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Branching with Hyperplanes in the Criterion Space: the Frontier Partitioner Algorithm for Biobjective Integer Programming

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Abstract

We present an algorithm for finding the complete Pareto frontier of biobjective integer programming problems. The method is based on the solution of a finite number of integer programs, each of them returning a Pareto optimal point. The feasible sets of the integer programs are built from the original feasible set, by adding cuts that separate efficient solutions. Providing the existence of an oracle to solve suitably defined single objective integer subproblems, the algorithm can handle biobjective nonlinear integer problems, in particular biobjective convex quadratic integer optimization problems. Our numerical experience on a benchmark of biobjective integer linear programming instances shows the efficiency of the approach in comparison with existing state-of-the-art methods. Further experiments on biobjective integer quadratic programming instances are reported.

keywords. Multiobjective Optimization; Integer Programming; Criterion Space Search

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1 Introduction

Most real-world optimization problems in the areas of applied sciences, engineering and economics involve multiple, often conflicting, goals. In the mathematical modelling of these problems, under the necessity of reflecting discrete quantities, logical relationships or decisions, integer and 0-1 variables need to be considered. We are in the context of multiobjective integer programming (MOIP) and the generic MOIP problem can be stated as follows:

$$\min_{x \in \mathcal{X} \cap \mathbb{Z}^n} y(x) = \min_{x \in \mathcal{X} \cap \mathbb{Z}^n} (y_1(x), \ldots, y_p(x)) \tag{MOIP}$$

where $\mathcal{X} \subseteq \mathbb{R}^n$ and $\mathcal{X} \cap \mathbb{Z}^n$ represents the feasible set in the decision space. The image of $\mathcal{X} \cap \mathbb{Z}^n$ under the vector-valued function $y : \mathbb{R}^n \to \mathbb{R}^p$ represents the feasible set in the criterion space, denoted by $\mathcal{Y} = \{z \in \mathbb{R}^p : z = y(x) \text{ for some } x \in \mathcal{X}\}$.

The challenging nature of MOIPs and the need of methods with guaranteed performance, motivated the development of exact approaches for multiobjective integer programming problems. Algorithms for multiobjective optimization can be divided into decision space search algorithms, i.e., approaches that search in the space of feasible solutions, and criterion space search algorithms, i.e., methods that search in the space of objective function values.

Among the decision space search algorithms, the first branch-and-bound algorithm for solving multiobjective mixed 0-1 integer programs was proposed by Mavrotas and Diakoulaki [24], who improved and extended their work in [23, 25]. In [32], the authors propose a branch-and-bound algorithm for multiobjective integer linear programming problems extending the bounding procedure introduced in [16]: the aim is to find separating hypersurfaces in the objective space between the upper and lower bound sets in order to prune the current node in the enumeration tree. More recently, Belotti and coauthors proposed advanced branch-and-bound algorithms for biobjective mixed-integer problems [2, 3]. They focus on the idea of finding the complete Pareto frontier for a relaxed subproblem, using this information to derive practical fathoming rules. We further mention [30, 33] as branch-and-bound algorithms for biobjective mixed integer linear programming problems.

Criterion space search algorithms find non-dominated points by addressing a sequence of single-objective optimization problems. Once a non-dominated point is computed, the dominated parts of the criterion space are removed and the algorithms go on looking for new non-dominated points. One of the first criterion space search algorithm is the one proposed in [11] from which we take the basic structure of our algorithm. Another important criterion space search algorithm is the one proposed in [34], improved in [20, 22]. Several contributions in this context have been given by Boland and coauthors [4, 5, 6, 7]. In our numerical experience, we compare our algorithm with the balanced box method proposed in [4]. The balanced box method maintains a priority queue with rectangles that are explored in order to detect nondominated points. In [13, 21] new theoretical insights on how to efficiently update the search region in branch-and-bound algorithms are given. In particular, in [13], assuming that a set of solutions is already known, an efficient algorithm to identify a minimal set of search zones that decompose

the search region is proposed.
In the application context, we mention works on biobjective minimum cost flow problems [26, 28, 31], on network routing problems [29], on the stable robotic flow shop scheduling problem [12] as well as the assignment problem with three objectives [27].
In this paper, we focus on biobjective integer programming, i.e. Problem (MOIP) with $p = 2$. We propose a branch-and-cut algorithm, called the Frontier Partitioner Algorithm (FPA), that belongs to the class of criterion space search algorithms.

Our contribution. We provide an Algorithm for biobjective integer problems with convergence rate $|\mathcal{Y}_N| + 2$ number of single objective integer programs, where $|\mathcal{Y}_N|$ is the number of non dominated points. This is the lowest convergence rate for this class of algorithms to the best of our knowledge.
The approach can be extended to nonlinear convex integer biobjective problems since the type of cuts introduced to partition the criterion space are linear in the criterion space and convex in the decision space as long as the objective functions of the problem are convex. This property allows us to tackle nonlinear convex integer biobjective problems, under the assumption that an oracle to solve the integer convex subproblems arising in the branching tree is available. The definition of our cuts relies on a positive scalar value, easily computable for several classes of problems. In particular, we are able to address quadratic convex integer biobjective problems as well as second order cone integer biobjective problems. As far as we know, the first general purpose method to tackle convex multiobjective integer programs is the heuristic algorithm proposed by Cacchiani and D'Ambrosio in [10]. In this respect, the Frontier Partitioner Algorithm gives a contribution in the context of exact methods for multiobjective nonlinear integer programs, defining a criterion space algorithm for nonlinear biobjective integer programs.

The paper is organized as follows. In Section 2, we give some basic definitions and concepts of multiobjective optimization, specifically adapted to biobjective integer optimization. We further report some assumptions we will need to define the algorithm as well as some scalarization techniques. In Section 3, we introduce and analyze our algorithm, providing convergence analysis, examples and a discussion on the assumptions. In Section 4 we introduce possible linear approximations of non linear inequalities and how to deal with numerical issues. Our numerical experience is presented in Section 5, where we compare our algorithm with the the balanced box method proposed in [4]. We further report results on biobjective integer quadratic programming instances. Section 6 concludes.

2 Notation and Preliminaries

We consider the biobjective integer programming BOIP, i.e. Problem (MOIP) with $p = 2$

$$\min_{x \in \mathcal{X} \cap \mathbb{Z}^n} (y_1(x), y_2(x)), \tag{BOIP}$$

where $\mathcal{X} \subseteq \mathbb{R}^n$ and the functions $y_1, y_2 : \mathbb{R}^n \to \mathbb{R}$ are continuous.

Definition 2.1. *A feasible solution $x \in \mathcal{X} \cap \mathbb{Z}^n$ is weakly efficient for Problem* (BOIP)*, if there is no feasible point $z \in \mathcal{X} \cap \mathbb{Z}^n$, such that $y_i(z) < y_i(x)$ for $i = 1, 2$. If $x \in \mathcal{X} \cap \mathbb{Z}^n$ is weakly efficient then $y(x)$ is called a weakly non-dominated point.*

Definition 2.2. *A feasible solution $x \in \mathcal{X} \cap \mathbb{Z}^n$ is efficient (or Pareto optimal) for Problem* (BOIP)*, if there is no feasible point $z \in \mathcal{X} \cap \mathbb{Z}^n$, such that $y_i(z) \leq y_i(x)$ for $i = 1, 2$ and $y(x) \neq y(z)$. If $x \in \mathcal{X} \cap \mathbb{Z}^n$ is efficient then $y(x)$ is called a non-dominated point. The set of all non-dominated points $\mathcal{Y}_N \subseteq \mathcal{Y}$ is called efficient frontier (or Pareto frontier).*

Definition 2.3. *The* ideal *objective vector of Problem* (BOIP) *is the vector $y^{id} \in \mathbb{R}^2$ defined component-wise as*

$$y_i^{id} = \min_{\mathcal{X} \cap \mathbb{Z}^n} y_i(x), \quad i = 1, 2. \tag{1}$$

In the following we use the following notation. Given a vector $x \in \mathbb{R}^n$, x_i is the $i-$th component and we use the sets $\mathbb{R}^n_{>0} = \{x \in \mathbb{R}^n : x_i > 0, i = 1, ..., n\}$, $\mathbb{R}^n_{\geq 0} = \{x \in \mathbb{R}^n : x_i \geq 0, i = 1, ..., n\}$ and $\mathbb{Z}^n_{\geq 0} = \{x \in \mathbb{Z}^n : x_i \geq 0, i = 1, ..., n\}$.

2.1 Assumptions

Our goal is to design an algorithm able to produce the entire Pareto frontier $\mathcal{Y}_N$ of Problem (BOIP). In order to achieve our aim we introduce a basic assumption on problem BOIP.

Assumption 2.4 (Existence of the ideal vector)**.** *We assume that the ideal objective vector y_i^{id}, $i = 1, 2$ exists and is finite.*

In the definition of our algorithm we need that it is assumed that a positive value exist that underestimate the distance between the image of two integer feasible points of (BOIP). We then give the following class of functions.

Definition 2.5 (Positive γ-function)**.** *A function $f : \mathbb{R}^n \to \mathbb{R}$ is a positive γ-functions over $\mathcal{X} \cap \mathbb{Z}^n$ if there exists a positive $\gamma \in \mathbb{R}$ such that $|f(x) - f(z)| \geq \gamma$, for all $x, z \in \mathcal{X} \cap \mathbb{Z}^n$ with $f(x) \neq f(z)$.*

We will widely use the following assumption.

Assumption 2.6 (Positive gap value)**.** *The functions $y_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2$ in Problem* (BOIP) *are positive γ-function as in definition 2.5.*

Note that, if γ_i were zero for some $i \in \{1, 2\}$ we would have that $y_i(x)$ is constant for all $x \in \mathcal{X} \cap \mathbb{Z}^n$. In Section 3.3 we will show that Assumption 2.6 is not restrictive and the algorithm can be applied to several classes of biobjective problems.
Under Assumption 2.6 and the Assumption 2.4, we can prove that the Pareto frontier is finite which is a commonly used assumption when defining exact algorithms for MOIP (see, e.g., [30, 4, 3]). We need this result to prove the convergence of our Frontier Partitioner Algorithm in Section 3.1.

Proposition 2.7. *Let Assumptions 2.4 and 2.6 hold. Then, the Pareto frontier $\mathcal{Y}_N$ is a finite set.*

Proof. Under Assumption 2.4, there exist $\hat{x}^i \in \arg\min_{\mathcal{X}\cap\mathbb{Z}^n} y_i(x),\ i = 1, 2.$
Hence there exist the values $M_1 = y_1(\hat{x}^2)$ and $M_2 = y_2(\hat{x}^1)$ such that the Pareto frontier $\mathcal{Y}_N \subseteq \mathcal{Y}$ is contained in the box $[y_1^{id}, M_1] \times [y_2^{id}, M_2]$ and hence it is bounded. Each objective function y_i can attain at most $\frac{M_i - y_i^{id}+1}{\gamma_i}$ distinct values, so that the cardinality of the Pareto frontier, obtained as the combination of the two, is finite and at most

$$\frac{(M_1 - y_1^{id} + 1)(M_2 - y_2^{id} + 1)}{\gamma_1\gamma_2}.$$

□

2.2 Scalarization techniques

To properly discuss our method we need a scalarization technique which allows to find a Pareto point of (BOIP), known as *scalarized problem*. We refer the interested reader to [15] for a complete overview of scalarization techniques.
We call a scalarized problem as Inner Scalarized program (ISP) and we will focus on three main scalarization problems reported below. To prove convergence of our FPA algorithm we will need an assumption on the scalarized problems generated during the partitioning of the region (Assumption 3.1) which requires at least that the (ISP) is bounded. Hence in the following for each scalarized problem, we show that this property holds.

Definition 2.8 (Weighted-sum ISP)**.** *Given (BOIP), the weighted-sum scalarization problem (ISP$_W$) is defined as*

$$\min_{x\in\mathcal{X}\cap\mathbb{Z}^n} \lambda_1 y_1(x) + \lambda_2 y_2(x), \tag{ISP$_W$}$$

where $\lambda_i \geq 0$, for $i = 1, 2$.

It is well known that under a proper choice of the weights the solution of the scalarized problem (ISP$_W$) leads to an efficient solution as reported in the following proposition.

