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A certain counterpart in dissipative setting of the Noether theorem with no dissipation pseudo-potentials

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We look at a mechanical dissipation inequality differing from the standard one by what we call a relative power, a notion that is appropriate in the presence of material mutations. We prove that a requirement of structural invariance for such an inequality under the action of diffeomorphism-based changes of observers (covariance) implies (i) the representation of contact actions in terms of the first Piola–Kirchhoff stress, (ii) local balances of standard and configurational actions, (iii) *a priori* constitutive restrictions in terms of free energy, and (iv) a representation of viscous-type stress components.

This article is part of the theme issue 'Foundational issues, analysis and geometry in continuum mechanics'.

1. Introduction

There are two views on the origin of balance equations in continuum mechanics, when we look directly at macroscopic scale:

— Foundational analyses developed by Truesdell's school, in particular Noll's work, established that balance equations are independent of constitutive relations. Their integral form (balance of momentum and its moment) can be postulated [1–3], or deduced from a requirement of invariance under rigid-body changes of observers for the power performed by external agencies over every body part, along any motion [4,5]. Integral balances of forces and couples are associated with the linear structure of the classical physical space, identified with $\mathbb{R}^k$.

— On the other hand, Noether's theorem states that local balance laws are associated with the action of diffeomorphisms leaving invariant a Lagrangian density [6].

In the second case, we need to specify the state variables occurring in the Lagrangian density *before* deriving the Euler–Lagrange equations that are a consequence of imposing the minimal action principle. The invariance required by Noether's theorem to the Lagrangian density is not restricted to the Euclidean space rigid structure is essential when integral balance equations are either postulated or derived from invariance of external power with respect to rigid-body-type changes of observer. This last invariance states that in Euclidean space integral balances of standard forces and couples are associated with the Killing fields of the metric (those along which the Lie derivative of the metric vanishes); in Euclidean space such fields are, in fact, translations and rotations. By contrast, Noether's theorem allows a more general ambient space, for example, the finite-dimensional differentiable Riemannian manifold of general relativity. However, appeal to the Lagrangian implies restriction to conservative mechanical systems; also, constitutive relations and balance laws do not result mutually independent.

Changes of observers play a structural role in deriving balance equations: in Noether's theorem the diffeomorphism leaving invariant the Lagrangian are themselves, in fact, changes of observers. Attention to their role allows us to reduce axioms with a consequent better control on mechanical models when we aim at extending their boundaries, driven by the will of considering non-standard effects. The description of complex materials (from liquid crystals to active metamaterials) shows clearly, in fact, that a mechanical model does not reduce at all only to the prescription of constitutive structures in a standard setting [7].

In elementary courses on classical mechanics of point particles, the analysis is commonly referred to frames of reference determined by the so-called fixed stars; they define privileged observers considered as external entities once and for all. However, even in that setting we consider also moving frames and recognize the associated presence of apparent forces.

Here, we avoid preferred observers, rather we focus on invariance with respect to changes of observers; such a perspective constitutes a tool (i) to discriminate among possible classes of constitutive choices (e.g. [1]), (ii) to reduce the necessary axioms for deciding and justifying balance equations (as it happens in the case of mixtures [8], and general complex materials [9]), (iii) to prove the splitting of total energy into kinetic and internal components, and the existence of mass as derived entities, rather than being primitive axioms [10]. Also, we detach progressively from considering only rigid-body-type changes of observers, being aware that in appropriate cases, we have to couple mechanics with electromagnetic effects that are not strictly connected only with rigid-body-type changes of observers (see [11, Part B]).

An observer is a family of reference frames assigned over all spaces selected to describe the morphology of a body and its motion. These 'environments' are not limited necessarily to the physical three-dimensional ambient space and the interval of time. This is clear in the mechanics of complex materials in which appropriate descriptors (say phase fields) bring at macroscopic scale information on the microscopic texture. They appear to be necessary when the microstructure is active in the sense that actions hardly representable through standard stresses appear. In this case, the differentiable manifold where such descriptors take values enters the list of spaces needed to describe the body morphology [7,9].

Here, the specific analysis of these issues is avoided and we consider first those bodies, called Cauchy bodies, with morphology adequately described by regions in the Euclidean space. Then, we refer to an ambient space that is a more general finite-dimensional Riemannian manifold.

Also, we account for macroscopic material mutations such as growing bulk defects. We picture the circumstance by resorting to infinitely many such configurations differing from one another only by the defect pattern. However, rather than referring to the whole family of reference configurations, we consider their infinitesimal generator determined by a differentiable vector field on a prototype reference configuration $\mathcal{B}$. With respect to such a field we introduce the notion of a *relative power* and first require (§3) its invariance under rigid-body-type changes in observers in *both* physical and reference spaces, considered to be distinguished—a key aspect.

As a consequence, we derive balances of standard and configurational actions from a unique source. The approach refines the proposal in references [12,13] and does not make use of the inverse mapping, which would be questionable because the growth of a defect destroys at least locally the commonly assumed one-to-one instantaneous correspondence between reference and current configurations. Also, our approach avoids a procedure adopted in references [14,15]: the introduction of a configurational stress and other *a priori* unknown actions that require a subsequent identification in terms of standard actions. At variance, here only driving forces and configurational couples not related to standard actions need to be introduced. They are power-conjugated with rupture and reformation of atomic (or molecular) bonds, and with changes of material symmetry (the couples). In other words, along the path followed here there is economy in the representation of actions, along the guidelines suggested by Ockham's razor, if we accept and use it being careful to avoid epistemic problems (for this last philosophical aspect see [16] and also [17,18]).

Then, we write a non-standard version of the energy balance including the relative power (§4). With it, we propose a variant of Marsden–Hughes's theorem that is different from what is discussed in reference [19], although the result is analogous: the proof that not only the balance of standard actions but also the configurational ones can be derived from requirements of invariance with respect to changes of observers that are not necessarily isometric but involve general (smooth) deformations of both ambient and reference spaces. This theorem, however, implies essentially conservative stresses.

We overcome such a limitation and account for dissipative components of the stress by considering (§6) a mechanical dissipation inequality (an isothermal version of the second law of thermodynamics), written including the relative power. For it, we require covariance, that is invariance in structure under diffeomorphism-based changes of observers. Such a requirement and appropriate regularity assumptions imply *alone*

(i) the representation of contact actions in terms of the first Piola–Kirchhoff stress,
(ii) local balances of standard and configurational actions,
(iii) *a priori* constitutive restrictions in terms of free energy,
(iv) a thermodynamically admissible representation of viscous stress components.

This is the main result presented here (§6); it generalizes to the presence of viscous stresses a theorem proven in reference [20] in the case of large strain multiplicative plasticity. Also, such a result answers the question on whether there is a counterpart of the Noether theorem in fully dissipative setting when a dissipation pseudo-potential is not evidently available. It can be extended to a multi-field setting up to including structured interfaces and junctions; the extension, however, is left to another work.

This paper, in fact, deals only with basic concepts. The attention is purely focused on foundational aspects, on concepts rather than formal details and analytical generalizations. This nature is not in contrast with applied attitudes or needs. A constructive discussion on the foundations, in fact, has not only academic interest for scholars working on the basic structure of physical theories: a clear understanding of the foundation of a theory—whatever it be—addresses applications and, perhaps primarily, can suggest the way to evolve the theory itself, by modifying the basic axioms where possible, with the consequent emergence of new potentialities (even if, psychologically, it is often reassuring to adopt existing treatments).

2. Notations and preliminaries

(a) Notations on algebraic structures

With $\mathcal{X}$ a finite-dimensional linear space over the field of real numbers, we indicate by $\{\mathbf{e}_i\}$ a basis in it and by $\{\mathbf{e}^j\}$ the corresponding basis in the dual counterpart of $\mathcal{X}$, namely $\mathcal{X}^*$, the space of linear maps over $\mathcal{X}$. Every $\mathbf{e}^j$ is defined to be such that $\mathbf{e}^i \cdot \mathbf{e}_j = \delta^i_j$, where δ^i_j is the Kronecker delta and the interposed dot indicates from now on the duality pairing; it associates with $\mathbf{e}^i$ and $\mathbf{e}_j$ the scalar given by the value over $\mathbf{e}_j$ taken by the linear map $\mathbf{e}^i$. The duality pairing corresponds to the standard scalar product when the metric is flat and trivial, meaning associated with a frame of reference that is orthonormal with respect to a given scalar product.

For $a \in \mathcal{X}$ and $b \in \mathcal{X}^*$, a second-rank linear operator from $\mathcal{X}$ onto itself is given by the dyad $a \otimes b$, defined to be such that for every $h \in \mathcal{X}$, $(a \otimes b)h = (b \cdot h)a$.

Let A be a linear operator from $\mathcal{X}$ onto itself. With respect to a basis $\{\mathbf{e}_i\}$ in $\mathcal{X}$, we have $A = A^i_j \mathbf{e}_i \otimes \mathbf{e}^j$, where we adopt here and throughout this paper the usual summation convention concerning repeated indices.

For A and B two linear maps from $\mathcal{X}$ onto itself (two tensors indeed), we indicate by $A \cdot B$ the scalar given by $A^i_j B^j_i$ and by AB the second rank tensor given by $AB = A^i_k B^k_j \mathbf{e}_i \otimes \mathbf{e}^j$. Analogous notations hold for linear maps from $\mathcal{X}$ onto its dual $\mathcal{X}^*$ and vice versa, or from $\mathcal{X}^*$ onto itself.

