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LINKING AND ITERATION SIGNS IN PROVING BY MATHEMATICAL INDUCTION

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We focus on the role of signs in the process of constructing proofs by mathematical induction of high-achieving post-graduate students. Using a multimodal semiotic perspective, speech, written inscription (symbols, drawings, etc.), and gestures are analysed, and two particular categories of signs are identified and observed: linking signs and iteration signs. We analyse what these signs reveal and how the students use them to formulate a conjecture and to structure the proof by mathematical induction.

INTRODUCTION

The analysis of signs offers an interesting access to mathematical thinking and has promoted the discovery of interesting processes with important didactical implications. In the last decades the semiotic analysis has been integrated by the study of gesture that has enriched research in different areas of mathematics education and, recently, the studies on argumentation and proof (see, for example, Edwards, 2010; Arzarello and Sabena, 2014; Krause, 2015; Sabena 2018). In particular, Arzarello and Sabena show that gestures can contribute "not only to the semantic content of mathematical ideas, but also to the logical structure that organizes them in mathematical arguments" (Arzarello & Sabena, 2014, p. 76). Along the same line, Krause (2015) analyses the gestures produced during an activity involving reasoning by induction by grade 10 students who had not studied mathematical induction at school and states that gestures "give visual access to the structure of a reasoning action" (Krause, 2015, p. 1432).

The study presented in this paper is part of a wider research on proving by mathematical induction of post-graduate, undergraduate and secondary students. In particular, in this paper, we focus on signs in post-graduate students' processes involved in the generation of a conjecture and of proof by induction.

THEORETICAL FRAMEWORK

In a multimodal perspective, we consider that thinking and learning processes involve simultaneously different kinds of signs (mathematical symbols, diagrams, sketches, language, gestures, etc.). Arzarello (2006) considers these different kinds of signs as an inseparable unit and defines a *semiotic bundle* as a dynamic structure consisting of different *semiotic sets* and relationships among them. Two main types of analysis are carried out on a semiotic bundle: a *synchronic analysis* of relationships between different kinds of signs activated simultaneously and a *diachronic analysis* of evolutions of signs activated over the time.

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In this paper, we analyse the semiotic bundle made of three semiotic sets - speech, written inscriptions (symbols, drawings, etc.) and gestures - in the production of a conjecture and of a proof by mathematical induction. The analysis of complex units of signs has enabled the identification of new interesting processes in argumentation and proof. In particular, Sabena (2018, p. 554) provides empirical evidence that "gestures may contribute to carrying out argumentations that depart from empirical stances and shift to a hypothetical plane in which generality is addressed". Sabena, Radford and Bardini (2005) observe that a deictic gesture used by a grade 9 student to point at a figure on the sheet becomes a gesture in the air and identify a crucial role of a progressive detachment of gestures from a sheet in generalization processes. Similarly, Krause (2016) proposes a classification of gestures in three levels (*concrete*, *potential*, and *general*) according to their detachment from a concrete inscription. Gestures of *level 1* refers *concretely* "to something actually represented in a fixed diagram" (e.g. pointing to the sheet). Gestures of *level 2 potentially* "depict new entities in an established diagram" but they need to be considered as embedded in it (e.g. gesture of rotating a figure). Gestures of *level 3* are *general* gestures performed in the gesture space. They are detached from a concrete level and their interpretation is general, i.e. not dependent on a "present referential frame" (Krause, 2016, p. 138).

In our study, we also refer to the classic distinction of gestures into *iconic*, *metaphoric*, *deictic* and *beats* (McNeil, 1992). We will use these classifications and synchronic and diachronic analyses to investigate processes of construction of a proof by induction.

Linking and Iteration Signs in Mathematical Induction

A proof by mathematical induction of a proposition $\forall n \in \mathrm{N}$, $\mathrm{P}(n)$ consists in a proof of the base case P(0) and of the inductive step $\forall n \in \mathrm{N}$, $\mathrm{P}(n) \rightarrow \mathrm{P}(n+1)$. Referring to the theory of natural numbers and to the logic theory, we know that the validity of the base case and of the inductive step guarantees that $\mathrm{P}(n)$ holds for all natural numbers. Usually, a non-formal explanation is that from the propositions P(0) and P(0)→P(1) it follows P(1) by modus ponens; from P(1) and P(1)→P(2) it follows P(2), and so on. In other words, this process can be iterated to cover all the natural numbers. In this paper we aim to investigate signs that reveal and support the construction of the inductive step and the iteration in the generation processes of a conjecture and of proof. Constructing the inductive step requires the consideration of two cases (P(n) and P(n+1)) and their relationships. The iteration requires the consideration of the possibility to repeat the inductive step. Thus, in particular, we look for and analyse:

- signs produced or used to refer to two or more entities (objects, mathematical objects, problems, situations, etc.) and to their relationships, where these entities are seen in connection with two consecutive natural numbers. For these we use the term *linking signs*;
- signs that refer to iteration, or that are composed by a repetition (in time or in space) of linking signs, or that refer to a repetition of them. For these we use the term *iteration signs*.

