Ca' Foscari University of Venice

Department of Economics

Working Paper

Federico Maglione

Multifractality in Finance: A deep understanding and review of Mandelbrot's MMAR

ISSN: 1827-3580
No. 05/WP/2015

Working Papers
Department of Economics
Ca' Foscari University of Venice
No. 05/WP/2015
ISSN 1827-3580

Multifractality in Finance: A deep understanding and review of Mandelbrot's MMAR

Federico Maglione
University of Venice at Ca' Foscari

Abstract

Benoît Mandelbrot, the father of Fractal Geometry, developed a multifractal model for describing price changes. Despite the commonly used models, such as the Brownian motion, the Mutifractal Model of Asset Return (MMAR) takes into account scale-consistency, long-range dependence and heavy tails, thus having a great flexibility in depicting the real-market peculiarities. In section 2 a review of the mathematics involved into multifractals is presented; Section 3 is addresses to the extension of multifractality towards stochastic processes, introducing the crucial concept of local Hölder exponent of a function. Finally, Section 4 deeply analyzes the mathematical properties of the scaling function which drives the "wildeness" of the process. The proof of Theorem 4.4 is unpublished and the generalization of Mandelbrot's results, which highlights a possible alternative motivation for the presence of heavy tails and a connection with the Extreme Value Theory. Section 5 is devoted to the analysis of the connection between the scaling function, Multifractal Formalism and Large Deviation Theory, suggesting possible ways in order to estimate the quantities involved. Finally in Section 6 the MMAR is presented, listing all the theorems that make it a suitable model for financial modelling.

Keywords

Multifractal processes, scaling function, multifractal spectrum, long-range dependence, heavy tails, MMAR, Extreme Value Theory.

JEL Codes

C10, C16, C65, G10, G17

Address for correspondence:
Federico Maglione
Via Trieste 19
35121 Padova (PD)
Phone: (++39) 347 3525005
e-mail: federico.mgln@gmail.com

1 Introduction

The development of this work is based on the ideas of Mandelbrot about market misbehavior. He, together with Laurent Calvet and Adlai Fisher, developed a model different from the ones prevailingly used on Classical Finance. It is based on fractals and multifractal measures. The model firstly appeared in 1996 (see [5], [6] and [7]). In 2003 it was developed by Mandelbrot himself in [17] and then implemented by the other two authors after Mandelbrot's death in [8].

The model, namely a **Multifractal Model of Asset Returns** (MMAR), is anchored on the assertion that market time is *relative* in a certain sense. Financial markets would work according to an intrinsic "trading time", distinct from the linear physical time. That time accelerates the clock in high-volatility periods and slows down during those moments of placidity. In mathematical terms, we can write an equation showing the relationship between the two time structures and use it in order to generate the same irregularities of real financial prices. That phenomenon highlights a very important involvement already known by the most part of the financial practitioners. They often refers to a "fast" market and "slow" market on the strength of their volatility's perception in that moment.

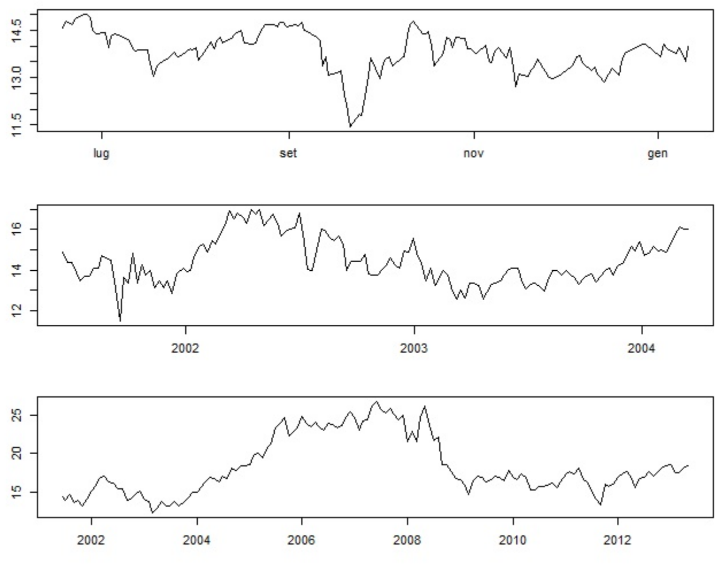

Figure 1: Which are the daily/weekly/mountly records?

Analogously, following a popular opinion, financial prices' patterns have all the same aspect: Without a caption about data's time-scale, none can state with certainty if the graph is taken by the last ten minutes, ten days, ten weeks and so long. Hence markets are *scale-invariant*. This quality defines the charts as fractal[1] curves and many powerful tools of mathematical analysis become available.

With regard to financial markets, scale-invariance means that the price evolution process can be described in terms of minutely, hourly, or daily recorded data, but the main property of the process, like the distribution of the price variations, will always be of the same general form, with only a scale parameter that needs to be adjusted for a change of the time scale.

However, before going on, we should wonder if we really need a further model to describe price changes. Thus, other models such as Brownian motions, fractional Brownian motions or Lévy-stable motions[2]. able to describe the behaviour of the financial markets? Unfortunately, they are not. Each one has got desirable features (scale-invariance, long-range dependence, heavy tails, discontinuities, volatility clustering), but no one possesses all.

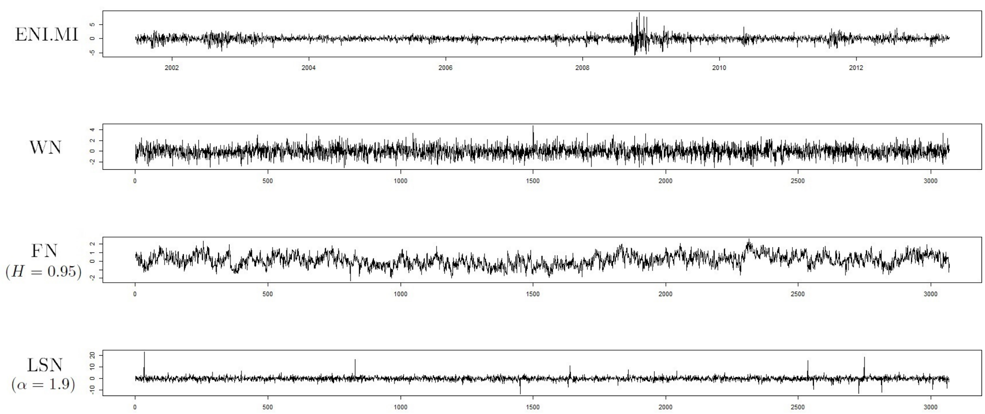

Figure 2: Who tells the truth?

[1]When we refer to a set F as a fractal, we will typically observe:

- F has a *fine* structure, that is it owns details on arbitrarily small scale;
- F is too *irregular* to be described in traditional geometrical language, both locally and globally;
- Often F has some form of *self-similarity*, not only exact but also approximate or statistical;
- Usually, its *fractal dimension* (defined in some way) is grater than its topological dimension;
- In most cases, a fractal is defined in a very simple way, usually *recursively*.

[2]From now on these processes will be identified with BM, FBM and LSM. If they are standard, we will put an 'S' before, e.g. SBM.

The MMAR aims at integrating all those features a financial time-series exhibits. As a matter of fact we will show that, starting from a stochastic process having long-range dependence (like a FBM with $\frac{1}{2} < H < 1$) and using a *multifractal measure*[3] in order to create a compounded process, we will be able to get a new process which has both a long-range dependence and heavy tails.

2 Binomial, multinomial and canonical measures

In order to understand the MMAR, which is based on multifractality, few preliminary notions such as *binomial* and *multinomial measures* are required. In the following, we will introduce the simplest multifractal, the (*Bernoulli*) *binomial measure* on the compact interval $[0, 1] \subset \mathbb{R}$. Furthermore, we will construct further multifractal measures which derive from the former.

The recursive construction of the binomial measure involves an initiator and a generator. The initiator is the interval $[0, 1]$ itself on which a unit of (probability) mass is uniformly spread. This interval will recursively split into halves, leading to, at the k-th stage, dyadic intervals of length 2^{-k}. The generator consists in a single parameter $0 < u_0 < 1$ and $u_0 \neq \frac{1}{2}$, named *multiplier*, which at each stage is spread over the halves of every dyadic interval, with unequal deterministic proportions.

Let u_0 be a multiplier and be u_1 its ones' complement. At stage $k = 0$, we start the construction with the uniform probability measure on $[0, 1]$, that is

$$f_0(t) = \begin{cases} \theta_0\left([0,1]\right) = 1 & \text{if } t \in [0,1] \\ 0 & \text{if } t \notin [0,1] \end{cases} .$$

At the step $k = 1$,[4] the measure θ_1 uniformly spread mass equal to u_0 on the subinterval

[3]Given a mass distribution $\theta : \mathbb{R}^n \to \mathbb{R}$ and an open n-ball $B(\mathbf{x}, r)$ of radius $r \in \mathbb{R}$, $\mathbf{x} \in \mathbb{R}^n$, a multifractal measure is an entity such that

$$\theta\big[B(\mathbf{x}, r)\big] \approx r^{\alpha} ,$$

with $\alpha \in \mathbb{R}$. Hence this definition involves a power-law, creating a power-law distribution (like Pareto's and Lévy-stable ones).

[4]We omit the case $t \notin [0, 1]$, since we are introducing a measure on $[0, 1]$.

$\left[0, \frac{1}{2}\right]$ and mass equal to u_1 on $\left[\frac{1}{2}, 1\right]$, that is

$$f_1(t) = \begin{cases} \theta_1\left(\left[0, \frac{1}{2}\right]\right) = u_0 & \text{if } t \in \left[0, \frac{1}{2}\right) \\ \theta_1\left(\left[\frac{1}{2}, 1\right]\right) = u_1 & \text{if } t \in \left[\frac{1}{2}, 1\right] \end{cases}.$$

Here, is trivial to see that the mass is preserved. In fact

$$\theta_1\left(\left[0, \frac{1}{2}\right]\right) + \theta_1\left(\left[\frac{1}{2}, 1\right]\right) = u_0 + u_1 = 1\,.$$

In step $k = 2$, the set $\left[0, \frac{1}{2}\right]$ is split into two subintervals, $\left[0, \frac{1}{4}\right]$ and $\left[\frac{1}{4}, \frac{1}{2}\right]$, which respectively receive a percentage u_0 and u_1 of the total mass $\theta_1\left(\left[0, \frac{1}{2}\right]\right)$. Applying the same procedure to the dyadic set $\left[\frac{1}{2}, 1\right]$, we obtain

$$f_2(t) = \begin{cases} \theta_2\left(\left[0, \frac{1}{4}\right]\right) = u_0 \cdot u_0 = u_0^2 & \text{if } t \in \left[0, \frac{1}{4}\right) \\ \theta_2\left(\left[\frac{1}{4}, \frac{1}{2}\right]\right) = u_0 \cdot u_1 = u_0 \cdot u_1 & \text{if } t \in \left[\frac{1}{4}, \frac{1}{2}\right) \\ \theta_2\left(\left[\frac{1}{2}, \frac{3}{4}\right]\right) = u_1 \cdot u_0 = u_0 \cdot u_1 & \text{if } t \in \left[\frac{1}{2}, \frac{3}{4}\right) \\ \theta_2\left(\left[\frac{3}{4}, 1\right]\right) = u_1 \cdot u_1 = u_1^2 & \text{if } t \in \left[\frac{3}{4}, 1\right] \end{cases}.$$

As we can see the total mass is preserved since

$$\sum_{i=0}^{2^2-1} \theta_2\left(\left[\frac{i}{2^2}, \frac{i+1}{2^2}\right]\right) = u_0^2 + 2 \cdot u_0 \cdot u_1 + u_1^2 = (u_0 + u_1)^2 = 1\,.$$

We want to ascertain that the procedure works even for $f_3(t)$.