Consider another finite-dimensional real linear space $\mathcal{Y}$, and a linear operator G from $\mathcal{X}$ to $\mathcal{Y}$; in short $G \in \mathrm{Hom}(\mathcal{X}, \mathcal{Y})$. Two other linear operators are associated with G. One is the transpose $G^\mathsf{T} \in \mathrm{Hom}(\mathcal{Y}, \mathcal{X})$; the other operator is the adjoint $G^* \in \mathrm{Hom}(\mathcal{Y}^*, \mathcal{X}^*)$. If $\dim\mathcal{X} = \dim\mathcal{Y} = \bar{\mathrm{m}}$ and G is non-singular, namely $\det G \neq 0$, its inverse G^{-1} belongs also to $\mathrm{Hom}(\mathcal{Y}, \mathcal{X})$ but is in general different from G^T; when $G^\mathsf{T} = G^{-1}$, itself G is said to be orthogonal and we write $G \in O(\bar{\mathrm{m}})$; in this case $\det G = \pm 1$; the subspace $SO(\bar{\mathrm{m}}) \subset O(\bar{\mathrm{m}})$ contains orthogonal tensors such that $G \in SO(\bar{\mathrm{m}})$ implies $\det G = 1$.

If g and $\tilde{g}$ are metrics in $\mathcal{X}$ and $\mathcal{Y}$ respectively, with g non-singular, for $G \in \mathrm{Hom}(\mathcal{X}, \mathcal{Y})$ we have $G^\mathsf{T} = g^{-1} G^* \tilde{g}$. If the two metrics are flat and trivial, G^* and G^T coincide.

$\mathrm{Sym}(\mathcal{X}, \mathcal{X})$ and $\mathrm{Skew}(\mathcal{X}, \mathcal{X})$ denote, respectively, the spaces of symmetric and skew-symmetric tensors that map $\mathcal{X}$ onto itself. We have $\mathrm{Hom}(\mathcal{X}, \mathcal{X}) = \mathrm{Sym}(\mathcal{X}, \mathcal{X}) \oplus \mathrm{Skew}(\mathcal{X}, \mathcal{X})$. Precisely, $A \in \mathrm{Hom}(\mathcal{X}, \mathcal{X})$ is such that $A = \mathrm{sym}A + \mathrm{skw}A$, with $\mathrm{sym}A = \frac{1}{2}(A + A^\mathsf{T})$ and $\mathrm{skew}A = \frac{1}{2}(A - A^\mathsf{T})$.

Consider a space–time field a with values $a(x, t)$ in a linear space. Assume that it is continuous everywhere but a fixed smooth surface Σ in space, oriented by the normal m. Define $a^\pm := \lim_{\epsilon \longrightarrow 0^+} a(x \pm \epsilon m)$. We commonly indicate by $[a]$ the jump across Σ, namely the difference $[a] := a^+ - a^-$. We say that a is PC^1 when it is C^1 outside Σ and suffers bounded jumps across Σ.

(b) Recalling Cauchy's theorem

Let Ω be any open set in $\mathbb{R}^k$ and $\mathfrak{b}$ any open bounded subset of Ω endowed with non-zero volume and Lipschitz boundary. We call $\mathfrak{b}$ a *proper domain* or *part* of Ω. A *balance law* over Ω states that the production $\mathcal{R}$ of a (scalar) extensive quantity in any proper domain $\mathfrak{b}$ of Ω is balanced by a flux $\mathcal{Q}$ of the same quantity through the boundary. Formally, we write

$$\mathcal{Q}_\mathfrak{b}(\partial\mathfrak{b}) = \mathcal{R}(\mathfrak{b}),$$

where $\mathcal{R}(\mathfrak{b})$ is a signed Radon measure over Ω and there is a density flux function $\hat{f}_\mathfrak{b} \in L^1(\partial\mathfrak{b})$ such that, for any open bounded subset $\mathfrak{C}$ of $\partial\mathfrak{b}$, the flux $\mathcal{Q}_\mathfrak{b}$ through $\mathfrak{C}$ is given by

$$\mathcal{Q}_\mathfrak{b}(\mathfrak{C}) = \int_\mathfrak{C} \hat{f}_\mathfrak{b}(x)\, \mathrm{d}\mathcal{H}^{k-1}(x),$$

where $\mathcal{H}^{k-1}$ is the $(k-1)$-dimensional Hausdorff measure.

Theorem 2.1. *[21, p. 3] With the above assumptions, let $|\hat{f}_\mathfrak{b}(x)| \le K$ for some constant K for any $x \in \partial\mathfrak{b}$ and all proper domains $\mathfrak{b}$.*

— *With each normal n in the $k-1$ sphere S^{k-1} is associated a bounded measurable function a_n on Ω with the following property: Given $\mathfrak{b}$, suppose that x is a point in $\partial\mathfrak{b}$ such that the normal to $\mathfrak{b}$ exists and is n. Assume also that x is a Lebesgue point for $\hat{f}_\mathfrak{b}$, with respect to $\mathcal{H}^{k-1}$, and that the* *upper derivative of $|\mathcal{R}|$ at x, with respect to Lebesgue measure, is finite. Then,*

$$\hat{f}_{\mathfrak{b}}(x) = a_n(x),$$

meaning that $\hat{f}_{\mathfrak{b}}$ depends on $\partial\mathfrak{b}$ only through the normal n.

— *There exists a vector field $\mathsf{a} \in L^\infty(\Omega; \mathbb{M}^{1\times k})$ such that, for any fixed $n \in S^{k-1}$,*

$$a_n(x) = \mathsf{a}(x)\cdot n$$

almost everywhere in Ω.

— *The vector function a satisfies the field equation*

$$div\,\mathsf{a} = \mathcal{R}$$

in the sense of distributions on Ω.

In short, and only for the sake of completeness, we recall that a measure is said to be a Radon one when it is defined on Borel sets of a Hausdorff topological space and is such that (1) the measure $\mathcal{R}(\mathcal{U})$ of any Borel set $\mathcal{U}$ is the supremum of values $\mathcal{R}(\mathcal{K})$ computed on all compact subsets $\mathcal{K}$ of $\mathcal{U}$, and (2) every point in the domain of $\mathcal{R}$ has a neighbourhood $\mathcal{U}$ such that $\mathcal{R}(\mathcal{U})$ is finite. The notion of upper derivative of a measure with respect to another refers to the Radon–Nikodyn theorem: it is the lim sup of a ratio between values of the two measures over non-degenerated closed balls.

Finally (and roughly speaking), Lebesgue points in the domain of a certain function f are those points around which f does not oscillate too much.

3. Mutant reference configurations and the relative power

(a) Configurations

Consider $\mathbb{R}^k$ and an isomorphic copy of it, namely $\tilde{\mathbb{R}}^k$, the isomorphism being simply given by the identification $i : \mathbb{R}^k \longrightarrow \tilde{\mathbb{R}}^k$. $\mathbb{R}^k$ is taken to be a reference space, while $\tilde{\mathbb{R}}^k$ is the physical space endowed with metric $\tilde{g}$. Typically, $k = 3$.

We indicate by $\mathcal{B}$ a fit region in $\mathbb{R}^k$, simply intended here as a connected proper domain, which we take as a reference configuration for a continuous body. Points $x \in \mathcal{B}$ are places occupied by material elements with internal structure not otherwise specified. A non-singular metric g is associated with $\mathcal{B}$.

Deformations are commonly taken as differentiable, orientation preserving, one-to-one maps $x \longmapsto y := \tilde{y}(x) \in \tilde{\mathcal{E}}^3$. We speak of motions when we consider time-parameterized families of deformations, namely

$$(x,t) \longmapsto y := \tilde{y}(x,t) \in \tilde{\mathbb{R}}^k,$$

with the time t ranging in an interval of the real line $\mathbb{R}$. The map $\tilde{y}$ is assumed to be twice differentiable with respect to time. We indicate by $\dot{y}$ the velocity field over $\mathcal{B}_c := \tilde{y}(\mathcal{B}, t)$, seen as a field defined over $\mathcal{B}$ by $\dot{y} := \partial\tilde{y}(x,t)/\partial t \in T_y\mathcal{B}_c$, where $T_y\mathcal{B}_c$ is the tangent space to $\mathcal{B}_c$ at y. The velocity field can be also considered as a time-dependent section of the tangent bundle $T\mathcal{B}_c = \sqcup_{y\in\mathcal{B}} T_y\mathcal{B}_c$, so that we have a field $(y,t) \longmapsto v := \tilde{v}(y,t)$ such that $\dot{y} = v$ at every y and t. As usual, we indicate by F the derivative $Dy = D\tilde{y}(x,t) = (\partial y^i/\partial x^A)\tilde{\mathbf{e}}_i \otimes \mathbf{e}^A$, where $\tilde{\mathbf{e}}_i$ is the i-th element of a basis in $\tilde{\mathbb{R}}^k$, $\mathbf{e}^A$ is the A-th element of a dual basis in $\mathbb{R}^k$. A standard nomenclature indicates F as a *deformation gradient*. In fact, we have $\nabla y = (Dy)g^{-1}$, that is, in components, $(\nabla y)^{iA} = (Dy)^i_C g^{CB}$.

From now on, we consider metrics to be time-independent.

(b) Observers

Observers are frames of reference assigned to all spaces selected to represent the body morphology and its motion. Here, in the definition of observer, we involve the assignment of reference frames on $\mathbb{R}^k$, $\tilde{\mathbb{R}}^k$, and the time scale.

By leaving invariant the time scale, we consider in *both* $\mathbb{R}^k$ and $\tilde{\mathbb{R}}^k$ *independent* rigid-body-type changes of observers, those governed by action of the semi-direct product $\mathbb{R}^k \ltimes SO(k)$. Let $\dot{y}$ be a velocity evaluated by an observer $\mathcal{O}$ in $\tilde{\mathbb{R}}^k$ and $\dot{y}'$ the corresponding value recorded by an observer $\mathcal{O}'$ that differs from $\mathcal{O}$ by a rigid-body motion. The pull-back of $\dot{y}'$ in the frame of reference defining $\mathcal{O}'$, namely $\dot{y}^\diamond := Q^\top \dot{y}'$, with $Q := Q(t) \in SO(k)$ is given by

$$\dot{y}^\diamond = c + q \times (y - y_0) + \dot{y},$$

where c and q are the instantaneous value of smooth maps $t \longmapsto c := \tilde{c}(t) \in \tilde{\mathbb{R}}^k$ and $t \longmapsto q := \tilde{q}(t) \in \tilde{\mathbb{R}}^k$, with $Q = \exp(q\times)$. The sum $c + q \times (y - y_0)$, where y_0 is an arbitrary fixed point selected in $\tilde{\mathbb{R}}^k$, is the infinitesimal generator of the $\mathbb{R}^k \ltimes SO(k)$ action over $\tilde{\mathbb{R}}^k$, with $q\times$ an element of the Lie algebra $\mathfrak{so}(k)$, a space of skew-symmetric tensor over $\tilde{\mathbb{R}}^k$.