Proposition 2.9 (Proposition 3.9 [15])**.** *Let $\lambda_1, \lambda_2 > 0$, then each solution of Problem* (ISP$_W$) *is an efficient solution for Problem* (BOIP)*.*

We observe that the converse is true only under proper convexity assumptions. Since integer multiobjective optimization problems are non-convex we cannot expect to find all efficient solutions by weighted sum scalarization as pointed out in Chapter 8 of [15]. Nevertheless as we mentioned above, we only need to ensure that (ISP$_W$) is bounded and this easily follows from Assumption 2.4 as reported in Remark 2.10 below.

Remark 2.10. *Under Assumption 2.4 problem (ISP$_W$) is bounded. Indeed, assume by contradiction that (ISP$_W$) is unbounded. Then, for all $M > 0$ there exists $x \in \mathcal{X} \cap \mathbb{Z}^n$ such that $\lambda_1 y_1(x) + \lambda_2 y_2(x) < -M$. We get a contradiction from Assumption 2.4, as $y_1(x) \geq y_1^{id}$, $y_2(x) \geq y_2^{id}$ and $\lambda_1, \lambda_2 \geq 0$.*

In the following we define two further scalarized problems that will be used in order to define an improved version of our algorithm.

Definition 2.11 (Lexicographic ISP)**.** *Given (BOIP) and a permutation* (i_1, i_2) *of the set* $\{1, 2\}$*, the* ***lexicographic scalarization problem*** *is defined as*

$$\min_{x \in \mathcal{X} \cap \mathbb{Z}^n} \left\{ y_{i_1}(x) : y_{i_2}(x) = y_{i_2}^{id} \right\},$$

where $y_{i_2}^{id} = \min_{x \in \mathcal{X} \cap \mathbb{Z}^n} y_{i_2}(x)$.

In the following we denote the lexicographic scalarization problem as:

$$\text{lex} \min_{x \in \mathcal{X} \cap \mathbb{Z}^n} (y_{i_1}(x), y_{i_2}(x)) \tag{ISP$_L$}$$

We note that addressing Problem (ISP$_L$) requires the solution of two single-objective mixed integer programming problems.
A well known result is the following.

Proposition 2.12 (Lemma 5.2 [15])**.** *Each solution of Problem* (ISP$_L$) *is an efficient solution for Problem* (BOIP)*.*

Also for the (ISP$_L$) we can prove boundedness under Assumption 2.4 as explained below in Remark 2.13.

Remark 2.13. *Under Assumption 2.4 problem (ISP$_L$) is bounded. In fact by Assumption 2.4 we know that there exists finite* $y_i^{id} = \min_{x \in \mathcal{X} \cap \mathbb{Z}^n} y_i(x)$*, for each* $i \in \{1, 2\}$*. This implies that* $y_{i_1}^{id} \leq \min_{x \in \mathcal{X} \cap \mathbb{Z}^n} \left\{ y_{i_1}(x) : y_{i_2}(x) = y_{i_2}^{id} \right\}$.

We further define a new scalarization which is a suitable weighted-sum strongly related to the lexicographic scalarization problem.

Definition 2.14 (Custom weighted-sum ISP)**.** *Let* $\hat{x} \in \mathcal{X} \cap \mathbb{Z}^n$ *be any solution of Problem* (ISP$_L$) *with respect to the permutation* (i_1, i_2) *of the set* $\{1, 2\}$*.*
The ***custom weighted-sum scalarization problem*** *is defined as*

$$\min_{x \in \mathcal{X} \cap \mathbb{Z}^n} \lambda_1^* y_1(x) + \lambda_2^* y_2(x), \tag{ISP*}$$

where $(\lambda_1^*, \lambda_2^*)$ *belongs to the set* $\{(\lambda_1, \lambda_2) \in \mathbb{R}^2_{>0} : \hat{x} \in \arg\min_{x \in \mathcal{X} \cap \mathbb{Z}^n} \lambda_1 y_1(x) + \lambda_2 y_2(x)\}$.

For sake of simplicity it is not explicitly required that $\lambda_1 + \lambda_2 = 1$, since this property can be gain by simple normalization.
The underlying idea of the custom weighted-sum scalarization is to reduce the computational complexity of the lexicografic rule, as one solution can be obtained by solving only one ILP instead of two as needed by (ISP$_L$).
Note however that the custom weighted-sum problem cannot be always defined, as the existence of the weights in the definition of Problem (ISP*) is not always guaranteed.

In the following we prove that this is the case when both Assumption 2.4 and Assumption 2.6 are satisfied. From an algorithmic point of view, we will see that the existence of such weights allows us to take advantage of the properties of both weighted-sum and lexicographic method and to significantly improve the complexity of our algorithmic framework.
Let us now describe a way to derive suitable weights for (ISP*). Given a permutation (i_1, i_2) of the set $\{1, 2\}$ let

$$\mathcal{P} = \arg\operatorname{lex}\min_{x \in \mathcal{X} \cap \mathbb{Z}^n} (y_{i_1}(x), y_{i_2}(x))$$

Let $\hat{x} \in \mathcal{P}$ and $y_i(\hat{x}) = \hat{y}_i$. For each $x \in \mathcal{P}$ we have that $y_i(x) = \hat{y}_i$. Further let

$$\bar{y}_{i_2} = \arg\operatorname{lex}\min_{x \in \mathcal{X} \cap \mathbb{Z}^n} (y_{i_2}(x), y_{i_1}(x)).$$

By proposition 2.12 both $\bar{y}$ and $\hat{y}$ are non-dominated points of (BOIP).
Let $\gamma = \min_{i=1,2} \{\gamma_i\} > 0$ and $\epsilon \in (0, \gamma)$. We define $\lambda^* \in \mathbb{R}^2_{>0}$ as

$$\lambda^* \ : \ \lambda^*_i = \begin{cases} \dfrac{\gamma - \epsilon}{\hat{y}_{i_1} - \bar{y}_{i_1}}, & \text{if } i = i_1 \\ 1, & \text{if } i = i_2 \end{cases} \tag{2}$$

We now show that under simple assumptions it is always possible to derive weights for (ISP*).

Theorem 2.15. *Given (BOIP), let Assumption 2.4 and Assumption 2.6 hold. Given a permutation (i_1, i_2) of the set $\{1, 2\}$ there exists weights $\lambda^* \in \mathbb{R}^2_{>0}$ such that $\lambda^*_{i_1} \leq \lambda^*_{i_2}$ and*

$$\arg\operatorname{lex}\min_{x \in \mathcal{X} \cap \mathbb{Z}^n} (y_{i_1}(x), y_{i_2}(x)) \ = \ \arg\min_{x \in \mathcal{X} \cap \mathbb{Z}^n} \lambda^{*\intercal} y(x)$$

Proof. Without loss of generality we fix $i_1 = 1$ and $i_2 = 2$. Define $\hat{y}$, $\bar{y}_1$ and $\lambda^* \in \mathbb{R}^2_{>0}$ as in (2). Let $x^* \in \arg\min_{x \in \mathcal{X} \cap \mathbb{Z}^n} \lambda^{*\intercal} y(x)$. Since $\lambda^* \in \mathbb{R}^2_{>0}$ the solution x^* does not change with a normalization coefficient, so that by proposition 2.12 $y(x^*) = y^*$ is a non-dominated point. Further we have that $\lambda^{*\intercal} y(x^*) \leq \lambda^{*\intercal} y(x) \ \ \forall \ x \in \mathcal{X} \cap \mathbb{Z}^n$, and, in particular, $\lambda^{*\intercal} (\hat{y} - y^*) \geq 0$ and from the definition of λ^* we get

$$\frac{\gamma - \epsilon}{\hat{y}_1 - \bar{y}_1} (\hat{y}_1 - y^*_1) + \hat{y}_2 - y^*_2 \geq 0. \tag{3}$$

Taken into account that both $\hat{y}$ and y^* are non-dominated points for (BOIP) and that $\hat{y}$ is in $\mathcal{P}$ only one of the following situations may occur

(i) $y^*_1 < \hat{y}_1$ and $y^*_2 > \hat{y}_2$

(ii) $y^*_1 = \hat{y}_1$ and $y^*_2 = \hat{y}_2$

We show that only case (ii) holds and this implies the theorem. Indeed, we need to prove that (i) cannot happen. Assume by contradiction that (i) holds. Then, we would have $\bar{y}_1 \leq y_1^* < \hat{y}_1$ implying

$$\hat{y}_1 - \bar{y}_1 \geq \hat{y}_1 - y_1^*. \tag{4}$$

Using (4) within (3) we get $\gamma - \epsilon + \hat{y}_2 - y_2^* \geq 0$. Since $y_2^* - \hat{y}_2 \geq \gamma$ we have $\gamma - \epsilon \geq \gamma$ and hence $\epsilon \leq 0$ which leads to a contradiction. ∎

We now prove that we have an infinite number of vector weights that can be used.

Proposition 2.16. *Given (BOIP), let Assumption 2.4 and Assumption 2.6 hold. Given a permutation (i_1, i_2) of the set $\{1, 2\}$ and given weights $\lambda^* \in \mathbb{R}^2_{>0}$ such that $\lambda^*_{i_1} \leq \lambda^*_{i_2}$ and*

$$\arg\operatorname{lex}\min_{x \in \mathcal{X} \cap \mathbb{Z}^n} (y_{i_1}(x), y_{i_2}(x)) \;=\; \arg\min_{x \in \mathcal{X} \cap \mathbb{Z}^n} \lambda^{*\intercal} y(x) \tag{5}$$

Then the vector weights $\tilde{\lambda} = (\alpha\lambda_1^*, \lambda_2^*)$ satisfy (5) for every $\alpha \in (0, 1]$.

Proof. Without loss of generality we set $i_1 = 1$ and $i_2 = 2$ and we call $\lambda^{\mathcal{X}} = \frac{1}{\lambda_2^*}\lambda^* = \left(\lambda_1^{\mathcal{X}}, 1\right)^\intercal$, since we know from Theorem 2.15 that $\lambda_1^* \leq \lambda_2^*$ and then $\lambda_1^{\mathcal{X}} \leq 1$. In the next we will call x^* a generic element of the set $\arg\min_{x \in \mathcal{X} \cap \mathbb{Z}^n} \lambda^{*\intercal} y(x)$.

Consider weights $\tilde{\lambda} = \left(\tilde{\lambda}_1, 1\right)^\intercal$. We want to show that if $0 < \tilde{\lambda}_1 < \lambda_1^*$ then

$$\arg\min_{x \in \mathcal{X} \cap \mathbb{Z}^n} \lambda^{\mathcal{X}\intercal} y(x) \;=\; \arg\min_{x \in \mathcal{X} \cap \mathbb{Z}^n} \tilde{\lambda}^\intercal y(x)$$

By contradiction suppose that it exists a feasible point $\tilde{x}$ such that $\tilde{\lambda}^\intercal y(\tilde{x}) \leq \tilde{\lambda}^\intercal y(x^*)$. This implies

$$y_2(x^*) - y_2(\tilde{x}) \geq \tilde{\lambda}_1 \left(y_1(\tilde{x}) - y_1(x^*)\right) \tag{6}$$

By the very definition of x^* we know that $\lambda^{\mathcal{X}\intercal} y(x^*) \leq \lambda^{\mathcal{X}\intercal} y(\tilde{x})$ and therefore

$$y_2(x^*) - y_2(\tilde{x}) \leq \lambda_1^{\mathcal{X}} \left(y_1(\tilde{x}) - y_1(x^*)\right) \tag{7}$$

Combining (6) and (7) we get $y_1(\tilde{x}) \geq y_1(x^*)$. From Proposition 2.12 we have that both $\tilde{x}$ and x^* are efficient points, then from the hypothesis on λ^* it must happens that $y_1(\tilde{x}) < y_1(x^*)$ and $y_2(\tilde{x}) > y_2(x^*)$, generating a contradiction. ∎

We now want to show that for a certain family of subsets of $\mathcal{X} \cap \mathbb{Z}^n$ it exists a vector of weights suitable for the subproblems defined over the subsets of this family.