$$f_3(t) = \begin{cases} \theta_3\left(\left[0, \frac{1}{8}\right]\right) = u_0 \cdot u_0 \cdot u_0 = u_0^3 & \text{if } t \in \left[0, \frac{1}{8}\right) \\ \theta_3\left(\left[\frac{1}{8}, \frac{1}{4}\right]\right) = u_0 \cdot u_0 \cdot u_1 = u_0^2 \cdot u_1 & \text{if } t \in \left[\frac{1}{8}, \frac{1}{4}\right) \\ \theta_3\left(\left[\frac{1}{4}, \frac{3}{8}\right]\right) = u_0 \cdot u_1 \cdot u_0 = u_0^2 \cdot u_1 & \text{if } t \in \left[\frac{1}{4}, \frac{3}{8}\right) \\ \theta_3\left(\left[\frac{3}{8}, \frac{1}{2}\right]\right) = u_0 \cdot u_1 \cdot u_1 = u_0 \cdot u_1^2 & \text{if } t \in \left[\frac{3}{8}, \frac{1}{2}\right) \\ \theta_3\left(\left[\frac{1}{2}, \frac{5}{8}\right]\right) = u_1 \cdot u_0 \cdot u_0 = u_0^2 \cdot u_1 & \text{if } t \in \left[\frac{1}{2}, \frac{5}{8}\right) \\ \theta_3\left(\left[\frac{5}{8}, \frac{3}{4}\right]\right) = u_1 \cdot u_0 \cdot u_1 = u_0 \cdot u_1^2 & \text{if } t \in \left[\frac{5}{8}, \frac{3}{4}\right) \\ \theta_3\left(\left[\frac{3}{4}, \frac{7}{8}\right]\right) = u_1 \cdot u_1 \cdot u_0 = u_0 \cdot u_1^2 & \text{if } t \in \left[\frac{3}{4}, \frac{7}{8}\right) \\ \theta_3\left(\left[\frac{7}{8}, 1\right]\right) = u_1 \cdot u_1 \cdot u_1 = u_1^3 & \text{if } t \in \left[\frac{7}{8}, 1\right] \end{cases},$$

and total mass still preserves

$$\sum_{i=0}^{2^3-1} \theta_3\left(\left[\frac{i}{2^3}, \frac{i+1}{2^3}\right]\right) = u_0^3 + 3 \cdot u_0^2 \cdot u_1 + 3 \cdot u_0 \cdot u_1^2 + u_1^3 = (u_0 + u_1)^3 = 1\,.$$

This procedure can generate an infinite sequence of measures.

At step $k+1$, we assume that the measure θ_k has been defined. To construct θ_{k+1}, consider a dyadic interval $\left[t, t+2^{-k}\right]$, where t is the *dyadic number* of the form:

$$t = (0.\eta_1\eta_2\ldots\eta_k)_2 = \left(\sum_{i=1}^{k} \eta_i \cdot 2^{-k}\right)_{10}$$

for a finite k and $\eta_1, \eta_2, \ldots, \eta_k \in \{0, 1\}$ (hence we are using the counting base $b = 2$). Then we uniformly spread a fraction u_0 and u_1 of the mass $\theta_k\left(\left[t, t+2^{-k}\right]\right)$ on the subintervals $\left[t, t+2^{-(k+1)}\right]$ and $\left[t+2^{-(k+1)}, t+2^{-k}\right]$. The repetition of this scheme to all the subintervals define the measure θ_{k+1}.

Let φ_0 and φ_1 denote the relative frequencies of 0's and 1's (that is $\varphi_1 = 1 - \varphi_0$) in the finite binary development $t = (0.\eta_1\eta_2\ldots\eta_k)_2$. The so-called *pre-binomial measure* in the dyadic interval $\left[t, t+2^{-k}\right]$, takes the value[5]

$$\theta_k\left(\left[t, t+2^{-k}\right]\right) = u_0^{k\cdot\varphi_0} \cdot u_1^{k\cdot\varphi_1} \,. \tag{1}$$

Because of the conservation of the mass at each stage, we can write

$$\sum_{i=0}^{2^k-1} \theta_k\left(\left[\frac{i}{2^k}, \frac{i+1}{2^k}\right]\right) = (u_0 + u_1)^k = 1 \,.$$

The iteration of the procedure generates an infinite sequence of random measure $\{\theta_k\}$ that weakly converges to the *binomial measure* θ, that is

$$\theta_k \xrightarrow{d} \theta \,.$$

In fact, setting the length $\Delta t = \left(t+2^{-k}\right) - t = 2^{-k}$, pre-binomial measure redistributes mass among the dyadic intervals, but the total mass is always preserved at each stage since $u_0 + u_1 = 1$.

Note that the binomial measure has important features common to many multifractal (measures): It is continuous but also a singular probability measure; it thus has no den-

[5] Take as an instance $\theta_3\left(\left[\frac{1}{8}, \frac{1}{4}\right]\right) = u_0^2 \cdot u_1$. Since $\frac{1}{8} = 0.125_{10}$ in base 10 (usual decimal representation), but in base 2 is equal to $t = 0.001_2$, we can see that the relative frequency of 0's is $\varphi_0 = \frac{2}{3}$ and that the relative frequency of 1's is $\varphi_1 = \frac{1}{3}$, using (1) we have

$$\theta_3\left(\left[\frac{1}{8}, \frac{1}{4}\right]\right) = u_0^{3\cdot\frac{2}{3}} \cdot u_1^{3\cdot\frac{1}{3}} = u_0^2 \cdot u_1 \,,$$

as expected.

sity[6].

This construction can receive several extensions. At each stage the interval can be split not in two but in $b \in \mathbb{N} : b > 2$ intervals of equal size. Subintervals, indexed from left to right by j $(0 \leq j \leq b-1)$, receive fraction of the total mass equal to $u_0, u_1, \ldots, u_{b-1}$. For conserving mass the multipliers have to be such that

$$\sum_{j=0}^{b-1} u_j = 1\,.$$

Thus the *multinomial measure* on the b-adic interval $\left[t, t+b^{-k}\right]$ follows the conservation rule

$$\sum_{i=0}^{b^k-1} \theta_k \left(\left[\frac{i}{b^k}, \frac{i+1}{b^k} \right] \right) = \left(\sum_{j=0}^{b-1} u_j \right)^k = 1\,.$$

Here, the measure is computed as

$$\theta_k\,(\Delta t) = \prod_{j=0}^{b-1} u_j^{k \cdot \varphi_j}\,. \tag{2}$$

where $\Delta t = b^{-k}$ and t is the b-adic number

$$t = (0.\eta_1 \eta_2 \ldots \eta_k)_b = \left(\sum_{i=1}^{k} \eta_i \cdot b^{-k} \right)_{10}$$

for a finite k and $\eta_1, \eta_2, \ldots, \eta_k \in \{0, 1, \ldots, b-1\}$, and φ_j are the relative frequencies of the digits of the representation in base b.

Moreover another discrete extension is easy to get, making the allocation of the mass random. Hence the multiplier of each subinterval is a sequence of *independent and identically distributed* (positive) random variables $\{U_j\}$. As for the previous cases, we need to assume that the mass is preserved *at each stage* of the construction, that is

$$\sum_{j=0}^{b-1} U_j = 1\,,$$

[6] As a matter of fact, since $u_0, u_1, \varphi_0, \varphi_1 \in (0,1)$, the limit

$$\lim_{k \to \infty} \theta_k \left(\left[t, t+2^{-k}\right] \right) = \lim_{k \to \infty} \left(u_0^{\varphi_0} \cdot u_1^{\varphi_1}\right)^k = 0$$

being $0 < u_0^{\varphi_0} \cdot u_1^{\varphi_1} < 1$.

leading to the obvious fact $0 \leq U_j \leq 1$. Taking the expectation of the sum, we get an expression for the expected value of the single random variable, that is[7]

$$\mathbb{E}(U) = \frac{1}{b}\,.$$

for all j. The resulting measure is called *microcanonical measure*. Given a date $t = (0.\eta_1\eta_2\ldots\eta_k)_b$ and a length $\Delta t = b^{-k}$, the measure of the b-adic cell $\left[t, t+b^{-k}\right]$ satisfies

$$\theta_k(\Delta t) = U_{\eta_1}\cdot U_{\eta_1\eta_2}\cdot\ldots\cdot U_{\eta_1\eta_2\ldots\eta_k} \tag{3}$$

where $\eta_1\eta_2\ldots\eta_k$ is one of the elements (it is a b-adic number) of the ordered selection with repetition[8] made with b digits. Because of its property, it follows

$$\theta_k(\Delta t)^q = \left(U_{\eta_1}\cdot U_{\eta_1\eta_2}\cdot\ldots\cdot U_{\eta_1\eta_2\ldots\eta_k}\right)^q = U^q_{\eta_1}\cdot U^q_{\eta_1\eta_2}\cdot\ldots\cdot U^q_{\eta_1\eta_2\ldots\eta_k}$$

for all $q \geq 0$. Taking the expectation of this expression and considering the fact that the multipliers are independent (and hence their q-th power are independent as well), we get

$$\begin{aligned}\mathbb{E}\Big[\theta_k(\Delta t)^q\Big] &= \mathbb{E}\left(U^q_{\eta_1}\cdot U^q_{\eta_1\eta_2}\cdot\ldots\cdot U^q_{\eta_1\eta_2\ldots\eta_k}\right) = \\ &= \mathbb{E}\left(U^q_{\eta_1}\right)\cdot\mathbb{E}\left(U^q_{\eta_1\eta_2}\right)\cdot\ldots\cdot\mathbb{E}\left(U^q_{\eta_1\eta_2\ldots\eta_k}\right)\,.\end{aligned}$$

Since, in addiction, the random variables are identically distributed, the previous expression becomes the following scaling rule

$$\begin{aligned}\mathbb{E}\Big[\theta_k(\Delta t)^q\Big] &= \mathbb{E}\left(U^q_{\eta_1}\right)\cdot\mathbb{E}\left(U^q_{\eta_1\eta_2}\right)\cdot\ldots\cdot\mathbb{E}\left(U^q_{\eta_1\eta_2\ldots\eta_k}\right) = \\ &= \underbrace{\mathbb{E}(U^q)\cdot\mathbb{E}(U^q)\cdot\ldots\cdot\mathbb{E}(U^q)}_{k\text{ times}} = \left[\mathbb{E}(U^q)\right]^k\,.\end{aligned}$$

Setting $\tau(q) \equiv -\log_b\left[\mathbb{E}(U^q)\right] - 1$ (which is named *scaling function*), the expression can be written as[9]

$$\mathbb{E}\Big[\theta_k(\Delta t)^q\Big] = \Delta t^{\tau(q)+1} \tag{4}$$

[7]Since the unit mass is preserved, we have $\mathbb{E}\left(\sum_{j=0}^{b-1} U_j\right) = \mathbb{E}(1)$, and hence

$$\mathbb{E}\left(\sum_{j=0}^{b-1} U_j\right) = \mathbb{E}(U_0) + \mathbb{E}(U_1) + \ldots + \mathbb{E}(U_{b-1}) = \underbrace{\mathbb{E}(U) + \mathbb{E}(U) + \ldots + \mathbb{E}(U)}_{b\text{ times}} = 1\,.$$

[8]Hence the set $\{\eta_1, \eta_2, \ldots, \eta_k\}$ can originate $D'_{b,k} = b^k$ ordered selections with repetition.

[9]In fact, since $\Delta t = b^{-k}$, we have

$$\begin{aligned}\mathbb{E}(U^q)^k &= \left(\frac{1}{b}\right)^{\log_{1/b}\left[\mathbb{E}(U^q)^k\right]} = \left(\frac{1}{b}\right)^{-k\cdot\log_b\left[\mathbb{E}(U^q)\right]} = \left(b^{-1}\right)^{-k\cdot\log_b\left[\mathbb{E}(U^q)\right]} = \\ &= \left(b^{-k}\right)^{-1\cdot\log_b\left[\mathbb{E}(U^q)\right]} = \Delta t^{-\log_b\left[\mathbb{E}(U^q)\right]} = \Delta t^{-\log_b\left[\mathbb{E}(U^q)\right]-1+1} = \Delta t^{\tau(q)+1}\,.\end{aligned}$$

which is the typical behaviour of a multifractal measure[10].

Finally we are able to introduce the last multifractal measure, which will be the one involved in the Multifractal Model of Asset Returns. If, given a sequence of independently and identically distributed (positive) random variables, each iteration conserves probability mass only "on average" in the sense that

$$\mathbb{E}\left(\sum_{j=0}^{b-1} U_j\right) = 1 \, ,$$

we obtain a less restrictive case on multipliers, leading just to the positivity of them $U_j \geq 0$[11].

The corresponding measure is then called *canonical measure* and its total mass, denoted as Υ, is generally random. As a matter of fact, if we consider such a measure that has started with mass 1 in $[0,1]$, and has continued over infinite many stages, because of the lack of an exact conservation, the ultimate mass is not identical to 1, but is a random variable Υ.

