Proving a conjecture on prime double square tiles

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Abstract

In 2013, while studying a relevant class of polyominoes that tile the plane by translation, i.e., double square polyominoes, Blondin Massé et al. found that their boundary words, encoded by the Freeman chain coding on a four letters alphabet, have specific interesting properties that involve notions of combinatorics on words such as palindromicity, periodicity and symmetry. Furthermore, they defined a notion of reducibility on double squares using homologous morphisms, so leading to a set of irreducible tile elements called *prime double squares*. The authors, by inspecting the boundary words of the smallest prime double squares, conjectured the strong property that no runs of two (or more) consecutive equal letters are present there. In this paper, we prove such a conjecture using combinatorics on words' tools, and setting the path to the definition of a fast generation algorithm and to the possibility of enumerating the elements of this class w.r.t. standard parameters, as perimeter and area.

Keywords: Discrete Geometry, Combinatorics on words, Tiling, Exact tile

1. Introduction

Algorithmic studies of planar tilings greatly benefit from the seminal works [14] and [15], where the decidability of the existence of planar tilings is addressed both with a given set of tiles and with a single one. In the first case, it has been shown that each Turing machine computation can be simulated by a planar tiling using a suitable set of tiles without rotations, starting from an initial partial configuration that models the input tape. On the other hand, if only one tile is provided, the computation becomes much easier. As a matter of fact in [15] the authors, with the aim of proving a conjecture by Shapiro [13], showed that a connected finite set of points in $\mathbb{Z}^2$ defines a *polyomino tile*, that is *exact* if and only if it admits a periodical tiling of the plane. This result is of great relevance since it limits the test for the exactness of a polyomino to a finite part of the plane. Relying on that, Beauquier and Nivat in [3] characterized the

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boundary of an exact polyomino, regarded as a word on a four letters alphabet, using the notions of rotation and conjugation proper of combinatorics on words, thus setting a strong connection between these two research fields.
In particular, the authors showed that the boundary word P of an exact polyomino can be factorized according to the equation $P = X_1X_2X_3\widehat{X}_1\widehat{X}_2\widehat{X}_3$, where, considering the word X as a path, the word $\widehat{X}$ refers to the coding of the same path travelled in the opposite direction. According to [3], at most one among X_1, X_2 and X_3 can be empty. Exact polyominoes with such a property are addressed as pseudo-squares, while pseudo-hexagons are those exact tiles in which all the factors X_i are non-empty words. The name is due to the property that such polyominoes tile the discrete plane by translation surrounding themselves with four, respectively six, copies of themselves.
It is easy to verify that an exact polyomino P can be used to tile the plane in different ways, in general, and that it can show both a pseudo-hexagon's and a pseudo-square's behaviour, also in the same tiling. Furthermore, the arrangement of the copies of P in each of its tilings has a periodical behaviour along one or two discrete directions, that are strictly related to the choice of the factors X_i in the decomposition of the word P, non-unique in general. See Fig. 1 for examples.

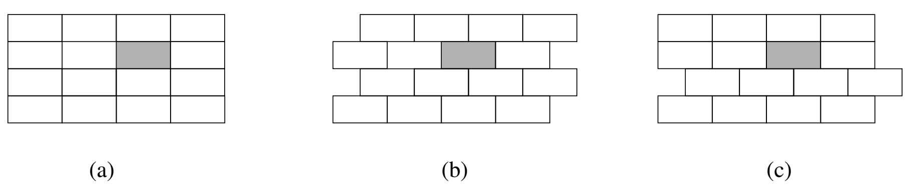

Figure 1: Three tilings of the plane with the two-cell polyomino showing different behaviours. In (a) the polyomino acts as a pseudo-square, since it is surrounded by four copies of itself. In (b) the polyomino acts as a pseudo-hexagon, since six copies surround each polyomino. In (c) both behaviours are present. Here, the dark polyomino is surrounded by five copies of itself, but this is not the case for each element of the tiling.

Relying on these strong geometrical properties, exact polyominoes have been considered under different perspectives. Several algorithms have been defined to improve the efficiency of their detection: moving from the simplest $\mathcal{O}(n^4)$ strategy to find a (possible) factorization of a n-length boundary word, in [9] the complexity has been reduced to $\mathcal{O}(n^2)$, finally reaching an optimal linear time strategy in [7] in case of pseudo-squares. Specific regularity properties of the tilings induced by exact polyominoes have been studied, and provided useful tools to study the periodicity of infinite binary patterns in the discrete plane, as well as the possibility of their decomposition into simpler configurations, see [1, 2, 8, 11]. On the other hand, combinatorial aspects of relevant subclasses of exact polyominoes have been considered with the aim of efficiently characterizing, enumerating and exhaustively generating them (few examples for all are [4, 5, 6]). In this work we carry on the study of tiles from this last perspective, and in particular we focus on the subclass of double square tiles.

We move from [4], where it was proved that an exact polyomino can tile the plane as a pseudo-square in at most two distinct ways, and that if two different pseudo-square factorizations of P exist, then no decomposition as a pseudo-hexagon does. The authors refer to these exact polyominoes as *double squares*. In Fig. 2, three double square polyominoes are provided together with their factorizations.
Furthermore, it is shown in [6] that double squares have specific boundary properties, that are exploited using equations on words. These properties lead to the definition of two operators that allow to exhaustively generate them, and to the notion of *double square reducibility* through the definition of suitable *homologous morphisms*. Among double squares, those that can be reduced without intermediate steps to the unit square constitute the subclass of *prime double squares*, and are pointed out for their relevance in the exhaustive generation of double squares polyominoes. Still in [6], the following conjecture was proposed:

Conjecture 35. Let w be the boundary word of a prime double square tile in a four letters alphabet Σ. Then, for any letter $\alpha \in \Sigma$, $\alpha\alpha$ is not a factor of w.

The authors of [6] put emphasis on the relevant role that *prime* double squares play in the exhaustive generation of all double square tiles: they defined an algorithm to generate all *prime* double squares starting from the unit square, relying on strong palindromicity properties of their boundary words. In this work we deepen the study of such tiles, and we prove Conjecture 35 by providing a strong combinatorial structure of the boundary word of any prime double square. Such results allow to take a step forward the characterization of prime double squares, and then to their exhaustive generation without repetitions and outliers, as intended in [6].
The paper is organized as follows: in the next section we recall basic definitions on combinatorics on words and some preliminary results to approach Conjecture 35. In Section 3, we give some properties of double squares' boundary words, mainly using the results from [6]. The proof of Conjecture 35 is in Section 4, while Section 5 is devoted to point out future combinatorial, algorithmic and geometrical research paths that originate from our main result.

2. Basic notions and previous results

In this section, we fix the notation and recall some basic results on exact polyominoes to introduce the study of Conjecture 35.
A *polyomino* is a subset of the square grid $\mathbb{Z}^2$ whose boundary is a continuous, closed and non-intersecting path. We describe polyominoes by coding their boundary through a word defined on the alphabet $\Sigma = \{0, 1, \overline{0}, \overline{1}\}$, whose elements correspond to the directions $\{\rightarrow, \uparrow, \leftarrow, \downarrow\}$ of steps made in the grid, respectively. We say that the letters 0 and $\overline{0}$, resp. 1 and $\overline{1}$, are *opposite*, since they represent opposite directions (see Example 2). We indicate by Σ^* the free monoid on Σ, i.e., the set of all words defined on the alphabet Σ, with ε the empty word, and $\Sigma^+ = \Sigma^* \setminus \{\varepsilon\}$. We therefore call *unit square* the polyomino coded by the word $U = 10\overline{1}\overline{0}$. Given a word $w \in \Sigma^*$, $|w|$ indicates its length,

$|w|_\alpha$ indicates the number of occurrences of the letter α in w, and w^n indicates the concatenation of n copies of the word itself. Finally, v is a *factor* of w if there exist $x, y \in \Sigma^*$ such that $w = xvy$. If $x = \varepsilon$ [resp. $y = \varepsilon$], then v is a *prefix* [resp. *suffix*] of w.
Boundary words can be considered as *circular* words, since the coding of the boundary of the polyomino can be defined up to the starting point.
Two words v and w are *conjugate*, say $w \equiv v$, if there exist two words x and y such that $v = xy$ and $w = yx$. The conjugacy is an equivalence relation, and the conjugacy class of a word w contains all its cyclic shifts. So, the boundary words of a polyomino, when fixing a traveling direction, form an equivalence class w.r.t. the conjugacy relation. In the sequel, we choose the clockwise traveling direction and each boundary word in the related equivalence class to identify a polyomino. Furthermore, the conditions of closure and non-intersection of a polyomino boundary word P can be stated as $|P|_\alpha = |P|_{\overline{\alpha}}$ and $|Q|_\alpha \neq |Q|_{\overline{\alpha}}$ for each Q proper factor of P, say $Q \subsetneq P$, and $\alpha \in \{0, 1\}$.

We define three operators on a word $w = w_1 w_2 \dots w_n \in \Sigma^*$:

1. the *opposite of* w, $\overline{w}$, is the word obtained by replacing each letter of w with its opposite;
2. the *reversal of* w, $\tilde{w}$, is defined as $\tilde{w} = w_n w_{n-1} \dots w_1$. A *palindrome* is a word s.t. $w = \tilde{w}$;
3. the *hat of* w, $\widehat{w}$, is the antimorphic involution given by the composition of the previous operations.

Example 1. *If we apply the previous operations to the word* $w = 101001\overline{0}1010\overline{1}010\overline{1}01$, *we get*

$$\begin{aligned} \overline{w} &= \overline{101001}0\overline{1010}1\overline{010}1\overline{01}, \\ \tilde{w} &= 10\overline{1}010\overline{1}0101\overline{0}100101, \\ \widehat{w} &= \overline{10}1\overline{010}1\overline{1010}0\overline{100101}. \end{aligned}$$

In this case $\tilde{w} \neq w$, *indeed the word* w *is not a palindrome.*

Finally, an *exact* polyomino is a polyomino that tiles the plane by translation. Beauquier and Nivat characterized exact polyominoes in relation to their boundary word, providing the following

Theorem 1 ([3]). *A polyomino* P *is exact if and only if there exist* $X_1, X_2, X_3 \in \Sigma^*$ *such that*

$$P = X_1 X_2 X_3 \widehat{X}_1 \widehat{X}_2 \widehat{X}_3,$$

where at most one of the words is empty. This factorization may be not unique.

