WEAK ENGEL ELEMENTS

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In memory of Francesco

Abstract. In this short note we define the notion of *weak Engel element* and *weak unipotent automorphisms*, and prove that, when such elements exist for a pro-p-group, some normal powerful subgroups cab ne constructed.

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1. Introduction

Let G be a group. An element $g \in G$ is said to be a *left Engel element* if, for every $x \in G$, there is an integer $n = n(x)$ such that $[x,_n g] = 1$. If n can be chosen uniformly, g is an *left n-Engel element* . Engel groups (resp. n-Engel groups) are groups where every element is a left Engel element (resp. left n-Engel) and, clearly, every nilpotent group belongs to this class. However there exist groups that are Engel but not nilpotent. Such groups are infinite, since for finite groups being Engel implies nilpotency. If p is a prime and A is an infinite elementary abelian p-group, the group $C_p \wr A$ is easily seen to be a non-nilpotent $(p+1)$-Engel group. However, in some cases, it is still possible to prove nilpotency of Engel groups. A relevant case is the one of profinite groups. Zel'manov has proved that torsion-free Engel profinite groups are nilpotent and, jointly with Wilson, they have shown that a profinite Engel group is locally nilpotent. References to these results can be found in [4], where the interested reader can also find a wealth of information about Engel groups and related topics.

In this short note we will be concerned with profinite groups, possessing elements (or automorphisms) satisfying a property that can be seen as a weak version of the Engel condition.

Definition 1. *Let G be a group. An element $g \in G$ is a* weak Engel element *if, for every normal elementary abelian quotient A/B, and every $xB \in A/B$, there exists $n = n(x)$ such that $[x,_n g] \in B$. If n can be chosen uniformly, we say that g is* weak n-Engel.

A similar definition can be given for profinite groups. In this case we ask that the subgroups A, B are open. It might sometimes be convenient to use a slightly more general definition.

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Definition 2. *Let G be a group. An automorphism α of G is a* weak unipotent automorphism *if α is a weak Engel element in $G\langle\alpha\rangle$. When the integers $n(x)$ can be chosen uniformly, we say that α is a* weak n-unipotent automorphism.

It is clear that a weak Engel element, induces an inner automorphism which is weakly unipotent.

In [3][Lemma 1], the author consider a normal subroup of a finite p-group, whose elements act trivially on elementary abelian section, showing that this subgroup is powerful. Using the definition we have just introduced, we have that this subgroup consists of weak Engel elements. In this note we will see that, when weak Engel elements exist, it is possible to construct several normal powerful subgroups. In order to make this claim precise, some definitions are needed. We therefore prefer to defer the statement of the main theorem to the next section. There is a well developed theory of powerful groups, so the existence of normal powerful subgroups is a strong property that can succesfully be used in the study of the group structure.

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2. Proofs

We start with a lemma about finite p-groups.

Lemma 1. *Let G be a solvable p-group of exponent p^ℓ and derived length d. If α is a weak n-unipotent automorphism of G, then the order of α is bounded in terms of p^ℓ, n and d.*

Proof. The proof is by induction on the derived length of G.

When G is abelian and $\ell = 1$, then $\alpha^{p^r} = 1$ whenever $p^r \geq n$. Since α acts on the filtration whose members are the subgroups G^{p^i} for $i = 0, \ldots, \ell$, we can invoke Lemma 16 of [1], since $[G,_\ell \alpha^{p^r}] = 1$ when r is chosen as indicated above. Assume now the claim holds for groups of derived length at most $d-1 > 1$, and let G have derived length d. The group $K = \langle \alpha^{f(p^\ell,n,d-1)} \rangle$ acts trivially on both $G/G^{(d-1)}$ and $G^{(d-1)}$. Thus $[G, K, K] = 1$ so that $[x, \beta^{p^\ell}] = [x, \beta]^{p^\ell} = 1$ for all $x \in G$ and $\beta \in K$. Therefore, setting $f(p^\ell, n, d) = f(p^\ell, n, d-1)p^\ell$, we get $[x, \alpha^{f(p^\ell,n,d)}] = 1$ for all $x \in G$ proving that $|\alpha|$ divides $f(p^l, n, d)$, as claimed. □

The next lemma shows that, when a a p-group G has an n-unipotent automorphism, then it is possible to construct several powerful subgroups of G.