Examples of linking signs can be found in usual algebraic manipulations. For instance, in the construction of the proof of the formula for the sum of the first n consecutive natural numbers it is common to use the sign (1+2+…+n)+(n+1). This sign links the case n with the case n+1 and prepares the proof of the inductive step. Some examples of the iteration signs are the verbal “and so on”, or the image of falling dominoes.

In this study, our goal is to look for the presence of linking and iteration signs, and to investigate what they reveal, in the process of generating a conjecture and a proof by induction, and considering not only mathematical symbols but a wider variety of signs, as speech, written inscriptions, and gestures.

METHODOLOGY

This is a qualitative study based on interviews in which students were asked to solve 4 problems and then to speak about mathematical induction. Data consist of audio, video recordings, and of written inscriptions produced by the students. The subjects were 1 high-achieving post-graduate student in the Master’s course in Mathematics and 4 doctoral students in Mathematics. They were interviewed individually by the second author of this paper, for approximately 70 minutes each. They were neither aware of our interest about their written inscriptions and gestures nor of our focus on proof by mathematical induction. In this paper we will refer to the following problem:

“*Consider a $2^n x 2^n$ chessboard. What is the maximum number of squares which can be tiled with L-shaped pieces composed of 3 squares each?”*

The solution is that it is possible to tile the entire $2^n x 2^n$ chessboard except for one square, for any natural number n. This can be proved by mathematical induction on n.

CASE ANALYSIS

Giuditta is a post-graduate student in the Master’s course in Mathematics. In the first 10 minutes of the interview she produces some drawings and recognises that for reasons of divisibility it is not possible to completely tile any chessboards. By minute 10:00 she has sketched an 8x8 chessboard (n=3) and determined a tessellation which covers every square except one. The interviewer then asks her if this property is also valid in other cases, for example in the case 16x16. In the transcript, Giu stands for Giuditta and with *italics* we describe gestures in the moments when they occur.

1	Giu:	16 by 16 (*with her left middle finger and the tip of the pen in the right hand she points to two vertices of the 8x8 chessboard drawing, Fig. 1a*).
2	Giu:	but, then I have another three (*she keeps her left middle finger on the vertex, and with the pen in the right hand she indicates respectively to the right, upper right, and above the drawing of the 8x8 chessboard, Fig. 1b,c,d*) of these (*she points with the pen to the drawing of the 8x8 chessboard*) squares here (*she moves the tip of the pen along the perimeter of three imaginary squares in the three places she has indicated before,* Fig. 2).

The synchronic analysis of the bundle produced in line 1 reveals an interesting element. In this moment, on the sheet there is the drawing of the 8x8 chessboard and no other written inscriptions referring to a 16x16 chessboard. Giuditta says “16 by 16” and at the same time points to two vertices of the drawing of the 8x8 chessboard (fig. 1a). She refers to something through her speech and to something else through her gesture: this is a case of *speech-gesture mismatch* and Goldin-Meadow (2003) highlights the cognitive potential of a mismatch in the representation of a new idea. In this case, pointing at the drawings of the 8x8 chessboard is co-timed to saying "16 by 16". The bundle and the mismatch offer Giuditta the possibility to represent simultaneously two different chessboards (8x8 and 16x16).

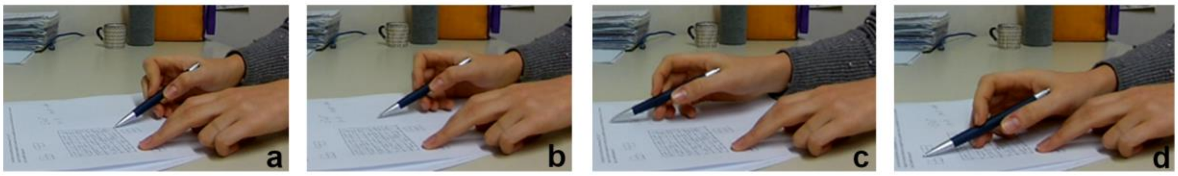

Figure 1: Gestures in line 2.