We refer to this decomposition as a *BN-factorization.* Starting from their BN-factorization(s), exact polyominoes can be further divided in two classes: *pseudo-hexagons*, if X_1, X_2 and X_3 are all non-empty words, and *pseudo-squares*, if one of the words is empty. We focus on *double square polyominoes*, i.e., those ones that admit two different BN-factorizations as a square, $P = AB\widehat{A}\widehat{B} \equiv$

$XY\widehat{X}\widehat{Y}$. Due to the presence of two BN-factorizations, double squares' boundary words can be written in the general form obtained from Corollary 6 in [7],

$$P = w_1 w_2 w_3 w_4 w_5 w_6 w_7 w_8, \tag{1}$$

where $A = w_1 w_2$, $B = w_3 w_4$, $\widehat{A} = w_5 w_6$, $\widehat{B} = w_7 w_8$ and $X = w_2 w_3$, $Y = w_4 w_5$, $\widehat{X} = w_6 w_7$, $\widehat{Y} = w_8 w_1$, with $w_1, \ldots, w_8$ non empty.
We now introduce the notion of *homologous morphism*. A morphism is a function $\varphi : \Sigma \to \Sigma^*$ s.t. $\varphi(\alpha\beta) = \varphi(\alpha)\varphi(\beta)$ with $\alpha, \beta \in \Sigma$, i.e., it preserves concatenation, and it is said to be *homologous* if $\varphi(\widehat{A}) = \widehat{\varphi(A)}$ for all $A \in \Sigma^*$, i.e., it preserves the hat operation. From now on we refer to homologous morphisms only. For each exact polyomino $P = AB\widehat{A}\widehat{B}$, we can define the trivial morphism that maps the unit square in P as $\varphi_P(1) = A$, $\varphi_P(0) = B$. In general, the boundary word of an exact polyomino can be obtained starting from the unit square through the composition of two or more morphisms (see Example 3). A double square is *prime* if its boundary word P is such that, for any homologous morphism φ, the equality $P = \varphi(U)$ implies that either $U = P$ or U is the boundary word of the unit square. This property can be rephrased saying that a double square is prime if its trivial morphism can not be obtained by composing two or more different morphisms.

Example 2. *In Fig. 2 are shown three prime double squares, whose boundary words are, from left to right,* $P_{(a)} = 1\bar{0}1\bar{0}10101\bar{0}1\bar{0}\overline{101010}$, $P_{(b)} = 1\bar{0}10101\bar{0}\overline{1010}$ *and* $P_{(c)} = 1\bar{0}1\bar{0}10101\bar{0}1\bar{0}\overline{1010}$.

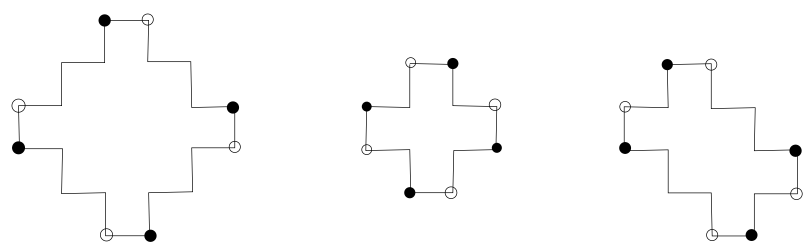

Figure 2: The *diamond tile* (a), the *cross tile* (b) and the *butterfly tile* (c) are three examples of prime double square tiles. Their double BN-factorizations are highlighted on the boundary with dots of different colors.

Example 3. *The double square*

$$P = 101001010010100\overline{101}00\overline{101}00\overline{1010010100101}00101\overline{00}101\overline{00}$$

is not prime. The trivial morphism, $\varphi_P(1) = 1010010100101$, $\varphi_P(0) = 00\overline{101}00\overline{101}00$, *can be decomposed as* $\varphi_1(1) = 10101$, $\varphi_1(0) = 0\bar{1}0\bar{1}0$ *and* $\varphi_2(0) = 00$, $\varphi_2(1) = 101$. *The composition* $\varphi_P = \varphi_1 \circ \varphi_2$ *maps the unit square in P using the diamond as intermediate step (as shown in Fig. 3).*

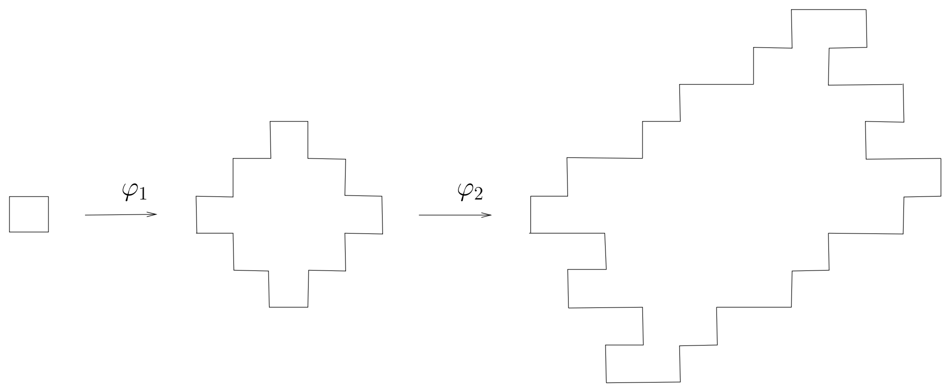

Figure 3: The figure shows the composition of two morphisms that lead to a non-prime double square.

Now, we have introduced all the notions to state a conjecture by Blondin Massé et al. in [6], that constitutes the focus of our work.

Conjecture 35 ([6]). *Let P be the boundary word of a double square and α a symbol of the alphabet Σ. If P is prime, then $\alpha\alpha$ is not a factor of P.*

Notation: for each $\alpha \in \Sigma$, we use the notation $\alpha\alpha \subseteq w$ (resp. $\alpha\alpha \not\subseteq w$) to indicate that the word w contains (resp. does not contain) the factor $\alpha\alpha$.
On the other hand, we indicate w to be *couple free* if no two consecutive occurrences of a same letter of Σ are present.
We conclude this section by stating two useful technical lemmas.

Lemma 1 ([6]). *Given a double square with BN-factorizations $P = AB\widehat{A}\widehat{B} \equiv XY\widehat{X}\widehat{Y}$, if P is prime then all its factors A, B, X, Y are palindrome.*

Lemma 2. *Given $a, b, u \in \Sigma^+$, if $b\tilde{u}a$ and $b\widehat{u}a$ are palindrome, then $b = \tilde{a}$ and u is palindrome. If $\tilde{a}u\tilde{b}$ and $\overline{b}\tilde{u}\overline{a}$ are palindrome, then the same result holds.*

Proof. By contradiction, let us suppose $|b| < |a|$ and $a = y\tilde{b}$ for some non-empty $y \in \Sigma^+$. Replacing, $b\tilde{u}y\tilde{b}$ and $b\widehat{u}y\tilde{b}$ are both palindrome if and only if $\tilde{u}y$ and $\widehat{u}y$ are palindrome at the same time, that is impossible since $u \neq \varepsilon$. If $|a| < |b|$ the proof is similar. Analogous arguments lead to the proof of the second statement. $\square$

3. Properties of the boundary word of prime double squares

In this section, we study the prime double squares' boundary words. In particular, we exploit some of their properties that, step by step, lead to write them in a useful general form.

Lemma 3 ([6]). *Let $P = w_1w_2w_3w_4w_5w_6w_7w_8$ be the BN-factorization of (the boundary word of) a prime double square as in (1). It holds that $w_{i+4} = \overline{w}_i$ for all $i = 1, \ldots, 4$.*

It follows that the boundary word of a prime double square can be written as

$$P = w_1 w_2 w_3 w_4 \overline{w}_1 \overline{w}_2 \overline{w}_3 \overline{w}_4. \tag{2}$$

Lemma 4. *Let $P = AB\widehat{A}\widehat{B}$ be a prime double square and $\alpha \in \Sigma$. If $\alpha\alpha \subseteq P$, then $\alpha\alpha \subseteq A$ or $\alpha\alpha \subseteq B$.*

Proof. By contradiction. Let us suppose that the BN-factorization splits $\alpha\alpha$, i.e., there exist $a_1, a_2 \in \Sigma^*$ such that $P = AB\widehat{A}\widehat{B} = (a_1\alpha)(\alpha a_2)\widehat{A}\widehat{B}$. Since P is a prime double square, its factors are both palindrome by Lemma 1, i.e., $A = (\alpha b_1 \alpha)$ and $B = (\alpha b_2 \alpha)$ for some $b_1, b_2 \in \Sigma^*$. Replacing A and B in P, we get

$$P = (\alpha b_1 \alpha)(\alpha b_2 \alpha)(\overline{\alpha}\widehat{b}_1\overline{\alpha})(\overline{\alpha}\widehat{b}_2\overline{\alpha}),$$

reaching a contradiction since the factor $\alpha\overline{\alpha}$ represents a closed path, not allowed in the boundary of a polyomino. □

Corollary 1. *From the previous lemma and the general form of P in (2), it directly follows that the term $\alpha\alpha$ can not be split between two consecutive factors w_i and w_{i+1}.*

Corollary 2. *If $P = w_1 \dots w_8$ is a prime double square, then $|w_i| \neq |w_{i+1}|$ for all $i = 1, \dots, 8$.*

Proof. By contradiction. Let be $|w_i| = |w_{i+1}|$ for some i. Since $w_i w_{i+1}$ is a BN-factor and P is prime, then, by Lemma 1, it is a palindrome, i.e., $w_i w_{i+1} = \tilde{w}_{i+1}\tilde{w}_i = w_i \tilde{w}_i$. It follows that the last letter of w_i and the first one of w_{i+1} match, i.e., we can split a term $\alpha\alpha$ between two BN-factors, in contradiction with Lemma 4. □

When considering the boundary word $P = w_1 \dots w_8$ of a generic (not necessarily prime) double square, the following property holds:

Property 1 ([6]). *For $i = 1, \dots, 8$, there exist unique words $u_i, v_i \in \Sigma^*$ and unique $n_i \geq 0$ such that*

$$\begin{cases} w_i = (u_i v_i)^{n_i} u_i, \\ \widehat{w}_{i-3} w_{i-1} = u_i v_i. \end{cases}$$

From Property 1 and Lemma 3, we can refine the generic form of the boundary word of a prime double square provided in (2).
First, we note that $w_{i+4} = \overline{w}_i$ and $w_i = (u_i v_i)^{n_i} u_i$ imply

$$\begin{aligned} u_{i+4} &= \overline{u}_i, \\ v_{i+4} &= \overline{v}_i, \\ n_{i+4} &= n_i, \end{aligned}$$

for $i = 1, \dots, 4$.