Lemma 2. *Let G be a finite p-group, $\alpha \in \mathrm{Aut}(G)$ be a weak n-unipotent automorphisms and L any α-invariant subgroup of G. Then there exists k such that $R = [L, H^k]$ is powerful.*

Proof. We treat the case $p \neq 2$ first. Let $k = f(p, n, 2)$ as defined in lemma 1, set $K = \langle \alpha^k \rangle$ and $R = [L, K] = [L, \alpha^k]$. Since α is a weak n-unipotent automorphism, then $[R, K] \leq R^{(2)}R^p$. Clearly $[R, K^g] \leq R^{(2)}R^p$ for every element $g \in L$. Therefore $R' = [R, [K, L]] \leq R^{(2)}R^p$. Dedekind's modular law gives $R' = R' \cap R^{(2)}R^p = R^{(2)}(R^p \cap R')$ but, since $R^{(2)}$ is contained in the Frattini subgroup of R', we get

$R' = R^p \cap R'$. Therefore $R' \leq R^p$, proving the claim. When $p = 2$ choose $k = f(4, n, 2)$ and apply the same argument. □

This lemma can be easily adapted to the case of finitely generated pro-p-groups. However it is better to consider a slightly more general setting, that we describe here below.

Definition 3. *Let G be a profinite group, and α an automorphism of G. We say that G is* α-regular *if it has a basis of normal open subgroups, whose members are α-invariant.*

If G is a finitely generated profinite group, then it is α-regular for every automorphism α, because it has only finitely many normal subgroups of fixed finite index (which are, by the way, open, as proved in [2]), which are therefore permuted by the automorphisms of G. If N_m is the intersection of the normal subgroups of index m of G, the set $\{N_m \mid m \in \mathbb{N}\}$ is a basis of open neighborhoods of 1 in G, hence G is α-regular.

Another important case is when α is inner. In this case G is clearly α-regular.

When considering a topological group U, for every subgroup A, we shall indicate by $cl_U(A)$ its topological closure in U. As usual a closed subgroup A is said to be *finitely generated* when A is the closure of a finitely generated discrete subgroup A_0.

Theorem 1. *Let G be an α-regular pro-p group, where α is weakly n-unipotent. Set $H = \langle \alpha \rangle$ Then the following facts hold*

(1) *There exists k such that, for every α-invariant subgroup L of G, the group $R = cl_G([L, H^k])$ is powerful.*
(2) *There exists $r = r(p, n)$ such that, for every $x \in G$, the group $L = cl_G(x^H)$ can be generated by r elements.*

Proof. Let $k = f(p, n, 2)$ or $f(4, n, 2)$ when $p = 2$, as defined in lemma 2, set $H = \langle \alpha \rangle$ and $K = H^k$. Let $\mathcal{B} = \{N_\lambda \mid \lambda \in \Lambda\}$ be a basis of normal H-invariant open subgroups of G and let L be an H-invariant subgroup of G. For each $\lambda \in \Lambda$ define $L_\lambda = LN_\lambda/N_\lambda$ and $R_\lambda = [L, K]N_\lambda/N_\lambda = [L_\lambda, K]$. We have $[L, K] \leq [L, H^k]N_\lambda$ so that $R_\lambda = [L, H^k]N_\lambda/N_\lambda = [L_\lambda, H^k]$. Each L_λ is a finite p-group and, by Lemma 2, R_λ is powerful. Therefore the group $R = cl_G([L, K])$, which is the inverse limit of the R_λ, is powerful.

The group H/K is cyclic of order bounded by k, which is a function of p and n. Choose $x \in G$ and set $L = cl_G(x^H)$. Clearly L is H-invariant. Stick to the notation established in the above paragraph. For each $\lambda \in \Lambda$, we have

$$[L_\lambda, K] \leq {L_\lambda}^{(2)} {L_\lambda}^p \leq \Phi(L_\lambda).$$

Choose $X = \{h_i \mid i = 1, \ldots, r\}$ a left tranversal for K in H and fix $\lambda \in \Lambda$. Given any $y \in K$, $h \in X$ and $x \in L_\lambda$, we have

$$x^{hy} = x^h[x^h, y] \equiv x^h \pmod{\Phi(L_\lambda)}.$$

Thus the set $X_\lambda = \{x^h N_\lambda \mid h \in X\}$ generates L_λ, modulo its Frattini subgroup so that $\langle X_\lambda \rangle = L_\lambda$. Since each L_λ can be generated by r elements, the same holds for their inverse limit L. □

This theorem applies, in particular, to pro-p groups possessing weak n-Engel elements.

References

[1] Carlo Casolo. Groups with all subgroups subnormal. *Note Mat.*, 28:1–153, 2008.

[2] Nikolay Nikolov and Dan Segal. On finitely generated profinite groups. I. Strong completeness and uniform bounds. *Ann. of Math. (2)*, 165(1):171–238, 2007.

[3] Aner Shalev. Characterization of p-adic analytic groups in terms of wreath products. *J. Algebra*, 145(1):204–208, 1992.

[4] Gunnar Traustason. Engel groups. In *Groups St Andrews 2009 in Bath. Volume 2*, volume 388 of *London Math. Soc. Lecture Note Ser.*, pages 520–550. Cambridge Univ. Press, Cambridge, 2011.

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