We consider an analogous action on the reference space $\mathbb{R}^k$. For any vector field defined over $\mathcal{B}$, with values $w \in \mathbb{R}^k$, we write w and w' for the vectors recorded by $\mathcal{O}$ and $\mathcal{O}'$, respectively, and write $w^\diamond$ for the pull-back of w' into $\mathcal{O}$. Once again we have

$$w^\diamond = \hat{c} + \hat{q} \times (x - x_0) + w,$$

with $\hat{c}$ and $\hat{q}$ the values in $\mathbb{R}^k$ of time-dependent smooth maps, and x_0 is an arbitrary point in $\mathbb{R}^k$.

(c) Material mutations and relative power

Consider any arbitrary part $\mathfrak{b}$ of $\mathcal{B}$ and its current image $\mathfrak{b}_c := \tilde{y}(\mathfrak{b}, t)$ in $\mathcal{B}_c$. Actions on $\mathfrak{b}_c$, seen as values of fields defined over $\mathfrak{b}$, are commonly classified into bulk and contact families; they are also defined by the power that they perform on the rate at which $\mathfrak{b}_c$ changes in time. We look at the power performed by external agencies, the one that we can in principle measure directly. We indicate it by $\mathcal{P}^{\text{ext}}_{\mathfrak{b}}(\dot{y})$, a functional defined by

$$\mathcal{P}^{\text{ext}}_{\mathfrak{b}}(\dot{y}) := \int_{\mathfrak{b}} b^\ddagger \cdot \dot{y} \,\mathrm{d}x + \int_{\partial \mathfrak{b}} \mathrm{t}_\partial \cdot \dot{y} \,\mathrm{d}\mathcal{H}^{k-1}(x),$$

and called an *external power* in Lagrangian (referential) representation. The covectors $b^\ddagger$ and t_∂ indicate, respectively, densities of bulk and contact standard actions, the former assumed to admit an additive decomposition into inertial, b^{in}, and non-inertial, b, components, the latter depending on the boundary $\partial\mathfrak{b}$, as recalled by the subscript ∂, in addition to x and t. In 1963, W. Noll remarked that if we raise at first principle a requirement of invariance for $\mathcal{P}^{\text{ext}}_{\mathfrak{b}}$ under rigid-body-type changes of observer, namely we require $\mathcal{P}^{\text{ext}}_{\mathfrak{b}}(\dot{y}) = \mathcal{P}^{\text{ext}}_{\mathfrak{b}}(\dot{y}^\diamond)$ for any choice of c and q, we obtain the integral balance of forces and the one of couples as derived entities rather than assuming them as first principles [4].

In Noll's analysis, $\mathcal{B}$ is considered to be fixed once and for all. Also, $\tilde{y}$ is assumed to be one-to-one [4,5,22]. If a defect evolves in the current configuration $\mathcal{B}_c$ with respect to $\mathcal{B}_c$ itself, the map $\tilde{y}$ is in general no more one-to-one. Precisely, at every instant t the current shape $\mathcal{B}_c$ is in one-to-one correspondence with another reference shape that differs from the initially chosen $\mathcal{B}$ by a *non-material* picture of the defect pattern. So, a *family of reference configurations*, all occupying $\mathcal{B}$ but differing just by a pre-image (under $\tilde{y}$) of the defect pattern, is related in principle with a defect evolution: a structural *mutation* of a body at the spatial scale under analysis. An analogous approach can be used to describe even growing bodies—they are mutant bodies too in a broader sense. In this case, we have to call upon a notion of body *content* (a concept formalized by R. Segev in reference [23]).

Families of reference configurations connected by maps [24], parameterized by time [14], or by measures [25] have been considered in various settings that involve structural material changes. The idea that a defect evolution could be described by altering the reference configuration appears already in Eshelby's work [26] and has its roots in the common notion of horizontal variations for integral functionals [27].

To simulate a multiplicity of reference configurations, we refer here to an infinitesimal generator of it, a differentiable vector field $x \longmapsto w \in \mathbb{R}^k$, which describes virtually the incoming mutation in the body structure.

In this setting, we define a power functional with the following features:

— For any part $\mathfrak{b}$ of $\mathcal{B}$, it describes the power of bulk and contact actions with respect to the relative velocity $\dot{y} - Fw$. The difference makes sense because Fw is the push-forward in $\tilde{\mathbb{R}}^k$ of $w \in \mathbb{R}^k$ along $\tilde{y}$.

— It accounts for variations in the energetic landscape induced in $\mathcal{B}$ by the material mutation.

For the second aspect, we need to introduce (presume or derive from an abstract setting) the notion of free energy, with density ψ. However—and this is a key point—*we do not presume any explicit constitutive structure for the energy at this stage*. We only affirm that ψ is of the type $\psi(x, t, \varsigma)$, where explicit time dependence implies possible ageing and ς indicates the list of state variables to be decided later, when the second law of thermodynamics is used to determine restrictions *a priori* to state functions. The list ς depends also on space and time. As a function $\psi(\cdot, t, \varsigma)$ of x alone, the energy takes into account inhomogeneity due to the redistribution of defects, along which material bonds can be broken and/or others formed. Forces f and couples μ are conjugated with the power developed in the process of annihilation and/or reformation of material bonds, which characterizes the mutation. Couples μ can be generated even in a conservative setting by changes of the material symmetry, for example when isotropy is broken. They are *undetermined* in principle

With f we only indicate actions having dissipative nature, and we assume that f admits the additive decomposition

$$f = f^{ba} + f^{da},$$

with f^{ba} the part due to the annihilation of bonds and f^{da} the indirect effects of bulk processes evolving far from thermodynamic equilibrium, for example what is due to the presence of viscous stresses.

— For f^{ba} we postulate

$$f^{ba} \cdot w \geq 0,$$

the equality sign holding only when $w = 0$. The assumption is compatible with a structure

$$f^{ba} = \hat{a}(\cdot) w,$$

with $\hat{a}(\cdot)$ a positive definite scalar function of the state variables, their rates, and other possible indicators of the annihilation process, depending on circumstances.

— f^{da} can be identified in terms of stress dissipative component, entropy and (possibly) chemical affinities once the second law of thermodynamics is used *in the standard way* to restrict *a priori* constitutive structures. When the setting is conservative, that is, we are in elastic, isothermal case, we have $f^{da} = 0$.

Sections of the tangent bundle $T\mathcal{B} = \sqcup_{x \in \mathcal{B}} T_x\mathcal{B}$ are velocity fields over $\mathcal{B}$, as w is, without any special status. Linear functions over $T\mathcal{B}$, namely elements of $T^*\mathcal{B}$, take values that are powers. In this sense, f and μ are configurational actions (see [14,15] for discussions about the notion of configurational forces); at every $x \in \mathcal{B}$ and t, they belong to the cotangent space $T_x^*\mathcal{B}$. At variance, the standard actions, expressed here in Lagrangian representation, belong at every x and t to $T^*_{\tilde{y}(x,t)}\mathcal{B}_c$ and perform power over the velocity, which belongs to $T_{\tilde{y}(x,t)}\mathcal{B}_c$.

With all these elements we define what we call a *relative power*: a functional over any part $\mathfrak{b}$, depending on $\dot{y}$ and w. We indicate it by $\mathcal{P}^{\mathrm{rel}}_{\mathfrak{b}}(\dot{y}, w)$ and presume that its values are given by the sum

$$\mathcal{P}^{\mathrm{rel}}_{\mathfrak{b}}(\dot{y}, w) := \mathcal{P}^{\text{rel-a}}_{\mathfrak{b}}(\dot{y}, w) + \mathcal{P}^{\mathrm{dis}}_{\mathfrak{b}}(w),$$

where

$$\mathcal{P}^{\text{rel-a}}_{\mathfrak{b}}(\dot{y}, w) := \int_{\mathfrak{b}} b^{\ddagger} \cdot (\dot{y} - Fw)\, \mathrm{d}x + \int_{\partial\mathfrak{b}} \mathfrak{t}_{\partial} \cdot (\dot{y} - Fw)\, \mathrm{d}\mathcal{H}^{k-1}(x)$$

is the *relative power of bulk and contact actions*, namely $b^{\ddagger}$ and $\mathfrak{t}_{\partial}$, already mentioned and defined by the power that they perform, while

$$\mathcal{P}^{\mathrm{dis}}_{\mathfrak{b}}(w) := \int_{\partial\mathfrak{b}} (n \cdot w)\psi\, \mathrm{d}\mathcal{H}^2 - \int_{\mathfrak{b}} (\partial_x \psi + f) \cdot w\, \mathrm{d}x + \int_{\mathfrak{b}} \mu \cdot \mathrm{curl}\, w\, \mathrm{d}x,$$

is what we call a *power of disarrangements*, where $\partial_x\psi$ is the *explicit derivative* of $\psi(x, t, \varsigma)$ with respect to x, holding fixed all the other entries of the energy, whatever they are. Such a definition of $\mathcal{P}^{\mathrm{dis}}_{\mathfrak{b}}(w)$ differs from what I have proposed previously in reference [12]. This new version seems to me more satisfactory from the viewpoint of a physical interpretation of the entities defined.