By the second equation in Property 1, we can write $u_i v_i$ depending on w_{i-1} and w_{i-3}, and replacing them in the first one we get

$$\begin{aligned} P =&(\tilde{w}_2\overline{w}_4)^{n_1}u_1(\tilde{w}_3 w_1)^{n_2}u_2(\tilde{w}_4 w_2)^{n_3}u_3(\widehat{w}_1 w_3)^{n_4}u_4\dots \\ &\dots(\widehat{w}_2 w_4)^{n_1}\overline{u}_1(\widehat{w}_3\overline{w}_1)^{n_2}\overline{u}_2(\widehat{w}_4\overline{w}_2)^{n_3}\overline{u}_3(\tilde{w}_1\overline{w}_3)^{n_4}\overline{u}_4. \end{aligned} \tag{3}$$

The following lemma is a rephrase of Lemma 11 in [6]:

Lemma 5 ([6]). *Given the boundary word P of a prime double square according to* (3)*, there are no two consecutive exponents n_1, n_2, n_3 and n_4 that are different from zero.*

Lemma 6. *The boundary word P of a prime double square, as in (3), can always be written in terms of the elements u_i only, for $i = 1, \dots, 4$.*

Proof. If $n_i = 0$ for all i, the thesis trivially holds. So, let us assume w.l.g. $n_1 > 0$ (the other cases can be treated similarly). By Lemma 5, it follows $n_2 = n_4 = 0$, and then $w_2 = u_2$ and $w_4 = u_4$. Since w_1 and w_3 are expressed in terms of w_2 and w_4 only, replacing in (3) the thesis follows. □

We finally get the generic form of the boundary word of a prime double square:

$$\begin{aligned} P =&(\tilde{u}_2\overline{u}_4)^{n_1}u_1 \vdots (\tilde{u}_3 u_1)^{n_2}u_2 \vdots (\tilde{u}_4 u_2)^{n_3}u_3 \vdots (\widehat{u}_1 u_3)^{n_4}u_4| \\ &(\widehat{u}_2 u_4)^{n_1}\overline{u}_1 \vdots (\widehat{u}_3\overline{u}_1)^{n_2}\overline{u}_2 \vdots (\widehat{u}_4\overline{u}_2)^{n_3}\overline{u}_3 \vdots (\tilde{u}_1\overline{u}_3)^{n_4}\overline{u}_4. \end{aligned} \tag{4}$$

We underline that (4) is always well defined since $n_1 = n_3 = 0$ or $n_2 = n_4 = 0$ (or both), see Lemma 5, and then the factors w_i in (3) can be expressed in terms of u_i only.

From now on, we use a vertical bar to indicate the half of the boundary word and dashed, vertical bars to separate two consecutive words w_i and w_{i+1}.

Theorem 2. *Let P be the boundary word of a prime double square as in (4), and $\alpha \in \Sigma$. If $\alpha\alpha \subseteq P$, then $\alpha\alpha$ is entirely contained in a factor u_i, i.e., the term $\alpha\alpha$ can not be split between two consecutive factors u_i.*

Proof. Let us suppose w.l.g. $\alpha\alpha \subseteq w_1 = (\tilde{u}_2\overline{u}_4)^{n_1}u_1$. By contradiction, we suppose to split the occurrences of α between two factors u_i. We analyze all possible cases for the values of n_1, n_3 and the factors u_i.

i) We start assuming $n_1 \geq 2$, that is the only case in which we can suppose $\alpha\alpha \subseteq \overline{u}_4\tilde{u}_2$. Since $\tilde{u}_2$ begins and $\overline{u}_4$ ends with α, then $\widehat{u}_2$ begins with $\overline{\alpha}$ and u_4 ends with $\overline{\alpha}$. In this case, we get the following factors in the boundary word P,

$$w_4|w_5 = \dots u_4|\widehat{u}_2\dots$$

and $\overline{\alpha\alpha}$ is split between two factors w_i, in contradiction with Lemma 4.

ii) Let $n_1 > 0$ and suppose $\alpha\alpha \subseteq \tilde{u}_2\overline{u}_4$. We remind that in this case $n_2 = n_4 = 0$ by Lemma 5. Since $\tilde{u}_2$ ends and $\overline{u}_4$ begins with α, then u_2 begins with α and u_4 begins with $\overline{\alpha}$. We analyze the factor $w_3 = (\tilde{u}_4 u_2)^{n_3} u_3$ and we study the exponent n_3. If $n_3 > 0$, then $\overline{\alpha}\alpha \subseteq w_3$ because of the presence of $\tilde{u}_4 u_2$, and the boundary of the polyomino intersects itself. Then, it holds $n_3 = 0$, and the BN-factor becomes $X = w_2 w_3 \overset{n_2=0}{=} u_2 u_3$. Since X is palindrome, u_3 ends with α and, being $n_4 = 0$, we get $\alpha\overline{\alpha} \subseteq w_3 w_4 = u_3 u_4$, and again the border intersects itself, reaching a contradiction.

iii) We suppose to have $n_1 > 0$ and $\alpha\alpha \subseteq \overline{u}_4 u_1$. Under this assumption, u_4 ends with the letter $\overline{\alpha}$ and u_1 begins with α. By Property 1, the factor w_1 has α as first letter as well; we then reach a contradiction with Corollary 1, since $\overline{\alpha}\overline{\alpha} \subseteq w_4 \overline{w}_1$.

Since we reached a contradiction for all possible cases, the proof is given. □

Since P is a circular word, from now on we suppose, w.l.g., that $|u_1| \leq |u_i|$ for $i = 2, 3, 4$. Assuming that, it is possible to define P expressed as in (4) in terms of two factors only, u_1 and u_3, and two words $k, p \in \Sigma^*$, with $k, p \neq \varepsilon$. It will be clear from the proof of Theorem 3 that $u_2 = k\tilde{u}_1$ and $u_4 = \widehat{u}_1\overline{p}$.
Relying on Lemmas 1 and 5, we get the following taxonomy according to the positive values of n_i, with $i = 1, \ldots, 4$:

a) $P = u_1 \vdots k\tilde{u}_1 \vdots u_3 \vdots \widehat{u}_1\overline{p} | \overline{u}_1 \vdots \overline{k}\widehat{u}_1 \vdots \overline{u}_3 \vdots \tilde{u}_1 p$. This form requires k and p palindrome.

b) $P = (u_1 k\tilde{u}_1 p)^{n_1} u_1 \vdots k\tilde{u}_1 \vdots u_3 \vdots \widehat{u}_1\overline{p} | (\overline{u}_1\overline{k}\widehat{u}_1\overline{p})^{n_1} \overline{u}_1 \vdots \overline{k}\widehat{u}_1 \vdots \overline{u}_3 \vdots \tilde{u}_1 p$. In this case, k and p are palindrome.

c) $P = u_1 \vdots (\tilde{u}_3 u_1)^{n_2} k\tilde{u}_1 \vdots u_3 \vdots \widehat{u}_1\overline{p} | \overline{u}_1 \vdots (\widehat{u}_3\overline{u}_1)^{n_2} \overline{k}\widehat{u}_1 \vdots \overline{u}_3 \vdots \tilde{u}_1 p$. In this case, p is palindrome.

d) $P = u_1 \vdots k\tilde{u}_1 \vdots (\overline{p}\overline{u}_1 k\tilde{u}_1)^{n_3} u_3 \vdots \widehat{u}_1\overline{p} | \overline{u}_1 \vdots \overline{k}\widehat{u}_1 \vdots (p u_1 \overline{k}\widehat{u}_1)^{n_3} \overline{u}_3 \vdots \tilde{u}_1 p$. In this case, k and p are palindrome.

e) $P = u_1 \vdots k\tilde{u}_1 \vdots u_3 \vdots (\widehat{u}_1 u_3)^{n_4} \widehat{u}_1\overline{p} | \overline{u}_1 \vdots \overline{k}\widehat{u}_1 \vdots \overline{u}_3 \vdots (\tilde{u}_1\overline{u}_3)^{n_4} \tilde{u}_1 p$. In this case, k is palindrome.

f) $P = (u_1 k\tilde{u}_1 p)^{n_1} u_1 \vdots k\tilde{u}_1 \vdots (\overline{p}\overline{u}_1 k\tilde{u}_1)^{n_3} u_3 \vdots \widehat{u}_1\overline{p} | (\overline{u}_1\overline{k}\widehat{u}_1\overline{p})^{n_1} \overline{u}_1 \vdots \overline{k}\widehat{u}_1 \vdots (p u_1 \overline{k}\widehat{u}_1)^{n_3} \overline{u}_3 \vdots \tilde{u}_1 p$.
In this case, k and p are palindrome.

g) $P = u_1 \vdots (\tilde{u}_3 u_1)^{n_2} k\tilde{u}_1 \vdots u_3 \vdots (\widehat{u}_1 u_3)^{n_4} \widehat{u}_1\overline{p} | \overline{u}_1 \vdots (\widehat{u}_3\overline{u}_1)^{n_2} \overline{k}\widehat{u}_1 \vdots \overline{u}_3 \vdots (\tilde{u}_1\overline{u}_3)^{n_4} \tilde{u}_1 p$.
In this case, k and p are generic, non-empty words in Σ^*.