The diachronic analysis allows us to look at the evolution of signs. In line 2, Giuditta produces signs connecting the chessboards. She keeps the left hand still on the drawing of the 8x8 chessboard (deictic gesture of level 1) and with the right hand she points to three places on the sheet (fig. 1b,c,d). Then she moves the tip of the pen along the sides of three imaginary squares in the three places she has just indicated. In summary, four 8x8 chessboards are represented: one by a written inscription, and three by speech and gesture (fig 1 and 2). These gestures represent something new into the inscription and are therefore gestures of level 2. The bundle speech-inscription-gesture represents a 16x16 chessboard composed by four 8x8 chessboards and, as a unit, can be considered a linking sign referring to the two chessboards and to their relationships. This linking sign, at this point, allows Giuditta to access the connections between the tessellation problem in the case n=3 (8x8) and in the case n=4 (16x16):

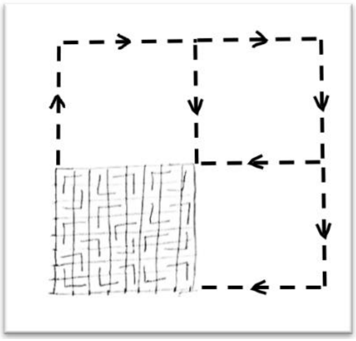

Fig. 2: Pointing with the left hand to the drawing of a 8x8 chessboard, Giuditta follows with a pen (without marking) the perimeter of 3 squares.

3	Giu:	And then there would be left out one, one, one and one (*she points to the drawing of the 8x8 chessboard on the sheet and to the other three she has in mind)* [omissis]. And so I would think to put three of them together, somehow. And then, there would always be one left out?

Giuditta conjectures that the 16x16 chessboard can be tiled except for one small square (a square 1x1) and imagines doing it by using the tessellation of the four 8x8 chessboards. In each of them, one small square would be left out, thus 4 squares in total, but three of them can be covered with an L-shape tile. Therefore, also the 16x16 chessboard would be tiled except for one little square. Her linking sign has a crucial role in the conjecture generation. In particular it enables Giuditta to anticipate the fact that the 16x16 chessboard can be tiled using the tessellation of the smaller one "somehow" (she doesn't know in which way and the conjecture is expressed as a question). At this point, Giuditta focuses on verifying her conjecture for n=1, n=2 and then for n=0. Differently from her reasoning in line 3, these cases are each tiled independently, without connections between them. Then she claims to be convinced of the truth of her conjecture. In argumentation process, new signs enrich the bundle:

4 Giu: So, what I was thinking (*the drawing of the 4x4 chessboard, Fig.3a, is extended into a new drawing, Fig. 3b*) was that to come, to move forward from n=1 (*she makes an arc-shaped gesture in the air from left to right, Fig.3c,d*) to n=2 (*with her left middle finger she points to a drawing of a 2x2 chessboard*) practically (*with the right hand she points specifically to three squares of the drawing of the 2x2 chessboard, see arrows in Fig. 3e*) I have to put another three identical little squares (*she draws two lines on the drawing in Fig. 3b obtaining the drawing of Fig. 3f*).

