}, \hat{p}) = \frac{\bar{t}}{\bar{x}\bar{C}} \alpha_s(\bar{C}\hat{n}, \bar{C}\hat{p}). \tag{2.18}$$

Concerning the Helmholtz problem, we choose

$$\bar{\beta} = \frac{1}{\bar{x}}, \qquad \bar{\Phi} = \frac{1}{\bar{x}}, \qquad \bar{\varepsilon}_* = \frac{1}{\bar{x}^2}.$$

After removing all the notational hats, the scaled system is stated in Box 1.

Box 1. Scaled van Roosbroeck-Helmholtz model for laser diodes.

Van Roosbroeck equations

$$-\nabla\cdot(\varepsilon\nabla\varphi) = C + p - n \tag{2.19a}$$

$$-\frac{\partial n}{\partial t} - \nabla\cdot(\mu_n n \nabla\varphi_n) = R + R_{\mathrm{s}} \qquad \text{in } \Omega\times(0,T), \tag{2.19b}$$

$$-\frac{\partial p}{\partial t} + \nabla\cdot(\mu_p p \nabla\varphi_p) = R + R_{\mathrm{s}} \tag{2.19c}$$

$$\varphi = \varphi_{\mathrm{D}} \qquad \varphi_n = \varphi_{n,\mathrm{D}}, \quad \varphi_p = \varphi_{p,\mathrm{D}} \qquad \text{on } D\times(0,T), \tag{2.19d}$$

$$\boldsymbol{\nu}\cdot(\varepsilon\nabla\varphi) = 0 \qquad \boldsymbol{\nu}\cdot\nabla\varphi_n = 0, \quad \boldsymbol{\nu}\cdot\nabla\varphi_p = 0 \qquad \text{on } \Gamma\times(0,T), \tag{2.19e}$$

$$\varphi_n(0) = \varphi_n^{\mathrm{I}}, \quad \varphi_p(0) = \varphi_p^{\mathrm{I}} \qquad \text{in } \Omega, \tag{2.19f}$$

Helmholtz eigenvalue problem

$$-(\nabla^2 + \varepsilon_*)\Phi = \lambda\Phi \qquad \text{in } \Omega, \tag{2.19g}$$

$$\Phi = 0 \qquad \text{on } \partial\Omega, \tag{2.19h}$$

$\lambda < 0$ principal eigenvalue, $\Phi > 0$ normalised principal eigenfunction

Coupling conditions

$$R_{\mathrm{s}} = \alpha_s(n,p)\frac{|\Phi|^2}{\beta}, \quad \beta = \sqrt{-\lambda}, \tag{2.19i}$$

$$\varepsilon_* = \varepsilon_*(n,p). \tag{2.19j}$$

3. Mathematical prerequisites

3.1. Notation and general assumptions

For a Banach space X, we denote its norm by $\|\cdot\|_X$ and the action of a bounded linear functional $\psi^* \in X^*$ on $\psi \in X$ by the dual pairing $\langle \psi^*, \psi \rangle_X$. If X is a Hilbert space, identified with its dual, then $\langle \cdot, \cdot \rangle_X$

represents the inner product in X. Specifically, for $X=\mathbb{R}^d$, we may also use the notation $\mathbf{a}\cdot\mathbf{b}$ for the scalar product of two vectors $\mathbf{a},\mathbf{b}\in\mathbb{R}^d$.

The space $\mathcal{B}(X;Y)$ shall represent bounded linear operators from a Banach space X to another Banach space Y. If both Banach spaces are equal, we use the simplified notation $\mathcal{B}(X):=\mathcal{B}(X;X)$. The space of compact operators on X is denoted by $\mathcal{B}_\infty(X)$. The notation $[X,Y]_\theta$ refers to the complex interpolation space between X and Y at index $\theta\in[0,1]$.

For a differentiable function ψ on a time interval with values in a Banach space, ψ' denotes its time derivative. The ∇-calculus applies in the distributional sense.

We use the following real-valued function spaces on a spatial domain $\Omega\subseteq\mathbb{R}^2$: for $p\in[1,\infty)$, L^p denotes the set of all Lebesgue measurable, p-integrable functions on Ω, and L^∞ denotes the space of essentially bounded functions on Ω. The norms in these spaces and the following spaces are usually denoted with a subscript. We use the simplified notation $\|\cdot\|$ instead of $\|\cdot\|_{L^2}$.

The space $W^{s,p}(\partial\Omega)$ is the Sobolev space of fractional order $s\in(0,1]$ and integrability exponent $p\in[1,\infty)$ on $\partial\Omega$. These definitions extend accordingly to measurable functions on subsets of $\partial\Omega$. For $p\in[1,\infty)$, we define the Sobolev space $W^{1,p}=\{\psi\in L^p:\ \nabla\psi\in L^p\}$, the subspace $W^{1,p}_\Gamma=\{\psi\in W^{1,p}:\ \psi|_D=0\}$ of functions with vanishing trace on D, and the subspace $W^{1,p}_0=\{\psi\in W^{1,p}:\ \psi|_{\partial\Omega}=0\}$ of function with vanishing trace on $\partial\Omega$. The dual of $W^{1,p}_\Gamma$ is denoted by $W^{-1,p'}_\Gamma$, where $1/p+1/p'=1$.

3.2. Function spaces and linear elliptic operators

In this section, we define three differential operators corresponding to model equations (2.19*a*) to (2.19*c*) and specify the functional spaces that will be used in the main result, along with assumptions on the boundary and initial conditions.

Assumption 3.1. We assume that ε is a bounded, measurable real function on Ω, and it is bounded below by a positive constant.

Definition 3.1 (Poisson operator). We define the Poisson operator $-\nabla\cdot\varepsilon\nabla$ from $W^{1,2}$ to $W^{-1,2}_\Gamma$ by

$$\langle-\nabla\cdot\varepsilon\nabla\psi_1\,,\,\psi_2\rangle_{W^{1,2}_\Gamma}:=\int_\Omega\varepsilon\nabla\psi_1\cdot\nabla\psi_2\,\mathrm{d}\mathbf{x},\quad\forall\psi_1\in W^{1,2},\psi_2\in W^{1,2}_\Gamma,$$

and we denote its restriction to $W^{1,2}_\Gamma$ by $\mathcal{P}_0:=-\nabla\cdot\varepsilon\nabla\big|_{W^{1,2}_\Gamma}$. We use the same symbol $\mathcal{P}_0$ to denote the maximal restriction of $-\nabla\cdot\varepsilon\nabla$ to any subspace that is continuously embedded in the domain $W^{1,2}_\Gamma$.

Definition 3.2 (Electron and hole continuity operators). For any function $n,p\in L^\infty$, we define the operators $-\nabla\cdot n\mu_n\nabla$ and $-\nabla\cdot p\mu_p\nabla$ from $W^{1,2}$ to $W^{-1,2}_\Gamma$ by

$$\langle-\nabla\cdot n\mu_n\nabla\psi_1\,,\,\psi_2\rangle_{W^{1,2}_\Gamma}:=\int_\Omega n\mu_n\nabla\psi_1\cdot\nabla\psi_2\,\mathrm{d}\mathbf{x},$$

and

$$\langle-\nabla\cdot p\mu_p\nabla\psi_1\,,\,\psi_2\rangle_{W^{1,2}_\Gamma}:=\int_\Omega p\mu_p\nabla\psi_1\cdot\nabla\psi_2\,\mathrm{d}\mathbf{x},$$

for $\psi_1\in W^{1,2}$, $\psi_2\in W^{1,2}_\Gamma$. The same definition applies to the operators $-\nabla\cdot\mu_n\nabla$ and $-\nabla\cdot\mu_p\nabla$. We denote the restrictions of $-\nabla\cdot\mu_n\nabla$ and $-\nabla\cdot\mu_p\nabla$ to the space $W^{1,2}_\Gamma$ by $a_n:W^{1,2}_\Gamma\to W^{-1,2}_\Gamma$ and $a_p:W^{1,2}_\Gamma\to W^{-1,2}_\Gamma$.