Theorem 3. *The above seven cases **a)** . . . **g)** entirely describe the possible forms in which the boundary word of a prime double square can be expressed.*

Proof. P is a prime double square, then its BN-factors $A = w_1w_2$, $B = w_3w_4$, $X = w_2w_3$ and $Y = w_4\overline{w}_1$ are all palindrome by Lemma 1; we further remind that u_i is a prefix of w_i by Property 1. Then, for any possible value of n_i and by the hypothesis on the mutual lengths $|u_i|$, in particular $|u_1|$ is minimum, we can always express $u_2 = k\tilde{u}_1$ and $u_4 = \widehat{u}_1\overline{p}$ for some proper words $k, p \in \Sigma^*$, see the BN-factors expressed as in (4). We notice that we can not deduce an analogous form for the word u_3, since we do not know the relative lengths w.r.t. u_2 and u_4. Let us now consider all the possible cases on the values n_i and study the boundary word we get for each one.

a) $n_1 = n_2 = n_3 = n_4 = 0$**,**

$$P = u_1 \vdots k\tilde{u}_1 \vdots u_3 \vdots \widehat{u}_1\overline{p} | \overline{u}_1 \vdots \overline{k}\widehat{u}_1 \vdots \overline{u}_3 \vdots \tilde{u}_1 p.$$

Since w_1w_2 and $w_4\overline{w}_1$ are palindrome, we get k and p palindrome. Furthermore, $k, p \neq \varepsilon$ to avoid to split a term $\alpha\alpha$ between two factors $u_1\tilde{u}_1$ or $\widehat{u}_1\overline{u}_1$, in contradiction with Theorem 2.

b) $n_1 > 0, n_2 = n_3 = n_4 = 0$**,**

$$P = (u_1\tilde{k}\tilde{u}_1 p)^{n_1} u_1 \vdots k\tilde{u}_1 \vdots u_3 \vdots \widehat{u}_1\overline{p} | (\overline{u}_1\widehat{k}\widehat{u}_1\overline{p})^{n_1} \overline{u}_1 \vdots \overline{k}\widehat{u}_1 \vdots \overline{u}_3 \vdots \tilde{u}_1 p.$$

Since w_1w_2 and $w_4\overline{w}_1$ are palindrome, we deduce that p and k are palindrome too. In particular, if n_1 is odd, the palindromicity of p follows from w_1w_2 and the palindromicity of k from $w_4\overline{w}_1$, vice versa if n_1 is even. Moreover, $p, k \neq \varepsilon$ as discussed in case **a)**.

c) $n_2 > 0, n_1 = n_3 = n_4 = 0$**,**

$$P = u_1 \vdots (\tilde{u}_3 u_1)^{n_2} k\tilde{u}_1 \vdots u_3 \vdots \widehat{u}_1\overline{p} | \overline{u}_1 \vdots (\widehat{u}_3\overline{u}_1)^{n_2} \overline{k}\widehat{u}_1 \vdots \overline{u}_3 \vdots \tilde{u}_1 p.$$

As in case **b)**, we deduce p palindrome from $w_4\overline{w}_1$. Again, $k, p \neq \varepsilon$.

d) $n_3 > 0, n_1 = n_2 = n_4 = 0$**,**

$$P = u_1 \vdots k\tilde{u}_1 \vdots (\widehat{p\overline{u}}_1 k\tilde{u}_1)^{n_3} u_3 \vdots \widehat{u}_1\overline{p} | \overline{u}_1 \vdots \overline{k}\widehat{u}_1 \vdots (\tilde{p} u_1\overline{k}\widehat{u}_1)^{n_3} \overline{u}_3 \vdots \tilde{u}_1 p.$$

Similarly, k is palindrome from w_1w_2 and p is palindrome from $w_4\overline{w}_1$. Moreover, they are not empty.

e) $n_4 > 0, n_1 = n_2 = n_3 = 0$**,**

$$P = u_1 \vdots k\tilde{u}_1 \vdots u_3 \vdots (\widehat{u}_1 u_3)^{n_4} \widehat{u}_1\overline{p} | \overline{u}_1 \vdots \overline{k}\widehat{u}_1 \vdots \overline{u}_3 \vdots (\tilde{u}_1\overline{u}_3)^{n_4} \tilde{u}_1 p.$$

We get that k is palindrome from w_1w_2. Again, $k, p \neq \varepsilon$.

f) $n_1, n_3 > 0, n_2 = n_4 = 0$,

$$P = (u_1\tilde{k}\tilde{u}_1 p)^{n_1} u_1 \vdots k\tilde{u}_1 \vdots (\widehat{p}\overline{u}_1 k\tilde{u}_1)^{n_3} u_3 \vdots \widehat{u}_1\overline{p} | (\overline{u}_1\widehat{k}\widehat{u}_1\overline{p})^{n_1}\overline{u}_1 \vdots \overline{k}\widehat{u}_1 \vdots (\tilde{p}u_1\overline{k}\widehat{u}_1)^{n_3}\overline{u}_3 \vdots \tilde{u}_1 p.$$

The word p is palindrome from $w_4\overline{w}_1$ while k is palindrome from w_1w_2, or vice versa, depending on the parity of n_1. Both are not empty.

g) $n_2, n_4 > 0, n_1 = n_3 = 0$,

$$P = u_1 \vdots (\tilde{u}_3 u_1)^{n_2} k\tilde{u}_1 \vdots u_3 \vdots (\widehat{u}_1 u_3)^{n_4}\widehat{u}_1\overline{p} | \overline{u}_1 \vdots (\widehat{u}_3\overline{u}_1)^{n_2}\overline{k}\widehat{u}_1 \vdots \overline{u}_3 \vdots (\tilde{u}_1\overline{u}_3)^{n_4}\tilde{u}_1 p.$$

As seen in the previous cases, $k, p \neq \varepsilon$. Moreover, in this case k and p are generic words in Σ^*. $\square$

4. Proof of the conjecture

We will prove Conjecture 35 relying on the results of the previous section, in particular the final forms **a)**...**g)** of the boundary word of a prime double square P (Theorem 3). We start analyzing case **a)**, where $n_i = 0$ for all i, and then we show how to generalize the results to the remaining cases. From now on, P is assumed to be prime.

Lemma 7. *Let P be as in case **a)**, and $\alpha \in \Sigma$. If $\alpha\alpha \subseteq u_1$, and u_3, k and p are couple free, then P is not a prime double square.*

Proof. Let us suppose there is only one occurrence $\alpha\alpha \subseteq u_1$, i.e., $u_1 = a\alpha\alpha b$ for some $a, b \in \Sigma^*$ and a and b couple free. Then

$$P = a\alpha\alpha b \vdots k\tilde{b}\alpha\alpha\tilde{a} \vdots u_3 \vdots \widehat{b}\overline{\alpha\alpha}\widehat{a}\overline{p} | \overline{a\alpha\alpha}\overline{b} \vdots \overline{k}\widehat{b}\overline{\alpha\alpha}\widehat{a} \vdots \overline{u}_3 \vdots \tilde{b}\alpha\alpha\tilde{a}p.$$

We are assuming that P is a prime double square, so w_2w_3 and w_3w_4 are palindrome being BN-factors (see Lemma 1), and since $\alpha\alpha \not\subseteq a$, b, k and u_3, we get

$$\begin{cases} k\tilde{b} = \tilde{u}_3 a, \\ u_3\widehat{b} = \overline{pa}, \\ k, p \text{ palindrome, since } P \text{ is in form } \mathbf{a)}. \end{cases}$$

It holds that $|k| = |p| = |u_3|$ and $\tilde{b}a$ and u_3 are both palindrome. We can prove these properties by considering the factors' lengths and proceeding by contradiction: let us suppose first $|k| < |u_3|$, so that $u_3 = yk$ for some $y \in \Sigma^+$. Replacing above, we obtain $b = \tilde{a}y$ and $yk\widehat{y}\overline{a} = \overline{pa}$. From this second condition and the palindromicity of $\overline{p}$, we get $y = \varepsilon$ and so $u_3 = k$, contradiction. Then $|u_3| < |k|$ holds, and there exists $k' \in \Sigma^+$ such that $k = \tilde{u}_3 k'$. Then, we obtain $a = k'\tilde{b}$ and $u_3\widehat{b} = \overline{p}\overline{k}'\widehat{b}$, that leads to $u_3 = \overline{p}\overline{k}'$. Finally, the factor $k = \tilde{u}_3 k' =$

$\widehat{k}'\overline{p}k'$ is palindrome if and only if $k' = \varepsilon$, reaching again a contradiction. So, $|k| = |u_3|$.
Moving on, it follows $|a| = |b|$, with $b = \tilde{a}$, and $k = \overline{p} = u_3$. Replacing in P, we get

$$P = a\alpha\alpha\tilde{a} \vdots u_3 a\alpha\alpha\tilde{a} \vdots u_3 \vdots \overline{a\alpha\alpha\widehat{a}}u_3|\overline{a\alpha\alpha\widehat{a}} \vdots \overline{u}_3\overline{a\alpha\alpha\widehat{a}} \vdots \overline{u}_3 \vdots a\alpha\alpha\tilde{a}\overline{u}_3.$$

Finally, P can be obtained from the boundary word of the cross, $Q = 1010\overline{1}0\overline{101}01\overline{0}$, through the morphism $\varphi(0) = u_3$, $\varphi(1) = a\alpha\alpha\tilde{a} = u_1$. It follows that P is not prime.
Let us now conclude the proof by considering the case where u_1 contains more occurrences of two consecutive equal letters, i.e., $u_1 = a\alpha\alpha c\beta\beta b$ with a and b couple free and $c \in \Sigma^*$. We underline that c is not couple free in general, and can contain an arbitrary number of consecutive equal letters as its factors. Replacing u_1 in P we get

$$P = a\alpha\alpha c\beta\beta b \vdots k\tilde{b}\beta\beta\tilde{c}\alpha\alpha\tilde{a} \vdots u_3 \vdots \widehat{\overline{b\beta\beta}}\widehat{\overline{c\alpha\alpha}}\widehat{\overline{a}}\overline{p}| \ldots \tilde{b}\beta\beta\tilde{c}\alpha\alpha\tilde{a}p.$$

By the palindromicity of w_2w_3 and the assumption that a, b, u_3 and k are couple free, we immediately argue $\alpha = \beta$, and then, following the same argument used in case of only one occurrence of $\alpha\alpha \subseteq u_1$, we get $u_3 = \overline{p} = k$ palindrome and $b = \tilde{a}$; moreover, c is palindrome too. Replacing in P, we get the final form

$$P = a\alpha\alpha c\alpha\alpha\tilde{a} \vdots u_3 a\alpha\alpha c\alpha\alpha\tilde{a} \vdots u_3 \vdots \overline{a\alpha\alpha c\alpha\alpha\widehat{a}}u_3| \ldots a\alpha\alpha c\alpha\alpha\tilde{a}\overline{u}_3,$$

that can be obtained from the cross through the morphism $\varphi'(0) = u_3$, $\varphi'(1) = a\alpha\alpha c\alpha\alpha\tilde{a} = u_1$, that easily generalizes φ. Even in this case, P is not prime. □

Remark 1. *The morphisms φ and φ' obtained in the proof of Lemma 7 are well defined and preserve the hat operation, since u_1 and u_3 are both palindrome.*

It is easy to verify that, as shown in the proof of Lemma 7, when considering more occurrences of couples of consecutive equal letters in a factor of P the morphisms, if any, that map a prime double square into P are a simple generalization of the morphism obtained by considering one occurrence of two equal letters only. It is sufficient to explicitly write the first and last occurrence of consecutive equal letters in the considered factor and then carry on a similar proof. For this reason, the next results will be provided adding this further hypothesis.