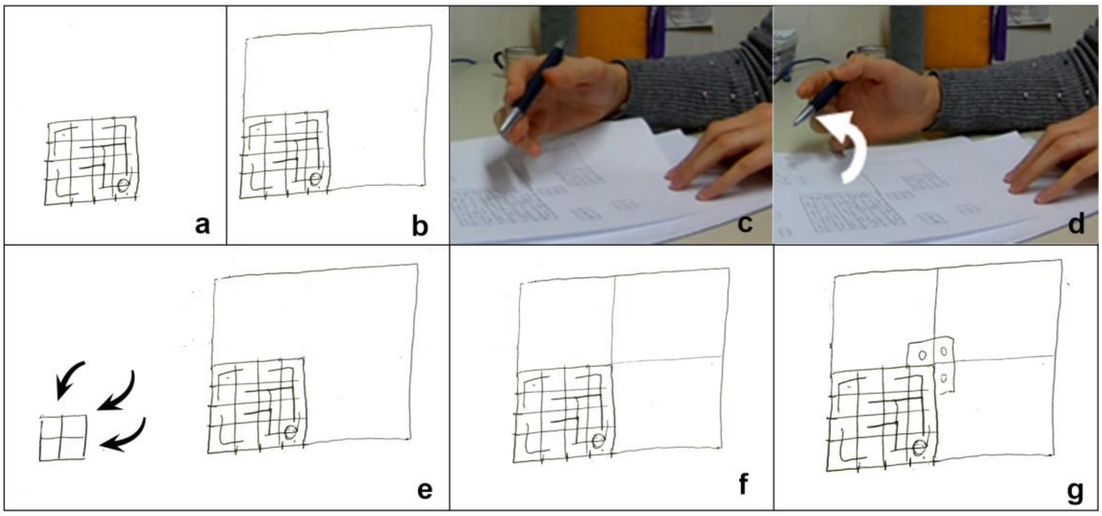

Figure 3: Gestures and written inscriptions in line 4 (a,b,c,d,e) and in line 5 (g). Fig. 3e indicates where Giuditta points to on the sheet.

In this excerpt, Giuditta produces three linking signs that become the object of her exploration. The first is the drawing of a big square (fig. 3b) as extension of the drawing of the chessboard 4x4 (already on the sheet, fig. 3a). The second is the gesture in the air from left to right (fig. 3c,d). The third is the bundle composed by the deictic gesture with her left middle finger pointing to the drawing of the 2x2 chessboard and the gesture made by the right hand referring to the action of adding three small 1x1 squares to build a 2x2 chessboard up from a single square. The gesture from left to right is iconic and refers to a path, but can also be interpreted as a metaphoric gesture of level 3. This gesture is detached from a concrete inscription and it is co-timed to the verbal "to move forward from n=1 to n=2". This gesture appears here for the first time and does not refer to any drawings, any chessboards or tessellations. With this, Giuditta doesn't refer to the specific aspects of the relationship between a smaller chessboard and a bigger one, neither to the relationship between tessellations. Rather, the gesture represents metaphorically the transition between two cases, i.e. the inductive step. The structure of the argumentation is thus emerging. The analysis of the bundle shows the genesis of linking signs with different levels of generality and in reference to different cases: the verbal “from n=1 to n=2”; the written inscription linking the drawings of the 4x4 and the 8x8 chessboards (from n=2 to n=3, see fig. 3a,b,f); the gesture (level 2) linking the drawing of the 2x2 and 1x1 chessboards (from n=1 to n=2, see fig. 3e) and the metaphorical gesture (level 3, see fig. 3c,d). Giuditta is progressively shifting her focus from the tessellation of some specific chessboards to the links between these tessellations. Now, the produced linking signs allow her to establish the inductive relationship. In fact, at this point Giuditta shows how she could tessellate the 8x8 chessboard (except for one square) using a tessellation of the 4x4 chessboard and placing a tile in the central part of the chessboard (fig. 3g). After a few minutes, she concludes:

5 Giu: And this, I can do it in general (*after a circular gesture around the drawing of a 4x4 chessboard, with the right hand she makes a spiral movement that widens as the right hand rises and concludes with spreading both the hands, Fig.4a,b,c,d,e and Fig. 4f for a summary*).

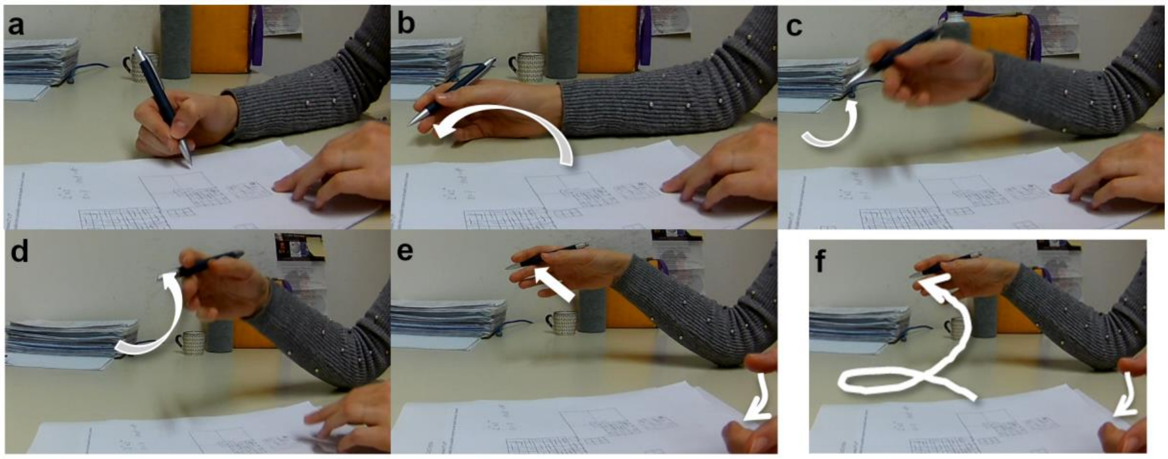

Figure 4: Gesture in line 5. The fig. 4f summarises the whole movement.