Proposition 3.1 (See [20, 21]). *There is a number $\hat q>2$ (depending on Ω, ε, μ_p, μ_n and Γ) such that for all $q\in[2,\hat q]$ the operators $\mathcal{P}_0:W^{1,q}_\Gamma\to W^{-1,q}_\Gamma$, $a_n:W^{1,q}_\Gamma\to W^{-1,q}_\Gamma$, and $a_p:W^{1,q}_\Gamma\to W^{-1,q}_\Gamma$ are homeomorphisms. Additionally, both a_n and a_p generate analytic semigroups on $W^{-1,q}_\Gamma$.*

From now on, we fix a number $q\in]2,\min(4,\hat q)[$ and define $r=\frac{q}{2}$. We define operators which will become useful later on.

Definition 3.3. For r defined as above, we introduce the operators

$$A_n:\psi\mapsto a_n\psi,\quad\psi\in\mathcal{D}_n:=\mathrm{dom}\,(A_n):=\left\{\psi\in W^{1,2}_\Gamma:a_n\psi\in L^r\right\},\tag{3.1}$$

$$A_p:\psi\mapsto a_p\psi,\quad\psi\in\mathcal{D}_p:=\mathrm{dom}\,(A_p):=\left\{\psi\in W^{1,2}_\Gamma:a_p\psi\in L^r\right\}.\tag{3.2}$$

$$A:\mathcal{D}\to L^r,\quad A:=\begin{pmatrix}A_n&0\\0&A_p\end{pmatrix},\ \mathcal{D}:=\mathrm{dom}\,(A)=\mathcal{D}_n\oplus\mathcal{D}_p\hookrightarrow L^r.\tag{3.3}$$

We observe that, if $\psi \in \mathcal{D}_n$, then $\boldsymbol{\nu} \cdot (\mu_n n \nabla \psi)|_\Gamma = 0$ and if $\psi \in \mathcal{D}_p$, then $\boldsymbol{\nu} \cdot (\mu_p p \nabla \psi)|_\Gamma = 0$ in the sense of distributions.

Finally, we state assumptions on the boundary and initial conditions.

Assumption 3.2 (on the boundary data). We assume that we can extend the Dirichlet functions φ_{D}, $\varphi_{n,\mathrm{D}}$ and $\varphi_{p,\mathrm{D}}$, appearing in the boundary conditions (2.19*d*), to the whole domain Ω. Keeping the same names, the functions φ_{D}, $\varphi_{n,\mathrm{D}}$ and $\varphi_{p,\mathrm{D}} \in W^{1,q}$ can be chosen to additionally satisfy

$$-\nabla \cdot \varepsilon \nabla \varphi_{\mathrm{D}} = 0, \quad -\nabla \cdot \mu_n \nabla \varphi_{n,\mathrm{D}} = 0, \qquad -\nabla \cdot \mu_p \nabla \varphi_{p,\mathrm{D}} = 0 \qquad \text{in } \Omega. \tag{3.4}$$

Assumption 3.3 (on the initial data). For the fixed $q \in]2, \min(4, \hat{q})[$, we assume that the initial values φ_n^{I} and φ_p^{I} belong to $W^{1,q}$. Moreover, there is a $\theta \in]1/2 + 1/q, 1[$ such that for each of the initial values φ_n^{I} and φ_p^{I} the differences $\varphi_n^{\mathrm{I}} - \varphi_{n,\mathrm{D}}$ and $\varphi_p^{\mathrm{I}} - \varphi_{p,\mathrm{D}}$ belong to the complex interpolation spaces $[L^r, \mathcal{D}_n]_\theta$ and $[L^r, \mathcal{D}_p]_\theta$ respectively.

We notice that for all $\theta \in]1/2 + 1/q, 1[$ the spaces $[L^r, \mathcal{D}_n]_\theta$, $[L^r, \mathcal{D}_p]_\theta$, compactly embed into $W^{1,q}_\Gamma \hookrightarrow L^\infty$.

3.3. Estimates for the reaction terms

The recombination terms R and R_{s} in equations (2.19*b*), (2.19*c*) and (2.19*i*) are expressed in terms of densities n and p. However, we choose the potentials as fundamental dependent variables and set $\boldsymbol{\varphi} := (\varphi, \varphi_n, \varphi_p)$, $\boldsymbol{\varphi} \in W^{1,q}$. With a little abuse of notation, we will write $R(\boldsymbol{\varphi})$, $\varepsilon_*(\boldsymbol{\varphi})$ and $\alpha_s(\boldsymbol{\varphi})$ instead of $R(n,p)$, $\varepsilon_*(n,p)$ and $\alpha_s(n,p)$, respectively.

Assumption 3.4. We assume that the functions $R(\boldsymbol{\varphi})$, $\varepsilon_*(\boldsymbol{\varphi})$ and $\alpha_s(\boldsymbol{\varphi})$ are continuous, and differentiable. Moreover, $\varepsilon_*(\boldsymbol{\varphi})$ is bounded below by a positive constant.

It is possible to characterise the smallest eigenvalue λ of the Helmholtz equation (2.19*g*) using the Rayleigh quotient [19, theorem 8.37].

Proposition 3.2. *The principal eigenvalue λ of the Helmholtz equation (2.19g) satisfies the minimisation problem:*

$$\lambda = \min_{\substack{\Phi \in W_0^{1,2} \\ \|\Phi\| \neq 0}} \frac{\langle \nabla \Phi, \nabla \Phi \rangle - \langle \varepsilon_* \Phi, \Phi \rangle}{\langle \Phi, \Phi \rangle} = \min_{\substack{\Phi \in W_0^{1,2} \\ \|\Phi\| = 1}} \left(\langle \nabla \Phi, \nabla \Phi \rangle - \langle \varepsilon_* \Phi, \Phi \rangle \right). \tag{3.5}$$

The following result deals with the dependence of the principal eigenvalue $\lambda < 0$ and of the normalised principal eigenfunction $\Phi > 0$ on the function $\varepsilon_*(\boldsymbol{\varphi})$. Let us denote by λ_0 the principal eigenvalue of the operator $-\nabla^2$ in $W_0^{1,2}$ with homogeneous Dirichlet boundary conditions. It is known that λ_0 is strictly positive [10, S6.5, theorem 1], [19, theorem 8.38].

If $\varepsilon_* = 0$, then the Helmholtz operator reduces to the negative Laplacian $-\nabla^2$, whose principal eigenvalue λ_0 is strictly positive. If ε_* is a positive constant, then the Helmholtz operator is a shifted negative Laplacian, and its eigenvalues are obtained by shifting the negative Laplacian eigenvalues by $-\varepsilon_*$. In particular, the principal eigenvalue becomes negative once the shift is sufficiently large, i.e. $\varepsilon_* > \lambda_0$. For a spatially varying coefficient function ε_*, this simple shift argument no longer applies, and the sign of the principal eigenvalue must instead be ensured by a more general spectral assumption.

Proposition 3.3. *For any bounded subset M of $W^{1,q}$ and for any $\boldsymbol{\varphi} \in M$, let $\underline{\varepsilon}_*(M) = \inf_{\boldsymbol{\varphi} \in M} \inf_{\mathbf{x} \in \Omega} \varepsilon_*(\boldsymbol{\varphi})$ and $\overline{\varepsilon}_*(M) = \sup_{\boldsymbol{\varphi} \in M} \sup_{\mathbf{x} \in \Omega} \varepsilon_*(\boldsymbol{\varphi})$. If*

$$\underline{\varepsilon}_*(M) > \lambda_0, \tag{3.6}$$

then: (i) the principal eigenvalue of the Helmholtz equation (2.19g) is simple and satisfies

$$\lambda < 0; \tag{3.7}$$

(ii) the corresponding eigenfunction can be uniquely chosen so that

$$\Phi > 0 \text{ in } \Omega \text{ and } \|\Phi\| = 1; \tag{3.8}$$

(iii) the eigenpair (λ, Φ) satisfies the estimates

$$\underline{\varepsilon}_*(M) - \lambda_0 \leqslant -\lambda \leqslant \overline{\varepsilon}_*(M) - \lambda_0, \tag{3.9}$$

$$\|\Phi\|^2_{W^{1,2}_0} \leqslant 1 + \lambda_0 + \overline{\varepsilon}_*(M) - \underline{\varepsilon}_*(M). \tag{3.10}$$