Theorem 3.1. *Assume that $\mathcal{P}^{rel}_{\mathfrak{b}}$ is invariant under rigid-body-type changes of observers in both physical and reference space, namely $\mathcal{P}^{rel}_{\mathfrak{b}}(\dot{y}, w) = \mathcal{P}^{rel}_{\mathfrak{b}}(\dot{y}^{\diamond}, w^{\diamond})$ for any choice of c, q, $\hat{c}$, $\hat{q}$, and $\mathfrak{b}$.*

— *Invariance implies the validity of integral balance laws of forces and couples, namely*

$$\int_{\mathfrak{b}} b^{\ddagger}\, \mathrm{d}x + \int_{\partial\mathfrak{b}} \mathfrak{t}_{\partial}\, \mathrm{d}\mathcal{H}^{k-1}(x) = 0, \tag{3.1}$$

and

$$\int_{\mathfrak{b}} (y - y_0) \times b^{\ddagger}\, \mathrm{d}x + \int_{\partial\mathfrak{b}} (y - y_0) \times \mathfrak{t}_{\partial}\, \mathrm{d}\mathcal{H}^{k-1}(x) = 0, \tag{3.2}$$

in $\tilde{\mathbb{R}}^k$, and the integral balances of configurational actions

$$\int_{\partial\mathfrak{b}} (\psi n - F^*\mathfrak{t}_{\partial})\, \mathrm{d}\mathcal{H}^{k-1}(x) - \int_{\mathfrak{b}} F^* b^{\ddagger}\, \mathrm{d}x - \int_{\mathfrak{b}} (\partial_x \psi + f)\, \mathrm{d}x = 0, \tag{3.3}$$

and

$$\int_{\partial\mathfrak{b}} (x - x_0) \times (\psi n - F^*\mathfrak{t}_{\partial})\, \mathrm{d}\mathcal{H}^{k-1}(x) - \int_{\mathfrak{b}} (x - x_0) \times F^* b^{\ddagger}\, \mathrm{d}x$$
$$- \int_{\mathfrak{b}} (x - x_0) \times (\partial_x \psi + f)\, \mathrm{d}x + \int_{\mathfrak{b}} 2\mu\, \mathrm{d}x = 0 \tag{3.4}$$

in $\mathbb{R}^k$.

— *Under the assumptions of theorem 2.1, there is a covector-valued function $\tilde{\mathfrak{t}}$ such that*

$$\mathfrak{t}_{\partial} = \mathfrak{t} := \tilde{\mathfrak{t}}(x, t; n), \tag{3.5}$$

with n the normal to $\partial\mathfrak{b}$ at x, in all points where it is uniquely defined, and

$$\tilde{\mathfrak{t}}(x, t; n) = -\tilde{\mathfrak{t}}(x, t; -n). \tag{3.6}$$

Also, there exists a map $(x, t) \longmapsto P = \tilde{P}(x, t)$, with $P = P^A_i \tilde{\mathbf{e}}^i \otimes \mathbf{e}_A$, such that

$$\mathfrak{t} = \tilde{\mathfrak{t}}(x, t; n) = P(x, t)n, \tag{3.7}$$

where n is taken as a covector.

— *Assume also that* $b^{\ddagger} = b^{in} + b$, *with the inertial component* b^{in} *such that*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathfrak{b}} \frac{1}{2}\rho\dot{y}^{\flat}\cdot\dot{y}\,\mathrm{d}x + \int_{\mathfrak{b}} b^{in}\cdot\dot{y}\,\mathrm{d}x = 0 \tag{3.8}$$

for any choice of the velocity field, where $\dot{y}^{\flat}$ *is the covector associated by the metric to* $\dot{y}$, *namely* $\dot{y}^{\flat} = \dot{y}\tilde{g}$. *Define* $\mathbb{P} := \psi I - F^*P$ *(we call it Hamilton–Eshelby's stress), with* I *the second-rank unit tensor. If the fields* $x \longmapsto P$ *and* $x \longmapsto \mathbb{P}$ *are of class* $PC^1(\mathcal{B}) \cap C^0(\bar{\mathcal{B}})$ *with discontinuity set a (fixed) smooth surface* Σ *in* $\mathcal{B}$, *oriented by the normal* m, *and the fields* $x \longmapsto b$, $x \longmapsto F^*b$, *and* $x \longmapsto \partial_x e$ *are continuous over* $\mathcal{B}$, *in the absence of inertial powerless terms, we have*

$$\mathrm{Div}P + b = \rho\ddot{y}^{\flat}, \tag{3.9}$$

$$\mathrm{skw}PF^* = 0, \tag{3.10}$$

$$\mathrm{Div}\mathbb{P} - F^*(b - \rho\ddot{y}^{\flat}) - \partial_x\psi = f \tag{3.11}$$

and

$$skw\mathbb{P} = -2\mu\times \tag{3.12}$$

in the bulk, and

$$[P]m = 0 \quad \text{and} \quad [\mathbb{P}]m = 0, \tag{3.13}$$

along Σ.

— *An extended version of the virtual power principle holds:*

$$\mathcal{P}^{rel}_{\mathfrak{b}}(\dot{y}, w) = \mathcal{P}^{rel-inn}_{\mathfrak{b}}(v, w), \tag{3.14}$$

where

$$\mathcal{P}^{rel\text{-}inn}_{\mathfrak{b}}(\dot{y}, w) := \int_{\mathfrak{b}} (P\cdot D\dot{y} + \mathbb{P}\cdot Dw + \mu\cdot curl w)\mathrm{d}x. \tag{3.15}$$

— *If the material is homogeneous, no driving force is present, and* $\mu = 0$, *the tensor* $\mathbb{P}$ *is symmetric and, in the absence of body forces,*

$$\int_{\partial\mathfrak{b}} \mathbb{P}n\,\mathrm{d}\mathcal{H}^2 = 0. \tag{3.16}$$

The proof is rather straightforward. In fact, the requirement of invariance implies $\mathcal{P}^{\mathrm{rel}}_{\mathfrak{b}}(c + q\times(y - y_0), \hat{c} + \hat{q}\times(x - x_0)) = 0$. Since c, q, $\hat{c}$, and $\hat{q}$ do not depend on x and are arbitrary, the previous identity implies the integral balances in theorem 3.1, distinguished in the spaces $\mathbb{R}^k$ and $\tilde{\mathbb{R}}^k$. Cauchy's theorem 2.1 thus applies to the integral balance of forces and determines the representation of t_{∂} in terms of the first Piola–Kirchhoff stress P. The balance (3.8) implies $b^{\mathrm{in}} = -\rho\ddot{y}^{\flat} + b^{\mathrm{pl}}$, where $\ddot{y}^{\flat}_i = \ddot{y}^j\tilde{g}_{ji}$ and b^{pl} is a powerless term. (Such a result allows one to define inertial frames of reference as those for which $b^{\mathrm{pl}} = 0$.) Then, use of the Gauss theorem under the assumed regularity implies the pointwise balances in both $\mathbb{R}^k$ and $\tilde{\mathbb{R}}^k$. The extended virtual power principle and the last identity are a matter of direct calculation.

Here is a list of comments to theorem 3.1.

— In avoiding to indicate the list of state variables, we maintain unaltered the distinction between derivation of balance equations and assignment of constitutive structures as separated steps of a conceptual hierarchy.

— If we restrict the treatment to a conservative setting, and to simple bodies, once the free energy is assigned to be $\psi = \psi(x, F)$, so the stress P is $P = \partial\psi/\partial F$, and the bulk forces are conservative too, equation (3.14) reduces to the integral version of the pointwise balance appearing in Noether's theorem referred to nonlinear elasticity of simple bodies, provided that ψ is invariant with respect to the action of diffeomorphisms in physical space and volume-preserving changes of the reference space.

— Requiring invariance for the relative power with respect to $\dot{y} \longmapsto \dot{y}^{\diamond}$ and $w \longmapsto w^{\diamond}$ is not related with any material symmetry compatible with a diffeomorphism determined by w. In fact, we do not impose any rule of related changes for stress and/or energy. We just test the *same* class of actions on velocity fields differing by rigid-body motions. So, we do not fall within conditions defining material symmetry classes (see [1, pp. 217-to-205], [10, pp. 157-to-161], [28, pp. 217-to-225]).

— The invariance requirement imposed to the relative power implies a standard condition in the physical space, while in the reference space it means essentially that configurational actions do not play a significant role when a defect—a bulk one in the case we are treating here—does not evolve relatively to the body. There is a link with the notion of horizontal variations for integral functionals (those involving the integration domain)[1] : in fact, w could be interpreted only as an *infinitesimal generator* of the action exerted by the group of volume-preserving diffeomorphisms that determine horizontal variations, not as a variation itself, which would be a flow associated with w. Per se w indicates only *a field superposed to* $\mathcal{B}$, where all the integrals are defined, and $\mathcal{B}$ *does not vary in time*; w *virtually* describes an incoming time-varying behaviour.

— Under appropriate conditions, the balance of configurational actions has been interpreted as the projection through inverse motion of the standard Cauchy balance of forces written in terms of the Piola–Kirchhoff stress (see [30–32]). The projection requires sufficiently smooth deformations. The necessary smoothness is not always granted. Consider for example standard nonlinear elastic bodies. Minimizers of the energy subjected to Dirichlet boundary conditions belong to the space $\mathrm{dif}^{p,1}$, $p > 1$, of weak diffeomorphisms [33]. Such a space is included in $\mathrm{dif}^{1,1}$, a subspace of the Sobolev space $W^{1,1}$, endowed with closure and compactness properties. It contains orientation preserving maps that allow self-contact without self-penetration and do not allow formation of fractures or holes (see also [34]). As Sobolev maps, they do not admit in general tangential derivatives everywhere. A consequence with physical interest is that, in trying to evaluate the first variation of the energy, we find only conditions implying the standard balance of forces in terms of the Cauchy stress[2] and configurational balances in terms of the Hamilton–Eshelby stress tensor. These equations emerge from distinct procedures. Further conditions are also needed to assure that minimizers admit inverse (see [34–36]).