Proposition 2.17 (λ-Inheritance)**.** *Given (BOIP), let Assumption 2.4 and Assumption 2.6 hold. Given a family Ω of subsets of $\mathcal{X} \cap \mathbb{Z}^n$ such that the Pareto frontier of $\min_{x \in H}(y_1(x), y_2(x)) \subseteq \mathcal{Y}_N$ and given a permutation (i_1, i_2) of the set $\{1, 2\}$, then the weight vector $\lambda^* \in \mathbb{R}^2_{>0}$ derived by (2) is such that $\lambda^*_{i_1} \leq \lambda^*_{i_2}$ and*

$$\arg\operatorname{lex}\min_{x \in \mathcal{H}}(y_{i_1}(x), y_{i_2}(x)) \;=\; \arg\min_{x \in \mathcal{H}} \lambda^{*\intercal} y(x), \quad \forall\ \mathcal{H} \in \Omega$$

Proof. For any set $\mathcal{H} \in \Omega$ we derive weights $\lambda^{\mathcal{H}}$ from (2). Since Assumption 2.4 and Assumption 2.6 hold these weights are such that

$$\arg\operatorname{lex}\min_{x\in\mathcal{H}}(y_{i_1}(x), y_{i_2}(x)) \;=\; \arg\min_{x\in\mathcal{H}} \lambda^{\mathcal{H}\intercal} y(x)$$

Since $\mathcal{H} \in \Omega$ then $\min_{x\in H}(y_1(x), y_2(x)) \subseteq \mathcal{Y}_N$. This implies that $\lambda_1^{\mathcal{H}} \geq \lambda_1^*$ and $\lambda_2^{\mathcal{H}} = \lambda_2^* = 1$. All the hypothesis of Proposition 2.17 are satisfied and therefore

$$\arg\operatorname{lex}\min_{x\in\mathcal{H}}(y_{i_1}(x), y_{i_2}(x)) \;=\; \arg\min_{x\in\mathcal{H}} \lambda^{*\intercal} y(x)$$

Since this is valid for any $\mathcal{H} \in \Omega$ the theorem is true. ∎

Refer now to (ISP) as a scalarized problem chosen among (ISP_W), (ISP_L) and (ISP*). Under Assumptions 2.4 and 2.6 by Proposition 2.7 the Pareto frontier is finite. Using Remarks 2.10 and 2.13, we have that any of the (ISP) problems either has an optimal solution which is an efficient solution for problem (BOIP), or it is infeasible and hence (BOIP) too. This property turns out to be crucial when proving well-posedness and convergence of `FPA` (see the proof of Theorem 3.7).
In the definition of our algorithm we can use also other scalarization techniques which guarantee that the scalarized problem either has a solution or it is infeasible. In particular compromise programming with a norm ℓ_p with $1 \leq p < \infty$ can also fit in this setting. However the scalarization techniques proposed above present the advantage that the corresponding scalarized problem belongs to the same class of the original (BOIP). In other words if BOIP has linear objective functions the scalarized problem is an ILP, if BOIP has quadratic objectives, the scalarized problem is an IQP and more in general the objective function of the scalarized problem maintains the structure of the original ones in (BOIP).

3 The Frontier Partitioner Algorithm

In this section we introduce the Frontier Partitioner Algorithm `FPA`. Convergence and finiteness of the algorithm is analyzed in Section 3.1.
The `FPA` uses a "divide and conquer" approach to explore the Pareto frontier of (BOIP). Starting from a non-dominated solution the method builds two subproblems where the chosen non-dominated point and all its dominated points are infeasible. Hence the key ingredients of `FPA` are

- the construction of subproblems using properly defined inequalities,
- the computation of non-dominated solutions at each node of the branching tree.

At a generic node k in the branching tree the subproblem (BOIP^k)

$$\min_{x\in\mathcal{X}^k\cap\mathbb{Z}^n} y(x) \qquad (\text{BOIP}^k)$$

is constructed, being $\mathcal{X}^k \subseteq \mathcal{X}$ the set obtained intersecting $\mathcal{X}$ with properly defined inequalities. For $k = 0$, i.e. at the root node, we define (BOIP^0)=(BOIP) and $\mathcal{X}^0 = \mathcal{X}$. For $k > 0$ the definition $\mathcal{X}^k$ is clarified below.
In order to compute a non-dominated solution of (BOIP^k) we can use any scalarization proposed in Section 2.2 to get (ISP^k). We need the following assumption on the scalarized problem (ISP^k).

Assumption 3.1 (Solvability of the scalarized problem)**.** *There exists an oracle that either returns an optimal solution of (ISP) or it certifies the infeasibility of problem (ISP^k).*

In other words an *oracle* is an algorithm able to solve (ISP^k). From the point of view of implementation using an oracle means calling a suitable solver.
In case (ISP^k) has a solution, two children nodes in the branching tree are produced. Let $\hat{x}^k \in \mathcal{X}^k \cap \mathbb{Z}^n$ be an optimal solution of $(ISP)^k$ and $\hat{y}^k = y(\hat{x}^k)$.

Let $\epsilon_i \in (0, \gamma_i]$, $i = 1, 2$ where γ_i satisfy 2.6. We consider the inequalities

$$y_i(x) \leq \hat{y}_i^k - \epsilon_i,\ i = 1, 2. \tag{8}$$

Remark 3.2. *The inequalities $y_i(x) \leq \hat{y}_i^k - \epsilon_i,\ i = 1, 2$ cut the non-dominated solution $\hat{y}^k$ and they are linear in the criterion space. Furthermore, they are convex (linear) in the decision space as long as the functions $y_i(x),\ i = 1, 2$ are convex (linear).*

From $(BOIP)^k$, using inequalities (8), we define the two children nodes of node k as follows:

$$\min_{x \in \mathcal{X}_1^k \cap \mathbb{Z}^n} y(x) \qquad \mathcal{X}_1^k = \mathcal{X}^k \cap \{x \in \mathbb{R}^n \,:\, y_1(x) \leq \hat{y}_1^k - \epsilon_1\},$$

$$\min_{x \in \mathcal{X}_2^k \cap \mathbb{Z}^n} y(x) \qquad \mathcal{X}_2^k = \mathcal{X}^k \cap \{x \in \mathbb{R}^n \,:\, y_2(x) \leq \hat{y}_2^k - \epsilon_2\}.$$

The Frontier Partitioner Algorithm produces iteratively a finite list of BOIPs.

Remark 3.3. *At a generic node $k > 0$ of the branching tree, the feasible region $\mathcal{X}^k$ is obtained from the original $\mathcal{X}$ adding at most two inequalities. Each inequality takes the form $y_i(x) \leq const_i$ with $i = 1, 2$, being $const_i = \min_{0 \leq j \leq k} \hat{y}_i^j - \epsilon_i$. Hence, letting m be the number of constraints defining $\mathcal{X}$, the number of constraints of $\mathcal{X}^k$ is at most $m + 2$ for all k, as there will be at most two non-redundant constraints.*

The scheme of the Frontier Partitioner Algorithm `FPA` is reported in Algorithm 1.

Algorithm 1: FPA algorithm

Input: $\mathcal{L} = \{(BOIP)^0\}$, $\mathcal{X}^0 = \mathcal{X}$, $\mathcal{Y}_N = \emptyset$, $\gamma_i > 0$, $\epsilon_i \in (0, \gamma_i], i = 1, 2$
Output: the Pareto frontier $\mathcal{Y}_N$ of $(BOIP)$
while $\mathcal{L} \neq \emptyset$ **do**
  Select a node $(BOIP)^k \in \mathcal{L}$ and delete it from $\mathcal{L}$
  Derive $(ISP)^k$ from $(BOIP)^k$
  Solve $(ISP)^k$.
  if $(ISP)^k$ *has a solution* $\hat{x}^k$ **then**
    Set $\mathcal{Y}_N = \mathcal{Y}_N \cup \{\hat{y}^k\}$, where $\hat{y}^k = y(\hat{x}^k)$
    Build $(BOIP)^k_i$, $i = 1, 2$ from $(BOIP)^k$

$$(BOIP)^k_i := \min\left\{y(x) : x \in \mathcal{X}^k_i \cap \mathbb{Z}^n\right\}$$

Where $\mathcal{X}^k_i := \mathcal{X}^k \cap \{x \in \mathbb{R}^n : y_i(x) \leq \hat{y}^k_i - \epsilon_i\},\ i = 1, 2,$

    Add the new nodes $(BOIP)^k_1$ and $(BOIP)^k_2$ to $\mathcal{L}$;
  end
end
Return $\mathcal{Y}_N$

We prove in Section 3.1 that under suitable assumptions `FPA` is well posed and terminates finitely, returning the entire Pareto frontier $\mathcal{Y}_N$.
The BOIPs generated are related to those that would be built applying the approach proposed in [11, 21, 13].
In particular in [11] the BOIPs are generated by using the weighted-sum scalarization method and the existence of an oracle for solving BOIP is assumed. In [21] the authors present a method which identifies a region where it is possible to find further non-dominated points. The region is updated iteratively each time a new non-dominated point if found. To this aim they construct a list $\mathcal{U}$ of *local upper bounds* and keep it updated according to the new non-dominated points found. It is mentioned how to use this list of local upper bounds in order to define an algorithm for multiobjective combinatorial optimization. At each iteration k, a local upper bound $u^k \in \mathcal{U}$ is selected and the subproblem $P(u^k)$ is built by adding to the feasible region $X \cap \mathbb{Z}^n$ the constraints $y(x) < u^k$. Problem $P(u^k)$ is solved, the list $\mathcal{U}$ is updated and a new efficient point is eventually found. In case $p = 2$, the inequalities $y(x) < u^k$ are strongly related to the inequalities introduced in (8), as $u^k \in \mathcal{U}$ is built by using components of non-dominated points found so far, so that they read as $y_i(x) < \hat{y}^j_i$ for some $j = 1, \ldots, k$ and for all $i = 1, 2$. We underline that `FPA` does not need to construct the list $\mathcal{U}$ and hence it does not need any algorithmic procedure to keep $\mathcal{U}$ updated and filter dominated local upper bounds as needed in [21, 13]. Furthermore, in [21] minor details on how to address the solution of the integer subproblem $P(u^k)$ are given. Indeed, handling strict inequalities is not allowed when using standard MISP softwares such e.g. CPLEX [19], Gurobi [18], SCIP [17], Couenne [1] or Bonmin [8]. In this respect, the definition of γ_i and of $\epsilon_i \in (0, \gamma_i],\ i = 1, 2$ in (8) is crucial to obtain scalarized problems which satisfy Assumption 3.1, namely for which an oracle exists using standard available softwares. The main contribution of the paper stays in the emebdding the new custom weighted-

sum scalarization procedure within the divide-and-conquer procedure. The use of the new scalarization allows to prove that exactly $|\mathcal{Y}_N|+2$ nodes are generated as reported in the next Section.

3.1 Convergence Analysis

As a first step in the convergence analysis of the Frontier Partitioner Algorithm, we prove that the cuts used in Algorithm 1 induce a partition of the decision space.

Proposition 3.4. *Let Assumption 2.6 holds. Then $\mathcal{X}_1^k \cap \mathcal{X}_2^k \cap \mathbb{Z}^n = \emptyset$ for all k in Algorithm 1.*

Proof. Let $\hat{x}^k \in \mathcal{X}^k \cap \mathbb{Z}^n$ be an efficient point for $(BOIP)^k$ corresponding to the non-dominated value $\hat{y}^k$. Assume by contradiction that $\mathcal{X}_1^k \cap \mathcal{X}_2^k \cap \mathbb{Z}^n \neq \emptyset$. Then $x \in \mathcal{X}_1^k \cap \mathcal{X}_2^k \cap \mathbb{Z}^n$ exists and satisfies $y_i(x) < y_i(\hat{x}_k)$, for $i = 1, 2$ as $\epsilon_i > 0$ by assumption. This contradicts the fact that $\hat{x}^k$ is an efficient solution for $(BOIP)^k$. ∎

Remark 3.5. *Under Assumption 2.6, we have that $y(x) \neq \hat{y}^k$ for any $x \in \mathcal{X}_1^k \cap \mathbb{Z}^n$ and any $x \in \mathcal{X}_2^k \cap \mathbb{Z}^n$. Therefore, the inequalities used to define $\mathcal{X}_i^k$, $i = 1, 2$ exclude all the efficient solutions $\hat{x}$ such that $\hat{y}^k = y(\hat{x})$.*
Since $|y_i(x) - \hat{y}_i^k| \geq \gamma_i$ for $i = 1, 2$, we have that

$$\mathcal{Y}_N^k \;=\; \{\hat{y}^k\} \;\cup\; \mathcal{Y}_N^{k,1} \;\cup\; \mathcal{Y}_N^{k,2}$$

where $\mathcal{Y}_N^{k,i}$, $i = 1, 2$ denote the Pareto frontier of the children nodes of $(BOIP)^k$.
Therefore, the Pareto frontier is recursively obtained as $\mathcal{Y}_N = \{\hat{y}^0\} \cup \mathcal{Y}_N^{0,1} \cup \mathcal{Y}_N^{0,2}$.