Proof. Using proposition 3.2, we have

$$\begin{aligned}\lambda &= \min_{\substack{\Phi \in W^{1,2}_0 \\ \|\Phi\|=1}} \left(\langle \nabla\Phi, \nabla\Phi\rangle - \langle \varepsilon_*\Phi, \Phi\rangle\right) \\ &\leqslant \min_{\substack{\Phi \in W^{1,2}_0 \\ \|\Phi\|=1}} \langle \nabla\Phi, \nabla\Phi\rangle - \underline{\varepsilon}_*(M) = \lambda_0 - \underline{\varepsilon}_*(M),\end{aligned}$$

which implies (3.7) and the first inequality in (3.9). In a similar way

$$\lambda \geqslant \min_{\substack{\Phi \in W^{1,2}_0 \\ \|\Phi\|=1}} \langle \nabla\Phi, \nabla\Phi\rangle - \overline{\varepsilon}_*(M) \geqslant \lambda_0 - \overline{\varepsilon}_*(M),$$

which implies the second inequality in (3.9). Moreover, λ is simple because the elliptic operator is self-adjoint and it has a positive eigenfunction Φ, [19, theorem 8.38]. Finally, if Φ is the normalised principal eigenfunction associated to λ, the minimum of the Rayleigh quotient is reached for Φ and we have:

$$\|\nabla\Phi\|^2 = \langle \nabla\Phi, \nabla\Phi\rangle = \lambda + \langle \varepsilon_*\Phi, \Phi\rangle \leqslant \lambda_0 - \underline{\varepsilon}_*(M) + \overline{\varepsilon}_*(M).$$

Recalling that $\|\Phi\| = 1$, we can conclude that $\|\Phi\|^2_{W^{1,2}_0} = 1 + \|\nabla\Phi\|^2$, which yields (3.10). □

Lemma 3.1. *Let M be any bounded subset of $W^{1,q}$, $K_0 := \left\{\varepsilon \in W^{1,q} \mid \inf_{\mathbf{x}\in\Omega} \varepsilon(\mathbf{x}) > \lambda_0\right\}$, and $\delta(M) = \inf_{\varepsilon \in K_0 \cap M}\{\lambda_2(\varepsilon) - \lambda_1(\varepsilon)\}$, with $\lambda_1(\varepsilon)$ and $\lambda_2(\varepsilon)$, respectively, the first and second smallest eigenvalue of the operator $\mathcal{T}(\varepsilon) = -\nabla^2 - \varepsilon$. Then for any $\varepsilon_1, \varepsilon_2 \in K_0 \cap M$ it holds*

$$\begin{aligned}|\lambda(\varepsilon_1) - \lambda(\varepsilon_2)| &\leqslant \|\varepsilon_1 - \varepsilon_2\|_{L^\infty} \\ \|\Phi(\varepsilon_1) - \Phi(\varepsilon_2)\| &\leqslant \frac{1}{\delta(M)}\|\varepsilon_1 - \varepsilon_2\|_{L^\infty}.\end{aligned}$$

Proof. Let $\varepsilon_1, \varepsilon_2 \in K_0 \cap M$, and consider the family of operators

$$\{\mathcal{T}(\theta) = \mathcal{T}(\varepsilon_1 + \theta(\varepsilon_2 - \varepsilon_1)) \equiv \mathcal{T}(\varepsilon_1) - \theta(\varepsilon_2 - \varepsilon_1)\}_{\theta\in[0,1]}.$$

Since $\mathcal{T}(\theta)$ is holomorphic as a function of the real parameter θ extended to the complex unit disc $|\theta| \leqslant 1$, we can use Kato's result that ensures that both the leading eigenvalue $\lambda(\theta)$ and the corresponding positive normalised eigenfunction $\Phi(\theta)$ are holomorphic functions, in particular differentiable [28]. The eigenpair $(\lambda(\theta), \Phi(\theta))$ satisfies per definition the eigenvalue equation

$$(\mathcal{T}(\theta) - \lambda(\theta))\Phi(\theta) = 0, \tag{3.11}$$

with the constraint

$$\langle \Phi(\theta), \Phi(\theta)\rangle = 1. \tag{3.12}$$

Notice that $(\lambda(\theta), \Phi(\theta))\big|_{\theta=0} = (\lambda(\varepsilon_1), \Phi(\varepsilon_1))$ and $(\lambda(\theta), \Phi(\theta))\big|_{\theta=1} = (\lambda(\varepsilon_2), \Phi(\varepsilon_2))$. We take the derivative of equations (3.11) and (3.12) with respect to θ, obtaining

$$(\mathcal{T}(\theta) - \lambda(\theta))\Phi'(\theta) = (\varepsilon_2 - \varepsilon_1 + \lambda'(\theta))\Phi(\theta), \tag{3.13}$$

$$\langle \Phi(\theta), \Phi'(\theta)\rangle = 0. \tag{3.14}$$

In particular we have $\Phi'(\theta) \in \operatorname{span}(\Phi(\theta))^\perp$. In order for equation (3.13) to have a solution $\Phi'(\theta)$, we need the solvability condition

$$\lambda'(\theta) = \langle \Phi(\theta), (\varepsilon_1 - \varepsilon_2)\Phi(\theta)\rangle. \tag{3.15}$$

Then, the solution can be written in the form

$$\begin{aligned}\Phi'(\theta) &= \sum_{k\geqslant 2} \left(\lambda_k(\theta)-\lambda(\theta)\right)^{-1} \left\langle \Phi_k(\theta), (\varepsilon_2-\varepsilon_1+\lambda'(\theta))\,\Phi(\theta)\right\rangle \Phi_k(\theta)\\ &= \sum_{k\geqslant 2} \left(\lambda_k(\theta)-\lambda(\theta)\right)^{-1} \left\langle \Phi_k(\theta), (\varepsilon_2-\varepsilon_1)\,\Phi(\theta)\right\rangle \Phi_k(\theta), \end{aligned} \tag{3.16}$$

where $\{\Phi_k(\theta)\}_{k\in\mathbb{N}}$ is a complete orthonormal basis of eigenfunctions, with $\Phi_1(\theta)=\Phi(\theta)$. Integrating equation (3.15) over $\theta\in[0,1]$, we get for some $\theta_*\in(0,1)$

$$\begin{aligned}|\lambda(\varepsilon_2)-\lambda(\varepsilon_1)| = |\lambda(1)-\lambda(0)| &= \left|\int_0^1 \lambda'(\theta)\right| = \left|\int_0^1 \left\langle \Phi(\theta), (\varepsilon_1-\varepsilon_2)\,\Phi(\theta)\right\rangle \mathrm{d}\theta\right|\\ &= \left|\left\langle \Phi(\theta_*), (\varepsilon_1-\varepsilon_2)\,\Phi(\theta_*)\right\rangle\right| \leqslant \|\varepsilon_1-\varepsilon_2\|_{L^\infty}.\end{aligned}$$

Similarly, from equation (3.16), we obtain for some other $\theta_*\in(0,1)$

$$\|\Phi(\varepsilon_2)-\Phi(\varepsilon_1)\| = \left\| \sum_{k\geqslant 2} \left(\lambda_k(\theta_*)-\lambda(\theta_*)\right)^{-1} \Phi_k(\theta_*) \left\langle \Phi_k(\theta_*), (\varepsilon_2-\varepsilon_1)\,\Phi(\theta_*)\right\rangle \right\|.$$