Lemma 8. *Let P be a double square as in case **a)**, and $\alpha \in \Sigma$. P is not prime when any of the following conditions verifies:*

i) *u_1 is couple free and $\alpha\alpha \subseteq p$, or*

ii) *u_1 is couple free and $\alpha\alpha \subseteq k$, or*

iii) *u_1 is couple free and $\alpha\alpha \subseteq u_3$.*

Proof. Since P is in form **a)**, then each exponent n_i in (4) equals 0, and w.l.g. we can assume that only one occurrence of $\alpha\alpha$ appears in p, respectively k and u_3, as shown in the proof of Lemma 7. We proceed with a full case analysis:

i) The factor p can be written as the palindrome $a\alpha\alpha\tilde{a}$. Since u_1 and a are couple free, and w_3w_4 is palindrome, it follows $u_3 = \overline{a\alpha\alpha}\tilde{b}$ for suitable $b, c \in \Sigma^*$ s.t. $\overline{a} = cb$. Then

$$P = u_1 \vdots k\tilde{u}_1 \vdots cb\overline{\alpha\alpha}\tilde{b} \vdots \widehat{u}_1 cb\overline{\alpha\alpha}\tilde{b}\tilde{c} | \overline{u}_1 \vdots \overline{k}\widehat{u}_1 \vdots \overline{c}\overline{b}\alpha\alpha\widehat{b} \vdots \tilde{u}_1\overline{c}\overline{b}\alpha\alpha\widehat{b}\widehat{c}.$$

By the palindromicity of w_2w_3, we argue that $k = b\overline{\alpha\alpha}d$ for some $d \in \Sigma^*$. We conclude that $d = \tilde{b}$, since k is palindrome by hypothesis. Replacing again in P, we get

$$P = u_1 \vdots b\overline{\alpha\alpha}\tilde{b}\tilde{u}_1 \vdots cb\overline{\alpha\alpha}\tilde{b} \vdots \widehat{u}_1 cb\overline{\alpha\alpha}\tilde{b}\tilde{c} | \overline{u}_1 \vdots \overline{b}\alpha\alpha\widehat{b}\widehat{u}_1 \vdots \overline{c}\overline{b}\alpha\alpha\widehat{b} \vdots \tilde{u}_1\overline{c}\overline{b}\alpha\alpha\widehat{b}\widehat{c}$$

and, by the palindromicity of w_2w_3 and w_3w_4, we observe that $\tilde{u}_1c$ and $\widehat{u}_1c$ are both palindrome, and then $c = \varepsilon$ and $u_1 = \tilde{u}_1$. We get the final form

$$P = u_1 \vdots b\overline{\alpha\alpha}\tilde{b}u_1 \vdots b\overline{\alpha\alpha}\tilde{b} \vdots \overline{u}_1 b\overline{\alpha\alpha}\tilde{b} | \overline{u}_1 \vdots \overline{b}\alpha\alpha\widehat{b}\overline{u}_1 \vdots \overline{b}\alpha\alpha\widehat{b} \vdots u_1\overline{b}\alpha\alpha\widehat{b}.$$

A morphism can now be defined mapping the cross in P, i.e., $\varphi(0) = b\overline{\alpha\alpha}\tilde{b} = \overline{a\alpha\alpha\widehat{a}} = u_3$, $\varphi(1) = u_1$. Then P is not prime.

ii) Assuming $\alpha\alpha \subseteq k$, the palindrome factor can be written as $k = a\alpha\alpha\tilde{a}$, with a couple free. The boundary word is

$$P = u_1 \vdots a\alpha\alpha\tilde{a}\tilde{u}_1 \vdots u_3 \vdots \widehat{u}_1\overline{p} | \overline{u}_1 \vdots \overline{a\alpha\alpha\widehat{a}}\widehat{u}_1 \vdots \overline{u}_3 \vdots \tilde{u}_1 p.$$

Using the same argument as in case i), and the palindromicity of w_2w_3, we get $u_3 = b\alpha\alpha\tilde{a}$ and $a = cb$, then

$$P = u_1 \vdots cb\alpha\alpha\tilde{b}\tilde{c}\tilde{u}_1 \vdots b\alpha\alpha\tilde{b}\tilde{c} \vdots \widehat{u}_1 b\alpha\alpha\tilde{b} | \overline{u}_1 \vdots \overline{c}\overline{b}\overline{\alpha\alpha}\widehat{b}\widehat{c}\widehat{u}_1 \vdots \overline{b}\overline{\alpha\alpha}\widehat{b}\widehat{c} \vdots \tilde{u}_1\overline{b}\overline{\alpha\alpha}\widehat{b}.$$

From the palindromicity of the BN-factor w_2w_3, it follows that $\tilde{c}\tilde{u}_1$ is palindrome, and then from the palindrome w_3w_4 we argue $c = \varepsilon$ and $u_1 = \tilde{u}_1$. In the end, the boundary word is

$$P = u_1 \vdots b\alpha\alpha\tilde{b}u_1 \vdots b\alpha\alpha\tilde{b} \vdots \overline{u}_1 b\alpha\alpha\tilde{b} | \overline{u}_1 \vdots \overline{b}\overline{\alpha\alpha}\widehat{b}\overline{u}_1 \vdots \overline{b}\overline{\alpha\alpha}\widehat{b} \vdots u_1\overline{b}\overline{\alpha\alpha}\widehat{b}.$$

A morphism mapping the cross in P can now be defined, i.e., $\varphi(0) = b\alpha\alpha\tilde{b} = a\alpha\alpha\tilde{a} = u_3$, $\varphi(1) = u_1$. Then P is not prime.

iii) We replace $u_3 = a\alpha\alpha b$, with a, b couple free, in the boundary word,

$$P = u_1 \vdots k\tilde{u}_1 \vdots a\alpha\alpha b \vdots \widehat{u}_1\overline{p}|\overline{u}_1 \vdots \overline{k}\widehat{u}_1 \vdots \overline{a\alpha\alpha}\overline{b} \vdots \tilde{u}_1 p,$$

and then proceed as in previous cases. Since w_2w_3 and k are palindrome, $k = \tilde{b}\alpha\alpha b$, while we deduce $\overline{p} = a\alpha\alpha\tilde{a}$ from the palindromicity of the BN-factor w_3w_4. We get

$$P = u_1 \vdots \tilde{b}\alpha\alpha b\tilde{u}_1 \vdots a\alpha\alpha b \vdots \widehat{u}_1 a\alpha\alpha\tilde{a}|\overline{u}_1 \vdots \widehat{b}\overline{\alpha\alpha}\overline{b}\widehat{u}_1 \vdots \overline{a\alpha\alpha}\overline{b} \vdots \tilde{u}_1\overline{a\alpha\alpha}\widehat{a},$$

and since $b\tilde{u}_1 a$ and $b\widehat{u}_1 a$ are both palindrome (see w_2w_3 and w_3w_4), it holds $b = \tilde{a}$ and $u_1 = \tilde{u}_1$ (Lemma 2).

The final form of the boundary word is

$$P = u_1 \vdots a\alpha\alpha\tilde{a}u_1 \vdots a\alpha\alpha\tilde{a} \vdots \overline{u}_1 a\alpha\alpha\tilde{a}|\overline{u}_1 \vdots \overline{a\alpha\alpha}\widehat{a}\overline{u}_1 \vdots \overline{a\alpha\alpha}\widehat{a} \vdots u_1\overline{a\alpha\alpha}\widehat{a}.$$

A morphism mapping the cross in P can now be defined, i.e., $\varphi(0) = a\alpha\alpha\tilde{a} = u_3$, $\varphi(1) = u_1$. Then P is not prime. $\square$

Corollary 3. *Let P be a double square as in case **a)**, and $\alpha \in \Sigma$. If $\alpha\alpha \subseteq P$ and u_1 is couple free, then $\alpha\alpha \subseteq p, k, u_3$.*

Lemma 9. *Let P be a double square as in case **a)**. P is not prime when any of the following conditions verifies:*

i) u_1 and p are not couple free, while u_3 and k are, or

ii) u_1 and k are not couple free, while u_3 and p are, or

iii) u_1, p and k are not couple free, while u_3 is.

Proof. Again, the proof proceeds by contradiction, assuming that only one occurrence of two consecutive equal letters is in the analyzed factors.

i) Let us assume $u_1 = a\alpha\alpha b$ and $p = c\beta\beta\tilde{c}$ (recall that p is palindrome), with $\alpha, \beta \in \Sigma$, and a, b, c that are couple free.

