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Numerical procedure for detecting the optimal stress state within the profile of a cracked arch

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Abstract. The method for identifying the line of thrust closest to the geometrical axis of an undamaged arch, developed by the authors in a publication in 2019 for the first time, is extended in this paper to the case of cracked arches. Cracks subdivide the arch into rigid portions linked in-between in correspondence to points at the extrados or the intrados, which can be interpreted as internal hinges. Under this assumption, the procedure is reformulated and extended to the cases of the one and two-hinged arch. In the paper, the procedure is presented in detail for the case of the one-hinged arch.

Introduction

The static indeterminacy of the equilibrium problem of the masonry arch is a well-known concept. In the 19th century, a great debate arose on this topic and the eighteenth-century theories for which the safety of an arch could be evaluated based on the condition of the existence of any line of thrust within the profile of the arch was replaced, little by little, by the emerging topic of detecting the true line of thrust within the arch, i.e. the actual state of stress. By removing the rigidity assumption, the true line of thrust can be identified by considering the elastic and inertial properties of the arch to formulate the congruency conditions.

However, some authors have proposed methodologies to work around the indeterminacy of the equilibrium problem by avoiding considering the masonry properties and its deformability. In this context it is worth mentioning the graphic procedure proposed by E. Mèry [1], where the elasticity theory affects his methodology only in the choice of the centers of pressure.

To work around the indeterminacy, particularly interesting is the research carried out by Winkler in [2], who, referring to the "*principle of least pressure"* enunciated by Moseley [3], stated that *"of all the thrust lines that can be built for the external loads, the true thrust line is the one that deviates least, on average, from the geometrical axis of the arch".* Although it cannot be proved that this is the true one, what is certain is that the thrust line closest to the geometrical axis is the optimal one because it provides the lowest bending and shearing stresses and the arch is subjected to a fairly uniform compression stress. The topic of detecting the thrust line closest to the geometrical axis was addressed, firstly, by Heyman [4] who proposed an iterative procedure by trials and errors and, later on, by the authors of this contribution who provided the closed form solution to the problem stating it as a minimization problem of the quadratic deviation of the centroids of the discrete elements composing the arch from the nodes of the thrust line [5,6]. The minimum of this function is obtained expressing the zeroing of its partial derivatives with respect to three redundant unknowns, chosen as the horizontal thrust and the ordinate of the first and the last node of the thrust line. A system of three equations is obtained, which provides the value of the three redundant unknowns and, backwards, the ordinate of all the nodes of the thrust line. It is worth noting that the development of closed-form equations, capable of accurately estimating the load-bearing capacity of structures, is strongly recommended as it is a useful tool for practitioners

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to perform a preliminary assessment of structure capacity [7].

In all the procedures mentioned above, only undamaged arches are considered, i.e. structures with a degree of statically indeterminacy equal to 3. However, arches can also be cracked or constructed with geometric defects, and, at the cracked or imperfect joints, hinges can have occurred. Nevertheless, no one, until now, has addressed this topic considering cracked arches. Leaving aside the three-hinged arch, which is a statically determined structure for which only one line of thrust exists, in this paper the numerical procedure developed by the authors for identifying the thrust line closest to the geometrical axis of an undamaged arch is extended to the case of the one and two-hinged arch and is presented, in detail, for the former case.

Procedure extended to cracked arches

Let us consider a continuous arch, subjected to a set of *n* vertical loads $F_1, F_2, F_3, \ldots, F_n$ (Fig. 1). Let us subdivide the arch into *n* elements in such a way as to apply these loads at the element centroids G_i. Let us also consider the occurrence of one or two hinges, C_1 and C_2, at the intrados/extrados of some lines of separation between the elements. The line of thrust is a polyline passing through the points P_i, vertically aligned to the element centroids, and through the hinges C_1 and C_2. It is worth noting that the line of thrust forms a cusp in correspondence to the nodes P_i and it is straight in correspondence to the hinges, where no external load is applied. For this reason, hereafter, nodes P_i, C_1 and C_2 are referred to as “cusp” nodes and “hinge” nodes, respectively.