In the following proposition, we prove that every node tackled in Algorithm 1 either can be pruned or it yields a yet unknown Pareto point.

Proposition 3.6. *Let $\hat{y} \in \mathcal{Y}_N^k$ be a non-dominated point of $(BOIP)^k$. Then, the child problem $(BOIP)_i^k$*

$$\begin{array}{ll} \min & y(x) \\ \text{s.t.} & x \in \mathcal{X}^k \cap \{y_i(x) \leq \hat{y}_i - \epsilon_i\} \\ & x \in \mathbb{Z}^n \end{array} \tag{9}$$

with $i = 1, 2$ is either infeasible or any of its optimal solutions is efficient for $(BOIP)^k$, leading to a new non-dominated point $\bar{y} \neq \hat{y}$.

Proof. If problem (9) is infeasible there is nothing to prove. W.l.o.g. let $i = 1$ and let $\bar{x}$ be a solution of (9) (case $i = 2$ can be proven identically). By contradiction, assume that $\bar{x}$ is not efficient for $(\text{BOIP})^k$. Then, $\tilde{x} \in \mathcal{X}^k \cap \mathbb{Z}^n$ exists such that $y_i(\tilde{x}) \leq y_i(\bar{x})$, for $i = 1, 2$ and $y(\tilde{x}) \neq y(\bar{x})$. In particular, we have that

$$y_1(\tilde{x}) \leq y_1(\bar{x}) \leq \hat{y}_1 - \epsilon_1$$

so that $\tilde{x}$ is feasible for (9). Since $y(\tilde{x}) \neq y(\bar{x})$, we necessarily have that either $y_1(\tilde{x}) < y_1(\bar{x})$ or $y_2(\tilde{x}) < y_2(\bar{x})$, contradicting the fact that $\bar{x}$ is efficient for (9). Furthermore,

we have that $y(x) \neq \hat{y}$ for any point x feasible for (9) as stated in Remark 3.5, so that $\bar{x}$ leads to a non-dominated point $y(\bar{x}) = \bar{y} \neq \hat{y}$. ∎

Now we are ready to prove the finite convergence of Algorithm 1.

Theorem 3.7. *Let Assumptions 2.4, 2.6 and 3.1 hold. Algorithm 1 returns the complete Pareto frontier $\mathcal{Y}_N$ of* (BOIP) *after generating exactly $2|\mathcal{Y}_N| + 1$ (BOIP^k).*

Proof. At each iteration k of the `FPA` a node $(BOIP)^k$ is chosen and the corresponding $(ISP)^k$ is built.
Assumption 3.1 allows us to solve Problem $(ISP)^k$. Using Remark 2.10, we have that either $(ISP)^k$ has a solution or it is infeasible. If $(ISP)^k$ is infeasible, we conclude that $\mathcal{X}^k \cap \mathbb{Z}^n$ does not contain any efficient point and the node $(BOIP)^k$ is pruned. Otherwise we have that the returned solution of $(ISP)^k$ is efficient for $(BOIP)^k$, giving us a non-dominated point $\hat{y}^k$. Using Proposition 3.6, we have that $\hat{y}^k$ belongs to the Pareto frontier of (BOIP). By Proposition 3.4 and Remark 3.5, the non-dominated point $\hat{y}^k \in \mathcal{Y}_N$ cannot be detected again by addressing any subsequent node in the branching tree and the inequalities induced by $\hat{y}^k$ do not cut any yet unknown Pareto point. Summarizing, whenever a node is addressed, either we get a new non-dominated point or we detect infeasibility and the node is pruned. Therefore since `FPA` produces exactly two children for each non-dominated point of (BOIP), we have that the branching tree has exactly $2|\mathcal{Y}_N|+1$ nodes (including the root node) so that exactly $2|\mathcal{Y}_N|+1$ (BOIP^k) are generated. ∎

We now show that the use of different scalarization techinques may lead to a different number of solutions of inner scalarized programs (ISP), namely to a different number of oracle calls, which represent the main computational burden. In particular we will show that the use of the custom weighted-sum problem allows us to define an improved version of `FPA`, called `FPA*`, able to detect the complete Pareto frontier after having addressed only $|\mathcal{Y}_N| + 2$ integer programs.
Before describing the special case of the FPA with the custom weighted-sum scalarization, we report the results that follows directly from Theorem 3.7. In particular for the FPA with (ISP) obtained by the weighted-sum scalarization technique we have the following results that has already been proved in [11].

Corollary 3.8. *If (ISP^k_W) as defined in* (2.12) *with strictly positive weights $\lambda_i > 0$ is used to define the scalarization program then Algorithm 1 returns the complete Pareto after having addressed $2|\mathcal{Y}_N| + 1$ ISPs of which $|\mathcal{Y}_N| + 1$ must have empty feasible set and after having called the oracle $2|\mathcal{Y}_N| + 1$ times.*

Proof. The proof is straightforward consequence of the construction of the (ISP_W) in the branching tree. ∎

When using (ISP_L) or (ISP^*) as scalarized problems we obtain a drastic reduction of the number of (BOIP^k) tackled by the algorithm.

Corollary 3.9. *If either (ISP_L) or (ISP^*) is used to define the scalarization, then at least one of the two subproblems ($BOIP^k_i$), $i = 1, 2$ has empty feasible set, so that the number of generated BOIPs is $|\mathcal{Y}_N| + 1$, where only one problem is empty.*

Proof. Every time a subproblem (ISP^k) is solved and a new Pareto point is found, the algorithm creates two subproblems $(BOIP)_i^k$, $i = 1, 2$, from $(BOIP)^k$ according the formulas $(BOIP)_i^k := \min\left\{y(x) : x \in \mathcal{X}_i^k \cap \mathbb{Z}^n\right\}$ and $\mathcal{X}_i^k := \mathcal{X}^k \cap \{x \in \mathbb{R}^n : y_i(x) \leq \hat{y}_i^k - \epsilon_i\}$ $i = 1, 2$. By the definition of lexicographic method we already know that $\mathcal{X}_{i_2}^k = \emptyset$, where (i_1, i_2) is the permutation of set $\{1, 2\}$ adopted by (ISP^k). As a consequence we can always deprecate one subproblem out of the branching procedure, leading to $|\mathcal{Y}_N| + 1$ problems of type (BOIP^k) and just one of them, i.e. the last, has empty feasible set. ∎

Remark 3.10. *If the lexicographic problem (ISP_L) is used as the scalarization technique in* `FPA`*, then Algorithm 1 returns the complete Pareto after* $2|\mathcal{Y}_N| + 1$ *oracle calls for the solution of single objective scalarized problems, with only one of them being infeasible. This is due to the fact that every time we tackle problem (ISP_L^k) we have to solve two integer problems if the feasible set is not empty or just one if the problem is infeasible - we need to check infeasibility.*

For the special case of linear BOIPs, a criterion space search algorithm that solves $2|\mathcal{Y}_N| - 1$ integer programs has been defined in [30] and, as far as we know, this was the best complexity result.
When (ISP*) is used as scalarization techniques it is possible to reduce drastically the number of oracle calls tackled by exploiting the properties of the scalarized problem.
The weights defined in (2) allow us to define an algorithm that detects the nondominated points of (BOIP) in decreasing order with respect to $y_{i_1}(x)$. This means that at every node of `FPA` only one subproblem $(BOIP)_i^k$ needs to be produced, as the other one is infeasible by construction.
In the following we present the `FPA*`, which is a particular version of `FPA` where we take advantage of the structure of (ISP*) to reduce the number of oracle calls as far as possible.

Algorithm 2: FPA*

Input: $(BOIP^0) = (BOIP)$, $\mathcal{X}^0 = \mathcal{X}$, $\mathcal{Y}_N = \emptyset$, $\gamma_i > 0$, $\epsilon_i \in (0, \gamma_i], i = 1, 2$
$\{i_1, i_2\}$ permutation of $(1, 2)$, $k = 0$
Derive the ideal vector y^{id} and custom weights as in (2) from (BOIP)
Output: the Pareto frontier $\mathcal{Y}_N$ of $(BOIP)$
Solve $(ISP^*)^0$ and let $\hat{x}^0$ be one of its solutions
while $\left(y_{i_1}(x^k) > y^{id}_{i_1}\right)$ **do**
  Derive $(ISP^*)^k$ from $(BOIP)^k$
  Solve $(ISP^*)^k$ and let $\hat{x}^k$ be one of its solutions
  Set $\mathcal{Y}_N = \mathcal{Y}_N \cup \left\{\hat{y}^k\right\}$, where $\hat{y}^k = y(\hat{x}^k)$
  Build $(BOIP)^{k+1}$ from $(BOIP)^k$
    $(BOIP)^{k+1} := \min\left\{y(x) : x \in \mathcal{X}^{k+1} \cap \mathbb{Z}^n\right\}$
    Where $\mathcal{X}^{k+1} := \mathcal{X}^k \cap \{x \in \mathbb{R}^n \,:\, y_{i_1}(x) \leq \hat{y}^k_{i_1} - \epsilon_i\}$,
  Set $k = k + 1$;
end
Return $\mathcal{Y}_N$

Theorem 3.11. *Let Assumptions 2.4, 2.6 and 3.1 hold. Algorithm 2 returns the complete Pareto frontier $\mathcal{Y}_N$ of* (BOIP) *after having addressed $|\mathcal{Y}_N|+2$ single-objective integer programs.*

Proof. Algorithm 2 is exactly `FPA` customized over (ISP*). By Corollary 3.9 if we use `FPA` we will address $|\mathcal{Y}_N|+1$ (ISP*)s, where exactly one has empty feasible set. By using `FPA*` we address the problems sequentially by changing the level of the i_1-th function. Since in the first step of the algorithm we derive the ideal vector, we know that the i_1-th function will end up at that value, in other words the last point found by the algorithm will be the extreme point with the lowest value of the i_1-th function. Since in the *while-loop* a stopping criterion on the i_1-th function is introduced, the algorithm will stop before tackling an empty subproblem. Under these considerations `FPA*` will stop after having addressed $|\mathcal{Y}_N|$ (ISP*)s.

Every time we solve the program (ISP*^{k}) we have to solve only one integer problem. According to Proposition 2.17 and Remark 3.5, if we use formulas (2) with a fixed permutation we need only to solve two integer problems to find custom weights valid for all the subproblems. In the end the algorithm will get exactly $|\mathcal{Y}_N| + 2$ single-objective integer programs and therefore $|\mathcal{Y}_N| + 2$ solver calls. ∎

3.2 Toy example

In this section we show the behavior of algorithm `FPA*` on the following simple example proposed in [15] (example 8.6)

$$\min_{x \in \mathcal{X} \cap \mathbb{Z}^2} (x_1, x_2), \tag{10}$$

where $\mathcal{X}$ is the polyhedral set defined as

$$\mathcal{X} = \{x \in \mathbb{R}^2_{\geq 0} : 2x_1 + 3x_2 \geq 11,\ x_1 \leq 4,\ x_2 \leq 4\}.$$

For this instance we have $\gamma_i = 1$ for $i = 1, 2$ and we set $\epsilon_i = 1$. The Pareto frontier is $\mathcal{Y}_N = \{(0, 4);\ (1, 3);\ (3, 2);\ (4, 1)\}$. As pointed out in [15] there is no setting of the weights $\lambda \in \mathbb{R}^2$ such that the point $(3, 2)$ is a solution of the weighted-sum problem. We show that our approach is able to detect the full Pareto frontier $\mathcal{Y}_N$.
The criterion space $\mathcal{Y}$ of Problem (10) is represented in Figure 1a. At every iteration, in order to produce new non-dominated points of Problem (10), we use as scalarization the custom weighted sum method with $\lambda^* = (\frac{1-\alpha}{4}, 1)^\intercal$ and $\alpha = 0.3$, so that we address the single-objective integer problem $(ISP*)^k$:

$$\min_{x \in \mathcal{X}^k \cap \mathbb{Z}^2} \frac{1-\alpha}{4} x_1 + x_2.$$

In Figure 1, we report the iterations of `FPA` on Problem (10). In particular Figure 1b reports the solution of $(ISP*)^0$.
The first non-dominated point found by `FPA` is $(4, 0)$.
Figure 1c reports the regions $\mathcal{X}_1^0 = \left\{x \in \mathbb{R}^2_{\geq 0} : 2x_1 + 3x_2 \geq 11,\ x_1 \leq 4,\ x_2 \leq 4, x_1 \leq 3\right\}$ and $\mathcal{X}_2^0 = \left\{x \in \mathbb{R}^2_{\geq 0} : 2x_1 + 3x_2 \geq 11,\ x_1 \leq 4,\ x_2 \leq 4, x_2 \leq -1\right\}$ of the two children nodes $(BOIP)_1^0$, $(BOIP)_2^0$ generated. As expressed in Theorem 3.11 $\mathcal{X}_2^0$ is empty and then automathically pruned. By solving $(ISP)_1^0$ in Figure 1d we get the non-dominated point $(3, 2)$. By reapplying the procedure in Figure 1e we get the point $(1, 3)$. Finally in Figure 1f we find the last Pareto point e we get the full Pareto front.