— The independent nature of the balance of configurational actions has been supported in reference [14] (see also [37]). Along this point of view, a balance of configurational actions has been postulated *a priori*. Then, its unspecified (at the beginning) ingredients, namely Hamilton–Eshelby stress and some configurational bulk forces, have been identified in terms of energy and standard actions, by referring to a version of the second law of thermodynamics, written on time-varying control volumes, selected in the reference space, and exploiting an invariance requirement of such a law under re-parameterization of the defect boundary (see [14,15]). In conservative setting, the independent nature of standard force balance (the one arising from vertical variations of the energy) from those arising from horizontal variations (the configurational balances without the dissipative driving forces) was already known [27, vol. I, pp. 152–153], [34].

— The requirement of relative power invariance implies *not only* standard and configurational integral balances *but also* allows the identification of configurational actions in terms of the standard ones in a direct way. The process does not make use of the inverse mapping and does not require *a priori* assumption of configurational stress and counterpart of $b^{\ddagger}$, with the need of their subsequent identification.

— In invoking the relative power invariance, we preserve the intrinsic independence of standard and configurational balances because the two velocity fields involved in the definition of $\mathcal{P}^{\mathrm{rel}}$ are independent from one another; also changes of observers in physical and reference spaces are independent too.

[1] As I have already mentioned, the idea of connecting horizontal variations with defect evolution in solids appears already in Eshelby's work [26] and is linked—there explicitly—with Noether's theorem, although the latter is originally proven in purely conservative setting. Extensions of it to some special non-conservative structures appear in [29].

[2] Under appropriate bounds for the derivatives of a polyconvex energy, the weak form of the balance of forces in terms of the Cauchy stress can be derived in conservative setting by the first variation of the energy, under some growth assumptions; a consequence is that the stress is locally L^1 in space.

(d) An alternative representation of the relative power: connection with the material momentum

At times, authors show a preference to write equation (3.11) in terms of what is called *material momentum*, defined by $\rho F^*\dot{y}$ (e.g. [30]), which is the pull back of $\rho\dot{y}$ in the reference configuration. A reason is that such an expression produces a version of equation (3.11) that allows in conservative setting a direct comparison with what is furnished by the Noether theorem as a consequence of horizontal variations, namely changes in the reference place produced by the special group of diffeomorphisms (e.g. [28]). On the other hand, it is just Noether's theorem, or better, the use of it made by J. D. Eshelby [26], which seems to have suggested in dissipative setting only *by analogy* the version of (3.11) that includes the material momentum. It reads

$$\mathrm{Div}\left(\mathbb{P}-\frac{1}{2}\rho(\dot{y}^\flat\cdot\dot{y})I\right)-F^*b-\partial_x\psi+\frac{1}{2}(\dot{y}^\flat\cdot\dot{y})D\rho=f-\overline{\rho F^*\dot{y}^\flat}^{\,\cdot}. \tag{3.17}$$

When the reference mass density ρ is uniform in space, we get

$$\mathrm{Div}\left(\mathbb{P}-\frac{1}{2}\rho(\dot{y}^\flat\cdot\dot{y})I\right)-F^*b-\partial_x\psi=f-\overline{\rho F^*\dot{y}^\flat}^{\,\cdot}. \tag{3.18}$$

The map between equations (3.11) and (3.17) is assured by the identity (e.g. [15, p. 47])

$$\rho F^*\ddot{y}=\overline{\rho F^*\dot{y}^\flat}^{\,\cdot}+D\left(\frac{1}{2}\rho(\dot{y}^\flat\cdot\dot{y})\right)-\frac{1}{2}(\dot{y}^\flat\cdot\dot{y})D\rho$$

or, when ρ is uniform in space, equation (3.11) transforms obviously into (3.18) thanks to the identity

$$\rho F^*\ddot{y}=\overline{\rho F^*\dot{y}^\flat}^{\,\cdot}+D\left(\frac{1}{2}\rho(\dot{y}^\flat\cdot\dot{y})\right).$$

If we want to interpret the circumstance in terms of relative power, we need to vary the definition of $\mathcal{P}^{\mathrm{rel}}$. Write $\tilde{\mathcal{P}}^{\mathrm{rel}}_{\mathfrak{b}}(\dot{y},w)$ for its new expression defined by

$$\tilde{\mathcal{P}}^{\mathrm{rel}}_{\mathfrak{b}}(\dot{y},w):=\mathcal{P}^{\text{rel-a-ni}}_{\mathfrak{b}}(v,w)+\mathcal{P}^{\mathrm{dis}}_{\mathfrak{b}}(w)+\mathcal{P}^{\text{rel-a-in}}_{\mathfrak{b}}(v,w),$$

where $\mathcal{P}^{\mathrm{dis}}_{\mathfrak{b}}(w)$ is the same power of disarrangements defined above, $\mathcal{P}^{\text{rel-a-ni}}_{\mathfrak{b}}(v,w)$ indicates the *relative power of non-inertial actions* (the superscript *ni* reminds this nature) defined by

$$\mathcal{P}^{\text{rel-a-ni}}_{\mathfrak{b}}(\dot{y},w):=\int_{\mathfrak{b}}b\cdot(\dot{y}-Fw)\,\mathrm{d}x+\int_{\partial\mathfrak{b}}\mathfrak{t}_\partial\cdot(\dot{y}-Fw)\,\mathrm{d}\mathcal{H}^{k-1}(x),$$

and, finally, $\mathcal{P}^{\text{rel-a-in}}_{\mathfrak{b}}(v,w)$ is the *inertial relative power* given by

$$\begin{aligned}\mathcal{P}^{\text{rel-a-in}}_{\mathfrak{b}}(v,w):=&\int_{\mathfrak{b}}b^{\mathrm{in}}\cdot(\dot{y}-Fw)\,\mathrm{d}x\\&+\int_{\mathfrak{b}}\rho\dot{y}^\flat\cdot\dot{F}w\,\mathrm{d}x-\int_{\partial\mathfrak{b}}\frac{1}{2}(\rho\dot{y}^\flat\cdot\dot{y})(n\cdot w)\,\mathrm{d}\mathcal{H}^{k-1}(x),\end{aligned}$$

when the density of mass is uniform in space, or

$$\begin{aligned}\mathcal{P}^{\text{rel-a-in}}_{\mathfrak{b}}(v,w):=&\int_{\mathfrak{b}}b^{\mathrm{in}}\cdot(\dot{y}-Fw)\,\mathrm{d}x\\&+\int_{\mathfrak{b}}\rho\dot{y}^\flat\cdot\dot{F}w\,\mathrm{d}x-\int_{\partial\mathfrak{b}}\frac{1}{2}(\rho\dot{y}^\flat\cdot\dot{y})(n\cdot w)\,\mathrm{d}\mathcal{H}^{k-1}(x)\\&+\int_{\mathfrak{b}}\frac{1}{2}(\dot{y}^\flat\cdot\dot{y})D\rho\cdot w\,\mathrm{d}x,\end{aligned}$$

when the mass is not uniformly distributed in space. The term

$$\int_{\mathfrak{b}}\rho\dot{y}^\flat\cdot\dot{F}w\,\mathrm{d}x$$

can be interpreted as a power that the momentum develops in the *apparent* change of w induced by the time-variation of F, which does not represent any incoming material (structural) mutation. The addendum

$$\int_{\partial\mathfrak{b}} \frac{1}{2}(\rho\dot{y}^\flat\cdot\dot{y})(n\cdot w)\,\mathrm{d}\mathcal{H}^{k-1}(x)$$

is a flow of kinetic energy across the boundary of a generic part $\partial\mathfrak{b}$ generated by w. Finally,

$$\int_{\mathfrak{b}} \frac{1}{2}(\dot{y}^\flat\cdot\dot{y})D\rho\cdot w\,\mathrm{d}x$$

is a production of kinetic energy due to inhomogeneous mass distribution—in a sense we could interpret it as a production of kinetic energy along variations of the mass distribution, due to a material mutation.

To understand better the difference between $\tilde{\mathcal{P}}^{\mathrm{rel}}_{\mathfrak{b}}(\dot{y},w)$ and the relative power $\mathcal{P}^{\mathrm{rel}}_{\mathfrak{b}}(\dot{y},w)$, defined previously, we have just to note that, *for any constant vector field* $\mathfrak{c}$,

$$\int_{\mathfrak{b}}\left(\rho\dot{y}^\flat\cdot\dot{F}\mathfrak{c}+\frac{1}{2}(\dot{y}^\flat\cdot\dot{y})D\rho\cdot\mathfrak{c}\right)\mathrm{d}x-\int_{\partial\mathfrak{b}}\frac{1}{2}(\rho\dot{y}^\flat\cdot\dot{y})(n\cdot\mathfrak{c})\,\mathrm{d}\mathcal{H}^{k-1}(x)=0,$$

as it is immediate to recognize by using the Gauss theorem on the last integral.

(e) Notes on the identification of f^{da}

Let us write the standard expression of the mechanical dissipation inequality—a way to call the Clausius–Duhem inequality in isothermal setting—just to allow an immediate comparison with its varied version presented later. It reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathfrak{b}}\psi\,\mathrm{d}x-\mathcal{P}^{\mathrm{ext}}_{\mathfrak{b}}(\dot{y})\le 0 \tag{3.19}$$

for every body part $\mathfrak{b}$ and any choice of the rates involved. Although the inequality is written with respect to the reference configuration, it refers only to phenomena occurring in the current configuration, the ones determined by a deformation. It is well known that such an inequality is a tool for (i) selecting admissible constitutive structures, (ii) furnishing an admissibility criterion for shock waves, (iii) restricting the possible choices of evolution laws for other singular surfaces moving relatively to the rest of the body.