Since the BN-factors w_2w_3 and w_3w_4 are palindrome and $\alpha\alpha, \beta\beta \nsubseteq a, b, c, k, u_3$, we deduce $\alpha = \beta$, $k\tilde{b} = \tilde{u}_3 a$ and $\widehat{c} = \overline{b}\tilde{u}_3$. Replacing in P we obtain

$$P = a\alpha\alpha b \vdots \tilde{u}_3 a\alpha\alpha\tilde{a} \vdots u_3 \vdots \widehat{b}\overline{\alpha\alpha}\widehat{a}u_3\widehat{b}\overline{\alpha\alpha}\overline{b}\tilde{u}_3|\overline{a\alpha\alpha}\overline{b} \vdots \widehat{u}_3\overline{a\alpha\alpha}\widehat{a} \vdots \overline{u}_3 \vdots \tilde{b}\alpha\alpha\tilde{a}\overline{u}_3\tilde{b}\alpha\alpha b\widehat{u}_3.$$

By the palindromicity of w_1w_2 and w_3w_4, it follows that $b\tilde{u}_3 a$ and $\widehat{a}u_3\widehat{b}$ are palindrome too. Since $b = \tilde{a}$ and $u_3 = \tilde{u}_3$ by Lemma 2, then P can be expressed as

$$P = a\alpha\alpha\tilde{a} \vdots u_3 a\alpha\alpha\tilde{a} \vdots u_3 \vdots \overline{a\alpha\alpha}\widehat{a}u_3\overline{a\alpha\alpha}\widehat{a}u_3|\overline{a\alpha\alpha}\widehat{a} \vdots \overline{u}_3\overline{a\alpha\alpha}\widehat{a} \vdots \overline{u}_3 \vdots a\alpha\alpha\tilde{a}\overline{u}_3 a\alpha\alpha\tilde{a}\overline{u}_3.$$

So, the morphism $\varphi(0) = u_3$, $\varphi(1) = a\alpha\alpha\tilde{a} = u_1$ maps the butterfly $Q = 1010\overline{1}0\overline{1}0\overline{1010}1\overline{0}1\overline{0}$ in P, preventing it from being prime.

As already observed, the case of more occurrences of couples of the same letter in u_1 and p can be treated similarly.

ii) Let $u_1 = a\alpha\alpha b$ and $k = c\beta\beta\tilde{c}$, with a, b, c couple free. Replacing in P, we obtain

$$P = a\alpha\alpha b \vdots c\beta\beta\tilde{c}\tilde{b}\alpha\alpha\tilde{a} \vdots u_3 \vdots \widehat{b}\overline{\alpha\alpha}\widehat{a\overline{p}} | \overline{a\alpha\alpha b} \vdots \overline{c\beta\beta}\widehat{c}\widehat{b}\overline{\alpha\alpha}\widehat{a} \vdots \overline{u}_3 \vdots \tilde{b}\alpha\alpha\tilde{a}p.$$

Since w_2w_3 and w_3w_4 are palindrome, and $\alpha\alpha, \beta\beta \not\subseteq a, b, c, p, u_3$, we deduce $\alpha = \beta$, $c = \tilde{u}_3 a$ and $\widehat{a\overline{p}} = \overline{b}\tilde{u}_3$, i.e.,

$$P = a\alpha\alpha b \vdots \tilde{u}_3 a\alpha\alpha\tilde{a}u_3\tilde{b}\alpha\alpha\tilde{a} \vdots u_3 \vdots \widehat{b}\overline{\alpha\alpha}\overline{b}\tilde{u}_3 | \overline{a\alpha\alpha b} \vdots \widehat{u}_3\overline{a\alpha\alpha}\widehat{a}\overline{u}_3\widehat{b}\overline{\alpha\alpha}\widehat{a} \vdots \overline{u}_3 \vdots \tilde{b}\alpha\alpha b\widehat{u}_3.$$

We further see that $\tilde{a}u_3\tilde{b}$ and $\overline{b}\tilde{u}_3\overline{a}$ are both palindrome (see w_2w_3 and $w_4\overline{w}_1$) then, by Lemma 2, $b = \tilde{a}$ and u_3 is palindrome. We finally get

$$P = a\alpha\alpha\tilde{a} \vdots u_3 a\alpha\alpha\tilde{a}u_3 a\alpha\alpha\tilde{a} \vdots u_3 \vdots \overline{a\alpha\alpha}\widehat{a}u_3 | \overline{a\alpha\alpha}\widehat{a} \vdots \overline{u}_3\overline{a\alpha\alpha}\widehat{a}\overline{u}_3\overline{a\alpha\alpha}\widehat{a} \vdots \overline{u}_3 \vdots a\alpha\alpha\tilde{a}\overline{u}_3.$$

Again, the morphism $\varphi(0) = u_3$, $\varphi(1) = a\alpha\alpha\tilde{a} = u_1$ maps the butterfly $Q = 1010\overline{1}0\overline{1}0\overline{1010}1\overline{0}1\overline{0}$ in P, preventing it from being prime.

iii) Let $u_1 = a\alpha\alpha b$, $k = c\beta\beta\tilde{c}$ and $p = d\gamma\gamma\tilde{d}$, with a, b, c, d couple free. Replacing in P, we obtain

$$P = a\alpha\alpha b \vdots c\beta\beta\tilde{c}\tilde{b}\alpha\alpha\tilde{a} \vdots u_3 \vdots \widehat{b}\overline{\alpha\alpha}\widehat{a}\overline{d}\overline{\gamma\gamma}\widehat{d} | \overline{a\alpha\alpha b} \vdots \overline{c\beta\beta}\widehat{c}\widehat{b}\overline{\alpha\alpha}\widehat{a} \vdots \overline{u}_3 \vdots \tilde{b}\alpha\alpha\tilde{a}d\gamma\gamma\tilde{d}.$$

As seen in the previous cases, from the palindromicity of the BN-factors w_2w_3 and w_3w_4 we deduce $\beta = \gamma = \alpha$, $c = \tilde{u}_3 a$ and $\widehat{d} = \overline{b}\tilde{u}_3$, and then $b = \tilde{a}$ and $u_3 = \tilde{u}_3$ (Lemma 2). The final form of the boundary word is

$$P = a\alpha\alpha\tilde{a} \vdots u_3 a\alpha\alpha\tilde{a}u_3 a\alpha\alpha\tilde{a} \vdots u_3 \vdots \overline{a\alpha\alpha}\widehat{a}u_3\overline{a\alpha\alpha}\widehat{a}u_3 |$$

$$\overline{a\alpha\alpha}\widehat{a} \vdots \overline{u}_3\overline{a\alpha\alpha}\widehat{a}\overline{u}_3\overline{a\alpha\alpha}\widehat{a} \vdots \overline{u}_3 \vdots a\alpha\alpha\tilde{a}\overline{u}_3 a\alpha\alpha\tilde{a}\overline{u}_3.$$

The morphism $\varphi(0) = u_3$, $\varphi(1) = a\alpha\alpha\tilde{a} = u_1$ maps the butterfly $Q = 101010\overline{1}0\overline{1}0\overline{101010}1\overline{0}1\overline{0}$ in P, preventing it from being prime. □

Remark 2. *Since the morphisms we defined in the proofs of Lemmas 8 and 9 map* $0, 1$ *in* u_1, u_3*, and* k, p *result to be equal, unless opposite, to* u_1 *or* u_3 *or their concatenation (as in Lemma 9), we can generalize these results when more couples of consecutive equal letters occur in one or more factors of* P*, using the same reasoning employed in the proof of Lemma 7.*

The analysis of the possible positions in P of the factor $\alpha\alpha$ continues, and three more lemmas are provided. We choose to set them separately since the proofs, even though all proceeding by contradiction, are different.

Lemma 10. *Let P be a double square as in case **a)**. If u_1 and u_3 are not couple free, while k and p are, then P is not prime.*

Proof. Let us assume that only one occurrence of a couple of consecutive equal letters is both in u_1 and u_3, say $u_1 = a\alpha\alpha b$ and $u_3 = c\beta\beta d$, with $\alpha, \beta \in \Sigma$, and a, b, c, d couple free. Replacing in P, we get

$$P = a\alpha\alpha b \vdots k\tilde{b}\alpha\alpha\tilde{a} \vdots c\beta\beta d \vdots \widehat{b}\overline{\alpha\alpha\widehat{a}p}|\overline{a\alpha\alpha}\overline{b} \vdots \overline{k}\widehat{b}\overline{\alpha\alpha\widehat{a}} \vdots \overline{c}\overline{\beta\beta d} \vdots \tilde{b}\alpha\alpha\tilde{a}p.$$

Looking at the palindrome w_2w_3 and w_3w_4, we deduce (respectively) $\beta = \alpha$ and $\beta = \overline{\alpha}$, that give a contradiction. Then, a double square P containing such configurations of couples $\alpha\alpha$ is not prime. If more than one occurrence of couples of equal letters are in u_1 or u_3 (or both), a similar proof holds. $\square$

Lemma 11. *The following statements hold:*

i) if u_1, u_3 and k are not couple free, while p is, then P is not a prime double square.

ii) If u_1, u_3 and p are not couple free, while k is, then P is not a prime double square.

Proof. Let us again assume that only one occurrence of a couple of consecutive equal letters is in the analyzed factors.

i) Let $u_1 = a\alpha\alpha b$, $u_3 = c\beta\beta d$ and $k = e\lambda\lambda\tilde{e}$, with $\alpha, \beta, \lambda \in \Sigma$ and a, b, c, d, e couple free. From the palindromicity of the BN-factors w_2w_3 and w_3w_4, we have $\lambda = \beta = \overline{\alpha}$, $e = \tilde{d}$, $c = \overline{pa}$, $\tilde{e}\tilde{b} = \tilde{c}a$. Moreover, we highlight that $\widehat{db}$ is palindrome. So, P can be written as

$$P = a\alpha\alpha b \vdots \tilde{d}\overline{\alpha\alpha}d\tilde{b}\alpha\alpha\tilde{a} \vdots \overline{pa\alpha\alpha}d \vdots \widehat{b}\overline{\alpha\alpha\widehat{a}p}|\overline{a\alpha\alpha}\overline{b} \vdots \widehat{d}\alpha\alpha\overline{d}\widehat{b}\overline{\alpha\alpha\widehat{a}} \vdots pa\alpha\alpha\overline{d} \vdots \tilde{b}\alpha\alpha\tilde{a}p.$$

Similarly, $d\tilde{b} = \widehat{a\overline{p}}a$ from w_2w_3 palindrome, and

$$P = a\alpha\alpha\tilde{a}\overline{pa\alpha\alpha}\widehat{a\overline{p}}a\alpha\alpha\tilde{a}\overline{pa\alpha\alpha}\widehat{db}\overline{\alpha\alpha\widehat{a}p}|\overline{a\alpha\alpha\widehat{a}}pa\alpha\alpha\tilde{a}p\overline{a\alpha\alpha\widehat{a}}pa\alpha\alpha\overline{d}\tilde{b}\alpha\alpha\tilde{a}p,$$

with $\widehat{db}$ palindrome and $d\tilde{b} = \widehat{a\overline{p}}a$.