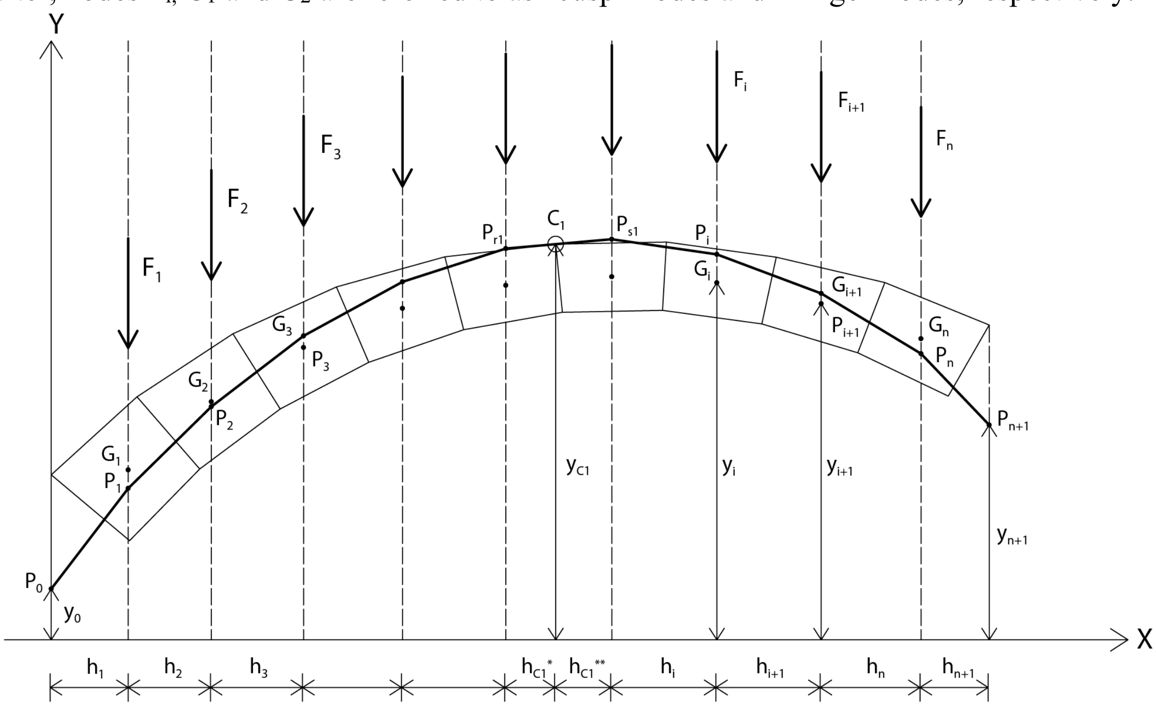

Fig. 1. Reference one-hinged arch.

To address the problem and identify the position of the nodes of the thrust line, the equilibrium conditions of both the “cusp” nodes and the “hinge” nodes are written. The equilibrium condition between the vertical components of the internal and external forces acting on the generic “cusp” node P_i of the thrust line is given by Eq. 1:

$$\left(\frac{-1}{h_i}\right) \cdot y_{i-1} + \left(\frac{h_i+h_{i+1}}{h_i \cdot h_{i+1}}\right) \cdot y_i + \left(\frac{-1}{h_{i+1}}\right) \cdot y_{i+1} = \frac{F_i}{H} \tag{1}$$

The equilibrium condition between the vertical components of the internal forces acting on the generic “hinge” node C_j ($1 \le j \le 2$) of the thrust line, located in-between the “cusp” nodes P_{r_j} and P_{s_j}, is given by Eq. 2:

$$\left(\frac{-1}{h_{c_j^*}}\right)\cdot y_{r_j}+\left(\frac{h_{c_j^*}+h_{c_j^{**}}}{h_{c_j^*}\cdot h_{c_j^{**}}}\right)\cdot y_{C_j}+\left(\frac{-1}{h_{c_j^{**}}}\right)\cdot y_{s_j}=0 \tag{2}$$