Using Bessel's inequality, we find

$$\|\Phi(\varepsilon_2)-\Phi(\varepsilon_1)\| \leqslant \frac{1}{|\lambda_2(\theta_*)-\lambda(\theta_*)|}\|\varepsilon_1-\varepsilon_2\|_{L^\infty} \leqslant \frac{1}{\delta(M)}\|\varepsilon_1-\varepsilon_2\|_{L^\infty},$$

which concludes the proof. □

Remark 3.1. The last result implies that the principal eigenpair $(\lambda(\varepsilon_*),\Phi(\varepsilon_*))$ of the Helmholtz equation $(-\nabla^2-\varepsilon_*)\Phi=\lambda\Phi$, can be viewed as a locally Lipschitz-continuous map

$$\varepsilon_* \mapsto (\lambda(\varepsilon_*),\Phi(\varepsilon_*)).$$

Since ε_* is a function of the potentials $\boldsymbol{\varphi}$, this establishes a map

$$\boldsymbol{\varphi} \mapsto (\lambda(\boldsymbol{\varphi}),\Phi(\boldsymbol{\varphi})) := (\lambda(\varepsilon_*(\boldsymbol{\varphi})),\Phi(\varepsilon_*(\boldsymbol{\varphi}))).$$

Since ε_* depends continuously on $\boldsymbol{\varphi}$, the last map is also locally Lipschitz-continuous. Thus, the stimulated recombination

$$R_s(\boldsymbol{\varphi}) = \alpha_s(n(\boldsymbol{\varphi}),p(\boldsymbol{\varphi}))\frac{|\Phi(\boldsymbol{\varphi})|^2}{\sqrt{-\lambda(\boldsymbol{\varphi})}}$$

in the van Roosbroeck equations has the same functional structure as the classical recombination term R, in the sense that both recombination terms depend directly on the potentials $\boldsymbol{\varphi}$.

Corollary 3.1. *Let M be any bounded subset of $W^{1,q}$,*

$$K_* := \left\{\boldsymbol{\varphi}\in W^{1,q} \mid \inf_{\mathbf{x}\in\Omega}\varepsilon_*(\boldsymbol{\varphi}(\mathbf{x}))>\lambda_0\right\},$$

and $\delta_*(M)=\inf_{\boldsymbol{\varphi}\in K_*\cap M}\{\lambda_2(\boldsymbol{\varphi})-\lambda_1(\boldsymbol{\varphi})\}$, with $\lambda_1(\boldsymbol{\varphi})$ and $\lambda_2(\boldsymbol{\varphi})$, respectively, the first and second smallest eigenvalue of the operator $-\nabla^2-\varepsilon_*(\boldsymbol{\varphi})$. Then for any $\boldsymbol{\varphi}_1,\boldsymbol{\varphi}_2\in K_*\cap M$ it holds

$$\begin{aligned}|\lambda(\boldsymbol{\varphi}_1)-\lambda(\boldsymbol{\varphi}_2)| &\leqslant \|\boldsymbol{\varphi}_1-\boldsymbol{\varphi}_2\|_{L^\infty}\\ \|\Phi(\boldsymbol{\varphi}_1)-\Phi(\boldsymbol{\varphi}_2)\| &\leqslant \frac{1}{\delta_*(M)}\|\boldsymbol{\varphi}_1-\boldsymbol{\varphi}_2\|_{L^\infty}.\end{aligned}$$

Next, we prove local Lipschitz continuity of the standard recombination term R and the stimulated recombination term R_s in the following two lemmas.

Lemma 3.2. *The semiconductor recombination term is a mapping $R\colon W^{1,q}\longrightarrow L^r$. Moreover, for any bounded subset $M\subset W^{1,q}$ there exists a constant c_M, such that*

$$\|R(\boldsymbol{\varphi}_1)-R(\boldsymbol{\varphi}_2)\|_{L^r} \leqslant c_M\|\boldsymbol{\varphi}_1-\boldsymbol{\varphi}_2\|_{W^{1,q}}, \quad \text{for all } \boldsymbol{\varphi}_1,\boldsymbol{\varphi}_2\in M. \tag{3.17}$$

Proof. Let M be a bounded subset of $W^{1,q}$, such that $\|\varphi\|_{W^{1,q}} \leqslant C_M$ for all $\varphi \in M$, for some positive constant C_M. Let $\varphi_1, \varphi_2 \in M$, which implies that $\|\varphi_1\|_{L^\infty}, \|\varphi_2\|_{L^\infty} \leqslant C_M$, since $W^{1,q} \hookrightarrow L^\infty$. We have

$$\begin{aligned}\|R(\varphi_1) - R(\varphi_2)\|_{L^r} &= \|\nabla_\varphi R(\varphi_1 + \theta(\varphi_2 - \varphi_1)) \cdot (\varphi_2 - \varphi_1)\|_{L^r} \\ &\leqslant \|\nabla_\varphi R(\varphi_1 + \theta(\varphi_2 - \varphi_1))\|_{L^q} \|\varphi_2 - \varphi_1\|_{L^q}\end{aligned}$$

with $\theta = \theta(\mathbf{x}) \in [0,1]$. Since,

$$|\varphi_1 + \theta(\varphi_2 - \varphi_1)| \leqslant (1-\theta)\|\varphi_1\|_{L^\infty} + \theta\|\varphi_2\|_{L^\infty} \leqslant C_M$$

we conclude that (3.17) holds with

$$c_M = \max_{|\varphi| \leqslant C_M} |\nabla_\varphi R(\varphi)| \ \mathrm{meas}(\Omega)^{1/q},$$

which is a well-defined constant by the Weierstrass theorem since $|\nabla_\varphi R(\varphi)|$ is continuous on the closed bounded set $\{|\varphi| \leqslant C_M\} \subset \mathbb{R}^3$. □

Lemma 3.3. *The stimulated recombination term is a mapping $R_s : W^{1,q} \longrightarrow L^r$. Moreover, for any bounded subset $M \subset W^{1,q}$ there exists a constant $c_{s,M}$, such that*

$$\|R_s(\varphi_1) - R_s(\varphi_2)\|_{L^r} \leqslant c_{s,M}\|\varphi_1 - \varphi_2\|_{W^{1,q}}, \quad \textit{for all } \varphi_1, \varphi_2 \in M. \tag{3.18}$$

Proof. Let M be a bounded subset of $W^{1,q}$, such that $\|\varphi\|_{W^{1,q}} \leqslant C_M$ for all $\varphi \in M$, for some positive constant C_M. Let $\varphi_1, \varphi_2 \in M$, which implies that $\|\varphi_1\|_{L^\infty}, \|\varphi_2\|_{L^\infty} \leqslant C_M$, since $W^{1,q} \hookrightarrow L^\infty$. We compute

$$\begin{aligned}\|R_s(\varphi_1) - R_s(\varphi_2)\|_{L^r} &= \left\| \frac{\alpha_s(\varphi_1)}{\beta(\varphi_1)}|\Phi(\varphi_1)|^2 - \frac{\alpha_s(\varphi_2)}{\beta(\varphi_2)}|\Phi(\varphi_2)|^2 \right\|_{L^r} \\ &\leqslant \left\| \frac{|\Phi(\varphi_1)|^2}{\beta(\varphi_1)}(\alpha_s(\varphi_1) - \alpha_s(\varphi_2)) \right\|_{L^r} + \left\| \alpha_s(\varphi_2)|\Phi(\varphi_1)|^2 \left(\frac{1}{\beta(\varphi_1)} - \frac{1}{\beta(\varphi_2)} \right) \right\|_{L^r} \\ &\quad + \left\| \frac{\alpha_s(\varphi_2)}{\beta(\varphi_2)} \left(|\Phi(\varphi_1)|^2 - |\Phi(\varphi_2)|^2 \right) \right\|_{L^r} \\ &\leqslant \frac{\||\Phi(\varphi_1)|^2\|_{L^\infty}}{\beta(\varphi_1)} \|\alpha_s(\varphi_1) - \alpha_s(\varphi_2)\|_{L^r} + \frac{\|\alpha_s(\varphi_2)\|_{L^q} \||\Phi(\varphi_1)|^2\|_{L^q}}{\beta(\varphi_1)\beta(\varphi_2)} |\beta(\varphi_1) - \beta(\varphi_2)| \\ &\quad + \frac{\|\alpha_s(\varphi_2)\|_{L^q}}{\beta(\varphi_2)} \||\Phi(\varphi_1)|^2 - |\Phi(\varphi_2)|^2\|_{L^q}.\end{aligned}$$