An alternative way to see Υ is to add the masses in the b subcells of length $\frac{1}{b}$. In the first stage, the j-th interval is given the mass U_j. Ultimately, it contains the mass $\Upsilon_j \cdot U_j$, the quantities U_j and Υ_j being statistically independent. Hence, the total sum of the partial masses can be written as

$$\sum_{j=0}^{b-1} \Upsilon_j \cdot U_j \overset{d}{\sim} \Upsilon \, ,$$

meaning they have the same distribution. Thus Υ is the fixed point of the operation of randomly weighted averaging using as weight the random quantities Υ_j (which are inevitably identically distributed).

[10]See the analogy with the definition of multifractal that is, for an open ball with centre $\mathbf{x} \in \mathbb{R}^n$ and radius r,

$$\theta[B(\mathbf{x}, r)] \approx r^{\alpha} \, .$$

Of course, since the microcanonical measure is a random measure, the relationship is valid for its expected value.

[11]Note that, since $\sum_{j=0}^{b-1} U_j = 1 \Rightarrow \mathbb{E}\left(\sum_{j=0}^{b-1} U_j\right) = 1$ (but the converse is not true), we have that the expectation of the random variables are all equal to

$$\mathbb{E}(U) = \frac{1}{b}$$

even in this case.

Thus, given a time $t = 0.\eta_1\eta_2\ldots\eta_k$, at the k-th stage, the canonical measure of a b-adic interval surely generate the same effect as the microcanonical measure, that is

$$\theta_k(\Delta t) = U_{\eta_1} \cdot U_{\eta_1\eta_2} \cdot \ldots \cdot U_{\eta_1\eta_2\ldots\eta_k}\,.$$

However, in contrast to the previous case, that is not all: Each stage is also subjected to the same process as has been for $[0,1]$. Therefore, the measure does *not* reduce to (3), but instead takes the form:

$$\theta_k(\Delta t) = \Upsilon_{\eta_1\eta_2\ldots\eta_k} \cdot \Big(U_{\eta_1} \cdot U_{\eta_1\eta_2} \cdot \ldots \cdot U_{\eta_1\eta_2\ldots\eta_k}\Big) \tag{5}$$

Since $\Upsilon_{\eta_1\eta_2\ldots\eta_k} \overset{d}{\sim} \Upsilon$ and it is independent from U_j for all j, similarly as for the microcanonical measure, we can write

$$\mathbb{E}\Big[\theta_k(\Delta t)^q\Big] = \mathbb{E}(\Upsilon^q) \cdot [\mathbb{E}(U^q)]^k\,,$$

where the new moment prefactor $\mathbb{E}(\Upsilon^q)$ reflects high frequency effects due to scales lower than b^{-k}.

Setting $c(q) \equiv \mathbb{E}(\Upsilon^q)$, we finally get the sought expression

$$\mathbb{E}\Big[\theta_k(\Delta t)^q\Big] = c(q) \cdot \Delta t^{\tau(q)+1}\,. \tag{6}$$

3 Multifractal processes and local Hölder exponents

Now the concept of multifractality can be extended to stochastic process. Because of the previous introduction on multifractal random measures, we find convenient defining *multifractal processes* in terms of their moments. Nevertheless, we have to remark that dealing with measures rather than stochastic process may be similar, but it is not the same thing. In the following, we will discuss about those discrepancies.

Definition 3.1. *Given a filtered probability space* $(\Omega, \mathcal{F}, \mathbf{P})$*, every real sto-chastic process* $\{Y(t)\}_{t\in[0,+\infty)}$ *defined on it is called a* multifractal process*, if it has stationary increments and satisfies the following properties:*

(a) $Y(0) = 0$ *almost surely;*

(b) *The expectation of its absolute increments raised to the q-th power are such that*

$$\mathbb{E}\big[|Y(t) - Y(s)|^q\big] = c(q) \cdot |t-s|^{\tau(q)+1}$$

where $q \in Q \subseteq \mathbb{R}$ *and* $c, \tau : Q \to \mathbb{R}$.

Thank to the definition, setting $s = 0$, we get

$$\mathbb{E}\left[|Y(t)|^q\right] = c(q) \cdot t^{\tau(q)+1} . \tag{7}$$

Such a definition of multifractal process extend the one of self-affine process whose BM and FBM belong to[12].

In order to study further properties of $\tau(q)$ and to be allowed to use some results of multifractal formalism introduced in the previous chapter, we need the definition of *local Hölder exponent* of a function. This is connected with Hölder function, but it characterizes the smoothness of a function at a *given point*.

Definition 3.2. *Let $g : \mathbb{R} \supseteq A \to \mathbb{R}$ be a function defined on a neighborhood of a given point t_0. The number*

$$\alpha(t_0) := \sup\left\{\beta > 0 : |g(t_0 + h) - g(t_0)| = O\left(|h|^\beta\right)\right\} \tag{8}$$

as $h \to 0$, is called the local Hölder exponent *of g at t_0.*

We note that $\alpha(t_0)$ is non-negative if and only if the function g is bounded around t_0. In this work, we will consider only this case. Local Hölder exponent is a notion that can be well applied to functions and measures, deterministic or stochastic, with some adjustment. It can be proven that BM and FBM have local Hölder exponent equal to $\frac{1}{2}$ and H respectively[13]). As a matter of fact, continuous stochastic process are characterized by a *unique* Hölder exponent. Differently, in the following we will examine the case when a continuum of Hölder exponent is allowed.

From Definition 3.2 it easy to see that, for a continuous function $g(t)$, the Hölder exponent at time $t = t_0$ can be computed as

$$\alpha(t_0) = \limsup_{h \to 0} \frac{\log|g(t_0 + h) - g(t_0)|}{\log|h|} .$$

This expression suggests a method for estimating the probability that a point, randomly chosen on the interval $[0, 1]$, will have a given Hölder exponent. An essential simplification

[12] A *self-affine process* is a process such that, given $a \in \mathbb{R}$

$$\{Y(a \cdot t)\}_{t \geq 0} \overset{d}{\sim} \{a^H \cdot Y(t)\}_{t \geq 0}$$

with $H \in \mathbb{R} : 0 < H < 1$.

[13] See [9].

for both, analytical and empirical study, is to replace the previous continuous limit by a discrete one.

Given the set $[0,1] \subset \mathbb{R}$, we iteratively subdivide the interval in b^k equal size pieces, k denoting the stage in the sequence of subdivision. At each stage, at point $t = t_i \in [0,1]$, we compute the finite quantities $\left|g\left(t_i + b^{-k}\right) - g\left(t_i\right)\right|$ for each b^k subdivision. We define *coarse Hölder exponent*, the following quantity

$$\alpha_k\left(t_i\right) := \frac{\log\left|g\left(t_i + b^{-k}\right) - g\left(t_i\right)\right|}{\log\left(b^{-k}\right)},$$

and hence the Hölder exponent at point t_i is given by

$$\alpha\left(t_i\right) := \liminf_{k\to\infty} \alpha_k\left(t_i\right).$$

From the definition we see that, varying the length of the interval $\Delta t = b^{-k}$, we can find different values of $\alpha\left(t_i\right)$. Partitioning the range of them into small non-overlapping intervals $(\alpha_j, \alpha_j + \Delta\alpha]$, and denoted by $N_k(\alpha_j, \Delta\alpha)$ the *number* of coarse Hölder exponents contained in each interval $(\alpha_j, \alpha_j + \Delta\alpha]$, as $k \to \infty$, the ratio $\frac{N_k(\alpha,\Delta\alpha)}{b^k}$ converges to the probability that a randomly selected point t_i has local Hölder exponent equal to $\alpha\left(t_i\right)$.

Even if this approach of representing the distribution of different Hölder exponents is correct, it will fail in a multifractal contest, since it is not *able* to distinguish between multifractal and unifractal processes. As a matter of fact, multifractals, allowing different values of α, typically have that a single Hölder exponent which predominates (called α^*), in the sense that the set of points $T_{\alpha^*} \subset [0,1]$ with exponent α^* "usurps" all of the *Lebesgue measure*. Differently, most of a *multifractal measure* concentrates on a set of instants with Hölder exponent differen from α^*. In order to distinguish between the two cases we need the following function.

Definition 3.3. *Given a function $g : [0,1] \supseteq A \to \mathbb{R}$, using the same iterative procedure used to compute coarse Hölder exponents and using the same notation, we define* multifractal spectrum[14]

$$f(\alpha) := \lim_{\Delta\alpha\to 0} \liminf_{k\to\infty} \frac{\log\left[N_k(\alpha,\Delta\alpha)\right]}{\log\left(\frac{1}{b^{-k}}\right)} = \lim_{\Delta\alpha\to 0} \liminf_{k\to\infty} \frac{\log\left[N_k(\alpha,\Delta\alpha)\right]}{\log\left(b^k\right)}. \tag{9}$$

If the limit exists and is positive (on a support larger than a single point), we call the function g a multifractal.

[14] This definition of multifractal spectrum has been developed for this particular purpose. For a general geometric definition see [9].

Of course the same definition can be applied substituting g with a random function, such as one of the (random) measures θ defined in Section 2.

This approach tells us that there exist different fractal local scaling behaviors as measured by different local coarse Hölder exponents, being so not uniform in general. In other words, $\alpha(t)$ is typically not constant in t but assume a whole range of values, thus imprinting a rich structure on the object of interest. This structure can be characterized either in geometrical terms making use of the concept of dimension, or in statistical terms based on sample moments. A tight connection between these two descriptions will emerge from the multifractal formalism.

In fact, many authors interpreted $f(\alpha)$ as the Hausdorff dimension of the set of points having local Hölder exponent equal to α. For any $\alpha \geq 0$, we can define a set T_α of instants with Hölder exponent α. As any subset of the real line, T_α has Hausdorff dimension $0 \leq \dim_{\mathcal{H}}(T_\alpha) \leq 1$[15].

Since we have shown that the microcanonical and canonical measure are self-similar random measures, using the multifractal formalism it can be proved that (see [9])

$$f(\alpha) = \dim_{\mathcal{H}}(T_\alpha) \tag{10}$$

holds true. Thanks to the same result, it also follows that $f(\alpha)$ is also the Legendre

[15]Let $\{A_i\}_{i \in \mathbb{N}}$ be a countable collection of sets of diameter at most equal to $\delta > 0$ that cover $A \subseteq \mathbb{R}^n$. This means that

$$A \subset \bigcup_{i=1}^{\infty} A_i \ : \ 0 \leq \operatorname{diam}(A_i) \leq \delta$$

for all $i \in \mathbb{N}$. If the previous statement holds, we say that $\{A_i\}_{i \in \mathbb{N}}$ is a *δ-cover* of A. Given $s \in \mathbb{R}_0^+$, for any $\delta > 0$ we define

$$\mathcal{H}_\delta^s(A) := \inf \left\{ \sum_{i=1}^{\infty} \left[\operatorname{diam}(A_i)\right]^s : \{A_i\}_{i \in \mathbb{N}} \text{ is a } \delta\text{-cover of } A \right\}.$$

Then the s-dimensional *Hausdorff measure* is the measure given by

$$\mathcal{H}^s(A) := \lim_{\delta \to 0^+} \mathcal{H}_\delta^s(A).$$

Moreover, we define *Hausdorff dimension* of a set $A \subseteq \mathbb{R}^n$ the following quantity:

$$\dim_{\mathcal{H}}(A) := \sup\{s : \mathcal{H}^s(A) > 0\} = \sup\{s : \mathcal{H}^s(A) = \infty\}.$$

transform[16] of the scaling function. In fact

$$f(\alpha) = \tau^{\star}(\alpha) = \inf_{q \in \mathbb{R}} \big\{ q \cdot \alpha - \tau(q) \big\}$$

with $\tau(q) \equiv -\log_b [\mathbb{E}(U^q)] - 1$ and $\tau^{\star}(\alpha)$ denotes its Legendre transform.

For our field of interest, closed-form expressions for $f(\alpha)$ depends on the densities of the random variables $\{U_j\}$ involved in the multifractal measure θ. However, in order to find them, an important theorem of *Large Deviation Theory* is required. We will examine that properties in the following sections.

4 The properties of the scaling function $\tau(q)$

In the following we will be interested in analyzing all the relevant properties of the scaling function $\tau(q)$; particularly, we are interested in its zeros and in its concavity/convexity.