We underline that the factor $w = \overline{pa\alpha\alpha\widehat{a}}pa\alpha\alpha\tilde{a}$ is such that $|w|_1 = |w|_{\overline{1}}$ and $|w|_0 = |w|_{\overline{0}}$; we then conclude that the boundary of the polyomino intersects itself, reaching a contradiction.

ii) Let $u_1 = a\alpha\alpha b$, $u_3 = c\beta\beta d$ and $p = e\gamma\gamma\tilde{e}$, with a, b, c, d, e couple free. From the palindromicity of w_2w_3 and w_3w_4, we have $\beta = \overline{\gamma} = \alpha$, $d = bk$, $\widehat{db} = \widehat{e\overline{a}}$ and $\overline{e} = c$. Moreover, we highlight that $\tilde{a}c = \tilde{c}a$ is palindrome. So, P can be written as

$$P = a\alpha\alpha b \vdots k\tilde{b}\alpha\alpha\tilde{a} \vdots c\alpha\alpha bk \vdots \widehat{b\overline{\alpha\alpha}a}c\alpha\alpha\tilde{c}|\overline{a\alpha\alpha\overline{b}} \vdots \overline{k\widehat{b}\alpha\alpha a}\vdots \overline{c\alpha\alpha\overline{bk}} \vdots \tilde{b}\alpha\alpha\tilde{a}\overline{c\alpha\alpha\widehat{c}},$$

with $\tilde{c}a$ palindrome and $\tilde{c}\overline{a} = bk\widehat{b}$ (see w_3w_4).

We underline that the factor $w = \widehat{a}c\alpha\alpha\tilde{c}\overline{a\alpha\alpha\overline{bk}} = \overline{b}k\tilde{b}\alpha\alpha bk\widehat{b}\overline{\alpha\alpha\overline{bk}}$ is such that $|w'|_1 = |w'|_{\overline{1}}$ and $|w'|_0 = |w'|_{\overline{0}}$, with $w' = \tilde{b}\alpha\alpha bk\widehat{b}\overline{\alpha\alpha\overline{bk}}$; we then conclude that the boundary of the polyomino intersects itself, reaching a contradiction. □

Lemma 12. *If u_1, u_3, p and k are not couple free, then P is not a prime double square.*

Proof. The result can be obtained providing a morphism φ that maps the cross in P. By contradiction, let $u_1 = a\alpha\alpha b$, $u_3 = c\beta\beta d$, $p = e\gamma\gamma\tilde{e}$ and $k = f\lambda\lambda\tilde{f}$, with a, b, c, d, e, f couple free. Replacing in P, we get

$$P = a\alpha\alpha b \vdots f\lambda\lambda\tilde{f}\tilde{b}\alpha\alpha\tilde{a} \vdots c\beta\beta d \vdots \widehat{b\overline{\alpha\alpha}a}\widehat{e\gamma\gamma e}|\overline{a\alpha\alpha\overline{b}} \vdots \overline{f\lambda\lambda}\widehat{\widehat{f}b}\overline{\alpha\alpha}\widehat{a} \vdots \overline{c\beta\beta d} \vdots \tilde{b}\alpha\alpha\tilde{a}e\gamma\gamma\tilde{e}.$$

From w_2w_3 and w_3w_4 palindrome, we get $\beta = \lambda = \overline{\gamma}$, $f = \tilde{d}$, $\overline{e} = c$, $\tilde{f}\tilde{b} = \tilde{c}a$ and $\widehat{a\overline{e}} = \overline{b}\tilde{d}$. The boundary word is now

$$P = a\alpha\alpha\tilde{a}c\beta\beta\tilde{c}a\alpha\alpha\tilde{a}c\beta\beta\tilde{c}\overline{a\alpha\alpha}\widehat{a}c\beta\beta\tilde{c}|\overline{a\alpha\alpha}\widehat{ac}\overline{\beta\beta}\widehat{c}\overline{a\alpha\alpha}\widehat{ac}\overline{\beta\beta}\widehat{c}a\alpha\alpha\tilde{a}\overline{c\beta\beta}\widehat{c}.$$

A non-trivial morphism can be defined as $\varphi(0) = c\beta\beta\tilde{c} = u_3$, $\varphi(1) = a\alpha\alpha\tilde{a} = u_1$, thus showing that P is not prime. □

We underline again that similar proofs lead to the definition of analogous morphisms when considering more couples of consecutive equal letters in u_1, u_3, k or p. The following example shows one of the possible situations:

Example 4. *We consider a generalization of the proof of Lemma 9, case* i)*, assuming that the non-couple free factors of the boundary word P have an arbitrary number of occurrences of consecutive equal letters. We assume by hypothesis that P is the boundary word of a prime double square, and for each factor we explicitly write the first and last occurrence of consecutive equal letters.*
We consider the factors $u_1 = a\alpha\alpha b\beta\beta c$ and $p = d\gamma\gamma e\gamma\gamma\tilde{d}$, with $\alpha, \beta, \gamma \in \Sigma$, a, c, d, k and u_3 couple free, e palindrome. Replacing in P, we obtain

$$P = a\alpha\alpha b\beta\beta c \vdots k\tilde{c}\beta\beta\tilde{b}\alpha\alpha\tilde{a} \vdots u_3 \vdots \widehat{c\overline{\beta\beta}b}\overline{\alpha\alpha}\widehat{a}\overline{d}\overline{\gamma\gamma e\gamma\gamma}\widehat{d}| \ldots$$

The palindromicity of w_2w_3, together with the hypothesis on the couple free factors, immediately lead to $\alpha = \beta$, $k\tilde{c} = \tilde{u}_3a$ and b palindrome, while the palindromicity of w_3w_4 gives $\gamma = \beta$, $d = \overline{u}_3\tilde{c}$ and $\overline{b}\widehat{\alpha\alpha}\widehat{a}\overline{d}\overline{\gamma\gamma e}$ palindrome. Replacing *again in P and imposing the palindromicity of w_1w_2 and w_3w_4, we find that $c\tilde{u}_3a$ and $\overline{b}\overline{\alpha\alpha}\widehat{a}u_3\widehat{c}\overline{\alpha\alpha e}$ are both palindrome. Focusing on the latter, more cases occur:*

- *$|e| < |b|$, so that $\overline{b} = \overline{e\alpha\alpha x}$ for some $x \in \Sigma^+$, and then $\overline{x\alpha\alpha}\widehat{a}u_3\widehat{c}$ is palindrome. Since $\widehat{a}u_3\widehat{c}$ is couple free, it must hold $|au_3c| \leq |x| < |b|$ and so $|u_3| < |u_1|$, in contradiction with the hypothesis on the mutual lengths of the factors u_i.*

- *$|b| < |e|$, so that $\overline{e} = \overline{x\alpha\alpha}\overline{b}$ for some $x \in \Sigma^+$, and then $\widehat{a}u_3\widehat{c}\overline{\alpha\alpha x}$ is palindrome. We further remind that e is palindrome too. First, we notice that $x \neq b$, otherwise we reach the same contradiction of the previous case. So, by the palindromicity of e, there exists a palindrome $y \in \Sigma^+$ such that $x = b\alpha\alpha y$ and $\widehat{a}u_3\widehat{c}\overline{\alpha\alpha}\overline{b}\overline{\alpha\alpha y}$ is palindrome. Being a, c and u_3 all couple free, two cases occur: $\overline{y} = \overline{c}\tilde{u}_3\overline{a}$ or $\overline{y} = \overline{z\alpha\alpha c}\tilde{u}_3\overline{a}$ for some $z \in \Sigma^+$, with $\overline{b}\overline{\alpha\alpha z}$ palindrome. This second case in analogous to the previous one, replacing $y = x$ and $z = y$. Then, we can assume w.l.g. $\overline{y} = \overline{c}\tilde{u}_3\overline{a}$ (otherwise, by iteration, we finally get that y is concatenation of the palindromes b, $\alpha\alpha$ and $c\widehat{u}_3a$). Therefore, we have shown that $\overline{c}u_3\overline{a}$ and $c\tilde{u}_3a$ are palindrome at the same time, that is, $c = \tilde{a}$ and u_3 is palindrome (by Lemma 2). Furthermore, u_1 is palindrome too. Replacing in the other factors, we find out that $p = \overline{u}_3u_1\overline{u}_3u_1\overline{u}_3$, $k = u_3$ and u_1, u_3 are palindrome. So, it is possible to define the non-trivial morphism $\varphi(0) = u_3$, $\varphi(1) = u_1$ that maps the double square $Q = 1010\overline{1}0\overline{1}0\overline{1}0\overline{1010}1\overline{0}1\overline{0}1\overline{0}$ in P, thus showing that it is not prime. More in general, $p = (\overline{u}_3u_1)^t\overline{u}_3$ for some $t \geq 1$ when the factor y is concatenation of more palindromes, and then $Q = 1010\overline{1}(0\overline{1})^t0\overline{1010}1(\overline{0}1)^t\overline{0}$.*

- *We can conclude that $b = e$, and so $\overline{c}u_3\overline{a}$ and cu_3a are palindrome at the same time, that is, $c = \tilde{a}$ and u_3 palindrome (by Lemma 2). Furthermore, u_1 is palindrome too. As in the previous case, it is possible to define the non-trivial morphism $\varphi(0) = u_3$, $\varphi(1) = u_1$ that maps the butterfly in P, thus showing that the double square is not prime.*

Since we reached a contradiction for all possible cases, it follows that P is not prime.