We estimate these three terms separately. In the first term, the difference

$$\|\alpha_s(\varphi_1) - \alpha_s(\varphi_2)\|_{L^r}$$

is bounded similarly to lemma 3.2. In the second term, we use the fact that

$$|\beta(\varphi_1) - \beta(\varphi_2)| = \frac{|\lambda(\varphi_1) - \lambda(\varphi_2)|}{\beta(\varphi_1) + \beta(\varphi_2)},$$

and for the the third term we use

$$\begin{aligned}\||\Phi(\varphi_1)|^2 - |\Phi(\varphi_2)|^2\|_{L^q} &= \|(\Phi(\varphi_1) + \Phi(\varphi_2))(\Phi(\varphi_1) - \Phi(\varphi_2))\|_{L^q} \\ &\leqslant \|\Phi(\varphi_1) + \Phi(\varphi_2)\|_{L^\infty} \|\Phi(\varphi_1) - \Phi(\varphi_2)\|_{L^q}.\end{aligned}$$

The proof is concluded by using proposition 3.3, lemma 3.1 and corollary 3.1. □

3.4. Formulation of the problem

In this section, we rewrite the problem in terms of the operators defined in section 3.2 to be able to introduce weak solutions. We start with the Poisson equation (2.19*a*), with boundary conditions (2.19*d*) and (2.19*e*), with $p, n \colon [0,T] \to W^{1,q}_\Gamma$.

Assumption 3.5. We assume that the doping profile C is a bounded, measurable real function on Ω.

Definition 3.4 (Weak solution of Poisson's problem). Let $n, p \in W_\Gamma^{-1,q}$ be given. We say that φ is a solution of Poisson's equation (2.19*a*), if

$$\varphi = \varphi_{\mathrm{D}} + \mathring{\varphi}, \tag{3.19}$$

and $\mathring{\varphi} \in W_\Gamma^{1,q}$ is the unique solution of

$$\mathcal{P}_0 \mathring{\varphi} = C - n + p. \tag{3.20}$$

Definition 3.5 (Precise formulation of the problem). The van Roosbroeck–Helmholtz system (2.19) admits a local in time solution, if there is a time $T > 0$, a triple $\boldsymbol{\varphi} = (\varphi, \varphi_n, \varphi_p)$, and a pair (λ, Φ), such that

$$(\varphi_n(0), \varphi_p(0)) = (\varphi_n^{\mathrm{I}}, \varphi_p^{\mathrm{I}}) \in W^{1,q}, \tag{3.21}$$

$$\mathring{\varphi} := \varphi - \varphi_{\mathrm{D}} \in C([0,T]; W_\Gamma^{1,q}) \cap C^1(]0,T[; W_\Gamma^{1,q}), \tag{3.22}$$

$$\mathring{\varphi}_n := \varphi_n - \varphi_{n,\mathrm{D}} \in C^1(]0,T[; L^r) \cap C(]0,T], \mathcal{D}) \cap C([0,T], [L^r, \mathcal{D}]_\theta), \tag{3.23}$$

$$\mathring{\varphi}_p := \varphi_p - \varphi_{p,\mathrm{D}} \in C^1(]0,T[; L^r) \cap C(]0,T], \mathcal{D}) \cap C([0,T], [L^r, \mathcal{D}]_\theta), \tag{3.24}$$

satisfy the Poisson equation and the continuity equations

$$\begin{cases} \mathcal{P}_0(\mathring{\varphi}(t)) = C - n(t) + p(t), & t \in [0,T] \\ -n'(t) + \nabla \cdot \mathbf{j}_n(t) = R(\boldsymbol{\varphi}(t)) + R_{\mathrm{s}}(\boldsymbol{\varphi}(t)), & t \in [0,T] \\ -p'(t) - \nabla \cdot \mathbf{j}_p(t) = R(\boldsymbol{\varphi}(t)) + R_{\mathrm{s}}(\boldsymbol{\varphi}(t)), & t \in [0,T] \end{cases} \tag{3.25}$$

where the carrier densities and current densities are given by

$$\begin{aligned} n(t) &= N_{\mathrm{c}} \mathcal{F}(\chi_n(t)), & p(t) &= N_{\mathrm{v}} \mathcal{F}(\chi_p(t)), \\ \mathbf{j}_n(t) &= -\mu_n n(t) \nabla \varphi_n(t), & \mathbf{j}_p(t) &= -\mu_p p(t) \nabla \varphi_p(t), \\ \chi_n(t) &= \varphi(t) - \varphi_n(t) - E_{\mathrm{c}}, & \chi_p(t) &= \varphi_p(t) - \varphi(t) + E_{\mathrm{v}}, \end{aligned} \tag{3.26}$$

with the properties

$$n, p \in C([0,T], L^\infty) \cap C^1(]0,T[, L^r]), \tag{3.27}$$

$$\mathbf{j}_n, \mathbf{j}_p \in C([0,T], L^q), \tag{3.28}$$

$$\nabla \cdot \mathbf{j}_n, \nabla \cdot \mathbf{j}_p \in C(]0,T], L^r), \tag{3.29}$$

and $(\lambda, \Phi) = (\lambda(\boldsymbol{\varphi}), \Phi(\boldsymbol{\varphi}))$, defined in remark 3.1, is the principal eigenpair of the Helmholtz equation

$$\left(-\nabla^2 - \varepsilon_*(\boldsymbol{\varphi})\right) \Phi = \lambda \Phi, \tag{3.30}$$

with

$$\Phi > 0 \text{ in } \Omega, \quad \|\Phi\| = 1. \tag{3.31}$$

4. Reformulation as a quasi-linear parabolic system

Our aim is now to prove the existence of a solution of the Helmholtz-van-Roosbroeck system, shown in Box 1, in the sense of definition 3.5, extending the approach used in [25] for the van-Roosbroeck system. The idea of the proof relies on solving first the Poisson equation for the electric potential φ, as a function of the quasi–Fermi potentials φ_n, φ_p, and use this representation of φ in the continuity equations, which can thus be written as a quasi-linear parabolic system. The idea of solving first the nonlinear Poisson equation was originally introduced by Gummel in an iterative scheme for the numerical solution of the van Roosbroeck system, known as Gummel's map [22]. In our setting, we do not rely on an iterative scheme. Instead, we view the nonlinear Poisson equation as a solution map, which allows us to reformulate the van Roosbroeck system as a quasi-linear parabolic system in a form suitable for the application of the existing theory [25, 26].

4.1. The nonlinear Poisson equation

By using the constitutive relations for the carrier densities in the Poisson equation, we obtain the nonlinear Poisson equation

$$\mathcal{P}_0\mathring{\varphi} = C - N_c\mathcal{F}(z_n + \mathring{\varphi}) + N_v\mathcal{F}(z_p - \mathring{\varphi}),$$

with $z_n = \varphi_D - \varphi_n - E_c$, $z_p = -\varphi_D + \varphi_p + E_v$, where $\mathring{\varphi}$ is described in section 3.4.

Theorem 4.1 (theorem 5.3, [25]). *The following statements are true.*

1. *For all $f \in W_\Gamma^{-1,2}$ and $\mathbf{z} = (z_n, z_p) \in L^\infty$ the nonlinear Poisson equation*

$$\mathcal{P}_0\mathring{\varphi} - N_c\mathcal{F}(z_n + \mathring{\varphi}) + N_v\mathcal{F}(z_p - \mathring{\varphi}) = f \tag{4.1}$$

   admits exactly one solution $\mathring{\varphi}$ with $\mathring{\varphi} \in W_\Gamma^{1,2}$ which we denote by $\mathscr{L}(f, \mathbf{z})$.