Let we take its explicit form

$$\tau(q) = -\log_b [\mathbb{E}(U^q)] - 1\,,$$

where U is one of the $\{U_j\}$ i.i.d. random variable with $j = 0, 1, ..., b-1$. Since these variable are discrete[17] the q-th moment can be also written as

$$\tau(q) = -\log_b \left(\sum_{j=0}^{b-1} \Pr\{U_j = u_j\} \cdot u_j^q \right) - 1$$

where u_j are the values taken by the random variables.

Theorem 4.1. *: The points* $(0, -1)$ *and* $(1, 0)$ *are respectively the intercept and a zero for the scaling function* $\tau(q)$.

Proof. Setting $q = 0$, we get

$$\tau(0) = -\log_b \left(\sum_{j=0}^{b-1} \Pr\{U_j = u_j\} \cdot u_j^0 \right) - 1 = -\log_b \left(\sum_{j=0}^{b-1} \Pr\{U_j = u_j\} \right) - 1 =$$
$$= -\log_b (1) - 1 = -1,$$

[16] Given a concave function $h : \mathbb{R} \to \mathbb{R}$, with $h \in C^2(\mathbb{R})$, we define the *Legendre transform* of $h(x)$ the function

$$h^{\star}(\alpha) := \inf_{x \in \mathbb{R}} \{\alpha \cdot x - h(x)\}\,.$$

[17] The sequence of random variables $\{U_j\}$ is now assumed to be discrete, but with little effort they can be associated with a continuous density. For that eventuality, see [6].

hence the point $(0,-1)$ is the intercept of the scaling function. Studying the first moment of the random variable, which corresponds to the value $q=1$, we find

$$\begin{aligned}\tau(1) &= -\log_b\left(\sum_{j=0}^{b-1}\Pr\{U_j=u_j\}\cdot u_j\right)-1 = -\log_b\left(\frac{1}{b}\right)-1 = \\ &= -(-1)-1=0,\end{aligned}$$

since $\mathbb{E}(U)=\frac{1}{b}$. Thus the point $(1,0)$ is one of the zeros of the function $\tau(q)$. □

On the existence of other zeros, it will be discussed later.

Theorem 4.2. : *The scaling function* $\tau(q)$ *is* non-decreasing *if the measure is microcanical; if the measure is canonical the function might exhibit* both increasing and decreasing *regions.*

Proof. Let we study the first derivatives of $\tau(q)$.

$$\begin{aligned}\tau'(q) &= \frac{\mathrm{d}\left[-\log_b\left(\sum_{j=0}^{b-1}\Pr\{U_j=u_j\}\cdot u_j^q\right)-1\right]}{\mathrm{d}q} = \\ &= -\frac{1}{\sum_{j=0}^{b-1}\Pr\{U_j=u_j\}\cdot u_j^q}\cdot\log_b(e)\cdot\sum_{j=0}^{b-1}\Pr\{U_j=u_j\}\cdot u_j^q\cdot\ln(u_j) = \\ &= -\log_b(e)\cdot\frac{\sum_{j=0}^{b-1}\Pr\{U_j=u_j\}\cdot u_j^q\cdot\ln(u_j)}{\sum_{j=0}^{b-1}\Pr\{U_j=u_j\}\cdot u_j^q}.\end{aligned}$$

Since $b\in\mathbb{N}\backslash\{0,1\}$, $-\log_b(e)$ is always negative. Moreover, since $U_j\geq 0$ for all j, all the moments of the random variable are positive (they would be exactly equal to zero only in the degenerate case, that is if the random variables were all null), and hence the denominator is positive. The only entity whose sign may vary is the numerator of the fraction, whose positivity/negativity is due to the quantities $\ln(u_j)$. If we consider the microcanical measure, that is $0\leq U_j\leq 1$, the logarithms are all negative, making the numerator negative as well. In this case, we find $\tau'(q)\geq 0$, being so *non-decreasing.* However, we have to underline that, if the measure is canonical, hence allowing $U_j\geq 0$, we are not be able to state, a priori, where the scaling function is increasing or decreasing. □

Theorem 4.3. : *Regardless the measure is microcanical or canonical, the scaling function* $\tau(q)$ *is* convcave.

Proof. : We decide to study the concavity/convexity of $\tau(q)$ through the second derivative.

$$\tau''(q) = \frac{\mathrm{d}^2\left[-\log_b\left(\sum_{j=0}^{b-1}\Pr\{U_j=u_j\}\cdot u_j^q\right)-1\right]}{\mathrm{d}q^2} =$$
$$= -\log_b(e)\cdot\frac{\sum_{j=0}^{b-1}\sum_{i=j+1}^{b-1}\Pr\{U_j=u_j\}\cdot\Pr\{U_i=u_i\}\cdot u_j^q\cdot u_i^q\cdot[\ln(u_j)-\ln(u_i)]^2}{\left(\sum_{j=0}^{b-1}\Pr\{U_j=u_j\}\cdot u_j^q\right)^2}.$$

Since the fraction is always positive, the second derivative is always negative (so, despite the monotonicity which may vary depending on the values taken by the random variables U_j, here we have found a "stronger" property). Hence $\tau''(q)<0$ and the scaling function is always *concave*. □

The last features we are interested in investigating is the existence of asymptotes for the function $\tau(q)$.

Theorem 4.4. *: Regardless the measure is microcanical or canonical, the scaling function $\tau(q)$ is* asymptotic linear *both for $q\to-\infty$ and $q\to+\infty$.*

We have to compute the following limits,

$$\lim_{q\to+\infty}\tau(q) = \lim_{q\to+\infty} -\log_b\left(\sum_{j=0}^{b-1}\Pr\{U_j=u_j\}\cdot u_j^q\right)-1 = \begin{cases}+\infty & \text{if } 0\le U_j\le 1\\ -\infty & \text{if } U_j\ge 0\end{cases}$$

and

$$\lim_{q\to-\infty}\tau(q) = \lim_{q\to-\infty} -\log_b\left(\sum_{j=0}^{b-1}\Pr\{U_j=u_j\}\cdot u_j^q\right)-1 = -\infty.$$

Because of these results, we might have at most two *oblique* asymptotes, one for $q\to-\infty$ and one when $q\to+\infty$. Hence, since the previous limits are necessary conditions but not sufficient, we have two compute

$$\lim_{q\to+\infty}\frac{\tau(q)}{q} = \lim_{q\to+\infty}\frac{-\log_b\left(\sum_{j=0}^{b-1}\Pr\{U_j=u_j\}\cdot u_j^q\right)-1}{q} =$$
$$= \lim_{q\to+\infty} -\frac{1}{\ln(b)}\cdot\frac{\ln\left(\sum_{j=0}^{b-1}\Pr\{U_j=u_j\}\cdot u_j^q\right)+1}{q} =$$
$$= \lim_{q\to+\infty} -\frac{1}{\ln(b)}\cdot\frac{\ln\left(\sum_{j=0}^{b-1}\Pr\{U_j=u_j\}\cdot e^{\ln\left(u_j^q\right)}\right)+1}{q} =$$
$$= \lim_{q\to+\infty} -\frac{1}{\ln(b)}\cdot\frac{\ln\left(\sum_{j=0}^{b-1}\Pr\{U_j=u_j\}\cdot e^{q\cdot\ln(u_j)}\right)+1}{q}.$$

Since, as $q \to +\infty$

$$\ln\left(\sum_{j=0}^{b-1} \Pr\{U_j = u_j\} \cdot e^{q \cdot \ln(u_j)}\right) \sim \ln\left(\Pr\{U_{max} = u_{max}\} \cdot e^{q \cdot \ln(u_{max})}\right) ,$$

where $U_{max} = \max_j\{U_j\}$, the previous limit becomes

$$\begin{aligned}\lim_{q\to+\infty} \frac{\tau(q)}{q} &= \lim_{q\to+\infty} -\frac{1}{\ln(b)} \cdot \frac{\ln\left(\Pr\{U_{max} = u_{max}\} \cdot e^{q \cdot \ln(u_{max})}\right) + 1}{q} = \\ &= \lim_{q\to+\infty} -\frac{1}{\ln(b)} \cdot \frac{\ln\left(\Pr\{U_{max} = u_{max}\}\right) + q \cdot \ln(u_{max}) + 1}{q} = \\ &= -\frac{\ln(u_{max})}{\ln(b)} = -\log_b(u_{max}) < \infty .\end{aligned}$$

To ascertain that there really is an oblique asymptote, we have to solve this further limit

$$\begin{aligned}\lim_{q\to+\infty} \tau(q) - \left[-\frac{\ln(u_{max})}{\ln(b)}\right] \cdot q &= \lim_{q\to+\infty} \tau(q) + \frac{\ln(u_{max})}{\ln(b)} \cdot q = \\ = \lim_{q\to+\infty} -\frac{\ln\left(\Pr\{U_{max} = u_{max}\} \cdot e^{q \cdot \ln(u_{max})}\right) + 1}{\ln(b)} &+ \frac{\ln(u_{max})}{\ln(b)} \cdot q = \\ = \lim_{q\to+\infty} -\frac{\ln\left(\Pr\{U_{max} = u_{max}\}\right) + q \cdot \ln(u_{max}) + 1}{\ln(b)} &+ \frac{\ln(u_{max})}{\ln(b)} \cdot q = \\ = -\frac{\ln\left(\Pr\{U_{max} = u_{max}\}\right)}{\ln(b)} - \frac{1}{\ln(b)} &= \\ = -\log_b\left(\Pr\{U_{max} = u_{max}\}\right) - \log_b(e) &< \infty .\end{aligned}$$

Thus, for $q \to +\infty$, the scaling function $\tau(q)$ is asymptotic to the straight line with equation

$$a_1(q) = -\log_b(u_{max}) \cdot q - \log_b\left(\Pr\{U_{max} = u_{max}\}\right) - \log_b(e) .$$

Let we examine the components of the equation: firstly, $-\log_b(e)$ is always negative; secondly, $-\log_b\left(\Pr\{U_{max} = u_{max}\}\right)$ is always positive (the least $\Pr\{U_{max} = = u_{max}\}$, the higher the summand). So the intercept of the asymptote $a_1(q)$ can be both negative or positive, depending on the odd of the event $U_{max} = u_{max}$. Eventually, the rate of growth $-\log_b(u_{max})$ is *positive* if we are dealing with microcanical measures ($0 \le U_{max} \le 1$), but becomes *negative* if the measure is canonical ($U_{max} \ge 1$).

The same procedure has to be repeated for $q \to -\infty$.

$$\begin{aligned}\lim_{q\to-\infty} \frac{\tau(q)}{q} &= \lim_{q\to-\infty} \frac{-\log_b\left(\sum_{j=0}^{b-1} \Pr\{U_j = u_j\} \cdot u_j^q\right) - 1}{q} = \\ &= \lim_{q\to-\infty} -\frac{1}{\ln(b)} \cdot \frac{\ln\left(\sum_{j=0}^{b-1} \Pr\{U_j = u_j\} \cdot e^{q \cdot \ln(u_j)}\right) + 1}{q} .\end{aligned}$$

Because, as $q \to -\infty$

$$\ln\left(\sum_{j=0}^{b-1} \Pr\{U_j = u_j\} \cdot e^{q \cdot \ln(u_j)}\right) \sim \ln\left(\Pr\{U_{min} = u_{min}\} \cdot e^{q \cdot \ln(u_{min})}\right) ,$$

where $U_{min} = \min_j\{U_j\}$, the limit becomes

$$\begin{aligned}\lim_{q\to-\infty} \frac{\tau(q)}{q} &= \lim_{q\to-\infty} -\frac{1}{\ln(b)} \cdot \frac{\ln\left(\Pr\{U_{min} = u_{min}\} \cdot e^{q \cdot \ln(u_{min})}\right) + 1}{q} = \\ &= \lim_{q\to+\infty} -\frac{1}{\ln(b)} \cdot \frac{\ln\left(\Pr\{U_{min} = u_{min}\}\right) + q \cdot \ln\left(u_{min}\right) + 1}{q} = \\ &= -\frac{\ln\left(u_{min}\right)}{\ln(b)} = -\log_b\left(u_{min}\right) < \infty\,.\end{aligned}$$

The limit for the existence of the intercept is

$$\begin{aligned}\lim_{q\to-\infty} \tau(q) - \left[-\frac{\ln\left(u_{min}\right)}{\ln(b)}\right] \cdot q &= \lim_{q\to-\infty} \tau(q) + \frac{\ln\left(u_{min}\right)}{\ln(b)} \cdot q = \\ &= -\log_b\left(\Pr\{U_{min} = u_{min}\}\right) - \log_b(e) < \infty\,.\end{aligned}$$

Hence, even for $q \to -\infty$, the scaling function $\tau(q)$ has another slant asymptote, that is

$$a_2(q) = -\log_b\left(u_{min}\right) \cdot q - \log_b\left(\Pr\{U_{min} = u_{min}\}\right) - \log_b(e)\,.$$

Here, the behaviour of the intercept is the same of the previous case. Nevertheless, the slope of the asymptote is always *positive*, since necessarily $0 \leq U_{min} \leq 1$, both in the microcanonical case and the canonical one.