Under the regularity assumptions for the stress field $x\longmapsto P(x)$ listed in theorem 3.1 to grant pointwise balance equations, we get the identity

$$\mathcal{P}^{\mathrm{ext}}_{\mathfrak{b}}(\dot{y})=\int_{\mathfrak{b}}P\cdot\dot{F}\,\mathrm{d}x$$

(see definition (3.15) and set $w=0$). The right-hand side integral is commonly called an *internal* (or *inner*) *power*. The previous relation allows us to write the mechanical dissipation inequality in the more common form

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathfrak{b}}\psi\,\mathrm{d}x-\int_{\mathfrak{b}}P\cdot\dot{F}\,\mathrm{d}x\le 0. \tag{3.20}$$

When we refer to non-isothermal processes, by using θ to indicate the absolute temperature, η for the entropy density and $\mathfrak{q}$ for the heat flux, linked to the entropy flux in the standard way by the factor $1/\theta$, the Clausius–Duhem inequality reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathfrak{b}}\psi\,\mathrm{d}x+\int_{\mathfrak{b}}\dot{\theta}\eta\,\mathrm{d}x+\int_{\mathfrak{b}}\frac{1}{\theta}\mathfrak{q}\cdot D\theta\,\mathrm{d}x-\int_{\mathfrak{b}}P\cdot\dot{F}\,\mathrm{d}x\le 0. \tag{3.21}$$

From these two versions of the Clausius–Duhem inequality, we can identify in the bulk configurational action f the component f^{da} due to the dissipative part of standard actions.

Case 1. *Standard viscoelastic materials in isothermal setting*. We consider first the inequality (3.20). Standard assumptions apply

— The first Piola–Kirchhoff stress P admits an additive decomposition into an energetic (linked to the free energy ψ) component, P^e, and a purely dissipative one, indicated by P^d, namely $P = P^e + P^d$.
— P^e has constitutive structure of the form $P^e = \hat{P}^e(x, F)$, while $P^d = \hat{P}^d(x, F, \dot{F})$.
— For the free energy, we take $\psi = \hat{\psi}(x, F)$. (Note that we could choose $\psi = \hat{\psi}(x, F, \dot{F})$ but the inequality (3.20) forbids the dependence of the free energy on $\dot{F}$—this is matter of standard textbooks.)

Insertion of these assumptions in (3.20) and arbitrariness of rate fields and $\mathfrak{b}$ imply the standard results

$$P^e = \frac{\partial \psi}{\partial F} \quad \text{and} \quad P^d \cdot \dot{F} \geq 0. \tag{3.22}$$

The latter inequality is compatible with $P^d = \hat{a}(\cdot)\dot{F}$, where $\hat{a}(\cdot)$ is a positive definite differentiable function of state. So, we can write

$$\partial_x \psi = D\psi - P^e DF^*,$$

intending with $P^e DF^*$ the covector of components $(P^e)_i^A F^i_{A,B}$ in which the comma indicates differentiation with respect to x^B. By inserting the previous identity in the pointwise balance (3.11) and putting $f^{ba} = 0$, we get

$$f^{da} = -P^d DF^*.$$

In the present example, we do not consider bond breaking. In any case, the structure of f^{ba} should be assigned independently, with the sole constraint given by its dissipative nature, expressed by the inequality $f^{ba} \cdot w \geq 0$ for every w; in it the equality sign holds only when $w = 0$.

Case 2. *Standard themo-viscoelastic materials*. In non-isothermal setting the list of state variables includes the temperature θ, so that we presume $\psi = \hat{\psi}(x, F, \theta)$, and, once again, the additive decomposition $P = P^e + P^d$, but now $P^e = \hat{P}^e(x, F, \theta)$ and $P^d = \hat{P}^d(x, F, \dot{F}, \theta)$. The procedure indicated in case 1, now exploiting the inequality (3.21), allows us to get the counterparts of relations (3.22), including temperature, and the additional standard relation $\eta = -\partial_\theta \psi$, so that we can write

$$\partial_x \psi = D\psi - P^e DF^* + \eta D\theta$$

and, from the balance (3.11), we get

$$f^{da} = -P^d DF^* - \eta D\theta,$$

after setting once again $f^{ba} = 0$.

Case 3. *Thermo-viscoelastic materials with internal variables*. In some cases, the detachment from thermodynamic equilibrium is not completely described by the sole account of temperature and deformation gradient time rate—examples can be found in the presence of chemical phenomena. In some circumstances and for some models, it is appropriate to introduce state variables, say $\alpha \in \mathbb{R}^m$, depending on space and time, such that there are no actions producing power in their rates, meaning they are *not observable*. This is a crucial aspect marking the difference with what is presented in the general model-building framework for the mechanics of complex materials, a scheme not discussed here for the sake of brevity. Also, the derivative of ψ with respect to α is a covector collecting chemical affinities and contributing only to the production of entropy. If we include thermal and viscous effects, the assumptions in previous cases vary simply in this way: $\psi = \hat{\psi}(x, F, \alpha)$, and $P = P^e + P^d$, with $P^e = \hat{P}^e(x, F, \theta, \alpha)$ and $P^d = \hat{P}^d(x, F, \dot{F}, \theta, \alpha)$.

By setting $\pi := \partial_\alpha \psi$, the machinery used in the previous examples leads to

$$f^{da} = -P^d DF^* - \eta D\theta + \pi D\alpha,$$

where $\pi D\alpha$ is a covector with A-th component given by $\pi_r (D\alpha^r)_A$. The index A refers to coordinates in the reference space, while r denotes coordinates in the space where the vector α takes values.

We find even more from the foundational and geometrical side if we imagine versions of the balance of energy and the Clausius–Duhem inequality including the relative power, as it is shown in the next two sections.

From now on, we exclude ageing for the sake of simplicity.

4. A variant of Marsden–Hughes's theorem including the relative power

When the list of state variables is decided from the outset in constructing a mechanical model, in conservative setting the Noether theorem is a fundamental tool for deriving balance laws. It can be also arranged to cover some dissipative effects when a pseudo-potential like the Rayleigh one is available.

Along this path, we could also adopt a Hamiltonian view on the basis of Poisson and (possibly) dissipative brackets (see [29,38–41]).

Another approach has been suggested by A. M. Green, R. S. Rivlin and P. Naghdi (see [42–44]): they required invariance under superposed rigid-body motions of the first law of thermodynamics, obtaining as a consequence pointwise balance equations [28, pp. 148–149].

This point of view suggested a result in nonlinear elasticity of simple bodies obtained by J. E. Marsden and T. R. J. Hughes [28, p. 165]. The structure of their analysis is the following:

— They considered the balance of energy for the nonlinear thermoelasticity of simple bodies.
— At variance of the approach in [42,43], such a balance is written in the current configuration.
— The energy is assumed to be a *function of the spatial metric alone*, besides the space variable y and time t.
— Synchronous changes of observers are considered: they are induced by smooth diffeomorphisms of the ambient space, which is identified with a *generic Riemannian manifold*.
— It is assumed that the internal energy density changes tensorially along a 'distortion' of the ambient space induced by the changes of observers adopted.

Balance equations in local form follow from a requirement of covariance, that is invariance in structure of the energy balance. From it, we obtain equality between a derivative of the energy with respect to the spatial metric and the stress tensor, modulo the density of mass and a factor two: this is what we call the Doyle–Ericksen formula, a result paving the way to relativistic elasticity. Bulk actions not necessarily associated with a potential are admitted. The local balance of mass emerges too as an ancillary result.

Such a theorem received renewed attention in [19] where the balance of energy is written in Lagrangian representation. To it, the common technique of vertical and horizontal variations (of fields and integral domain, respectively) is applied. Both standard and configurational balances emerge (for the specification of regularity assumptions justifying the formal use of variations, at least in the static case, see [34]). Marsden–Hughes's theorem has been also discussed with reference to mixtures first in [8].

Here, we prove a version of Marsden–Hughes's theorem that includes the notion of relative power. We work in isothermal setting for the sake of simplicity, but we add a remark explaining how we can tackle a non-isothermal case. The point of view adopted is the following:

— The ambient space is a finite-dimensional differentiable Riemannian (simply connected) manifold $\mathfrak{S}$ that is geodesically complete and endowed with metric $\tilde{g}$. Motions are one-to-one differentiable maps $(x,t) \longmapsto y := \tilde{y}(x,t) \in \mathfrak{S}$. Once again, we write F for $D\tilde{y}(x,t)$. $\mathcal{B}$ is also considered to be a generic differentiable Riemannian (simply connected) manifold with boundary oriented by the normal n almost everywhere and with dimension k not greater than the one of $\mathfrak{S}$. $\mathcal{B}$ is endowed with a metric g. Connections considered in both manifolds are compatible with the pertinent metrics.

— The first law of thermodynamics adopted here for mutant elastic bodies reads[3]

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathfrak{b}} e\,\mathrm{d}x - \mathcal{P}^{\mathrm{rel}}_{\mathfrak{b}}(\dot{y},w) = 0 \tag{4.1}$$

for any k-dimensional Riemannian (simply connected) submanifold $\mathfrak{b}$ of $\mathcal{B}$. In the previous equation, e is the *internal energy density*. It appears also in $\mathcal{P}^{\mathrm{rel}}_{\mathfrak{b}}(\dot{y},w)$ where it replaces the free energy ψ.

— Synchronous changes of observers are involved.

(i) A time-parameterized class of diffeomorphims $h_t \in \mathrm{Diff}(\mathfrak{S},\mathfrak{S})$, with $t \in [0,+\infty)$ and $h_0 = \text{identity}$, acts over the ambient space. Dependence on time t is taken to be smooth. The pertinent infinitesimal generator of the actions of such a class is a vector field $v' := (\mathrm{d}/\mathrm{d}s)h_t|_{t=0}$.

(ii) Another time-parameterized class of diffeomorphisms acts over the reference manifold, namely we have $\hat{h}_t \in \mathrm{SDiff}(\mathcal{B},\mathcal{B})$, with t in the same interval above and $\hat{h}_0 = \textit{identity}$. So, $\det D\hat{h}_t = 1$, which implies that the changes of observer on $\mathcal{B}$ preserve mass; in other words, every observer must record the same type of material for a certain body (about this principle see discussions in reference [45]). The infinitesimal generator of their action is a vector field $w' := (\mathrm{d}/\mathrm{d}s)\hat{h}_t|_{t=t_0}$. As fields, both v' and w' are bounded.