2. *The maximal restriction of the operator from equation (4.1) to the range space $W_\Gamma^{-1,q}$ has the domain $W_\Gamma^{1,q}$. Moreover, if M is a bounded subset of $W_\Gamma^{-1,q} \oplus L^\infty$, then the set $\{\mathscr{L}(f, \mathbf{z}) : (f, \mathbf{z}) \in M\}$ is bounded in $W_\Gamma^{1,q}$.*

3. *The mapping $\mathscr{L} : W_\Gamma^{-1,q} \oplus L^\infty \to W_\Gamma^{1,q}$ is continuously differentiable.*

4.2. Derivation of the quasi-linear system

We start by differentiating Poisson's equation (2.19*a*) with respect to time

$$\mathcal{P}_0\partial_t\mathring{\varphi} = \partial_t p - \partial_t n = -\nabla\cdot\mathbf{j}_p - \nabla\cdot\mathbf{j}_n$$

or

$$\partial_t\varphi := \partial_t\mathring{\varphi} = \mathcal{P}_0^{-1}\left(-\nabla\cdot\mathbf{j}_p - \nabla\cdot\mathbf{j}_n\right).$$

Next, we use the constitutive relations for the carrier densities and insert the above expression for $\partial_t\varphi$

$$\begin{aligned}\partial_t n &= N_c\mathcal{F}'(\chi_n)\left(\mathcal{P}_0^{-1}\left(-\nabla\cdot\mathbf{j}_n - \nabla\cdot\mathbf{j}_p\right) - \partial_t\varphi_n\right)\\ \partial_t p &= N_v\mathcal{F}'(\chi_p)\left(-\mathcal{P}_0^{-1}\left(-\nabla\cdot\mathbf{j}_n - \nabla\cdot\mathbf{j}_p\right) + \partial_t\varphi_p\right).\end{aligned}$$

We then insert these expressions into the continuity equations for the currents

$$\begin{aligned}N_c\mathcal{F}'(\chi_n)\left(\mathcal{P}_0^{-1}\left(-\nabla\cdot\mathbf{j}_n - \nabla\cdot\mathbf{j}_p\right) - \partial_t\varphi_n\right) - \nabla\cdot\mathbf{j}_n &= -(R + R_s)\\ N_v\mathcal{F}'(\chi_p)\left(-\mathcal{P}_0^{-1}\left(-\nabla\cdot\mathbf{j}_n - \nabla\cdot\mathbf{j}_p\right) + \partial_t\varphi_p\right) + \nabla\cdot\mathbf{j}_p &= -(R + R_s).\end{aligned}$$

Rewriting yields

$$\begin{aligned}\partial_t\varphi_n - \left(\mathcal{P}_0^{-1} + \frac{1}{N_c\mathcal{F}'(\chi_n)}\right)(-\nabla\cdot\mathbf{j}_n) - \mathcal{P}_0^{-1}\left(-\nabla\cdot\mathbf{j}_p\right) &= \frac{R + R_s}{N_c\mathcal{F}'(\chi_n)}\\ \partial_t\varphi_p - \left(\mathcal{P}_0^{-1} + \frac{1}{N_v\mathcal{F}'(\chi_p)}\right)\left(-\nabla\cdot\mathbf{j}_p\right) - \mathcal{P}_0^{-1}(-\nabla\cdot\mathbf{j}_n) &= -\frac{R + R_s}{N_v\mathcal{F}'(\chi_p)}.\end{aligned}$$

We can express both equations in vector-matrix form as follows

$$\partial_t\boldsymbol{\Psi} - (I + Z(\boldsymbol{\varphi}))E(\boldsymbol{\varphi})\nabla\cdot\mathbf{J} = \mathbf{Y}(\boldsymbol{\varphi}),$$

with

$$\boldsymbol{\Psi} = \begin{pmatrix}\varphi_n\\ \varphi_p\end{pmatrix}, \qquad Z(\boldsymbol{\varphi}) = \begin{pmatrix}\mathcal{P}_0^{-1}N_c\mathcal{F}'(\chi_n) & N_v\mathcal{P}_0^{-1}\mathcal{F}'(\chi_p)\\ \mathcal{P}_0^{-1}N_c\mathcal{F}'(\chi_n) & N_v\mathcal{P}_0^{-1}\mathcal{F}'(\chi_p)\end{pmatrix} \tag{4.2}$$

$$E(\boldsymbol{\varphi}) = \begin{pmatrix}\frac{1}{N_c\mathcal{F}'(\chi_n)} & 0\\ 0 & \frac{1}{N_v\mathcal{F}'(\chi_p)}\end{pmatrix}, \quad \nabla\cdot\mathbf{J} = \begin{pmatrix}-\nabla\cdot\mathbf{j}_n\\ -\nabla\cdot\mathbf{j}_p\end{pmatrix}, \tag{4.3}$$

$$\mathbf{Y}(\boldsymbol{\varphi}) = E(\boldsymbol{\varphi})\begin{pmatrix}1\\ -1\end{pmatrix}(R(\boldsymbol{\varphi}) + R_s(\boldsymbol{\varphi})), \tag{4.4}$$

$$\chi_n = \varphi - \varphi_n - E_c, \quad \chi_p = \varphi_p - \varphi + E_v, \tag{4.5}$$

and I the identity matrix. It remains to express $\mathbf{J}$ in terms of $\mathbf{\Psi}$ as well because we want the operator $(I+Z)E$ to act on $\nabla\mathbf{\Psi}$ instead of $\nabla\cdot\mathbf{J}$. We have $-\mathbf{j}_n = n\mu_n\nabla\varphi_n$ and $-\mathbf{j}_p = p\mu_p\nabla\varphi_p$, so we can write

$$\nabla\cdot\mathbf{J} = \nabla\cdot G(\boldsymbol{\varphi})\,\mu\nabla\mathbf{\Psi},$$

with

$$G(\boldsymbol{\varphi}) = \begin{pmatrix} N_c\mathcal{F}(\chi_n) & 0 \\ 0 & N_v\mathcal{F}(\chi_p) \end{pmatrix}, \qquad \mu = \begin{pmatrix} \mu_n & 0 \\ 0 & \mu_p \end{pmatrix}. \tag{4.6}$$

Finally, we express $\mathring{\varphi}$ in terms of $\mathbf{\Psi} = (\varphi_n, \varphi_p)^T$, using the solution of the nonlinear Poisson equation and the operator $\mathscr{L}$, as described in theorem 4.1,

$$\mathring{\varphi}(\mathbf{\Psi}) := \mathscr{L}\left(C, -\varphi_n + \varphi_D - E_c, \varphi_p - \varphi_D + E_v\right). \tag{4.7}$$

In this way, we get

$$\partial_t\mathbf{\Psi} - (I+Z(\mathbf{\Psi}))E(\mathbf{\Psi})\nabla\cdot G(\mathbf{\Psi})\mu\nabla\mathbf{\Psi} = \mathbf{Y}(\mathbf{\Psi}), \tag{4.8}$$

with

$$\begin{aligned} Z(\mathbf{\Psi}) &:= Z\left(\mathring{\varphi}(\mathbf{\Psi})+\varphi_D,\ \varphi_n,\ \varphi_p\right), \\ E(\mathbf{\Psi}) &:= E\left(\mathring{\varphi}(\mathbf{\Psi})+\varphi_D,\ \varphi_n,\ \varphi_p\right), \\ G(\mathbf{\Psi}) &:= G\left(\mathring{\varphi}(\mathbf{\Psi})+\varphi_D,\ \varphi_n,\ \varphi_p\right), \\ \mathbf{Y}(\mathbf{\Psi}) &:= \mathbf{Y}\left(\mathring{\varphi}(\mathbf{\Psi})+\varphi_D,\ \varphi_n,\ \varphi_p\right). \end{aligned}$$

Now, the equation (4.8) is of the form discussed in [26]. It should be complemented by the initial-boundary conditions

$$\mathbf{\Psi}(\mathbf{x},0) = \mathbf{\Psi}_0^I(\mathbf{x}) := \begin{pmatrix} \varphi_n^I(\mathbf{x}) \\ \varphi_p^I(\mathbf{x}) \end{pmatrix}, \text{ in } \Omega, \tag{4.9}$$

$$\mathbf{\Psi} = \mathbf{\Psi}_D := \begin{pmatrix} \varphi_{n,D} \\ \varphi_{p,D} \end{pmatrix} \text{ on } D, \quad \boldsymbol{\nu}\cdot\nabla\mathbf{\Psi} := \begin{pmatrix} \boldsymbol{\nu}\cdot\nabla\varphi_n \\ \boldsymbol{\nu}\cdot\nabla\varphi_p \end{pmatrix} = \mathbf{0} \text{ on } \Gamma. \tag{4.10}$$

4.3. Solution of the quasi-linear parabolic system

In this section, we prove that equation (4.8) admits a unique local in time solution by applying the existence theorem in [26]. The assumptions of the theorem are summarised next.