Then, we summarize all the properties of the scaling function that have been found:

- The point $(1, 0)$ is a zero for $\tau(q)$;
- The point $(0, -1)$ is the intercept of $\tau(q)$;
- If the measure is microcanonical, the function is non-decreasing;
- The function is concave;
- The function is asymptotical linear. Particularly, when $q \to +\infty$

  $$\tau(q) \sim -\log_b\left(u_{max}\right) \cdot q - \log_b\left(\Pr\{U_{max} = u_{max}\}\right) - \log_b(e)\,,$$

  and

  $$\tau(q) \sim -\log_b\left(u_{min}\right) \cdot q - \log_b\left(\Pr\{U_{min} = u_{min}\}\right) - \log_b(e)\,,$$

  when $q \to -\infty$.

These properties allow us to infer another property of $\tau(q)$ if a canonical measure is involved. Since the function is always concave and also asymptotic, for $q \to +\infty$, to the straight line which has a negative slope, thus it must have a "cap" form, such as $\bigcap$. Moreover, since the function as a zero for $q = 1$, there must exist another one, which is usually addressed as q_{crit}, that is

$$q_{crit} := \{q > 1 : \tau(q) = 0\}$$

which is present, we remark, *only* for canonical measures. This entity has a great impact on the finiteness of the moments of the measure. As a matter of fact Mandelbrot in [17] showed that, for $q > 1$, the moments of the measure are finite if and only if $\tau(q) > 0$. That eventuality occurs only for $1 < q < q_{crit}$.

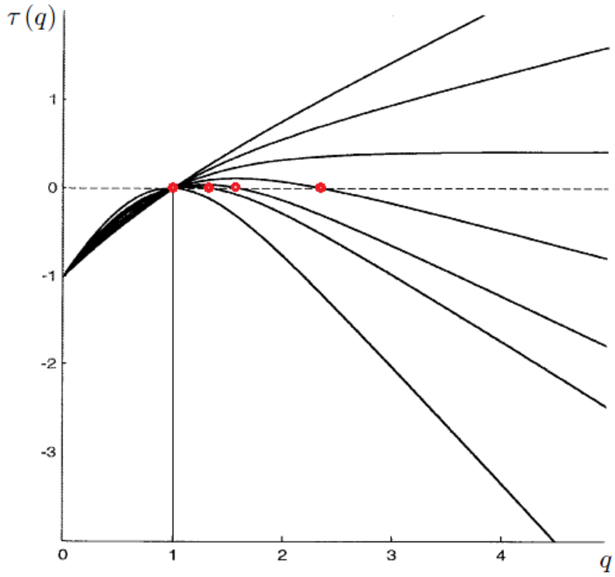

Figure 3: Possible shapes of $\tau(q)$. Red dots are different q_{crit}.

As a matter of fact, since the q-th moment of a canonical measure is given by

$$\mathbb{E}\Big\{[\theta\,(\Delta t)]^q\Big\} = \mathbb{E}\,(\Upsilon^q) \cdot \Delta t^{\tau(q)+1}\,,$$

its finiteness may depends only on either $\tau(q)$ or $\mathbb{E}\,(\Upsilon^q)$. Since, for finite q, $\tau(q)$ is surely finite, hence the "infiniteness" can be achieved only by a particular behaviour of the random variable Υ which is the random mass on the interval $[0, 1]$ (see Section 2).

Indeed, Guivarc'h proved in [10] that Υ has Paretian tails and allows infinite moments, for those values $q \geq q_{crit}$. Thus, the random variable Υ follows a Pareto's distribution[18] of exponent q_{crit}, that is

$$\Pr\{\Upsilon > \upsilon\} = \left(\frac{\upsilon}{\upsilon_{min}}\right)^{-q_{crit}} .$$

5 Three faces of $f(\alpha)$

First of all we focus on the shape of $f(\alpha)$; then we will discuss as this entity can be interpreted form a financial and mathematical point of view.

Since for the scaling function the following asymptotic relations hold true

$$\tau(q) \sim -\log_b(u_{max}) \cdot q - \log_b\left(\Pr\{U_{max} = u_{max}\}\right) - \log_b(e) \quad \text{as} \quad q \to +\infty$$

and

$$\tau(q) \sim -\log_b(u_{min}) \cdot q - \log_b\left(\Pr\{U_{min} = u_{min}\}\right) - \log_b(e) \quad \text{as} \quad q \to -\infty\,,$$

thus, the slope of the oblique asymptotes are respectively

$$\alpha_{max} = -\log_b(u_{min}) > 0 \quad \text{and} \quad \alpha_{min} = -\log_b(u_{max}) \lesseqgtr 0\,.$$

The relation max/min is inverted since the the slope of the asymptote for $q \to -\infty$ is grater than the one of the asymptote for $q \to +\infty$. Moreover, these two values give the bounds of the support of the multifractal spectrum, that is

$$f : \mathbb{R} \supset [\alpha_{min}, \alpha_{max}] \to \mathbb{R}\,,$$

since they are the least and the highest value α can take. Hence, if the scaling function $\tau(q)$ is defined on the entire real line, its asymptotic linear behaviour implies the spectrum $f(\alpha)$ to be defined only on a closed set of values. Furthermore, using Young's Inequality[19]

[18] Given a real valued random variable defined on the support $[x_{min}, +\infty) \subset \mathbb{R}^+$, we say it is Paretian distributed if and if only its cumulative distribution function is so defined

$$F_X(x) := 1 - \left(\frac{x}{x_{min}}\right)^{-\omega}$$

where ω is a real positive parameter. To say that a random variable follows the Pareto's distribution we write $X \overset{d}{\sim} \mathcal{P}(\omega, x_{min})$.

[19] *Young's inequality* states that, given two concave functions $f, h : \mathbb{R} \to \mathbb{R}$ such that $f(\alpha) = h^\star(\alpha)$, thus

$$\alpha \cdot x \geq f(\alpha) + h(x)\,.$$

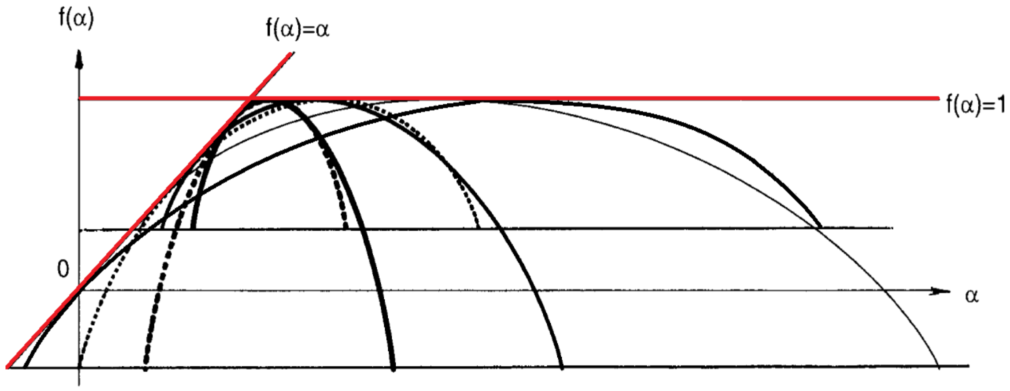

Figure 4: Possible shapes of $f(\alpha)$ for different $\tau(q)$.

and the fact that $\tau(0) = -1$ and $\tau(1) = 0$, we can find

$$\alpha \cdot 0 \geq f(\alpha) + \tau(0) \quad \rightarrow \quad f(\alpha) \leq 1$$

and

$$\alpha \cdot 1 \geq f(\alpha) + \tau(1) \quad \rightarrow \quad f(\alpha) \leq \alpha \, .$$

Figure 4 illustrates it. Then we will investigate about the coordinates of the maximum of $f(\alpha)$.

Now we take into account the three possible interpretations of $f(\alpha)$[20].

Given the microcanonical measure,

$$\theta\left(\Delta t\right) = U_{\eta_1} \cdot U_{\eta_1\eta_2} \cdot \ldots \cdot U_{\eta_1\eta_2\ldots\eta_k}$$

where $\Delta t = b^{-k}$ and t is a b-adic number, let we consider its coarse Hölder exponent

$$\begin{aligned} \alpha_k\left(t\right) := \frac{\log\theta\left(\left[t, t+b^{-k}\right]\right)}{\log\left(b^{-k}\right)} &= \frac{\log_b\left(U_{\eta_1} \cdot U_{\eta_1\eta_2} \cdot \ldots \cdot U_{\eta_1\eta_2\ldots\eta_k}\right)}{\log_b\left(b^{-k}\right)} = \\ &= -\frac{1}{k} \cdot \left[\log_b\left(U_{\eta_1}\right) + \log_b\left(U_{\eta_1\eta_2}\right) + \ldots + \log_b\left(U_{\eta_1\eta_2\ldots\eta_k}\right)\right] . \end{aligned} \tag{11}$$

Since the measure is randomly generated because of the multipliers $U_{\eta_1}, U_{\eta_1\eta_2}, \ldots, U_{\eta_1\eta_2\ldots\eta_k}$, if we choose a fixed interval $[t, t+\Delta t]$, the coarse exponents $\alpha_k(t)$ are identically distributed

[20] That is, $f(\alpha)$ can be interpreted as the multifractal spectrum, the Hausdorff dimension of the set of time having α as Hölder exponent T_α and the Legendre transform of $\tau(q)$.

across the b-adic cells, and hence can be viewed as the realizations of a random variable α_k. In the following, we will use this random variable to study the multifractal spectrum $f(\alpha)$ of the measure θ, since it can be directly derived from the asymptotic distribution of α_k.

By equation (11) and thanks to the way we defined the microcanonical measure, the coarse Hölder exponent is the sample sum of k i.i.d. random variables. Because of it, the distribution of α_k can be analyzed via limit theorems: **(a)** Strong Law of Large Numbers, **(b)** Central Limit Theorem, and **(c)** Large Deviation Theory, which will be then introduced.

(a) Strong Law of large numbers

By the Strong Law of Large Numbers, the sequence of random variable $\{\alpha_k\}_{k\in\mathbb{N}}$ converge almost surely to $\mu_\alpha = \mathbb{E}\big[-\log_b(U)\big]$, that is

$$\Pr\left\{\lim_{k\to\infty} \alpha_k = \mu_\alpha\right\} = 1\,.$$

Moreover, since $\mathbb{E}(U) = \frac{1}{b}$, by Jensen's inequality, we have $\mu_\alpha \geq 1$. As k increases, we expect that almost all coarse Hölder exponents are contained in a neighborhood of μ_α. However, the *other* coarse Hölder exponents are also important, and may more than the ones around the mean. As a matter of fact, if we consider (11), we can see that the measure can be expressed as

$$\theta\left(\left[t, t+b^{-k}\right]\right) = \left(b^{-k}\right)^{\alpha_k(t)} = \Delta t^{\alpha_k(t)}\,,$$

Let Θ denote the set of b-adic intervals with Hölder exponents grater than $\frac{\mu_\alpha+1}{2}$, for large values of k, almost all intervals belong to Θ, but their mass

$$\sum_{t\in\Theta}\theta\left(\left[t, t+b^{-k}\right]\right) = \sum_{t\in\Theta}\Delta t^{\alpha_k(t)} \leq b^k \cdot \Delta t^{\frac{\mu_\alpha+1}{2}} = b^{-k\cdot\frac{\mu_\alpha-1}{2}}\,,$$

tends to zero as k goes to infinity. Since the mass must conserve, it has to be in those few b-adic intervals which do not belong to Θ. Since with the Law of Large Numbers we are dealing with centre of the α_k's distribution, information on these "rare events" is presumably contained in the tail of the random variable α_k.

In other words, if α_k differs from μ_α, then $(\alpha_j, \alpha_j + \Delta\alpha]$ will not contain μ_α for small $\Delta\alpha$ and the chance to observe other coarse exponents which lie in $(\alpha_j, \alpha_j + \Delta\alpha]$ will decrease exponentially fast with rate given by $f(\alpha)$.