3.3 Discussion on the assumptions

In this section we present some classes of functions that easily satisfy Assumption 2.6. As a first step we look for sufficient conditions to have $\gamma > 0$ and $\epsilon \in (0, \gamma]$ computable whenever $f(x) \neq f(z)$ for all $x, z \in \mathcal{X} \cap \mathbb{Z}^n$ (Assumption 2.6).

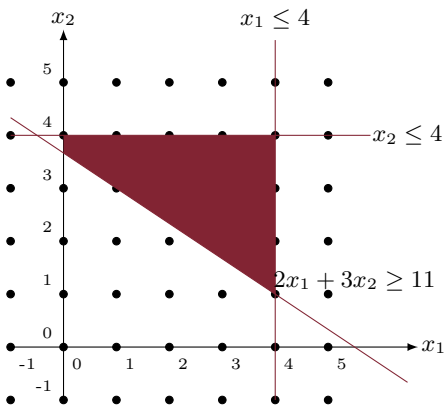
Proposition 3.12. *Assume that $f : \mathbb{Z}^n \to \mathbb{Z}$. Then $\gamma \geq 1$.*

Proof. Since the image of $\mathcal{X} \cap \mathbb{Z}^n$ under f is a subset of $\mathbb{Z}$, we have that $|f(x) - f(z)| \geq 1$, for all $x, z \in \mathcal{X} \cap \mathbb{Z}^n$ such that $f(x) \neq f(z)$. ∎

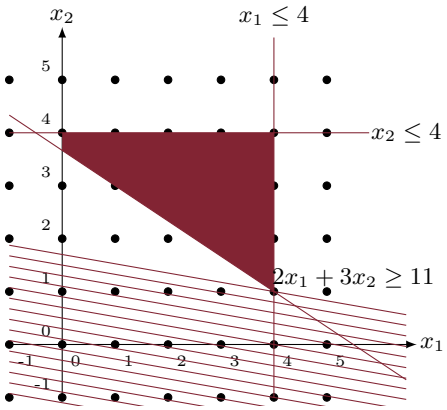
Remark 3.13. *As a matter of example of functions satisfying the condition in Proposition 3.12 we have all the polynomials with integer coefficients and in particular $f(x) = c^\intercal x$ with $c \in \mathbb{Z}^n$ and $f(x) = x^\intercal Q x + c^\intercal x$ with $Q \in \mathbb{Z}^{n \times n}$ and $c \in \mathbb{Z}^n$.*

We now look for larger classes of functions for which $\gamma > 0$. We focus on functions defined on rational domains.

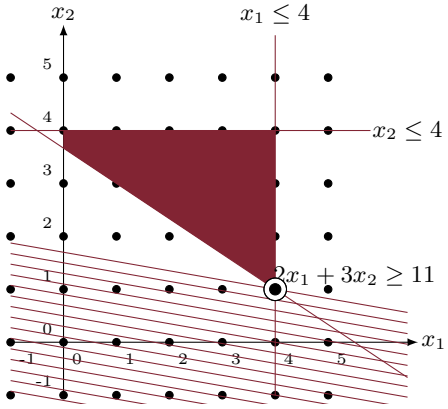
Proposition 3.14. *Let $f(x) : \mathbb{Z}^n \to \mathbb{R}$ be a polynomial with rational coefficients, $f(x) \neq 0$. Then $r \in \mathbb{N}$, $r \neq 0$ exists so that $\gamma \geq \frac{1}{r}$.*



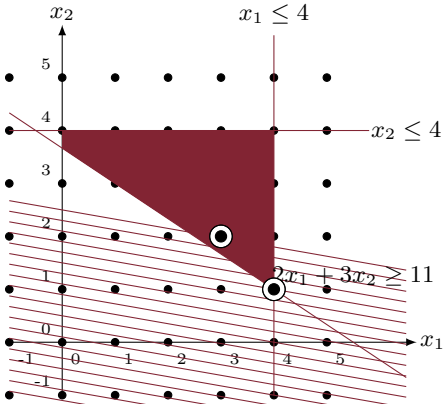
(a) the criterion space of the instance



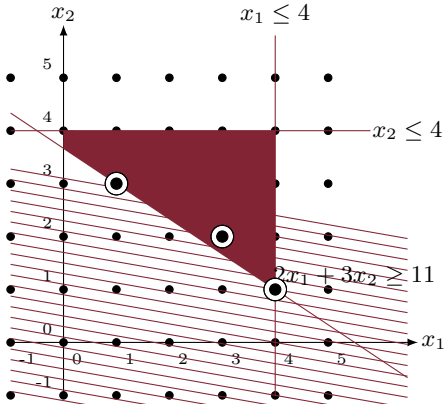
(b) first non-dominated point found



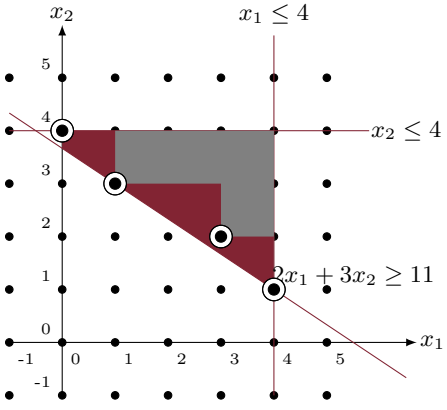
(c) cuts in the criterion space



(d) second non-dominated point found



(e) third non-dominated point found



(f) the complete Pareto frontier

Figure 1: The Frontier Partitioner Algorithm applied to Problem (10)

Proof. Let $f(x) = \sum_{k=0}^{s} \alpha_k q_k(x)$, where $q_k(x) = \prod_{i=1}^{n} x_i^{m_{ik}}$ and $m_{ik} \in \mathbb{N}$, for $k = 0, \ldots, s$, $i = 1, \ldots, n$. We have $q_k(x) \in \mathbb{Z}$, for $k = 0, \ldots, s$ and for all $x \in \mathbb{Z}^n$. Since $r \in \mathbb{N}$, $r \neq 0$ exists such that $r\alpha_k \in \mathbb{Z}$, $k = 0, \ldots, s$, we have that $g(x) = rf(x)$ satisfies the assumption of Proposition 3.12 and

$$|f(x) - f(z)| \geq \frac{1}{r}, \quad \forall\, x, z \in \mathcal{X} \cap \mathbb{Z}^n : f(x) \neq f(z).$$

□

Remark 3.15. *Proposition 3.14 holds when $f(x) = x^\intercal Q x + c^\intercal x$, where $Q \in \mathbb{Q}^{n \times n}$ and $c \in \mathbb{Q}^n$.*

Remark 3.16. *Of course Proposition 3.14 holds when $f(x)$ is a rational linear function: $f(x) = c^\intercal x$, $c \in \mathbb{Q}^n$.*

Note that the value $r \in \mathbb{N}$ used in Proposition 3.14 is easily calculable as the least common multiple of the denominators of the rational coefficients.

Proposition 3.17. *Let $f(x) = \|Ax + b\|_2$ and assume that $A \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^n$ and that*

$$\bar{v} = \max_{x \in \mathcal{X} \cap \mathbb{Z}^n} \|Ax + b\|_2^2 \in \mathbb{R}_+ < \infty.$$

Then $\gamma \geq \sqrt{\bar{v} + 1} - \sqrt{\bar{v}}$.

Proof. Since $A \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^n$ we have that $(Ax + b) \in \mathbb{Z}^m$ for all $x \in \mathcal{X} \cap \mathbb{Z}^n$. Therefore for all $x, z \in \mathcal{X} \cap \mathbb{Z}^n$ such that $f(x) \neq f(z)$ we get

$$|f(x) - f(z)| = |\|Ax + b\|_2 - \|Az + b\|_2| \geq |\|v\|_2 - \|w\|_2|$$

with $v, w \in \mathbb{Z}^m$ such that $v \neq w$ and they differ exactly for one component, which is the least difference possible. We can assume w.l.o.g. that $v_i = w_i$ for all $i \neq j$ and $w_j = v_j + 1$ and we finally get

$$|f(x) - f(z)| \geq \left| \sqrt{\sum_{i=1}^{m} v_i^2} - \sqrt{\sum_{i=1}^{m} v_i^2 + 2v_j + 1} \right| \geq \sqrt{\|v\|^2 + 1} - \sqrt{\|v\|^2}.$$

Let $g(x) = \|Ax + b\|_2^2$, the function $\sqrt{g + 1} - \sqrt{g}$ is monotonically decreasing hence it attains its minimum value at its upper bound $\bar{v}$ and

$$|f(x) - f(z)| \geq \sqrt{\bar{v} + 1} - \sqrt{\bar{v}}, \quad \forall\, x, z \in \mathcal{X} \cap \mathbb{Z}^n : f(x) \neq f(z).$$

□

In Table 1, we report some classes of objective functions that can be considered when using integer programming solvers such as CPLEX [19], Gurobi [18], SCIP [17], Couenne [1] or Bonmin [8], in order to deal with problem $(ISP)^k$. In particular,

- if both $y_i(x)$ $i = 1, 2$ are linear, then $(ISP)^k$ is an Integer Linear Program (ILP)

- if one $y_i(x)$ is written as $\|Ax+b\|_2$, then $(ISP)^k$ is an Integer Second Order Cone Program (ISOCP)
- if one $y_i(x)$ is convex quadratic, then $(ISP)^k$ is a Quadratically Constrained Quadratic Integer Program (QCQIP)
- if one $y_i(x)$ is general convex, then $(ISP)^k$ is a Convex Integer Program (CIP).

$y_i(x) =$	$\gamma \geq$	oracle
$c^\intercal x$ with $c \in \mathbb{Z}^n$	1	*ILP*
$c^\intercal x$ with $c \in \mathbb{Q}^n$	$\frac{1}{r}$	*ILP*
$\|Ax+b\|_2$ with $A \in \mathbb{Z}^{n\times m}$, $b \in \mathbb{Z}^m$	$\sqrt{\bar{v}+1}-\sqrt{\bar{v}}$	*ISOCP*
$x^\intercal Qx + c^\intercal x$ with $Q \succeq 0$, $Q \in \mathbb{Z}^{n\times n}$, $c \in \mathbb{Z}^n$	1	*QCQIP*
$x^\intercal Qx + c^\intercal x$ with $Q \succeq 0$, $Q \in \mathbb{Q}^{n\times n}$, $c \in \mathbb{Q}^n$	$\frac{1}{r}$	*QCQIP*
$: \mathbb{Z}^n \to \mathbb{Z}$, convex	1	*CIP*

Table 1: Classes of functions that satisfy Assumptions 2.6 and 3.1. In the table we denote with r the least common multiple of the denominators of the rational coefficients used in Proposition 3.14. We denote with $\bar{v}$ the value defined in Proposition 3.17.