— Changes of observers described above imply that, at $t = 0$,

$$\dot{y} \longrightarrow \dot{y}^{\diamond} := \dot{y} + v' \tag{4.2}$$

and

$$w \longrightarrow w^{\diamond} := w + w'. \tag{4.3}$$

Moreover, in the presence of a discontinuity surface Σ for the deformation gradient, a surface not moving relatively to the body, we assume that both v' and w' are continuous across Σ.

— For bulk and contact actions we presume time-parameterized variations $b^{\ddagger\prime}_t$ and $\mathrm{t}'_{\partial;t}$ such that at $t = 0$, the instant where we evaluate (4.2) and (4.3), we have

$$b^{\ddagger\prime}_0 = b^{\ddagger} \quad \text{and} \quad \mathrm{t}'_{\partial;0} = \mathrm{t}_{\partial}$$

(compare [28, p. 163]).

— We write $e = \rho\tilde{e}$, with ρ the *constant* referential *mass density*, and assume that *e changes tensorially* under the action of *both* h_t and $\hat{h}_t$ (in the standard version of Marsden–Hughes's theorem it is required that the energy changes tensorially only with respect to h_t because in that formulation the reference space is not involved). Formally, indicating by the superscript $\diamond$ values after the changes of observer, as in the previous section, the requirement of tensoriality means that

$$\rho\tilde{e}^{\diamond}(x',t) = \rho\tilde{e}(x,t,\hat{h}^*_t g, h^*_t\tilde{g}),$$

where $\hat{h}^*_t$ and h^*_t indicate pull-back, and $x' = \hat{h}_t(x)$ (compare with [28, p. 165], namely $e^{\diamond}(y',t) = e(y,t,\xi^*_t g(y))$, with $y' := \xi_t(y)$, ξ_t a diffeomorphism analogous to h_t). With $\tilde{e}$ a

[3]In the standard balance of energy used in Marsden–Hughes's theorem, the rate of kinetic energy appears explicitly and bulk actions are non-inertial. Here, in formula (4.1), inertial effects are included in the bulk actions.

differentiable function of its arguments, the assumption implies

$$\rho\frac{\mathrm{d}\tilde{e}^{\diamond}}{\mathrm{d}t}=\rho\frac{\mathrm{d}\tilde{e}}{\mathrm{d}t}+\rho\frac{\partial\tilde{e}}{\partial g}\cdot\mathfrak{L}_{w'}g+\rho\frac{\partial\tilde{e}}{\partial\tilde{g}}\cdot\mathfrak{L}_{v'}\tilde{g}, \tag{4.4}$$

where $\mathfrak{L}_{w'}g$ and $\mathfrak{L}_{v'}\tilde{g}$ are, respectively, autonomous Lie derivatives[4] of g and $\tilde{g}$ along the vector fields w' and v'.

Theorem 4.1. *If the balance of energy (4.1) remains invariant under changes of observers induced by h and $\hat{h}$ as described above, the discontinuity surface considered previously has no relative motion with respect to the body, and assumptions of the Cauchy theorem on bulk and surface terms hold, we get the representation of contact actions in terms of the Piola–Kirchhoff stress P and, under PC^1 regularity for P as a function of space, all local balance equations in theorem 3.1 hold true (provided the substitution of ψ with e), together with the following relations:*

$$P=2\rho\frac{\partial\tilde{e}}{\partial\tilde{g}}F^{-*} \tag{4.5}$$

and

$$sym\mathbb{P}=2\rho\frac{\partial\tilde{e}}{\partial g}. \tag{4.6}$$

Proof. Set

$$\mathfrak{E}(\dot{y},w;\mathfrak{b}):=\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathfrak{b}}\rho\tilde{e}\mathrm{d}x-\mathcal{P}^{\mathrm{rel}}_{\mathfrak{b}}(\dot{y},w)=\int_{\mathfrak{b}}\rho\frac{\mathrm{d}\tilde{e}}{\mathrm{d}t}\,\mathrm{d}x-\mathcal{P}^{\mathrm{rel}}_{\mathfrak{b}}(\dot{y},w).$$

The invariance requirement reads

$$\mathfrak{E}^{\diamond}(\dot{y}^{\diamond},w^{\diamond};\mathfrak{b})-\mathfrak{E}(\dot{y},w;\mathfrak{b})=0, \tag{4.7}$$

where $\mathfrak{E}^{\diamond}$ indicates that the time derivative of the energy is given by relation (4.4). The above identity is then

$$\int_{\mathfrak{b}}\left(\rho\frac{\partial\tilde{e}}{\partial g}\cdot\mathfrak{L}_{w'}g+\rho\frac{\partial\tilde{e}}{\partial\tilde{g}}\cdot\mathfrak{L}_{v'}\tilde{g}\right)\mathrm{d}x-\mathcal{P}^{\mathrm{ext}}_{\mathfrak{b}}(v',w')=0, \tag{4.8}$$

where

$$\begin{aligned}\mathcal{P}^{\mathrm{rel}}_{\mathfrak{b}}(v',w')=&\int_{\mathfrak{b}}b^{\ddagger}\cdot v'\,\mathrm{d}x+\int_{\partial\mathfrak{b}}(\mathfrak{t}_{\partial}\cdot v'-(\rho\tilde{e}n-F^{*}\mathfrak{t}_{\partial})\cdot w')\,\mathrm{d}\mathcal{H}^{k-1}(x)\\&-\int_{\mathfrak{b}}\left(\rho\frac{\partial\tilde{e}}{\partial x}+f\right)\cdot w'\,\mathrm{d}x+\int_{\mathfrak{b}}\mu\cdot\mathrm{curl}w'\,\mathrm{d}x.\end{aligned}$$

Set $w'=0$. The identity (4.7) then reads

$$\int_{\mathfrak{b}}\rho\frac{\partial\tilde{e}}{\partial g}\cdot\mathfrak{L}_{v'}g\,\mathrm{d}x-\int_{\mathfrak{b}}b^{\ddagger}\cdot v'\,\mathrm{d}x-\int_{\partial\mathfrak{b}}\mathfrak{t}_{\partial}\cdot v'\,\mathrm{d}\mathcal{H}^{k-1}(x)=0.$$

Since we presume that bulk and surface terms satisfy assumptions in theorem 2.1, from the previous identity we get (i) the dependence of $\mathfrak{t}_{\partial}$ on the boundary $\partial\mathfrak{b}$ only through the normal where it is well-defined, (ii) the action-reaction classical relation, and (iii) the linearity of contact actions with respect to n in terms of the first Piola–Kirchhoff stress P. Consequently, by using the

[4]Consider a time-parameterized family of diffeomorphisms φ_t of the space onto itself. Write $\varphi_{t,s}$ for $\varphi_t\circ\varphi_s^{-1}$ (the inverse being defined by time-reversal, namely $\varphi_s^{-1}=\varphi_{-s}$), and evaluate the vector field $\mathsf{v}:=(\mathrm{d}\varphi_{t,s}/\mathrm{d}t)|_{t=s}$, $\varphi_{s,s}=$ identity. Take a differentiable tensor-valued function over the space, say $y\longmapsto B(y)$. We can evaluate a derivative of B along the vector field v—write $\mathfrak{L}_{\mathsf{v}}B$ for it—by defining it by $\mathfrak{L}_{\mathsf{v}}B:=\mathrm{d}/\mathrm{d}t(\varphi_{t,s*}B)|_{t=s}$. In particular, if B is a metric g, we compute

$$(\mathfrak{L}_{\mathsf{v}}g)_{ij}=\mathsf{v}_{i|j}+\mathsf{v}_{j|i},$$

where the vertical bar in subscript position denotes covariant derivative.

Gauss theorem, in the presumed PC^1 setting, the equality (4.8) becomes

$$\begin{aligned}&\int_{\mathfrak{b}}\left(\rho\frac{\partial\tilde{e}}{\partial g}\cdot\mathfrak{L}_{w'}g+\rho\frac{\partial\tilde{e}}{\partial\tilde{g}}\cdot\mathfrak{L}_{v'}\tilde{g}\right)\mathrm{d}x-\int_{\mathfrak{b}}\left((b^{\ddagger}+\mathrm{Div}P)\cdot v'+(\mathrm{Div}\mathbb{P}-\rho\partial_x\tilde{e}-f)\cdot w'\right)\mathrm{d}x\\&\quad-\int_{\mathfrak{b}}(P\cdot Dv'+\mathbb{P}\cdot Dw'+2(\mu\times)\cdot\mathrm{skw}Dw')\,\mathrm{d}x\\&\quad-\int_{\mathfrak{b}\cap\Sigma}([P]n\cdot v'+[\mathbb{P}]n\cdot w')\,\mathrm{d}x=0,\end{aligned}$$

for any choice of v' and w', and their gradients. Once again, $\partial_x\tilde{e}$ indicates the explicit derivative of $\tilde{e}$ with respect to x, maintaining fixed the state variables. The last integral above follows by considering the continuity of v' and w' across Σ. The arbitrariness of v' and w' implies the validity of (3.9), (3.11) and (3.13).