(b) Central Limit Theorem

Assuming that $-\log_b(U)$ has finite variance σ_α^2, we can apply the (Lindeberg-Lévy) Central Limit Theorem, which implies that

$$\sqrt{k} \cdot \frac{\alpha_k - \mu_\alpha}{\sigma_\alpha} \xrightarrow{d} \mathcal{N}(0,1)\,.$$

If $N_k(\alpha)$ stands for the number of coarse Hölder exponents[21] the equal to α_k, in terms of histograms, we have

$$\frac{N_k(\alpha)}{b^k} \sim \frac{1}{\sqrt{2 \cdot \pi \cdot \frac{\sigma_\alpha^2}{k}}} \cdot e^{-\frac{(\alpha-\mu_\alpha)^2}{2 \cdot \sigma_\alpha^2 / k}}$$

as $k \to \infty$. After some arrangements and taking the logarithms[22], we found that the multifractal spectrum

$$f(\alpha) \sim 1 - \frac{1}{2 \cdot \ln(b)} \cdot \left(\frac{\alpha - \mu_\alpha}{\sigma_\alpha}\right)^2 \tag{12}$$

is locally quadratic around the most probable exponent μ_α.

(c) Large Deviation Theory

Since Large Deviation Theory is not frequently present in textbook on Probability Theory, we give a brief introduction of it. This branch of Probability Theory deals with

[21] For simplicity, we write $N_k(\alpha)$ instead of $N_k(\alpha, \Delta\alpha)$.

[22] In fact, as $k \to \infty$

$$N_k(\alpha) \sim \frac{b^k}{\sqrt{2 \cdot \pi \cdot \frac{\sigma_\alpha^2}{k}}} \cdot e^{-\frac{(\alpha-\mu_\alpha)^2}{2 \cdot \sigma_\alpha^2 / k}}\,.$$

Taking the logarithms

$$\ln\left[N_k(\alpha)\right] \sim \ln\left(b^k\right) - \ln\left(\sqrt{2 \cdot \pi \cdot \frac{\sigma_\alpha^2}{k}}\right) - \frac{(\alpha - \mu_\alpha)^2}{2 \cdot \frac{\sigma_\alpha^2}{k}}\,,$$

and diving by $\ln\left(b^k\right)$

$$f(\alpha) = \frac{\ln\left[N_k(\alpha)\right]}{\ln\left(b^k\right)} \sim 1 - \frac{\ln\left(\sqrt{2 \cdot \pi \cdot \frac{\sigma_\alpha^2}{k}}\right)}{\ln\left(b^k\right)} - \frac{\frac{k \cdot (\alpha-\mu_\alpha)^2}{2 \cdot \sigma_\alpha^2}}{\ln\left(b^k\right)}\,.$$

Since the second summand goes to zero as $k \to \infty$, we find

$$f(\alpha) \sim 1 - \frac{\frac{k \cdot (\alpha-\mu_\alpha)^2}{2 \cdot \sigma_\alpha^2}}{k \cdot \ln(b)} = 1 - \frac{1}{2 \cdot \ln(b)} \cdot \frac{(\alpha - \mu_\alpha)^2}{\sigma_\alpha^2}\,.$$

the decay of the probability of increasingly unlikely events. It was introduced by the Swedish mathematician Harald Cramér, generally more famous for his model on insurance ruin.

Suppose that $X_1, X_2, \ldots$ are independent and identically distributed random variables with mean $\mathbb{E}(X) = \mu_X$ and variance $\mathbb{V}(X) = \sigma_X^2 < +\infty$. If denote the k-th partial sum by

$$S_k := \sum_{i=1}^{k} X_i \,.$$

In order to prove the Strong Law of Large Number, the hypothesis of the Borel-Cantelli Lemma is required to hold, that is

$$\sum_{k=1}^{\infty} \Pr\left\{ \frac{S_k}{k} - \mu_X > \Delta\alpha \right\} < \infty \,.$$

with $\Delta\alpha > 0$. However, if such odds are of the order of $\frac{1}{n}$, that is

$$\Pr\left\{ \frac{S_k}{k} - \mu_X > \Delta\alpha \right\} \approx \frac{1}{n}$$

the Borel-Cantelli Lemma is not applicable anymore, since they are not summable, and the Strong Law of Large Numbers fails. Large Deviation Theory finds out exactly how fast the large deviation probabilities decay. This depends on finer features of the random variable X than merely the finiteness of its variance. Our initial focus is on random variables satisfying

$$\ln\Big[\mathbb{E}\left(e^{q\cdot X}\right)\Big] < \infty$$

where $q \in \mathbb{R}$ and the entire quantity is usually addressed as *cumulant generating function*[23]. In this case the large deviation probabilities decay exponentially and *Cramér's theorem* tells us exactly how fast.

Theorem 5.1. (**Cramér's theorem**) *Let $X_1, X_2, \ldots$ be independent and identically distributed random variables with mean μ_X, and finite cumulant generating function for all $q \in \mathbb{R}$. Then, we have*

$$\lim_{k\to\infty} \frac{1}{k} \cdot \ln\left[\Pr\left\{\frac{S_k}{k} > x\right\}\right] = \inf_{q\in\mathbb{R}} \left\{q \cdot x - \ln\left[\mathbb{E}\left(e^{q\cdot X}\right)\right]\right\}$$

for any $x > \mu_X$, and

$$\lim_{k\to\infty} \frac{1}{k} \cdot \ln\left[\Pr\left\{\frac{S_k}{k} < x\right\}\right] = \inf_{q\in\mathbb{R}} \left\{q \cdot x - \ln\left[\mathbb{E}\left(e^{q\cdot X}\right)\right]\right\}$$

for any $x < \mu_X$.

[23] It is just the logarithm of the *moment generating function*.

We can now apply Cramér's theorem to the random variable

$$\alpha_k := -\frac{1}{k} \cdot \left[\log_b \left(U_{\eta_1}\right) + \log_b \left(U_{\eta_1\eta_2}\right) + ... + \log_b \left(U_{\eta_1\eta_2...\eta_k}\right)\right] ,$$

that is the k-th partial sum of the i.i.d. random variables $-\log_b(U)$. Thus, we find

$$\lim_{k\to\infty} \frac{1}{k} \cdot \ln \left[\Pr\left\{\alpha_k > \alpha\right\}\right] = \inf_{q\,\in\,\mathbb{R}} \left\{q \cdot \alpha - \ln\left[\mathbb{E}\left(e^{-q\cdot\log_b(U)}\right)\right]\right\}$$

for any $\alpha > \mu_\alpha$. Calling this quantity as $\delta(\alpha)$, we find[24]

$$\delta(\alpha) = \inf_{q\,\in\,\mathbb{R}} \left\{q \cdot \alpha + \log_b \left[\mathbb{E}\left(U^q\right)\right]\right\} .$$

Since we have set $\tau(q) \equiv -\log_b \left[\mathbb{E}\left(U^q\right)\right] - 1$, the previous expression can be written as

$$\delta(\alpha) = \inf_{q\,\in\,\mathbb{R}} \left\{q \cdot \alpha - \tau(q)\right\} - 1 , \tag{13}$$

which shows that $\delta(\alpha) + 1$ is the Legendre transform of $\tau(q)$.

Now, let we come back to the construction of the multifractal spectrum $f(\alpha)$. As we have done before, we subdivide the interval $[0, 1]$ into b^k cells of length $\Delta t = b^{-k}$.

As $k \to \infty$, the following heuristic expression

$$\frac{1}{k} \cdot \ln\left[\frac{N_k\left(\alpha_j\right)}{b^k}\right] \sim \frac{1}{k} \cdot \ln\Big(\Pr\left\{\alpha_j < \alpha_k \leq \alpha_j + \Delta\alpha\right\}\Big) \tag{14}$$

has to be postulated[25]. Since

$$\begin{aligned}\Pr\left\{\alpha_j < \alpha_k \leq \alpha_j + \Delta\alpha\right\} &= \Pr\left\{\alpha_k \leq \alpha_j + \Delta\alpha\right\} - \Pr\left\{\alpha_k \leq \alpha_j\right\} = \\ &= \Pr\left\{\alpha_k > \alpha_j\right\} - \Pr\left\{\alpha_k > \alpha_j + \Delta\alpha\right\} = \\ &= \Pr\left\{\alpha_k > \alpha_j\right\} \cdot \left(1 - \frac{\Pr\left\{\alpha_k > \alpha_j + \Delta\alpha\right\}}{\Pr\left\{\alpha_k > \alpha_j\right\}}\right) ,\end{aligned}$$

[24] Due to independence, it follows

$$\begin{aligned}\delta(\alpha) &= \inf_{q\,\in\,\mathbb{R}}\left\{q \cdot \alpha - \ln\left[\mathbb{E}\left(e^{-q\cdot\log_b(U)}\right)\right]\right\} = \inf_{q\,\in\,\mathbb{R}}\left\{q \cdot \alpha - \ln(b) \cdot \log_b\left[\mathbb{E}\left(e^{-q\cdot\log_b(U)}\right)\right]\right\} = \\ &= \inf_{q\,\in\,\mathbb{R}}\left\{q \cdot \alpha - \log_b\left[\mathbb{E}\left(e^{-q\cdot\log_b(U)\cdot\ln(b)}\right)\right]\right\} = \inf_{q\,\in\,\mathbb{R}}\left\{q \cdot \alpha - \log_b\left[\mathbb{E}\left(b^{-q\cdot\log_b(U)}\right)\right]\right\} = \\ &= \inf_{q\,\in\,\mathbb{R}}\left\{q \cdot \alpha - \log_b\left[\mathbb{E}\left(b^{-\log_b(U^q)}\right)\right]\right\} = \inf_{q\,\in\,\mathbb{R}}\left\{q \cdot \alpha + \log_b\left[\mathbb{E}\left(U^q\right)\right]\right\} .\end{aligned}$$

[25] This expression holds exactly, for any k, for binomial and multinomial measures since the coarse Hölder exponents are discrete. In more general cases, like the microcanonical measure, it has to be postulated.

expressing in term of tail behaviour, we can apply Large Deviation Theory. Since $\delta(\alpha_j + \Delta\alpha) < \delta(\alpha_j)$, applying (13) we find[26]

$$\frac{\Pr\{\alpha_k > \alpha_j + \Delta\alpha\}}{\Pr\{\alpha_k > \alpha_j\}} \sim \left(b^k\right)^{\frac{\delta(\alpha_j+\Delta\alpha)-\delta(\alpha_j)}{\ln(b)}}$$

which hence vanishes as $k \to \infty$. Thus the previous expression becomes

$$\Pr\{\alpha_j < \alpha_k \leq \alpha_j + \Delta\alpha\} \sim \Pr\{\alpha_k > \alpha_j\}$$

as k goes to infinity. Hence (14) can be rewritten as

$$\frac{1}{k} \cdot \ln\left[\frac{N_k(\alpha_j)}{b^k}\right] \sim \frac{1}{k} \cdot \ln\big(\Pr\{\alpha_k > \alpha_j\}\big) \sim \delta(\alpha_j)\ , \tag{15}$$

for any $\alpha_j > \mu_\alpha$. Identical argumentations are valid for $\alpha_j < \mu_\alpha$.

Thus, the Large Deviation Theory approach to multifractal spectrum gives us further information. As a matter of fact, we have shown that $f(\alpha)$ is the limit of

$$\frac{1}{k} \cdot \log_b(\Pr\{\alpha_k > \alpha\}) + 1 \qquad \text{if } \alpha > \mu_\alpha$$

and

$$\frac{1}{k} \cdot \log_b(\Pr\{\alpha_k < \alpha\}) + 1 \qquad \text{if } \alpha < \mu_\alpha\,.$$

We note particularly that $f(\alpha) \leq 1$, and it increases for $\alpha < \mu_\alpha$ and decrease for $\alpha > \mu_\alpha$. Hence $\alpha = \mu_\alpha$ is a maximum and $f(\alpha)$ is a concave function. Morover, since

$$f(\mu_\alpha) = \inf_{q\in\mathbb{R}}\big\{q \cdot \mu_\alpha - \tau(q)\big\}$$

and it is minimal when $q = 0$, thus $f(\mu_\alpha) = -\tau(0) = 1$, having set the function $\tau(q) \equiv -\log_b[\mathbb{E}(U^q)] - 1$. This result receives a simple interpretation in terms of fractal dimension: The set of instants with exponent μ_α has *Lebesgue measure* equal to one (that is what the Strong Law of Large Number states).