By collecting the equilibrium conditions of all nodes of the thrust line, a system of ($n+n_c$) linear equations in (n+3) unknowns is obtained (Eq. 3), where n_c is the number of internal hinges:

$$\begin{cases}\left(\frac{-1}{h_1}\right)\cdot y_0+\left(\frac{h_1+h_2}{h_1\cdot h_2}\right)\cdot y_1+\left(\frac{-1}{h_2}\right)\cdot y_2=\frac{F_1}{H}\\ \left(\frac{-1}{h_2}\right)\cdot y_1+\left(\frac{h_2+h_3}{h_2\cdot h_3}\right)\cdot y_2+\left(\frac{-1}{h_3}\right)\cdot y_3=\frac{F_2}{H}\\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\ \left(\frac{-1}{h_i}\right)\cdot y_{i-1}+\left(\frac{h_i+h_{i+1}}{h_i\cdot h_{i+1}}\right)\cdot y_i+\left(\frac{-1}{h_{i+1}}\right)\cdot y_{i+1}=\frac{F_i}{H}\\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\ \left(\frac{-1}{h_{r_j}}\right)\cdot y_{r_j-1}+\left(\frac{h_{r_j}+h_{c_j^*}}{h_{r_j}\cdot h_{c_j^*}}\right)\cdot y_{r_j}=\frac{F_{r_j}}{H}+\left(\frac{1}{h_{c_j^*}}\right)\cdot y_{C_j}\\ \left(\frac{-1}{h_{c_j^*}}\right)\cdot y_{r_j}+\left(\frac{h_{c_j^*}+h_{c_j^{**}}}{h_{c_j^*}\cdot h_{c_j^{**}}}\right)\cdot y_{C_j}+\left(\frac{-1}{h_{c_j^{**}}}\right)\cdot y_{s_j}=0\\ \left(\frac{h_{c_j^{**}}+h_{s_j+1}}{h_{c_j^{**}}\cdot h_{s_j+1}}\right)\cdot y_{s_j}+\left(\frac{-1}{h_{s_j+1}}\right)\cdot y_{s_j+1}=\frac{F_{s_j}}{H}+\left(\frac{1}{h_{c_j^{**}}}\right)\cdot y_{C_j}\\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\ \left(\frac{-1}{h_n}\right)\cdot y_{n-1}+\left(\frac{h_n+h_{n+1}}{h_n\cdot h_{n+1}}\right)\cdot y_n+\left(\frac{-1}{h_{n+1}}\right)\cdot y_{n+1}=\frac{F_n}{H}\end{cases} \tag{3}$$

In the system of Eq. 3 the unknowns are the ordinates y_i of the n+2 cusp nodes P_i of the thrust line, comprising the former one, P_0, and the latter one, P_{n+1}, and the constant thrust, H. Therefore, as expected, the problem is statically undetermined to ($3-n_c$) degree, because ∞^{3-n_c} lines of thrust in equilibrium with the external loads can be identified. In fact, the requirement that the line of thrust passes through one or two hinges reduces the indeterminacy of the problem by one ($n_c=1$) or two ($n_c=2$) degrees.

The numerical technique developed to solve the problem requires the choice of three parameters, as follows: primarily, the n_c ordinates y_{C_j} of the hinges (known terms) are selected; successively, the remaining ($3-n_c$) parameters (unknowns) that are chosen are the horizontal thrust H and the ordinate y_0 of the first node of the thrust line. These parameters are then isolated within the system of Eq. 3, which is rewritten, in matrix form, rearranging the system putting the equation(s) of the hinge node(s) in the last row(s).