Giuditta does not write anything and she uses very few words: “and this, I can do it in general”. However, her gesture reveals the structure of argumentation and give us access to her reasoning. The gesture is articulated in four components.

The first component is the same gesture she has produced several times since line 1 when she linked the 8x8 and the 16x16 chessboards; now this gesture represents the action of constructing the 8x8 chessboard using the 4x4 chessboards.

The second component consists of contracting the previous gesture and moving away her right hand from the sheet in two directions: upwards and outwards. The upward direction takes the gesture from level 2 to level 3. It is the first time that Giuditta produces this gesture in the air. The shift through levels and her words indicate the generality of the actions of tessellation. Moreover, the gesture grows wider away from her body to indicate the construction of bigger chessboards (in mathematical terms, n is increasing). Until now, the left hand has remained still with a finger of the drawing of the 4x4 chessboards (which could represent the starting point of the recurrence; in fact she has already directly verified the cases of the smaller chessboards).

The third component consists in moving the right hand to the right - making the metaphoric gesture of a link, as seen in figure 3c,d - and moving the left hand to the left: the link between the chessboards of different sizes, represented before by an iconic gesture, here becomes an inductive step represented by a metaphoric gesture. These first three components, consisting of a sequence of different linking signs, constitute a unique iteration sign, which in its complete form is a gesture of level 2-3: it starts on the sheet, in which the base of the induction is represented, and rapidly moves away from the sheet becoming a gesture of the level of the general (level 3).

Finally, the fourth component consists in keeping her hands still in the air, as if they contain the space in which the iteration gesture took place. This space, to use an expression of McNeil (1992, p. 173) when describing an iconic gesture that indicates a point in space, is not empty but "full of conceptual significance". In our case, this space is the location that contains the argumentation and its logical structure.

CONCLUDING REMARKS

The multimodal perspective and the notion of semiotic bundle (Arzarello, 2006) has allowed us to identify and to analyse linking and iterative signs, and to observe and study the genesis of a proof by mathematical induction. Our analysis confirms the results presented in other studies (Arzarello & Sabena, 2014; Krause, 2015; Sabena, 2018) regarding the role of gestures in providing a logical structure to argumentation.

In the first excerpt, the speech-gesture mismatch (synchronic analysis) shows that the subject focuses simultaneously on two cases (8x8 and 16x16 chessboards). The bundle evolves and new signs are produced (diachronic analysis) to connect the two objects. The bundle is composed by different kinds of signs with mutual relationships. Only when we consider the bundle as a unit, we can see the linking sign representing a 16x16 chessboard as composed by 8x8 chessboards. This and other signs lead the subject to establish the connection between the problem of tessellating a chessboard and the same problem on a bigger chessboard, and then to construct the inductive step.

During the production of the argumentation, a repetition of linking signs produces an iterative sign and the complete detachment of the gesture from the sheet shows the transition to the general (Krause, 2016). The gesture contracts progressively, from iconic (referring to the extension of a chessboard into a bigger one) to metaphoric (referring to the inductive step), from level 2 (level of concrete) to level 3 (level of general). The iterative sign reveals that Giuditta constructs the entire recurrence even if it is not formally necessary (having proved the base case and the inductive step). The still hands at the end show the transition of argumentation from process to object.

The contraction of linking signs reveals a change of the focus. For Radford, "contraction is the mechanism for reducing attention to those aspects that appear to be relevant […] We need to forget to be able to focus" (Radford, 2008, p. 94). The contraction of Giuditta's gesture shows that she "forgets" the tessellation and focuses on the relationships between tessellations. Following Radford (2003), the contraction of linking signs is a process of objectification of the inductive step.

Moreover, the repetition of linking signs is an example of *catchment*. According to McNeill (2005), a catchment is due to the recurrence of consistent visuospatial imagery in the speaker's thinking, and indicates and provides the discourse cohesion. Arzarello and Sabena (2014) show that catchments contribute to support the students in structuring a mathematical argumentation. Our analysis seems to confirm their results.

Finally, further research is necessary to identify linking signs in symbolic manipulation and to study the evolution of linking signs within the bundle from the proving process to the written proof.

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