Remark 3.18. *For the classes of functions mentioned above, we can set $\epsilon_i \in (0, \gamma_i]$ to the values reported in Table 1, as they represent valid lower bounds on γ_i.*

4 Computational aspects

Tackling non linear problems leads to further issues than the ones of integer linear programming. This is due to the fact that non linear functions have often a tough structure which is not well addressed by standard software. On the other hand they can also lead to numerical instabilities.
In the following we propose two way to lighten the computational burden. In particular in Subsection 4.1 we discuss about approximations of non linear sets, whereas in Subsection 4.2 we describe possible procedure to deal with numerical instabilities, presenting a new algorithm derived from a combination of `FPA` and `FPA*`.

4.1 Circumventing nonlinear inequalities

In the Frontier Partitioner Algorithm new nodes are built adding inequalities to the feasible set in the decision space. More specifically, at a generic node k, the set $\mathcal{X}^k \cap \mathbb{Z}^n$ is intersected with the following set:

$$C = \{x \in \mathbb{R}^n \,:\, y_i(x) \leq \hat{y}_i^k - \epsilon_i\}, \tag{11}$$

being $i = 1$ or $i = 2$ and $\hat{y}^k$ the non-dominated point found at node k. When $y_i(x)$ is convex nonlinear we are introducing a nonlinear cut, as the set in (11) is defined according to a nonlinear inequality. However, we do not necessarily need to add nonlinear inequalities: for our purposes, it would suffice to define a valid formulation for the integer set $\{x \in \mathbb{Z}^n : y_i(x) \leq \hat{y}_i - \epsilon_i\}$, or, in other words, it would suffice to find a matrix $A \in \mathbb{R}^{m \times n}$ and a vector $b \in \mathbb{R}^m$ such that

$$\{x \in \mathbb{Z}^n : Ax \leq b\} = \{x \in \mathbb{Z}^n : y_i(x) \leq \hat{y}_i - \epsilon_i\}.$$

However, from a practical point of view, it is not yet clear how to easily generate a valid formulation. It is the purpose of this section to investigate on the use of linear inequalities within `FPA`.

Under the assumption that $y_i : \mathbb{R}^n \to \mathbb{R}$ is convex and continuously differentiable, we have that

$$\nabla y_i(\hat{x}^k)^\intercal(\hat{x}^k - x) \geq y_i(\hat{x}^k) - y_i(x) \geq \epsilon_i.$$

Therefore, we can think of defining $\mathcal{X}_i^k$ intersecting $\mathcal{X}^k$ with the halfspace

$$\{x \in \mathbb{R}^n : \nabla y_i(\hat{x}^k)^\intercal(\hat{x}^k - x) \geq \epsilon_i\}.$$

The resulting `FPA`, may eventually not terminate with the entire Pareto frontier, as we are not guaranteed to cut all the efficient solutions associated to the current non-dominated point. On one hand we loose the exactness of the method, on the other we have the advantage of dealing only with linear constraints.
For the specific class of problems where the objectives are strictly convex quadratic forms, we can prove the following result:

Proposition 4.1. *Let $y_i(x) = x^\intercal Q x$, with $Q \succ 0$. Then*

$$C \cap \mathbb{Z}^n \subseteq C^1 \cap C^\infty$$

where

$$C^1 = \left\{x \in \mathbb{Z}^n : \|Q^{1/2}x\|_1 \leq \frac{1}{\sqrt{n}}\sqrt{\hat{y}_i - \epsilon_i}\right\}$$

and

$$C^\infty = \left\{x \in \mathbb{Z}^n : \|Q^{1/2}x\|_\infty \leq \sqrt{\hat{y}_i - \epsilon_i}\right\}$$

Proof. We have that

$$\begin{aligned}\{x \in \mathbb{Z}^n : y_i(x) \leq \hat{y}_i - \epsilon_i\} &= \{x \in \mathbb{Z}^n : x^\intercal Q x \leq \hat{y}_i - \epsilon_i\} \\ &= \{x \in \mathbb{Z}^n : \|Q^{1/2}x\|_2^2 \leq \hat{y}_i - \epsilon_i\} \\ &= \{x \in \mathbb{Z}^n : \|Q^{1/2}x\|_2 \leq \sqrt{\hat{y}_i - \epsilon_i}\} = C \cap \mathbb{Z}^n\end{aligned}$$

Using the relations between norms $\sqrt{n}\|x\|_2 \leq \|x\|_1$ and $\|x\|_2 \geq \|x\|_\infty$ we have that

$$C \cap \mathbb{Z}^n \subseteq C^1 = \{x \in \mathbb{Z}^n : \|Q^{1/2}x\|_1 \leq \tfrac{1}{\sqrt{n}}\sqrt{\hat{y}_i - \epsilon_i}\}$$

$$C \cap \mathbb{Z}^n \subseteq C^\infty = \{x \in \mathbb{Z}^n : \|Q^{1/2}x\|_\infty \leq \sqrt{\hat{y}_i - \epsilon_i}\}$$

hence we get the result. ∎
Note that both C^1, C^∞ are defined by $2n$ linear inequalities. Hence, in the specific case of problems where the objective functions are strictly convex quadratic forms, $(BOIP)^k$ can be generated using these $4n$ linear inequalities. Again, the resulting `FPA` will be a heuristic approach, as we are not guaranteed of cutting all the efficient solutions associated to the the non-dominated points found so far. However, we have the advantage of dealing, at every node, with a quadratic integer programming problem, that can be handled more efficiently with respect to a quadratically constrained quadratic integer programming problem (see, e.g., [9]).

4.2 Managing numerical instabilities

`FPA*` offers a really tight bound with respect to the number of integer problems to be solved in order to get the Pareto front. From a numerical point of view it works very well for linear biojective integer problems, but it suffers when the functions become nonlinear. This is due mostly to the inability of available software to solve nonlinear problems when one of the two weights is near to zero. This is immediately clear when dealing with quadratic problems. In fact even if singularly the quadratic matrices in the objectives are positive semidefinite (then theoretically solvable) their sum can sometimes be rejected by the software and marked as non convex. From our experience it seems that the algorithms responsible to certify semidefiteness end up finding minimum eivenvalues of the order of -10^{-16} and therefore neglecting convexity.
We proposed two strategies to deal with numerical instabilities:

- iterative weights updating
- `FPA` and `FPA*` combination.

The first adjustment can be made quickly with almost no computational cost. We just need to put ourselves in the hypothesis of Theorem 3.11. We can update the weights of the single (ISP$^*k$) at each iteration, by maintaining an approximation of the ideal vector of each subproblem. For further details about why this will lead to higher weights we redirect to the proof of Proposition 2.17, where the central point is that the starting weights are small than the ones of the subproblems. Since in this context the two objectives are following a specific order, to reduce the slope of the weights we just need to consider a lower bound to the ideal point y_l^k for each (BOIP^k). For the root node - $k = 0$ - we just take the ideal $y_l^0 = y^i d$ as defined in Definition 2.3. For $k > 0$ we put $(y_l^k)_{i_1} = (y_l^0)_{i_1}$ and $(y_l^k)_{i_2} = y_{i_2}^{k-1}$. At each step we are returning a new non dominated point, which is the extreme point of (BOIP^k) in the corner defined by the lexicographic order adopted (i_1, i_2). With the procedure above we can iteratively reduce the imbalance between weights.

The second strategy adopted attempts to mix up `FPA` with weighted sum and `FPA*`. The procedure is simple and it is described in Algorithm 3.
We would like to use the custom weighted sum scalarization as soon as possible. In other words when addressing a new subproblem (BOIP^k) we calculate the weights needed to

obtain (ISP*). Since we are trying to reduce as possible numerical difficulties we include the imbalance reduction discussed above, in this way we associate a point y_l^k which is an approximation of the ideal point of (BOIP^k). Starting from y_l^k we can derive the weights w needed by the custom weighted sum method by normalizing the ones in (2), opportunely checking the order of the functions used. If $w_{min} = \min\{w_1, w_2\}$ is too small numerical issues could arise, therefore we just apply Algorithm 1 with a more suitable scalarization, say weighted sum method (ISP$_W$) with weights $(0.5, 0.5)^\intercal$. In the case w_{min} satisfies our tollerance, the algorithm applies (ISP*) directly, solving fewer problems.
The number of oracle calls of Algorithm (3) lies between $|\mathcal{Y}_N|+3$ and $2|\mathcal{Y}_N|+1$. It really depends on the nature of the problem and on the value of the tolerance, as discussed in Section 5.

Algorithm 3: Stable (FPA `SFPA`)

Input: $(BOIP)$, $\gamma_i > 0$, $\epsilon_i \in (0, \gamma_i], i = 1, 2$, $\delta \in (0, 1)$
Output: the Pareto frontier $\mathcal{Y}_N$ of $(BOIP)$
Initializzation: $(BOIP)^0 = (BOIP)$, $\mathcal{L} = \{(BOIP)^0\}$, $\mathcal{X}^0 = \mathcal{X}$, $\mathcal{Y}_N = \emptyset$
Find: y^{id}, **then put** $y_l^0 = y^{id}$
while $\mathcal{L} \neq \emptyset$ **do**
  Select a node $(BOIP)^k \in \mathcal{L}$ and delete it from $\mathcal{L}$
  Using y_l^k, **calculate** w by normalizing (2) following the order (i_1, i_2) **if** $\min\{w_{i_1}, w_{i_2}\} \leq \delta$ **then**
    Derive (ISP_W^k) from $(BOIP)^k$ with weights $(0.5, 0.5)^\intercal$
    Solve (ISP_W^k).
    if *(ISP_W^k) has a solution* $\hat{x}^k$ **then**
      Set $\mathcal{Y}_N = \mathcal{Y}_N \cup \{\hat{y}^k\}$, where $\hat{y}^k = y(\hat{x}^k)$
      Build $(BOIP)_i^k$, $i = 1, 2$ from $(BOIP)^k$
        $(BOIP)_1^k := \min\left\{y(x) : x \in \mathcal{X}_1^k \cap \mathbb{Z}^n\right\}$
        Put $(y_l^{k_1})_1 = (y_l^k)_1$ and $(y_l^{k_1})_2 = y_2^k$
        $(BOIP)_2^k := \min\left\{y(x) : x \in \mathcal{X}_2^k \cap \mathbb{Z}^n\right\}$
        Put $(y_l^{k_2})_2 = (y_l^k)_2$ and $(y_l^{k_2})_1 = y_1^k$
        Where $\mathcal{X}_i^k := \mathcal{X}^k \cap \{x \in \mathbb{R}^n : y_i(x) \leq \hat{y}_i^k - \epsilon_i\}$, $i = 1, 2$,
      Add the new nodes $(BOIP)_1^k$ and $(BOIP)_2^k$ to $\mathcal{L}$ respectively with $y_l^{k_1}$ and $y_l^{k_2}$;
    end
  else
    Derive $(ISP*)^k$ from $(BOIP)^k$
    Solve $(ISP*)^k$.
    if *$(ISP*)^k$ has a solution* $\hat{x}^k$ **then**
      Set $\mathcal{Y}_N = \mathcal{Y}_N \cup \{\hat{y}^k\}$, where $\hat{y}^k = y(\hat{x}^k)$
      Build $(BOIP)_{i_1}^k$ from $(BOIP)^k$
        $(BOIP)_{i_1}^k := \min\left\{y(x) : x \in \mathcal{X}_{i_1}^k \cap \mathbb{Z}^n\right\}$
        Put $(y_l^{k_{i_1}})_{i_1} = (y_l^k)_{i_1}$ and $(y_l^{k_{i_1}})_{i_2} = y_{i_2}^k$
        Where $\mathcal{X}_{i_1}^k := \mathcal{X}^k \cap \{x \in \mathbb{R}^n : y_{i_1}(x) \leq \hat{y}_{i_1}^k - \epsilon_i\}$, $i = 1, 2$,
      Add the new nodes $(BOIP)_{i_1}^k$ to $\mathcal{L}$;
    end
  end
end
Return $\mathcal{Y}_N$

Adopting one scalarization technique or the other can significantly change not only the theoretical performances, but also the overall software interaction. There are two major aspects we want to focus on:

- certification of emptiness
- warm starts.