Referring to the case of the one-hinged arch ($n_c=1$) and to the case of the two-hinged arch ($n_c=2$), the matrix system takes the form of Eq. 4 and Eq. 5, respectively:

$$[D]\{Y\}=\{T_1\}\cdot k+\{T_2\}\cdot y_0+\{T_3\}\cdot y_{C_1} \tag{4}$$

$$[D]\{Y\}=\{T_1\}\cdot k+\{T_2\}\cdot y_{C_1}+\{T_3\}\cdot y_{C_2} \tag{5}$$

where k=1/H.

For lack of space and for the sake of clarity, in the following lines the numerical procedure is presented and discussed only for the one-hinged arch. Hence, referring to Eq. 4, entries of vectors and matrices are shown below:

$$
D=\begin{bmatrix}
\left(\frac{h_1+h_2}{h_1\cdot h_2}\right) & \left(\frac{-1}{h_2}\right) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\left(\frac{-1}{h_2}\right) & \left(\frac{h_2+h_3}{h_2\cdot h_3}\right) & \left(\frac{-1}{h_3}\right) & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
0 & 0 & \left(\frac{-1}{h_i}\right) & \left(\frac{h_i+h_{i+1}}{h_i\cdot h_{i+1}}\right) & \left(\frac{-1}{h_{i+1}}\right) & 0 & 0 & 0 & 0 & 0\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
0 & 0 & 0 & 0 & \left(\frac{-1}{h_{r_j}}\right) & \left(\frac{h_{r_j}+h_{c_j^*}}{h_{r_j}\cdot h_{c_j^*}}\right) & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & \left(\frac{h_{c_j^{**}}+h_{s_j+1}}{h_{c_j^{**}}\cdot h_{s_j+1}}\right) & \left(\frac{-1}{h_{s_j+1}}\right) & 0 & 0\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \left(\frac{-1}{h_n}\right) & \left(\frac{h_n+h_{n+1}}{h_n\cdot h_{n+1}}\right) & \left(\frac{-1}{h_{n+1}}\right)\\
0 & 0 & 0 & 0 & 0 & \left(\frac{-1}{h_{c_j^*}}\right) & \left(\frac{-1}{h_{c_j^{**}}}\right) & 0 & 0 & 0
\end{bmatrix},
$$

$$
Y=\begin{Bmatrix} y_1\\ y_2\\ \cdots\\ y_i\\ \cdots\\ y_{r_j}\\ y_{s_j}\\ \cdots\\ y_n\\ y_{n+j}\end{Bmatrix},\qquad
T_1=\begin{Bmatrix} F_1\\ F_2\\ \cdots\\ F_i\\ \cdots\\ F_{r_j}\\ F_{s_j}\\ \cdots\\ F_n\\ 0\end{Bmatrix}.\quad
T_2=\begin{Bmatrix} 1/h_1\\ 0\\ \cdots\\ 0\\ \cdots\\ 0\\ 0\\ \cdots\\ 0\\ 0\end{Bmatrix},\quad
T_3=\begin{Bmatrix} 0\\ 0\\ \cdots\\ 0\\ \cdots\\ \frac{1}{h_{c_j^*}}\\ \frac{1}{h_{c_j^{**}}}\\ \cdots\\ 0\\ -\left(\frac{h_{c_j^*}+h_{c_j^{**}}}{h_{c_j^*}\cdot h_{c_j^{**}}}\right)\end{Bmatrix}. \tag{6}
$$

The solution of Eq. 4 is provided by Eq. 7:

$$\{Y\}=\{R_1\}\cdot k+\{R_2\}\cdot y_0+\{R_3\}\cdot y_{C_1} \tag{7}$$

where:

$$\{R_1\}=[D]^{-1}\cdot\{T_1\},\qquad \{R_2\}=[D]^{-1}\cdot\{T_2\},\qquad \{R_3\}=[D]^{-1}\cdot\{T_3\}. \tag{8}$$