We conclude this section remarking that, throughout the dissertation, we have given different notions of multifractal spectrum $f(\alpha)$, and that we have linked it with other quantities by the occasions. Summarizing $f(\alpha)$ can be viewed as:

- The limit of a renormalized histogram of coarse Hölder exponents

$$f(\alpha) = \lim_{\Delta\alpha\to 0} \liminf_{k\to 0} \frac{\log[N_k(\alpha, \Delta\alpha)]}{\log(b^k)}\ ;$$

[26]By Large Deviation Theory we have $\ln[\Pr\{\alpha_k > \alpha\}] \sim k \cdot \delta(\alpha)$, as $k \to \infty$.

- The fractal dimension of the set of instants with Hölder exponent equal to α

$$f(\alpha) = \dim_{\mathcal{H}} (T_\alpha) \ ;$$

- The limit provided by Large Deviation Theory

$$f(\alpha) = \begin{cases} \lim_{k\to\infty} \frac{1}{k} \cdot \log_b (\Pr\{\alpha_k > \alpha\}) + 1 & \text{if } \alpha > \mu_\alpha \\ \\ \lim_{k\to\infty} \frac{1}{k} \cdot \log_b (\Pr\{\alpha_k < \alpha\}) + 1 & \text{if } \alpha < \mu_\alpha \end{cases} .$$

However, it is crucial to highlight that these definition *may* coincide. But there may be also discrepancies. For a deep analysis of such an eventuality see [11].

However, the previous explanations are based on microcanonical measure. It will be easy to extend the to the more general canonical case. As a matter of fact, considering the canonical measure, the coarse Hölder exponent is given by

$$\alpha_k(t) := \frac{\log \theta\left(\left[t, t+b^{-k}\right]\right)}{\log\left(b^{-k}\right)} = \frac{\log_b\left[\Upsilon \cdot \left(U_{\eta_1} \cdot U_{\eta_1\eta_2} \cdot \ldots \cdot U_{\eta_1\eta_2\ldots\eta_k}\right)\right]}{\log_b\left(b^{-k}\right)} =$$
$$= -\frac{1}{k} \cdot \log_b(\Upsilon) - \frac{1}{k} \cdot \left[\log_b\left(U_{\eta_1}\right) + \log_b\left(U_{\eta_1\eta_2}\right) + \ldots + \log_b\left(U_{\eta_1\eta_2\ldots\eta_k}\right)\right] .$$

Since when $k \to \infty$ the summand $-\frac{1}{k} \cdot \log_b(\Upsilon) \to 0$, hence the latter does not affect $f(\alpha)$. Then all the previous results on multifractal spectrum are still valid.

6 The MMAR

The Multifractal Model of Asset Returns appeared for the first time in the three paper series [5], [6], and [7], introducing the concept of multifractality to economics. This model attempts to describe price changes, accounting for several features of financial data: Long memory, fat tails and scale invariance. The authors especially criticized the GARCH-type representations, the latter assuming that the conditional distribution of the return (in respect to the information available until today) has a finite, time-varying second moment. Neither long-range dependence nor scale invariance can be described with them. Furthermore, being scale invariance the equivalence between representations of the model at different time-scales, the authors remark that the absence of an invariance under scaling implies that, in empirical works, the researcher adds an *additional restriction* to the model when choosing the time-scale of the data.

In this way, temporal heterogeneity is introduced by time-varying conditional second moments in a discrete time framework. Conversely, multifractality introduces a new source of heterogeneity through time-varying local regularity in the price path. The concept of local Hölder exponent is able to describe local regularity. Multifractal processes bridge the gap between locally Gaussian diffusions and jump-diffusions by allowing a multiplicity of Hölder exponents.

As a matter of fact, the MMAR generate heavy tails and a divergent variance directly in the directing process of log-returns. Prices of financial asset are seen as a *multiscaling process* with long memory and heavy tails. Persistence in volatility is given by the use of a *random trading time*, generated as the cumulative distribution function of a random multifractal measure (which, we remark, can be seen as a random variable).

Additionally, the authors tested and simulated the model generating very realistic sample paths. It might open the door to new theoretical and applied approaches to asset pricing and risk valuation. For the proofs of the theorems listed hereinafter, see [5], [6], and [7].

As we have already sketched, trading time plays a notable robe in transmitting multifractality to financial records. We do need a preliminary definition regarding a particular class of stochastic processes.

Definition 6.1. *Let $\{Z(t)\}_{t\in[0,+\infty)}$ be a stochastic process and $\theta(t)$ an increasing function of the time t. The process*

$$X(t) := Z\left[\theta(t)\right] \tag{16}$$

is called a compound process.

Since t denotes the clock physical time, the function $\theta(t)$ reproduces the so-called trading time. We are now able to state the theoretical assumption of the MMAR.

Definition 6.2. *Let $X(t)$ be the stochastic process describing rates of log-return, that is*

$$X(t) := \ln S(t) - \ln S(0) \tag{17}$$

where $\{S(t)\}_{t\in[0,T]}$ is the stochastic process leading the price of the financial asset, where $T \in \mathbb{R}^+$ is the final time[27]*. The MMAR bases on the following three hypotheses:*

[27] The reason of the introduction of a final time T, rather than taking into account a period $[0,+\infty)$, is due to multifractality which may exist only on bounded intervals. See Section 3.

H1 $\{X(t)\}_{t\in[0,T]}$ *is a compound process*

$$X(t) = B_H[\theta(t)]$$

where $B_H(t)$ *is a fractional Brownian motion with Hurst exponent* $H \in \mathbb{R} : 0 < H < 1$, *and* $\theta(t)$ *is a stochastic measure that leads trading time;*

H2 *The trading time* $\theta(t)$ *is the cumulative distribution function of a multifractal measure defined on* $[0, T]$[28]. *Thus,* $\theta(t)$ *is a multifractal process with continuous, non-decreasing paths, and stationary increments;*

H3 $\{B_H(t)\}_{t\in[0,T]}$ *and* $\{\theta(t)\}_{t\in[0,T]}$ *are independent.*

Hence trading time plays a crucial role in the MMAR. Moreover, compounding allows direct modelling of a process' variability without affecting the direction of the increments or their correlation.

We first note that $\theta(0)$ must be null due to the definition of $X(t)$, which imposes $X(0) = 0$. Moreover **H2** imposes that $\theta(t)$ be the cumulative distribution function of a self-similar random measure, such as the microcanonical or the canonical.

It is quite natural to expect that trading time $\theta(t)$ "transfers" multifractality to $X(t)$, and that scaling functions $\tau_\theta(q)$ and $\tau_X(q)$ may be related. The following theorem expresses this intuition.

Theorem 6.3. *Under hypotheses* **H1**, **H2**, **H3**, *the process* $X(t)$ *is multifractal, with stationary increments and scaling function*

$$\tau_X(q) = \tau_\theta(Hq)\,. \tag{18}$$

Hence, choosing a fractional Brownian motion as compounder and a multifractal measure to deform time, we are able to "spread" multiscaling to the process of the asset return $X(t)$. This is one of the most important property of the MMAR. In the following we analyze the other features of such a process.

Since $\mathbb{E}\big[|X(t)|^q\big]$ depends on $\mathbb{E}\left[\theta(t)^{Hq}\right]$, it is finite if and if only the process $\theta(t)$ has finite moment of order Hq. Hence, the trading time *controls* the moment of the return $X(t)$.

[28] All the measure introduced in Section 2 are defined on $[0, 1]$. With little effort, they can be extended to the set of times $[0, T]$, setting $\Delta t = b^{-k} \cdot T$, being t the usual b-adic number.

Due to **H2**, the trading time is generated by a self-similar random measure. It is crucial to highlight that, whether the measure is microcanonical or canonical, the moments possess very different features. As a matter of fact, microcanonical measures have a fixed mass on the interval $[0,T]$ on which are defined. So $\theta(t)$ is bounded, and the compound process $X(t)$ has finite moments of all order. As Mandelbrot said in [5] and [15], it generates *mild* randomness, with relatively thin tails.

On the other hand, being canonical measure depending on the random variable Υ, it permits the model to have divergent moments. The corresponding process $X(t)$ will be then *wild*. Overall, the MMAR has enough flexibility to allow for a wide variety of tail behaviour, both thin and fat.

We are now interested in martingale property of the MMAR. The following result follows.

Theorem 6.4. *If* $\{B_H(t)\}_{t\in[0,T]}$ *is a standard Brownian motion* $\left(H=\frac{1}{2}\right)$*, the following properties hold true:*

1. *If* $\mathbb{E}\left[\theta(t)^{\frac{1}{2}}\right]$ *is finite, then* $\{X(t)\}_{t\in[0,T]}$ *is a martingale with respect to its natural filtration;*

2. *If* $\mathbb{E}\left[\theta(t)\right]$ *is finite, the increments of* $\{X(t)\}_{t\in[0,T]}$ *are uncorrelated.*

This result is based on the martingale property of the Brownian motion. As it may be easily imagined, it does not extend when $H\neq\frac{1}{2}$. In this case, the following holds.

Theorem 6.5. *If* $\{B_H(t)\}_{t\in[0,T]}$ *is a fracional Brownian motion, and* $\mathbb{E}\left[\theta(t)^{2H}\right]$ *is finite, the autocovariance function at points* t_1,t_2 *of the incremental process* $V(t):=X(t+\Delta t)-X(t)$*, with* $t\geq\Delta t$*, is equal to*

$$\mathcal{C}_V(t_1,t_2)=c_\theta(2H)\cdot\frac{\sigma^2}{2}\cdot\left(|t_1-t_2+\Delta t|^{\tau_\theta(2H)+1}+|t_1-t_2-\Delta t|^{\tau_\theta(2H)+1}-2\cdot|t_1-t_2|^{\tau_\theta(2H)+1}\right),$$

where $\sigma^2=\mathbb{E}\left[B_H(1)^2\right]$.

If in addiction we postulate $t_1\geq t_2+\Delta t$, all the absolute values can be removed becoming

$$c_\theta(2H)\cdot\frac{\sigma^2}{2}\cdot\left[(t_1-t_2+\Delta t)^{\tau_\theta(2H)+1}+(t_1-t_2-\Delta t)^{\tau_\theta(2H)+1}-2\cdot(t_1-t_2)^{\tau_\theta(2H)+1}\right]$$

Consistently, if $H = \frac{1}{2}$ (that is a SBM), we find that the autocovariance function of the incremental process vanishes. In fact $\tau_\theta\left(2 \cdot \frac{1}{2}\right) = \tau_\theta(1) = 0$, and hence

$$\begin{aligned}\mathcal{C}_V(t_1, t_2) &= c_\theta(1) \cdot \frac{\sigma^2}{2} \cdot \left[(t_1 - t_2 + \Delta t)^1 + (t_1 - t_2 - \Delta t)^1 - 2 \cdot (t_1 - t_2)^1\right] \\ &= c_\theta(1) \cdot \frac{\sigma^2}{2} \cdot (t_1 - t_2 + \Delta t + t_1 - t_2 - \Delta t - 2 \cdot t_1 + 2 \cdot t_2) = 0\,.\end{aligned}$$

Thus the persistence of $\mathcal{C}_V$ is determined by the sign of $\tau_\theta(2H)$. Since the scaling function $\tau_\theta(q)$ has the same sign of $q-1$, when $H > \frac{1}{2}$ (the FBM has long-range dependence) we find that $\tau_\theta(2H)$ is positive, since $2H - 1 > 0$, and then the MMAR has long memory[29]; on the other hand, if $H < \frac{1}{2}$, the converse is true.

We finally examine the dependence in the absolute values of return, which indicates that the MMAR exhibits persistence in volatility. It is convenient to define the *autocovariance function of the q-th power* of the increments, that is

$$\mathcal{A}_X(t_1, t_2, q) := \mathcal{C}_{|V|^q}(t_1, t_2) = \mathbb{E}\big[|V(t_1)|^q \cdot |V(t_2)|^q\big]\,,$$

where $V(t) := X(t + \Delta t) - X(t)$ is the incremental process. The following theorem is related to this function.

Theorem 6.6. *If $\{B_H(t)\}_{t \in [0,T]}$ is a fracional Brownian motion, and $\mathbb{E}\big[\theta(t)^{Hq}\big]$ is finite, the compound process satisfies*

$$\mathcal{A}_X(t_1, t_2, q) \geq \mathcal{A}_\theta(t_1, t_2, Hq) \cdot \mathbb{E}\big[|B_H(1)|^q\big]^2\,,$$

for all $0 \leq q < q_{crit}$, where the equality holds when $H = \frac{1}{2}$.