Assumption 4.1. For any bounded set $M \subset W^{1,q}$ there exist positive constants E_M, G_M, Z_M and Y_M such that the mappings

$$\begin{aligned} &E\colon [0,T]\times W^{1,q} \to L^\infty, && G\colon [0,T]\times W^{1,q} \to W^{1,q} \\ &Z\colon [0,T]\times W^{1,q} \to \mathcal{B}_\infty(L^p), && \mathbf{Y}\colon [0,T]\times W^{1,q} \to L^p \end{aligned} \tag{4.11}$$

satisfy the conditions

$$\begin{aligned} &\|E(\mathbf{\Psi}) - E(\check{\mathbf{\Psi}})\|_{L^\infty} \leqslant E_M\|\mathbf{\Psi}-\check{\mathbf{\Psi}}\|_{W^{1,q}}, \\ &\|G(\mathbf{\Psi}) - G(\check{\mathbf{\Psi}})\|_{W^{1,q}} \leqslant G_M\|\mathbf{\Psi}-\check{\mathbf{\Psi}}\|_{W^{1,q}}, \\ &\|Z(\mathbf{\Psi}) - Z(\check{\mathbf{\Psi}})\|_{\mathcal{B}_\infty(L^p)} \leqslant Z_M\|\mathbf{\Psi}-\check{\mathbf{\Psi}}\|_{W^{1,q}}, \\ &\|\mathbf{Y}(\mathbf{\Psi}) - \mathbf{Y}(\check{\mathbf{\Psi}})\|_{L^p} \leqslant Y_M\|\mathbf{\Psi}-\check{\mathbf{\Psi}}\|_{W^{1,q}}, \end{aligned}$$

for all $\mathbf{\Psi}, \check{\mathbf{\Psi}} \in M$, and the diagonal components of the matrices E and G satisfy

$$\min_{k=1,2}\ \inf_{\mathbf{\Psi}\in M}\ \operatorname*{essinf}_{\mathbf{x}\in\Omega}\ E_{kk}(\mathbf{\Psi})(\mathbf{x}) > 0,$$

$$\min_{k=1,2}\ \inf_{\mathbf{\Psi}\in M}\ \operatorname*{essinf}_{\mathbf{x}\in\Omega}\ G_{kk}(\mathbf{\Psi})(\mathbf{x}) > 0.$$

We define the notion of weak the solution of the equation (4.8), noting that $-\nabla\cdot G(\boldsymbol{\Psi})\mu\nabla = G(\boldsymbol{\Psi})A$, where A is defined in (3.3).

Definition 4.1. Let $\mathcal{D}$ be the space defined in (3.3) and V a Banach space such that $\mathcal{D}\hookrightarrow V\hookrightarrow W^{1,q}$. We say the evolution equation (4.8), with initial condition $\boldsymbol{\Psi}(0)=\boldsymbol{\Psi}^{\mathrm{I}}\in W^{1,q}$ has a unique local solution $\boldsymbol{\Psi}=\boldsymbol{\Psi}_D+\mathring{\boldsymbol{\Psi}}$ with respect to V if $\boldsymbol{\Psi}^{\mathrm{I}}-\boldsymbol{\Psi}_D\in V$ implies the existence of a number $T>0$ such that the initial value problem

$$\partial_t\mathring{\boldsymbol{\Psi}}(t)+\left[I+Z\left(\boldsymbol{\Psi}_D+\mathring{\boldsymbol{\Psi}}\right)\right]E\left(\boldsymbol{\Psi}_D+\mathring{\boldsymbol{\Psi}}\right)G\left(\boldsymbol{\Psi}_D+\mathring{\boldsymbol{\Psi}}\right)A\mathring{\boldsymbol{\Psi}}(t)=\mathbf{Y}\left(\boldsymbol{\Psi}_D+\mathring{\boldsymbol{\Psi}}\right)+J\left(\mathring{\boldsymbol{\Psi}}\right), \tag{4.12}$$

$$\mathring{\boldsymbol{\Psi}}(0)=\boldsymbol{\Psi}^{\mathrm{I}}-\boldsymbol{\Psi}_D, \tag{4.13}$$

admits a unique solution

$$\mathring{\boldsymbol{\Psi}}\in C^1\left(]0,T[,L^r\right)\cap C\left(]0,T],\mathcal{D}\right)\cap C\left([0,T],V\right). \tag{4.14}$$

For $\mathring{\boldsymbol{\Psi}}\in W^{1,q}_{\Gamma}$ the term J is given by

$$J\left(\mathring{\boldsymbol{\Psi}}\right):=\left[I+Z\left(\boldsymbol{\Psi}_D+\mathring{\boldsymbol{\Psi}}\right)\right]E\left(\boldsymbol{\Psi}_D+\mathring{\boldsymbol{\Psi}}\right)\nabla\cdot G\left(\boldsymbol{\Psi}_D+\mathring{\boldsymbol{\Psi}}\right)\mu\nabla\boldsymbol{\Psi}_D.$$

The following result was proven in [26].

Proposition 4.1. *Let assumption 4.1 be satisfied. For each $\gamma\in]\frac{1}{2}+\frac{1}{q},1[$ the initial value problem for our evolution equation (4.8), with initial value $\boldsymbol{\Psi}^{\mathrm{I}}\in W^{1,q}$ has a unique local solution $\boldsymbol{\Psi}$, in the sense of definition 4.1, with respect to the complex interpolation spaces $V:=[L^p,\mathcal{D}]_\gamma$.*

To apply this result we need to verify that the assumption 4.1 hold for system (4.8).

Lemma 4.1. *Let assumptions 3.1, 3.4, 3.2, 3.3, 3.5 be satisfied. Then the mappings $Z,E,G,\mathbf{Y}$, satisfy assumption 4.1.*

Proof. In order to prove this result, we use the results in [25, section 6] with a few adjustments, because we consider a domain that does not contain oxide and the boundary conditions and the doping profile do not vary with time. The Lipschitz continuity of the recombination terms R and R_{s}, which is needed to prove the estimates for $\mathbf{Y}$, is ensured by the lemmas 3.2 and 3.3, under assumption 3.4. □

5. Main result

In this section, we prove the main result of the paper, that is, local existence and uniqueness of the van Roosbroeck–Helmholtz model, adjusting [25, theorem 7.4].

Theorem 5.1. *Under assumptions 3.1, 3.4, 3.2, 3.3, 3.5, the van Roosbroeck–Helmholtz system (2.19) admits a unique, local solution in time, in the sense of definition 3.5.*

Proof. The lemma 4.1 ensures that assumption 4.1 hold. It follows that the quasi-linear parabolic system (4.8) admits a unique solution $\boldsymbol{\Psi}=(\varphi_n,\varphi_p)$, local in time, in the sense of definition 4.1. It remains to show that this solution of the evolution equation (4.8) maps back to a solution of the original equations for the semiconductor laser (2.19).

Recalling remark 3.1 and theorem 4.1, we introduce the functions

$$\varphi=\varphi_{\mathrm{D}}+\mathscr{L}\left(C,-\varphi_n+\varphi_{\mathrm{D}}-E_{\mathrm{c}},\varphi_p-\varphi_{\mathrm{D}}+E_{\mathrm{v}}\right),$$
$$(\lambda,\Phi)=\left(\lambda(\boldsymbol{\varphi}),\Phi(\boldsymbol{\varphi})\right),\qquad\text{with }\boldsymbol{\varphi}=\left(\varphi,\varphi_n,\varphi_p\right).$$

We wish to prove that $\boldsymbol{\varphi}=(\varphi,\varphi_n,\varphi_p)$, (λ,Φ) is a solution of the van Roosbroeck–Helmholtz model (2.19), in the sense of definition 3.5. Conditions (3.23) and (3.24) are identical to (4.12), while (3.21) follows from the properties of the operator $\mathscr{L}$. The remaining properties (3.27), (3.28) and (3.29) can be proven as in [25, Theorem 7.4]. Finally, the Helmholtz equation (3.30) with constraint (3.31) is satisfied by construction. □

6. Conclusion

We have established the local-in-time existence and uniqueness of weak solutions to a coupled van Roosbroeck–Helmholtz system for semiconductor lasers under physically motivated assumptions. Central

to our analysis is the derivation of local Lipschitz bounds for the stimulated recombination operator, which enable the application of the abstract theory of quasi-linear parabolic equations in Banach spaces. This result provides the first rigorous mathematical foundation for a drift-diffusion-Helmholtz model incorporating stimulated emission and thus offers a solid basis for further analytical and numerical investigations of semiconductor laser dynamics. We point out that it is possible to incorporate additional modelling features such as oxide layers, spatially varying Robin or Neumann boundary conditions, as well as time-dependent boundary data and doping profiles. A solid and physically meaningful derivation of the present model will be the topic of future research.