In order to identify, among the ∞^{3-n_c} lines of thrust (n_c=1 in the case at hand), the one closest to the geometrical axis and passing through the hinge node, the squared distance function between the ordinates of the nodes of the thrust line and the ordinates of the centroids of the elements is computed and minimized with respect to the (3-n_c) redundant unknowns, n_c parameters being known terms:

$$S=\sum_{i=1}^{n+n_c}(\Delta y_i)^2=\sum_{i=1}^{n+n_c}(y_i-y_{Gi})^2=\left(\{R_1\}\cdot k+\{R_2\}\cdot y_0+\{R_3\}\cdot y_{C_1}-\{Y_G\}\right)^2 \tag{9}$$

Function S is minimized by expressing the partial derivatives of it, with respect to the two unknowns, equal to zero:

$$
\begin{aligned}
&\frac{\partial S}{\partial k}(y_0,k)=0\\
&\frac{\partial S}{\partial y_0}(y_0,k)=0
\end{aligned} \tag{10}
$$

By doing so, the system of two linear equations that follows is obtained:

$$\begin{cases} \{R_1\}^2 \cdot k + \{R_1\}\{R_2\} \cdot y_0 = \{R_1\}\left(\{Y_G\} - \{R_3\} \cdot y_{C_1}\right) \\ \{R_1\}\{R_2\} \cdot k + \{R_2\}^2 \cdot y_0 = \{R_2\}\left(\{Y_G\} - \{R_3\} \cdot y_{C_1}\right) \end{cases} \tag{11}$$

and it is then put in matrix form:

$$\begin{bmatrix} \{R_1\}^2 & \{R_1\}\{R_2\} \\ \{R_1\}\{R_2\} & \{R_2\}^2 \end{bmatrix} \cdot \begin{Bmatrix} k \\ y_0 \end{Bmatrix} = \begin{Bmatrix} \{R_1\}\left(\{Y_G\} - \{R_3\} \cdot y_{C_1}\right) \\ \{R_2\}\left(\{Y_G\} - \{R_3\} \cdot y_{C_1}\right) \end{Bmatrix} \tag{12}$$

and, more compactly, re-written as:

$$[N]\{P\} = \{W\} \tag{13}$$

The solution of Eq. 13 provides the value of the two unknowns, k and y_0, stored in vector P. The ordinates of the nodes of the line of thrust are finally obtained by back-substituting these values into Eq. 7.

Nevertheless, it is worth noting that the solution of Eq. 13 is not unique because it depends on the squared distance function in Eq. 9, which is affected by the value of the ordinate $y_{G(n+1)}$ of the centroid of the element *n+1*, which does not exist within the arch. It is possible to explore different lines of thrust passing through the hinge, with different shape and distance from the geometrical axis of the arch, by setting different values of $y_{G(n+1)}$. To identify the line of thrust passing through the hinge and closest to the geometrical axis, a procedure for trials and errors is conceived which converges when the value of $y_{G(n+1)}$ corresponding to the minimum of the squared distance function, computed for the *n* elements composing the arch, is obtained. At convergence, the two unknowns in Eq. 13 and the ordinates of the nodes of the thrust line are obtained.

Numerical explanatory example

The simple case of a portion of a segmental barrel vault, 100 cm deep, with a span length of 600 cm, a rise of 210.1 cm, 45 cm thick and an angle of embrace of 140° (t/R=0.13) is analyzed in this section. The vault is subdivided into six elements of the same size (*n=6*), with a unit weight of 18kN/m^3. Except for the self-weight, no additional load is applied on the vault.