A very surprising consequence of this theorem is that the return process has long memory in the absolute value of its increments, but if $H = \frac{1}{2}$ (that is a SBM), the returns display both *uncorrelated increments* and *persistence in volatility.*

The last theoretical property of the MMAR is about its multifractal spectrum.

[29] Here the definition of long memory requires more attention. As a matter of fact, unlike the fractional Brownian motion case, the MMAR is multifractal only on *bounded* intervals of times (and hence t cannot goes to infinity, if we want to preserve that property). In this case, long memory can be informally defined stating that the longest (apparent) cycle has approximatively the same length as the interval of definition. In this sense, the c.d.f. of a canonical measure has long memory.

Theorem 6.7. *Under hypotheses* **H1**, **H2**, **H3***, the process* $X(t)$ *has multifractal spectrum* $f_X(\alpha)$ *equal to*

$$f_X(\alpha) = f_\theta\left(\frac{\alpha}{H}\right) . \tag{19}$$

This theorem shows that the MMAR is multifractal since the trading time is allowed to have a continuum of coarse Hölder exponents, thus transferring multifractality through compounding. Moreover, long memory has an interesting geometric interpretation: When the return process is multiscaling, several sets T_α have non-integer fractal dimension $f(\alpha) \in (0, 1)$. Their elements necessarily cluster in certain regions of the interval of definition, so explaining the alternation of periods of large and small price changes. The set T_α is also statistically self-similar, in the sense that after proper rescaling, subsets of T_α have statistically the same relative placement of points than the original T_α. Therefore, the knowledge of T_α in one period owns important information of T_α in later periods. This property is another "face" of the long memory involved in the process.

7 A quick comparison

In this final section we again take into account the most famous models used for describing price changes (that is BM, FBM, and LSM), in order to compare their properties with those of MMAR. Moreover, we will see that the model we have introduced is the only one able to contemplate a great variety of suitable properties.

First of all, Theorem 6.4 states that when $H = \frac{1}{2}$ the process describing asset returns is a martingale, being so the future unpredictable from the knowledge of past prices. At the same time, Theorem 6.6 guarantees long memory in the absolute value of returns. That is, there is not only *time heterogeneity* in the size of log-returns, but also that heterogeneity is present on *any time scale* at which we decide to investigate the data, that is they may be daily, weekly, monthly, or yearly returns[30].

Moreover, the MMAR incorporates important features observed in financial time series, including long tails and invariance under scaling. Multifractality is defined by a set of restrictions on the process' moments as the time scale of observation change (that is

[30] The simultaneous presence of the martingale property associated with a memory of any sort might be surprising to many. However, the MMAR is not the only one model to have provided for it: Several authors have developed the so called FIGARCH models, which present both unpredictability of the future and memory. Nevertheless, the MMAR is the only one model to own that property, being scale-invariant as well.

the parameters of the model vary, but the structure of it does not modify). Furthermore, it is integrated in the model through *trading time*, a random distortion of clock time that leads to strong changes in volatility.

In addiction, the MMAR allows the possibility for returns to be uncorrelated, but does not impose it: The model has got enough flexibility to satisfy the martingale property in some cases and and long memory in its increments. But, perhaps, the main advantage of this model over alternatives is the property of scale-invariance. Because of this property, the whole information contained at different frequencies can be used to identify and test the model.

The following table summarizes what just stated, with the convenience to report part of Table 1.1 allowing the comparison with the SBM, FBM, and LSM.

Table 3.1: Comparison of the MMAR with BM, FBM and LSM.

	SBM	FBM	LSM	MMAR
Distribution of increments	Gaussian	Gaussian	L-stable	Gaussian
Independence of increments	✓	×	✓	✓[d]
Stationarity of increments	✓	✓	✓	✓
Mean of the process	0	0	∞ or 0[a]	0
Covariance of the process	finite	finite	infinite	finite
Semi-martingale	✓	×	✓	✓[d]
Self-affinity/uniscaling	✓	✓	✓[b]	✓
Multiscaling	×	×	×	✓
Long-range dependence	×	✓[c]	×	✓[c]

Continued on next page

Continued from previous page

	SBM	FBM	LSM	MMAR
Heaviness of the tails	$\times$	$\times$	$\checkmark$	$\checkmark^e$

[a] See [4].

[b] See [9].

[c] The process has long-range dependence if and only if $\frac{1}{2} < H < 1$.

[d] The process has independent increments and is a martingale if and only if $H = \frac{1}{2}$.

[e] Tails become heavy for those q-th absolute moments with $q \geq q_{crit}$.

The compatibility with the martingale property of returns and the long memory are given by Theorem 6.4 and Theorem 6.6 respectively. Otherwise, scale-invariance and multiscaling are consequence of the definition of multifractal processes. They respectively correspond to the time-invariance and non-linearity of the scaling function $\tau(q)$. Moreover, the multiscaling properties of the MMAR contrast with the unique scale contained in all the previous financial models.

The main disadvantage of the MMAR is the dearth of applicable statistical methods, although the authors of it – Mandelbrot, Calvet and Fisher – introduced few innovative tools able to be used for models being both time-invariant and scale-invariant.

Furthermore, the connection with the multifractal spectrum is also crucial. As a matter of fact, the former, $f(\alpha)$, can be seen as a renormalized density obtained as the limits of histograms. In an alternative, $f(\alpha)$ is viewed as the fractal dimension of the set of instants T_α with local Hölder exponent α. The statistical self-similarity of the set T_α is closely related to the long memory. In addiction, for a large class of multifractal measures, the spectrum can be explicitly derived from Large Deviation Theory through the Legendre transform. This allow the researcher to relate an empirical estimate of the spectrum back to a particular construction of the multifractal. On this direction goes the other paper of the authors [7].

The consequences on the MMAR can be so interpreted. The heterogeneity of the local scales along the price process is entirely given by the trading time $\theta(t)$. Moreover, the multifractal spectrum of the *price* is derived from the of the *trading time* by a very simple transformation.

Other practical aspects are of great interest to financial economics. Those are mainly the consequences of using a multifractal process and multifractality as underlying assumptions: The clear non-classic Brownian (mis)behaviour of the multifractal processes has enormous implication over the risk-adverse investors' decisions and the pricing of derivatives.

8 Conclusion and future development

Local Hölder exponent is a notion that can be well applied to functions and measures, deterministic or stochastic, with some adjustment. As a matter of fact, continuous stochastic process are characterized by a *unique* Hölder exponent. Differently, for the MMAR a continuum of Hölder exponent is allowed.

Defining log-returns in the following way

$$X(t, \Delta t) := X(t) - X(t - \Delta t),$$

where $X(t)$ is defined as in (17), we have

$$|X(t, \Delta t)| \sim \Delta t^{\alpha(t)} \tag{20}$$

for all t, as $\Delta t \to 0$. In the standard Brownian motion and in the standard fractional Brownian motion cases, the local variation are always proportional to $\Delta t^{\frac{1}{2}}$ and to Δt^H respectively. On the other hand, multifractal processes generate variety in local regularity while filling with a *continuum* of values of $\alpha(t)$. The Legendre transform of $\tau(q)$ (addressed as multifractal spectrum) is the main tool for describing the distribution of local Hölder exponents.

Because of the asymptotic relations of $\tau(q)$, the slope of the asymptotes α_{max} and α_{min} give the bounds of the support of the multifractal spectrum, that is

$$\tau^\star : \mathbb{R} \supset [\alpha_{min}, \alpha_{max}] \to \mathbb{R}\,,$$

since they are the least and the highest value α can take. Hence, if the scaling function $\tau(q)$ is defined on the entire real line, its asymptotic linear behaviour implies the multifractal spectrum $\tau^\star(\alpha)$ to be defined only on a closed set of values.

Since (20) holds, the largest the range of possible $\alpha(t)$, the riskiest is the asset since the coarse Hölder exponents can take more values. It conveys a more variability of the log-returns, and hence an higher uncertainty in the magnitude of future price variations. Moreover, the least the value of α_{min}, which is related to the random variable U_{max}, the riskiest the asset should be considered.

The asymptotic behavior of the scaling function and the connection with the αs through the Legendre transform should lead to reconsider the information connected with the higher moments of the log-returns' distribution. As a matter of fact, the right asymptote of $\tau(q)$ might have positive or negative slope according to the intrinsic riskiness of the asset. Since this quantity is linked to the higher moments of the distribution, more consistent estimation techniques[31] ought to be developed being α_{min} the most ruinous local exponent that can occur.

Moreover, a more in-depth study of the link with the distribution of U_{max} is surely necessary. Since U_j are considered to be random, a proper distribution should be chosen *a priori*[32] for practical purpose. Subsequently, the distribution of $\max\{U_j\}$ would require special attention since the maximum value of u_{max} defines α_{min}.

Finally, since asymptotic behaviour of the scaling function is related to the maximum value of the random generators, the explanation given for the presence of, and the subsequent capability to generate, heavy tails in the process might be related with the Extreme Value Theory. A more deep analysis of this connection could be a possible way to develop and discover new properties of the MMAR.

[31] The authors in [7] suggested a procedure for the values of the scaling function based on multiple regressions at different time-scales

$$\mathbb{E}\Big[\ln P(q,\Delta t)\Big] - \ln(T) = \ln c_X(q) + \tau_X(q)\cdot \ln(\Delta t)\,,$$

where $P(q,\Delta t) := \sum_{i=1}^{N}\big|X(i\Delta t,\Delta t)\big|^q$, having divided the interval $[0,T]$ into N subintervals of length Δt.

[32] In [6], the authors analyzed different possibilities for the multipliers' distribution. Mandelbrot proved that, in order to have canonical measure and hence allowing fat tails, the most immediate distribution to be chosen is the log-normal.

References

[1] Andreoli A., Caravenna F., Dai Pra P. and Posta G. (2012), *Scaling and Multiscaling in Financial Series: A Simple Model*, Advances in Applied Probability, Vol. 44, No. 4, pp. 1018-1051.

[2] Baldovin F. and Stella A. L. (2007), *Scaling and Efficiency Determine the Irreversible Evolution of a Market*, PNAS, Vol. 104, No. 50, 19741-19744.

[3] Baldovin F. and Stella A. L. (2008), *Role of Scaling in the Statistical Modelling of Finance*, Pramana Journal of Physics, Vol. 71, No. 2, pp. 341-352.

[4] Bianchi M. L., Fabozzi F. J., Kim Y. S. and Rachev S. T. (2011) *Financial Models with Levy Processes and Volatility Clustering*, Wiley.

[5] Calvet L., Fisher A. and Mandelbrot B. (1996), *A Multifractal Model of Asset Returns*, Cowles Foundation for Research in Economics, Discussion Paper No. 1164.

[6] Calvet L., Fisher A. and Mandelbrot B. (1996), *Large Deviations and the Distribution of Price Changes*, Cowles Foundation for Research in Economics, Discussion Paper No. 1165.

[7] Calvet L., Fisher A. and Mandelbrot B. (1996), *Multifractality of Deutschemark/US Dollar Exchange Rates*, Cowles Foundation for Research in Economics, Discussion Paper No. 1165.

[8] Calvet L. and Fisher A. (2012), *Extreme Risk and Fractal Regularity in Finance*, Working Papers Series.

[9] Falconer K. (2003), *Fractal Geometry: Mathematical Foundations and Applications*, Wiley.

[10] Guivarc'h Y. (1987), *Remarques sur les solutions d'une Ã©quation fonctionnelle non linÃ©aire de B. Mandelbrot*, C.R.A.S., SÃ©rie I , pp. 139-141.

[11] Mandelbrot B. (1989), *Multifractal Measures, Expecially fot the Geophysicist*, Pure and Applied Geophysics , Vol.131, pp 5-42.

[12] Mandelbrot B. (1999), *Survey of Multifractality in Finance*, Cowles Fundation for Research in Economics, Discussion Paper No. 1238.

[13] Mandelbrot B. (1999), *Multifractals and 1/f Noise: Wild Self-Affinity in Physics*, Springer.

[14] Mandelbrot B. (2001), *Gaussian Self-Affinity and Fractals*, Springer.

[15] Mandelbrot B. (2010), *Fractals and Scaling in Finance: Discontinuity, Concentration, Risk*, Springer.

[16] Mandelbrot B. (2006), *The Misbehavior of Markets: A Fractal View of Financial Turbulence*, Basic Books.

[17] Rachev S. T. (2003), *Handbook of Heavy Tailed Distributions in Finance*, North Holland.