Theorem 4. *If P is the boundary word of a prime double square s.t. $n_i = 0$ for all i, then P is couple free.*

Proof. The proof directly follows from the previous lemmas, where a complete analysis of the possible positions of consecutive equal letters inside the form **a)** of P is carried on. Assuming that the factor u_1 is couple free, Lemma 8 shows that if two consecutive equal letters occur in P, then the double square is not prime. So, we need to suppose that $\alpha\alpha \subseteq u_1$ for some $\alpha \in \Sigma$. Let us now consider the factor u_3: following the proofs of Lemmas 10 and 11, it holds that, in case u_3 is not couple free, an occurrence of two consecutive equal letters is present in both factors k and p or in none of them, otherwise P is not the boundary word of a polyomino. In the first case, Lemmas 9 and 12 show how to define a morphism so that P is a non-prime double square. In the second case, Lemma 7 proves that the existence of a prime double square such that $\alpha\alpha \subseteq u_1$ and u_3, k and p are all couple free is not possible. $\square$

Generalizing Theorem 4 to the forms ***b)*** *...* ***g)*** *of the word* P

We indicate here, for each form of the boundary word of a prime double square from **a)** to **g)**, the BN-factors involved in the lemmas leading to the proof of Theorem 4. As a matter of fact, it is possible to follow the lemmas' sequence according to the new indicated factors to obtain the related version of Theorem 4 for each remaining case from **b)** to **g)**. Hence, Conjecture 35 is proved in Theorem 5. The BN-factors are the following:

Case a) $A = u_1 k \tilde{u}_1$, $B = u_3 \widehat{u}_1 \overline{p}$, $X = k\tilde{u}_1 u_3$, $Y = \widehat{u}_1 \overline{pu}_1$.

Case b) $A = (u_1 k \tilde{u}_1 p)^{n_1} u_1 k \tilde{u}_1$, $B = u_3 \widehat{u}_1 \overline{p}$, $X = k\tilde{u}_1 u_3$, $Y = \widehat{u}_1 \overline{p}(\overline{u}_1 \overline{k} \widehat{u}_1 \overline{p})^{n_1} \overline{u}_1$.

Case c) $A = u_1 (\tilde{u}_3 u_1)^{n_2} k \tilde{u}_1$, $B = u_3 \widehat{u}_1 \overline{p}$, $X = (\tilde{u}_3 u_1)^{n_2} k \tilde{u}_1 u_3$, $Y = \widehat{u}_1 \overline{pu}_1$.

Case d) $A = u_1 k \tilde{u}_1$, $B = (\overline{pu}_1 k \tilde{u}_1)^{n_3} u_3 \widehat{u}_1 \overline{p}$, $X = k\tilde{u}_1 (\overline{pu}_1 k \tilde{u}_1)^{n_3} u_3$, $Y = \widehat{u}_1 \overline{pu}_1$.

Case e) $A = u_1 k \tilde{u}_1$, $B = u_3 (\widehat{u}_1 u_3)^{n_4} \widehat{u}_1 \overline{p}$, $X = k\tilde{u}_1 u_3$, $Y = (\widehat{u}_1 u_3)^{n_4} \widehat{u}_1 \overline{pu}_1$.

Case f) $A = (u_1 k \tilde{u}_1 p)^{n_1} u_1 k \tilde{u}_1$, $B = (\overline{pu}_1 k \tilde{u}_1)^{n_3} u_3 \widehat{u}_1 \overline{p}$, $X = k\tilde{u}_1 (\overline{pu}_1 k \tilde{u}_1)^{n_3} u_3$, $Y = \widehat{u}_1 \overline{p}(\overline{u}_1 \overline{k} \widehat{u}_1 \overline{p})^{n_1} \overline{u}_1$.

Case g) $A = u_1 (\tilde{u}_3 u_1)^{n_2} k \tilde{u}_1$, $B = u_3 (\widehat{u}_1 u_3)^{n_4} \widehat{u}_1 \overline{p}$, $X = (\tilde{u}_3 u_1)^{n_2} k \tilde{u}_1 u_3$, $Y = (\widehat{u}_1 u_3)^{n_4} \widehat{u}_1 \overline{pu}_1$.

Focusing on case **a)**, we notice that A and Y are palindrome by construction, while B and X begin or end with the words k, p or u_3. As seen in all proofs of the previous lemmas, when including a couple $\alpha\alpha$ in P we compare u_3, p and k by the palindromicity of B and X. Through this procedure, we can write k and p as a concatenation of u_3 and u_1 (as highlighted in Remark 2). Then, after replacing, through the palindromicity of the BN-factors A and Y we deduce that $\tilde{u}_1 = u_1$ and $\tilde{u}_3 = u_3$ are both palindrome, and finally provide a well-defined homologous morphism φ that allows us to reach a contradiction and conclude the proof.

We can generalize this procedure analyzing the BN-factors in the other cases:

b) B and X allow to compare k, p and u_3, while A and Y give the palindromicity of u_1, u_3.

c) A and B allow to compare k, p and u_3, while X and Y give the palindromicity of u_1, u_3.

d) B and X allow to compare k, p and u_3, while A and Y give the palindromicity of u_1, u_3.

e) B, X and Y allow to compare k, p and u_3, while A gives the palindromicity of u_1, u_3.

f) B and X allow to compare k, p and u_3, while A and Y give the palindromicity of u_1, u_3.

g) B and Y allow to compare k, p and u_3, while A and X give the palindromicity of u_1, u_3.

After these observations, we can conclude that the results obtained in case **a)** admit a generalization to all the remaining cases. So, we can state the final theorem and solve Conjecture 35:

Theorem 5. *If P is the boundary word of a prime double square and $\alpha \in \Sigma$, then $\alpha\alpha \nsubseteq P$.*

The following example is related to case **g)**, when u_1 is not couple free while u_3, p and k are. In this case we are able, similarly to case **a)**, to define a morphism that leads to the non primality of P.

Example 5. *Let us consider P having the form **g)**. We assume that u_1 is not couple free, while u_3, p and k are. We can map in P the double square $Q = 1(01)^{n_2}010(\overline{1}0)^{n_4}\overline{1}0\overline{1}(\overline{01})^{n_2}\overline{010}(1\overline{0})^{n_4}1\overline{0}$, depicted in Fig 4, through the morphism $\varphi(0) = u_3 = k = \overline{p}$, $\varphi_1 = u_1$. The proof is similar to Lemma 7.*

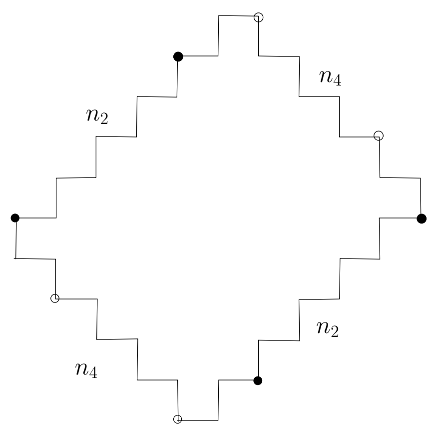

Figure 4: The polyomino Q can be used as an intermediate step to define the trivial morphism from the unit square to a double square P in form **g)**, with $n_2 = 4$, $n_4 = 3$ and s.t. $\alpha\alpha \subseteq u_1$ only. Notice that varying the values n_2 and n_4 is equivalent to extend or reduce the length of the sides of the polyomino (between dots), in this case starting from the butterfly.

5. Conclusion

In this paper, we consider Conjecture 35 in [6], and we prove it by showing through a full case analysis that there are no couples of consecutive equal letters in the boundary word of a prime double square polyomino. The study of

double square tiles would greatly benefit from the characterization of their prime elements: in particular, such a result would constitute a step toward the definition of an algorithm that directly generates all of them, without repetitions or pruning steps. The final aim is to enhance the performances of the algorithm defined in [6], by reducing the size of the explored space. As a matter of fact, it could also be interesting moving the spot to pseudo-hexagon polyominoes and performing a similar inspection.
From a combinatorial perspective, Theorem 5 may lead to the definition of a general form characterizing the boundary of a prime double square, coded as a word on a four letters alphabet. Usually, similar expressions lead to the definition of a growth law according to some parameters, like perimeter or area. Consequently, such a result could open a further research line, considering the possibility of the complete characterization and generation of all double squares, as well as their enumeration.

References

[1] Ascolese, M., Frosini, A., *On the Decomposability of Homogeneous Binary Planar Configurations with Respect to a Given Exact Polyomino*, Lect. Notes Comput. Sci., 13493: 139-152 (2022)

[2] Battaglino, D., Frosini, A., Rinaldi, S., *A decomposition theorem for homogeneous sets with respect to diamond probes*, Computer Vision and Image Understanding, 117(4): 319-325 (2013)

[3] Beauquier, D., Nivat, M., *On translating One Polyomino To Tile the Plane*, Discrete Comput. Geom. 6: 575-592 (1991)

[4] Blondin Massé, A., Brlek, S., Labbé, S., *A parallelogram tile fills the plane by translation in at most two distinct ways*, Discrete Appl. Math. 160 (7–8): 1011–1018 (2012)

[5] Blondin Massé, A., Frosini, A., Rinaldi, S., Vuillon, L., *On the shape of permutomino tiles*, Discrete Appl. Math., 161: 2316-2327 (2013)

[6] Blondin Massé, A., Garon, A., Labbé, S., *Combinatorial properties of double square tiles*, Theor. Comput. Sci. 502: 98-117 (2013)

[7] Brlek, S., Provençal, X., Fédou, J.M., *On the tiling by translation problem*, Discrete Appl. Math. 157: 464–475 (2009)

[8] Frosini, A., Nivat, M., *Binary Matrices under the Microscope: A Tomographical Problem*, Theoret. Comput. Sci. 370: 201-217 (2007)

[9] Gambini, I., Vuillon, L., *An algorithm for deciding if a polyomino tiles the plane by translations*, Theoret. Inform. Appl. 41: 147-155 (2007)

[10] Nivat, M.: Invited talk at the International Colloquium on Automata, Languages and Programming (ICALP), Bologna (1997)

[11] Nivat, M., *Sous-ensembles homogènes de $\mathbb{Z}^2$ et pavages du plan*, C.R. Acad. Sci. Paris. Ser. I, 335: 83-86 (2002)

[12] Schattschneider, D., *Will it tile? Try the Conway Criterion!*, Mathematics Magazine 53: 224–233 (1980)

[13] Shapiro, H.D., *Theoretical Limitations on the Efficient Use of Parallel Memories*, IEEE Trans. Comput. C-27: 421-428 (1978)

[14] Wang, H., *Games, Logic and Computers*, Scientific American 213: 98-107 (1965)

[15] Wijshoff, H.A.G., van Leeuwen, J., *Arbitrary versus periodic storage schemes and tesselations of the plane using one type of polyomino*, Inf. Control. 62: 1–25 (1984)