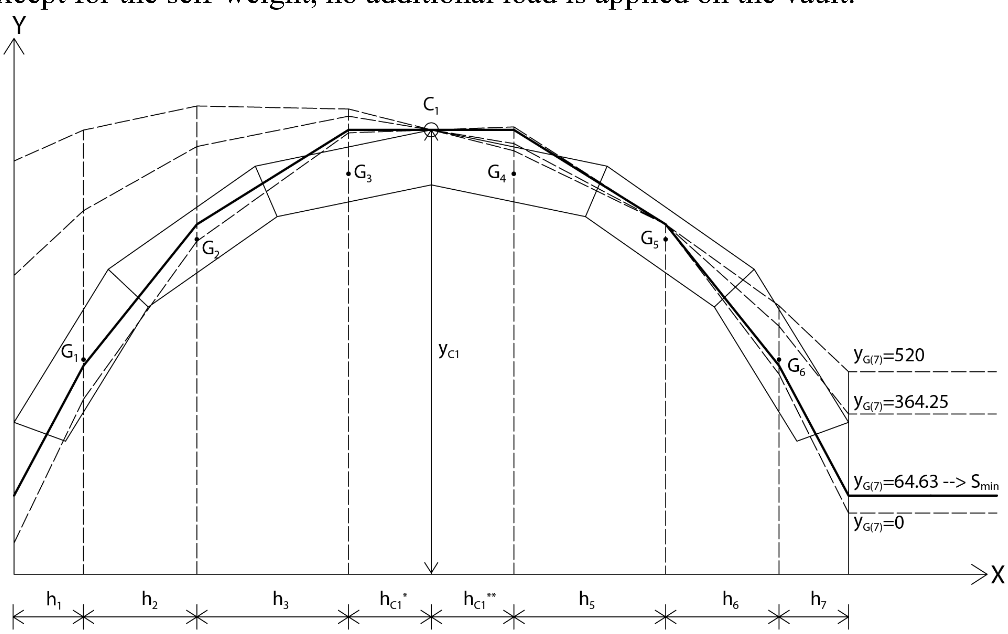

Fig. 2. Explanatory example; set of lines of thrust passing through the extrados hinge at the key section and identification of the one which is closest to the geometrical axis.

The numerical procedure proposed is used to identify the line of thrust closest to the geometrical axis of the structure passing through the extrados hinge at the key section (Fig. 2).

Fig. 2 allows for the exploring of the set of lines of thrust passing through the hinge at the key, corresponding to different values of the squared distance function, and the one which is also closest to the geometrical axis, corresponding to the minimum of the squared distance function which is found to be equal to *min(S)*=64.63 cm^2 by applying Eq. 9 iteratively. For that value, the ordinate $y_{G(7)}$ = 64.63 cm of the centroid of the fictitious 7^{th} element is obtained.

In detail, in order to allow the interested reader the possibility of reproducing the example numerically on his own, we report the main data and entries of the input vectors built by the procedure:

$$h_1 = h_7 = 57.01\,\text{cm}, h_2 = h_6 = 93.02\,\text{cm}, h_3 = h_5 = 124.47\,\text{cm}, h_{c_1^*} = h_{c_1^{**}} = 67.78\,\text{cm}, y_{C_1} = 364.25\,\text{cm}$$

$$\{T_1\}^t = \{16.96|16.96|16.96|16.96|16.96|16.96|0\}\ \text{kN}$$

$$\{T_2\}^t = \{0.0175|0|0|0|0|0|0\}\ \text{cm}^{-1}$$

$$\{T_3\}^t = \{0|0|0.0148|0.0148|0|0|-0.02951\}\ \text{cm}^{-1}$$

The ordinates of the nodes of the line of thrust closest to the geometrical axis and passing through the hinge at the key, computed for the centroid ordinate $y_{G(7)}$ = 64.63 cm of the fictitious 7^{th} element are finally shown in the vector that follows, obtained by applying Eq. 7:

$$\{Y\}^t = \{64.63|171.05|286.80|364.25|286.80|171.05|64.63\}\ \text{cm}$$

The corresponding thrust line is drawn in Fig. 2 with the thick continuous polyline.

Conclusions

In this paper the numerical procedure developed by the authors for identifying the line of thrust closest to the geometrical axis of an undamaged arch has been reformulated and extended to the case of cracked arches. The case of the one-hinged arch has been addressed and the related numerical procedure presented in detail. The case of the two-hinged arch will be addressed in an extended paper in the near future and, most certainly, presented, in Palermo, at the Aimeta 2022 conference in person by the corresponding author of this contribution.

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