The first point can be exploited easily. By looking at the results in Theorems 3.7 it could seem that the choice on the scalarization technique is no big deal. By the way certification of emptiness is a huge problem in integer programming. In fact it is common to spend more time trying to certify infeasibility than to solve an analogous feasible problem. Generally speaking it depends a lot on the structure of the instance we are addressing and which one will perform better it is not clear apriori. Of course if there are not numerical instabilities then the custom weighted sum scalarization performs usually a lot better.
On the other hand it is known that to start from a feasible solution (or partially feasible) can help significantly the computational time. The speedup is usually around 15%. If we are using Algorithm 1 with weighted sum method (ISP_W) as scalarization technique, then warm start and partial tree recovery lose their advantage after solving few problems. This is due to the fact that the regions addressed by the subproblems are less related the one to the other the deeper we go in the branching tree. This forced us to remove warm starts when using weighted sum. Surprisingly, when dealing with custom weighted sum and lexicographic method, warm starts become game changers. The feasible region of a subproblem is in fact always contained in the one of the father and most importantly they are always addressed sequentially.

5 Numerical results

To test the performance of our algorithms, we considered biobjective integer linear instances (see Section 5.1) and biobjective integer convex quadratic instances (see Section 5.2). Algorithm `FPA`, `FPA*` and `SFPA` are implemented in Java and uses as oracles respectively the MILP and MIQP solvers of `CPLEX 12.7.1` to address the scalarized problem $(ISP)^k$ at each node. All our experiments were carried out on an Intel Core i7 processor running at 2.40 GHz. All running times were measured in CPU seconds.

5.1 Numerical experiments on linear instances

The generic biobjective integer linear programming problem is modeled as

$$\begin{aligned} \max \quad & (c_1^\intercal x,\, c_2^\intercal x) \\ \text{s.t.} \quad & Ax \leq b \\ & x \in \mathbb{Z}^n_{\geq 0} \end{aligned}$$

We consider two classes of instances. The first one is available at `http://home.ku.edu.tr/~moolibrary/`, where instances have three, four and five objectives. In our experiments, we took only the first two objectives. Parameters are $c_i \in \mathbb{Z}^n$, $i = 1, 2$, $A \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^m$. In particular, c_{ij} is generated in the ranges $[-100, -1]$

with probability 0.2 and $[0, 100]$ with probability 0.8, $j = 1, \dots, n$ and $i = 1, 2$. The coefficients a_{kl} are generated in the ranges $[-100, -1]$ with probability 0.1, $[1, 100]$ with probability 0.8 and $a_{kl} = 0$ with probability 0.1. The right-hand side b_k is generated randomly in the range 100 and $\sum_{l=1}^{n} a_{kl}$.

The second class of instances has been randomly generated with $c_i \in \mathbb{Z}^n$, $i = 1, 2$, $A \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^m$ and it is publicly available at `https://github.com/GiorgioGrani/Biobjective_Instances`. We produced 97 instances: 58 of them have a number of constraints which is the 83% of the number of variables and their coefficients are chosen such that c_{ij} is generated in the ranges $[-100, -1]$ with probability 0.2 and $[0, 100]$ with probability 0.8, $j = 1, \dots, n$; $i = 1, 2$. The coefficients a_{kl} are generated in the ranges $[-100, -1]$ with probability 0.05, $[1, 100]$ with probability 0.9 and $a_{kl} = 0$ with probability 0.05. The right-hand side b_k is generated randomly in the range $[0, \sum_{l=1}^{n} a_{kl}]$. The remaining 39 instances have exactly 10 constraints each and their coefficients are chosen such that c_{ij} is generated in the range $[-100, -1]$ with probability 0.02, in the range $[0, 100]$ with probability 0.08 and they are set to zero with probability 0.9, $j = 1, \dots, n$; $i = 1, 2$. The coefficients a_{kl} are generated in the ranges $[-100, -1]$ with probability 0.2, $[1, 100]$ with probability 0.6 and $a_{kl} = 0$ with probability 0.2. The right-hand side b_k is generated randomly in the range $[0, \sum_{l=1}^{n} a_{kl}]$.
Note that, for both classes of instances, the condition in Proposition 3.12 is satisfied, so that $\gamma_i \geq 1$ and we can set $\epsilon_i = 1$ for $i = 1, 2$.
In order to assess the performance of the algorithms considered, we make use of performance profiles, as proposed in [14]. Given our set of solvers $\mathcal{S}$ and a set of problems $\mathcal{P}$, we compare the performance of a solver $s \in \mathcal{S}$ on problem $p \in \mathcal{P}$ against the best performance obtained by any solver in $\mathcal{S}$ on the same problem. To this end, we define the performance ratio $r_{p,s} = t_{p,s} / \min\{t_{p,s'} : s' \in \mathcal{S}\}$, where $t_{p,s}$ is the computational time, and we consider a cumulative distribution function $\rho_s(\tau) = |\{p \in \mathcal{P} : \ r_{p,s} \leq \tau\}|/|\mathcal{P}|$. The performance profile for $s \in S$ is the plot of the function ρ_s.

We compared four algorithms: (`FPA*`), `FPA` with weighted sum (`FPA-W`), `SFPA` and the Balanced Box Method (`BBM`). `BBM` ha been proposed in [4] and it is one of the most recent and well behaving algorithms for biobjective problems, it uses lexicographic method and it divides the region into rectangles. We did not include the `FPA` with lexicographic method since `BBM` exploits a similar idea, so it would be redundant.
In order to study the effect of changing the weight vector in the definition of (ISP_W^k), we consider three different settings for $\lambda \in \mathbb{R}^2$ within `FPA-W`, namely

$$\lambda \in \left\{ \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}, \begin{pmatrix} 0.25 \\ 0.75 \end{pmatrix}, \begin{pmatrix} 0.1 \\ 0.9 \end{pmatrix} \right\}.$$

In Table 2 and Table 3, we report the results obtained applying `FPA-W`, where (ISP_W^k) is defined using one of weight vectors above for every k. For example, in the first columns of Table 2 and Table 3 we report the results obtained when (ISP_W^k) is defined using $\lambda = (0.9, 0.1)^\intercal$.
The instances are grouped according to the cardinality of the Pareto frontier $|\mathcal{Y}_N|$. In Table 2 we report the results obtained on the first group of instances, namely those

$\lvert\mathcal{Y}_N\rvert$	# inst.	$\lambda = (0.1, 0.9)^\intercal$ avg	min	max	$\lambda = (0.25, 0.75)^\intercal$ avg	min	max	$\lambda = (0.5, 0.5)^\intercal$ avg	min	max
≤ 10	66	0.7	0.1	11.6	0.7	0.1	8.2	0.7	0.1	9.2
$> 10, \leq 20$	71	8.4	0.2	105.7	8.1	0.2	103.2	8.4	0.2	102.3
$> 20, \leq 30$	37	47.6	0.4	395.9	46.5	0.4	392.3	50.3	0.3	414.8
$> 30, \leq 40$	19	29.0	0.5	130.5	29.3	0.5	132.1	31.1	0.5	155.3
$> 40, \leq 50$	16	221.0	0.6	894.8	220.1	0.6	906.6	222.1	0.7	872.8
$> 50, \leq 65$	10	600.5	7.2	2313.3	582.6	7.3	2252.4	612.8	7.0	2508.3

Table 2: Comparison on biobjective integer linear programming instances (first group) - `FPA` applied with different weight vectors.

$\lvert\mathcal{Y}_N\rvert$	# inst.	$\lambda = (0.1, 0.9)^\intercal$ avg	min	max	$\lambda = (0.25, 0.75)^\intercal$ avg	min	max	$\lambda = (0.5, 0.5)^\intercal$ avg	min	max
≤ 70	13	13.9	0.9	80.0	13.4	0.9	76.4	27.7	0.8	253.0
$> 70, \leq 150$	16	54.9	5.1	172.1	51.7	5.1	180.8	57.3	5.2	176.4
$> 150, \leq 300$	19	213.2	10.3	2092.8	207.2	10.1	2058.1	211.7	10.0	2021.8
$> 300, \leq 500$	26	297.4	34.5	963.8	285.9	29.6	949.5	329.9	29.9	1574.3
$> 500, \leq 1000$	21	1122.8	58.8	3669.2	1073.2	59.0	3657.9	1185.8	55.2	3607.7

Table 3: Comparison on biobjective integer linear programming instances (second group) - `FPA` applied with different weight vectors.

available at `http://home.ku.edu.tr/~moolibrary/`. In Table 3 we report the results obtained on the second group of instances, namely those available at `https://github.com/GiorgioGrani/Biobjective_Instances`.
In each table, we report the number of instances (# inst.) belonging to a specific range of $\lvert\mathcal{Y}_N\rvert$, the average (avg), the minimum (min) and the maximum (max) CPU time in seconds needed to detect the entire Pareto frontier. The results obtained using different weight vectors are all very similar and there is no clear winner. However, looking at the performance profiles shown in Figure 2, we can see that `FPA-W` applied with the weight vector $\lambda = (0.1, 0.9)^\intercal$ gives slightly better results with respect to the others.

In Table 4 and Table 5, we report a comparison between `BBM`, `FPA*` and `SFPA`. As before, we group the instances according to the cardinality of the Pareto frontier $\lvert\mathcal{Y}_N\rvert$ and the class of instances. In each table, we report the number of instances (# inst.) belonging to a specific range of $\lvert\mathcal{Y}_N\rvert$, the average (avg), the minimum (min), the maximum (max) CPU time in seconds needed to detect the entire Pareto frontier. According to what

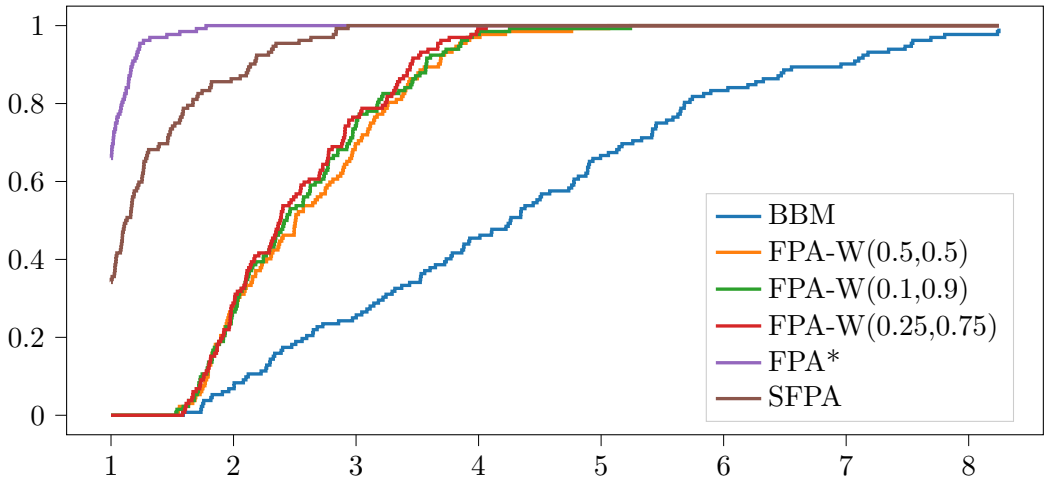


Figure 2: Performance profiles with respect to CPU time on the first group of instances.

said in Subsection 4.2 it arises that there is no clear winner between `BBM` and `FPA-W` with respect to the second group of instances. The cause is that empty problems are sometimes easier to solve than non empty ones (first group of instances) and sometimes they are not (second group of instances). By the way, as expected from theory, `FPA*` outperform `FPA-W`, `BBM` and `SFPA`.
In Figures 2 and 3 performance profiles are reported for the first and the second group of instances respectively. Performance profiles confirms what expressed above.
Another important aspect is the number of oracle calls for each algorithm. In Figures 4 and 5, box plots are reported with respect to the number of oracle calls. For `FPA-W` we have chosen the weight vector $(0.1, 0.9)^\intercal$ since it preforms slightly better than the others. It is clear how much `FPA*` saves solver calls. It is also evident that `SFPA` oscillates between the performances of `FPA*` and `FPA-W` depending on the problem structure.