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Data availability statement

No new data were created or analysed in this study.

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References

[1] Agrawal G P and Dutta N K 2013 *Semiconductor Lasers* (Springer)
[2] Ali G, Farrell P and Rotundo N 2025 Forward lateral photovoltage scanning problem: Perturbation approach and existence-uniqueness analysis *J. Math. Anal. Appl.* **541** 128725
[3] Alí G and Rotundo N 2010 An existence result for elliptic partial differential–algebraic equations arising in semiconductor modeling *Nonlinear Anal. Theory Method Appl.* **72** 4666–81
[4] Alí G, Bartel A and Rotundo N 2015 Index-2 elliptic partial differential-algebraic models for circuits and devices *J. Math. Anal. Appl.* **423** 1348–69
[5] Baccarani G, Odeh F, Gnudi A and Ventura D 1994 A critical review of the fundamental semiconductor equations *Semiconductors*, Volume 59 of *IMA Volumes in Mathematics and its Applications* (Springer) pp 19–32
[6] Bandelow U 2009 Advanced mathematical modeling of nonlinear optoelectronic and photonic components. *Habiliation thesis*, Humboldt-Universität zu
[7] Bandelow U, Gajewski H and Hünlich R 2005 *Fabry-Perot Lasers: Thermodynamics-Based Modeling* (Springer) pp 63–85
[8] Brunk M and Jüngel A 2008 Simulation of thermal effects in optoelectronic devices using coupled energy-transport and circuit models *Math. Models Methods Appl. Sci.* **18** 2125–50
[9] Chuang S L 2012 *Physics of Photonic Devices* (Wiley)
[10] Evans L C 1998 *Partial Differential Equations volume 19 of Graduate Studies in Mathematics* vol 19 (American Mathematical Society)
[11] Farrell P, Kayser S and Rotundo N 2021 Modeling and simulation of the lateral photovoltage scanning method *Comput. Math. Appl.* **102** 248–60
[12] Farrell P, Rotundo N, Doan D H, Kantner M, Fuhrmann J and Koprucki T 2017 Mathematical methods: Drift-diffusion models *Handbook of Optoelectronic Device Modeling and Simulation* vol 2, ed J Piprek (CRC Press) chapter 50, pp 733–71
[13] Gajewski H 1985 On existence, uniqueness and asymptotic behavior of solutions of the basic equations for carrier transport in semiconductors *Zeitschrift für Angewandte Mathematik und Mechanik* **65** T459–61
[14] Gajewski H and Gröger K 1986 On the basic equations for carrier transport in semiconductors *J. Math. Anal. Appl.* **113** 12–35
[15] Gajewski H and Gröger K 1989 Semiconductor equations for variable mobilities based on boltzmann statistics or fermi–dirac statistics *Mathematische Nachrichten* **140** 7–36
[16] Gajewski H, Liero M, Nürnberg R and Stephan H 2016 WIAS-TeSCA–two-dimensional semiconductor analysis package (Weierstraß-Institut für Angewandte Analysis und Stochastik) (https://doi.org/10.20347/WIAS.TECHREPORT.14)
[17] Gamba I M 1997 Stationary transonic solutions of a one-dimensional hydrodynamic model for semiconductors *Commun. PDE* **22** 635–55
[18] Cercignani C, Gamba I M and Levermore C D 1997 High field approximations to a Boltzmann-Poisson system boundary conditions in a semiconductor *Applied Math Letters* **10** 111–8

[19] Gilbarg D and Trudinger N S 2001 *Elliptic Partial Differential Equations of Second Order Classics in Mathematics.* (Springer) Reprint of the 1998 edn
[20] Gröger K 1989 A $W^{1,p}$–estimate for solutions to mixed boundary value problems for second order elliptic differential equations *Math. Ann.* **283** 679–87
[21] Gröger K and Rehberg J 1989 Resolvent estimates in $W^{1,p}$ for second order elliptic differential operators in case of mixed boundary conditions *Math. Ann.* **285** 105–13
[22] Gummel H K 1964 A self-consistent iterative scheme for one-dimensional steady state transistor calculations *IEEE Trans. Electron Devices* **11** 455–65
[23] Hess O and Kuhn T 1996 Spatio-temporal dynamics of semiconductor lasers: theory, modelling and analysis *Prog. Quantum Electron.* **20** 85–179
[24] Jerome J W 1996 *Analysis of Charge Transport: A Mathematical Study of Semiconductor Devices* (Springer)
[25] Kaiser H-C, Neidhard H and Rehberg J 2009 Classical solutions of drift–diffusion equations for semiconductor devices: the two-dimensional case *Nonlinear Anal. Theory Methods Appl.* **71** 1584–605
[26] Kaiser H-C, Neidhardt H and Rehberg J 2006 Classical solutions of quasilinear parabolic systems on two dimensional domains *Nonlinear Anal. Theory Methods Appl.* **13** 287–310
[27] Kantner M, Höhne T, Koprucki T, Burger S, Wünsche H-J, Schmidt F, Mielke A and Bandelow U 2020 Multi-dimensional modeling and simulation of semiconductor nanophotonic devices *Semiconductor Nanophotonics: Materials, Models and Devices* (Springer) pp 241–83
[28] Kato T 1995 Perturbation theory for linear operators *Classics in Mathematics* 2nd edn (Springer)
[29] Markowich P A 1986 *The Stationary Semiconductor Device Equations* (Springer)
[30] Markowich P A, Ringhofer C and Schmeiser C 1990 *Semiconductor Equations* (Springer)
[31] Mielke A 2011 A gradient structure for reaction–diffusion systems and for energy-drift-diffusion systems *Nonlinearity* **24** 1329
[32] Mock M S 1983 *Analysis of Mathematical Models of Semiconductor Devices* (Boole Press)
[33] Numai T 2015 *Fundamentals of Semiconductor Lasers (Springer Series in Optical Sciences)* vol 99 (Springer) pp 89–186
[34] Ohtoshi T, Yamaguchi K, Nagaoka C, Uda T, Murayama Y and Chinone N 1987 A two-dimensional device simulator of semiconductor lasers *Solid-State Electron.* **30** 627–38
[35] Piprek J 2013 *Semiconductor Optoelectronic Devices: Introduction to Physics and Simulation* (Elsevier)
[36] Pomplun J, Burger S, Schmidt F, Schliwa A, Bimberg D, Pietrzak A, Wenzel H and Erbert G 2010 Finite element simulation of the optical modes of semiconductor lasers *Phys. Status Solidi b* **247** 846–53
[37] Selberherr S 1984 *Analysis and Simulation of Semiconductor Devices* (Springer)
[38] Van Roosbroeck W 1950 Theory of the flow of electrons and holes in germanium and other semiconductors *Bell Syst. Tech. J.* **29** 560–607
[39] Wenzel H 2013 Basic aspects of high-power semiconductor laser simulation *IEEE J. Sel. Top. Quantum Electron.* **19** 1–13
[40] Wenzel H and Zeghuzi A 2017 High-power lasers *Handbook of Optoelectronic Device Modeling and Simulation* (CRC Press) pp 15–58
[41] Wünsche H J 1991 Modellierung optoelektronischer Bauelemente. *Tagung NUMSIM'91*