5.2 Numerical experiments on quadratic instances

The generic biobjective integer quadratic programming problem is modeled as

$$\begin{aligned} \min \quad & (x^\intercal Q_1 x + c_1^\intercal x,\ x^\intercal Q_2 x + c_2^\intercal x) \\ \text{s.t.} \quad & Ax \leq b \\ & x \in \mathbb{Z}^n_{\geq 0}. \end{aligned}$$

We considered a subgroup of the first class of instances used in Section 5.1 (namely those available at `http://home.ku.edu.tr/~moolibrary/`), where we took only the instances with less than or equal to 60 variables. In the objective functions we added the quadratic term $x^\intercal Q_i x$, with $Q_i \succeq 0$, $Q_i \in \mathbb{Z}^{n\times n}$, $i = 1, 2$. To obtain positive semidefinite matrices Q_i we considered the matrices $L_i \in \mathbb{R}^{n\times h}$, where h is integer and randomly chosen in $[1, n]$. The generic element l of the matrix L_i is chosen to be 0 with probability 0.8 and to be 1 with probability 0.2. Finally we get $Q_i = L_i L_i^\intercal$

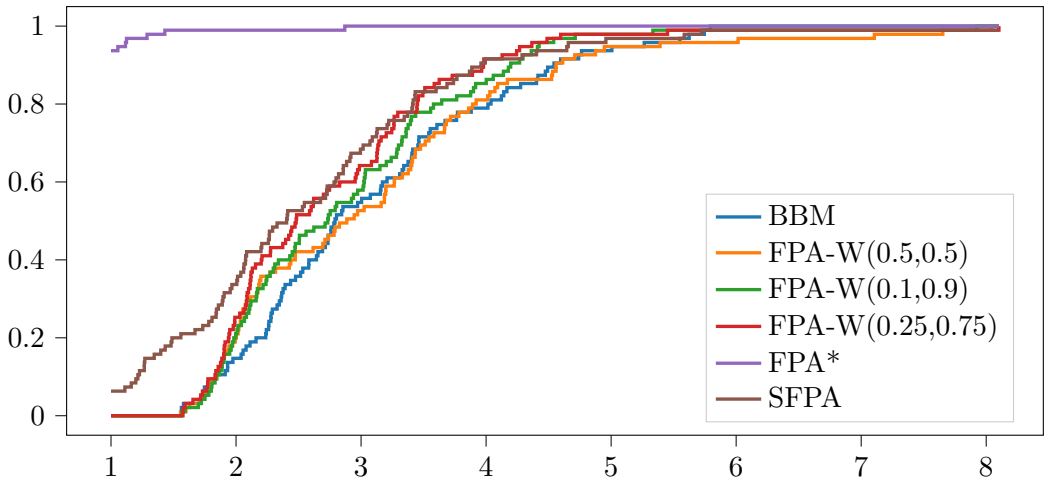


Figure 3: Performance profiles with respect to CPU time on the second group of instances.

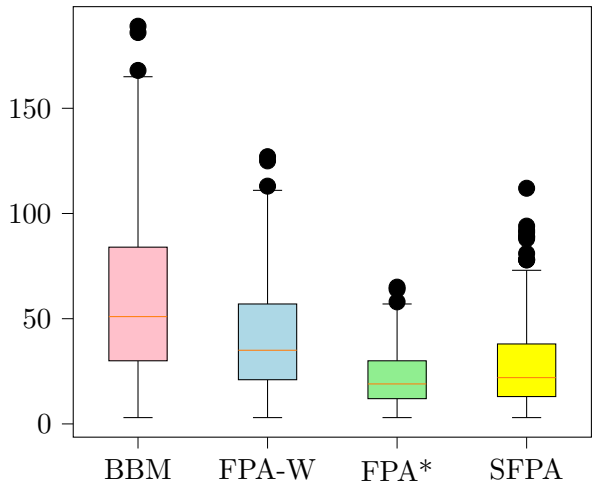


Figure 4: Box plot with respect to the number of oracle calls in the first group of instances.

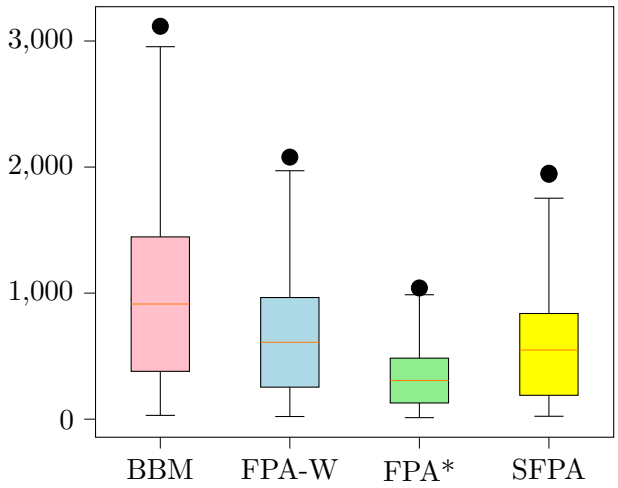


Figure 5: Box plot with respect to the number of oracle calls in the second group of instances.

$\lvert\mathcal{Y}_N\rvert$	# inst.	BBM			FPA*			SFPA		
		avg	min	max	avg	min	max	avg	min	max
≤ 10	66	1.0	0.1	15.9	0.3	0.1	2.7	0.3	0.1	3.3
$> 10, \leq 20$	71	13.6	0.2	134.7	3.2	0.1	35.2	4.0	0.1	50.2
$> 20, \leq 30$	37	76.2	0.3	544.0	16.6	0.2	146.2	24.2	0.2	263.5
$> 30, \leq 40$	19	49.0	0.4	255.1	9.7	0.3	46.3	11.0	0.3	41.2
$> 40, \leq 50$	16	299.3	0.5	1167.2	76.1	0.4	325.4	149.3	0.4	710.7
$> 50, \leq 65$	10	830.4	9.6	3298.1	237.6	3.1	996.7	278.0	5.5	1142.1

Table 4: Comparison on biobjective integer linear programming instances (first group).

$\lvert\mathcal{Y}_N\rvert$	# inst.	BBM			FPA*			SFPA		
		avg	min	max	avg	min	max	avg	min	max
≤ 70	13	13.1	0.6	55.3	6.2	0.4	31.9	11.3	0.8	33.7
$> 70, \leq 150$	16	57.5	5.7	175.1	19.4	3.3	64.3	34.4	5.0	116.9
$> 150, \leq 300$	19	198.2	8.9	1885.9	62.0	5.7	589.0	190.5	8.3	1,683.1
$> 300, \leq 500$	26	333.3	32.8	1153.0	95.4	15.2	393.4	254.8	31.2	1,073.3
$> 500, \leq 1000$	21	1076.6	65.3	3752.4	350.7	30.9	1216.1	978.1	44.4	2751.3

Table 5: Comparison on biobjective integer linear programming instances (second group).

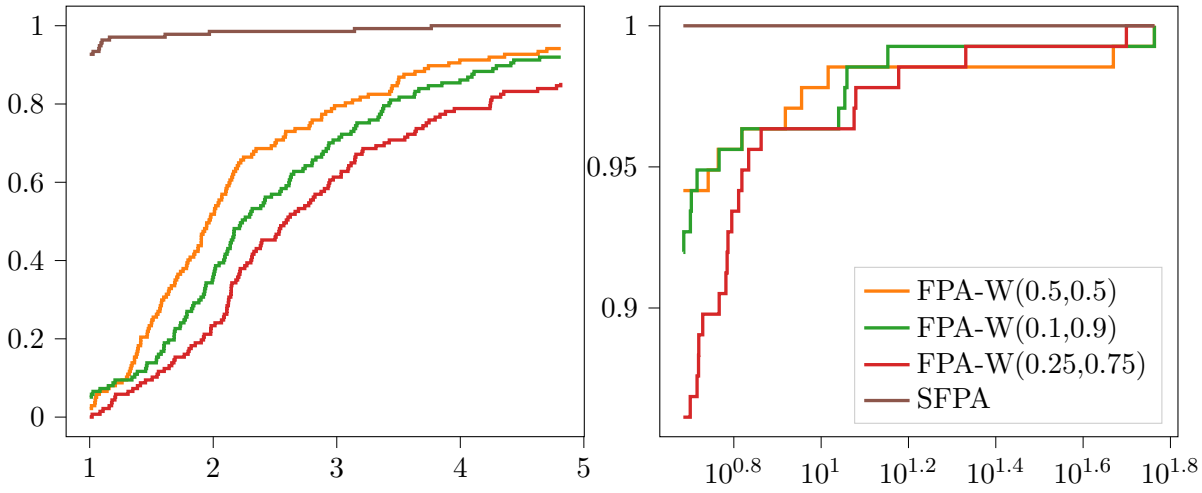


Figure 6: Performance profile on quadratic instances.

which is positive semidefinite. As before, c_{ij} is generated in the ranges $[-100, -1]$ with probability 0.2 and $[0, 100]$ with probability 0.8, $j = 1, \ldots, n$; $i = 1, 2$. The coefficients a_{kl} are generated in the ranges $[-100, -1]$ with probability 0.1, $[1, 100]$ with probability 0.8 and $a_{kl} = 0$ with probability 0.1. The right-hand side b_k is generated randomly in the range 100 and $\sum_{l=1}^{n} a_{kl}$. Note that the condition in Proposition 3.12 is satisfied, so that $\gamma_i \geq 1$ and we can set $\epsilon_i = 1$ for $i = 1, 2$. The test problems are publicly available at `https://github.com/GiorgioGrani/Biobjective_Instances`.
In Table 6, we group the instances according to the cardinality of the Pareto frontier $|\mathcal{Y}_N|$. We consider six different ranges and for each range of $|\mathcal{Y}_N|$, we report the number of instances (# inst.) belonging to that range and the average CPU time in seconds needed to detect the entire Pareto frontier by `FPA*` (FPA time (s)) or by `FPA-W` (FPA-W time (s)) with different weights.
We considered 1 hour of CPU time as time limit for the solution of one instance.
We notice that in the case of quadratic instances the choice of the weights can affect numerical stability of `FPA*`. For this reason we used Algorithm 3 instead of pure `FPA*`.
In Figure 6 is reported the performance profile with respect to CPU time for the quadratic instances.

6 Conclusions

We presented a criterion space search algorithm able to detect the entire Pareto frontier of biobjective integer programming problems. Using a suitable scalarization technique, at every node of the branching tree, a single-objective integer programming problem is addressed.Using as scalarization technique the weighted sum method with a particular choice of the weights we were able to push down to $|\mathcal{Y}_N| + 2$ the number of integer

$\lvert\mathcal{Y}_N\rvert$	# inst.	$\lambda = (0.1, 0.9)^\intercal$ avg	min	max	$\lambda = (0.25, 0.75)^\intercal$ avg	min	max
≤ 10	36	3.4	0.0	18.9	3.4	0.0	19.0
$> 10, \leq 20$	39	25.0	0.4	70.4	27.5	0.4	74.1
$> 20, \leq 30$	30	448.3	8.8	5772.8	587.8	4.5	8203.3
$> 30, \leq 40$	31	271.4	0.7	1729.3	303.3	0.8	1618.8
$> 40, \leq 50$	4	2122.2	454.5	3767.0	2219.6	407.4	6387.8
$> 50, \leq 65$	12	803.6	106.6	3680.5	953.6	105.1	3632.4
$\lvert\mathcal{Y}_N\rvert$	# inst.	$\lambda = (0.5, 0.5)^\intercal$ avg	min	max	SFPA avg	min	max
≤ 10	36	3.8	0.0	18.9	1.5	0.0	9.9
$> 10, \leq 20$	39	30.3	0.4	75.3	13.4	0.3	49.9
$> 20, \leq 30$	30	712.0	4.3	8851.7	96.6	2.0	992.3
$> 30, \leq 40$	31	401.2	0.9	2402.3	111.8	0.4	558.3
$> 40, \leq 50$	4	2806.5	501.6	7371.5	895.2	130.3	1705.6
$> 50, \leq 65$	12	1557.6	113.9	7714.2	439.0	66.2	2133.1

Table 6: Comparison on biobjective integer quadratic programming instances - `FPA` applied with different weight vectors and `SFPA`.

problems to be solved in order to get the full Pareto front, where $|\mathcal{Y}_N|$ is the total number of non dominated points in the criterion space. This is, in our knowledge, the best known bound for this class of algorithms.
The inequalities we introduced avoid the detection of known Pareto points and rely on the definition of a parameter easily calculable for many classes of instances, including integer convex quadratic instances.
The approach presented works for non linear integer problems, as far as we are able to satisfy some properties on the function and to provide an oracle able to find a solution. In particular we applied it to integer convex quadratic programs with rational entries, which are tackled by commercial software. Non linear problems can present numerical issues when scalarized, for this reason we proposed an effective Algorithm able to overcome numerical instabilities but maintaining competitiveness.
We supported our statement by giving both theoretical and computational proof about efficacy and efficiency of the algorithm.

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