We compute

$$\begin{aligned}\mathbb{P}\cdot Dw'&=\mathrm{sym}\mathbb{P}\cdot\mathrm{sym}Dw'+\mathrm{skw}\mathbb{P}\cdot\mathrm{skw}Dw'\\&=\frac{1}{2}\mathrm{sym}\mathbb{P}\cdot\mathfrak{L}_{w'}g+\mathrm{skw}\mathbb{P}\cdot\mathrm{skw}Dw'\end{aligned}\tag{4.9}$$

and

$$\begin{aligned}P\cdot Dv'&=P\cdot(D_yv')F=PF^*\cdot\mathrm{sym}D_yv'+PF^*\cdot\mathrm{skw}D_yv'\\&=\frac{1}{2}PF^*\cdot\mathfrak{L}_{v'}\tilde{g}+PF^*\cdot\mathrm{skw}D_yv',\end{aligned}\tag{4.10}$$

with D_y the spatial derivative with respect to y so that $D(\cdot)=D_y(\cdot)Dy=D_y(\cdot)F$. The arbitrariness of $\mathrm{skw}D_yv'$ implies equation (3.10), while the one of $\mathrm{skw}D_yv'$ implies (3.12). Finally, we get

$$\int_{\mathfrak{b}}\left(\left(\rho\frac{\partial\tilde{e}}{\partial g}-\frac{1}{2}\mathrm{Sym}\mathbb{P}\right)\cdot\mathfrak{L}_{w'}g+\left(\rho\frac{\partial\tilde{e}}{\partial\tilde{g}}-\frac{1}{2}PF^*\right)\cdot\mathfrak{L}_{v'}\tilde{g}\right)\mathrm{d}x=0.$$

The arbitrariness of $\mathfrak{L}_{w'}\hat{g}$ and $\mathfrak{L}_{v'}g$ allows us to get (4.6) and (4.5). This concludes the proof. ■

— The relation (4.5) is a referential version of the Doyle–Ericksen formula (see [46]), which prescribes that

$$\sigma=2\rho_a\frac{\partial\tilde{e}}{\partial\tilde{g}},$$

while (4.6) is a version of the Rosenfeld–Belinfante theorem in classical field theories. In the previous formula, ρ_a indicates the current mass density. If the target manifold has the same dimension of $\mathcal{B}$, we have $\rho=(\det F)\rho_a$.

— Previous theorem requires prescribing explicitly the internal energy. So, at equilibrium we could appeal directly to a minimality requirement for the energy, obtaining balances of forces (3.9) and (3.13_a) by the energy first variation around a minimizer (provided the necessary bounds for the energy and its derivatives—see [33]), while the local configurational balance (3.11) would follow from horizontal variations of the energy, the one obtained by varying virtually a reference place. The reason of using the first law of thermodynamics (in the form of Marsden–Hughes's theorem, although in the variant that we use here) is only the possibility of considering non-conservative bulk forces. Viscous-like stresses are not involved in theorem 4.1. They are, in fact, not related with energy. So, if we would accept the standard decomposition $P=P^e+P^d$ where, once again, the superscript e stands for *energetic* and d for *dissipative* (see [47]), and we would insert it in the first law of thermodynamics, by developing the same calculations leading to theorem 4.1, linking only P^e with the energy through (4.5), we would find that P^dF^* would be powerless, a result in contrast with the evidence that in the presence of viscous effects P^d is intrinsically dissipative, meaning that $P^d\cdot\dot{F}\geq 0$, the equality holding only when $\dot{F}=0$. Including the occurrence of dissipative stresses thus requires to call upon the second law—even only in the version of mechanical dissipation inequality—and to prove a theorem like the previous one. Such a proof is the main result of this paper.

— If we refer to Marsden–Hughes's theorem [28], to its material version discussed in reference [19] *without* relative power, or to what we presented here, in all these cases we are not postulating *a priori* the balance equations in weak form as we do in looking at the principle of virtual power, which requires to introduce from the beginning an inner power, so to postulate in advance the existence of a stress tensor, not deriving it.

— The extension of theorem 4.1 to the non-isothermal case is straightforward. The balance of energy changes into

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathfrak{b}} e\,\mathrm{d}x - \mathcal{P}^{\mathrm{rel}}_{\mathfrak{b}}(\dot{y}, w) - \int_{\mathfrak{b}} \mathfrak{r}\,\mathrm{d}x + \int_{\partial\mathfrak{b}} \mathfrak{a}_{\partial}\,\mathrm{d}\mathcal{H}^{k-1}(x) = 0, \tag{4.11}$$

where $\mathfrak{r}$ is a heat source that is a function of space and time, while $\mathfrak{a}_{\partial}$ a heat flux that depends also on the oriented boundary $\partial\mathfrak{b}$. In this case, the energy is assumed to depend on temperature θ. An additional assumption is required: under changes of observers, $\mathfrak{r}$ and $\mathfrak{a}_{\partial}$ become $\mathfrak{r}'_t$ and $\mathfrak{a}'_{\partial,t}$ and are such that

$$\mathfrak{r}'_0 = \mathfrak{r} \quad \text{and} \quad \mathfrak{a}'_{\partial,0} = \mathfrak{a}_{\partial}.$$

Previous analyses apply. Then, one comes back to the definition (4.11) and by Cauchy's theorem realizes that there exists a vector field $\mathfrak{q}$ depending on space and time alone such that $\mathfrak{a}_{\partial} = \mathfrak{q}\cdot n$, with n the normal to $\partial\mathfrak{b}$. The rest is a matter of regularity assumptions, divergence theorem and localization once we assume that the integral balance (4.11) holds for any k-dimensional submanifold $\mathfrak{b}$ of $\mathcal{B}$.

5. Notes on formula (4.6)

The identity (4.6) shows that variations of the metric in $\mathcal{B}$ indicate material mutations and induce configurational stress. To see this from another viewpoint, we look back to the free energy density ψ and consider for it a structure of the type

$$\psi = \hat{\psi}(x, g, F).$$

Assume now that ψ *is invariant under the action of the entire group of diffeomorphisms from* $\mathcal{B}$ *onto itself and also that the first Piola–Kirchhoff stress has only energetic nature—no dissipative component is present.*

Consider sufficiently smooth curves $t \longmapsto \hat{h}_t$, with $\hat{h}_0 = identity$, in $\mathrm{Diff}(\mathcal{B},\mathcal{B})$. Call once again w' the derivative $(\mathrm{d}/\mathrm{d}t)\hat{h}_t|_{t=0}$. Note that we do not require here that $\det Dh_t = 0$ at every t as in the previous section. This choice implies that $\hat{h}_t$ induces a material change because, in selecting $\hat{h}_t$ such that $\det D\hat{h}_t \neq 0$, the mass density is altered so that $\hat{h}_t$ is not a material isomorphism. It is not per se a change of observer on the material manifold $\mathcal{B}$ because we always presume (although often implicitly) that different observers must perceive the *same* type of material, so that changes of frames in $\mathcal{B}$ must be material isomorphisms in Noll's sense. Such an aspect does not emerge often because we commonly assume that different observers record the same material manifold. However, the issue enters into play indirectly when we compute the horizontal variations of the energy and select for them the special group of diffeomorphisms.

Under the action of $\mathrm{Diff}(\mathcal{B},\mathcal{B})$, the energy changes as

$$\psi \longrightarrow \psi_{\hat{h}_t} = (\det D\hat{h}_t)\psi(D\hat{h}_t^{-*} g D\hat{h}_t^{-1}, FD\hat{h}_t^{-1}).$$

The requirement of invariance implies formally that

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi_{\hat{h}_t}|_{t=0} = 0.$$

From it, we get

$$\psi\,(g, F)\,trDw' - \frac{\partial\psi}{\partial g}\cdot\left((Dw')^{*} g + gDw'\right) - \frac{\partial\psi}{\partial F}\cdot FDw' = 0,$$

by taking into account that $(\mathrm{d}/\mathrm{d}t)D\hat{h}_t^{-1}|_{t=0} = -Dw'$. Moreover, since g is symmetric, we obtain

$$\left(\psi I - F^* \frac{\partial \psi}{\partial F} - 2\frac{\partial \psi}{\partial g} g\right) \cdot Dw' = 0$$

for any choice of Dw'. The identity

$$P = \frac{\partial \psi}{\partial F}$$

coming from the Clausius–Duhem inequality, thanks to the arbitrariness of Dw, implies

$$\mathbb{P} = 2\frac{\partial \psi}{\partial g} g,$$

where $\mathbb{P}$ stands for $\psi I - F^* P$ and has components

$$\mathbb{P}_B^A = 2\left(\frac{\partial \psi}{\partial g}\right)^{AC} g_{CB}.$$

By multiplying by g^{-1} (with components g^{AB}) from the right, we finally get

$$\mathbb{P}^{AB} = \mathbb{P}_C^A g^{CB} = 2\left(\frac{\partial \psi}{\partial g}\right)^{AB}.$$

6. Covariance of the mechanical dissipation inequality and dissipative stress components

In discussing a variant of the Marsden–Hughes theorem, we have adopted an argument the structure of which goes in this way: The balance of energy (4.1) as evaluated by the first observer, say $\mathcal{O}$, is something like $A = 0$, with A indicating all terms in the left-hand side in equation (4.1). Another observer, say $\mathcal{O}'$, writes a balance of energy of the type $A' = 0$. After pulling-back in the frame defining the first observer, we get $A^\circ = 0$, with $A^\circ = A + A^\dagger$ (pull-back is intrinsic in the use of Lie derivatives in assumption (4.4) and the definition of v' and w'). So, the invariance requirement reduces to imposing $A^\dagger = 0$ for any choice of the rate fields v' and w'. The theorem itself implies also that the stress has only an energetic component, in the sense that it is fully determined by the internal energy.

To consider dissipative stress components, we need to require covariance of the Clausius–Duhem inequality. Write in short $B \leq 0$ for it, as the observer $\mathcal{O}$ evaluates it, with B involving the free energy and some power, depending on the scheme that we are following. The inequality is then $B' \leq 0$ for an observer $\mathcal{O}'$. The pull-back of B' into $\mathcal{O}$ gives rise to another inequality, say $B^\bullet \leq 0$, with $B^\bullet = B + B^\dagger$, which involves the rates v' and w', namely the infinitesimal generators of the change $\mathcal{O} \longrightarrow \mathcal{O}'$ induced by the Lie groups $\mathrm{Diff}(\mathfrak{S}, \mathfrak{S})$ and $\mathrm{SDiff}(\mathcal{B}, \mathcal{B})$, acting over the ambient (physical) manifold and the reference one, respectively.

If we start from $\mathcal{O}'$ and consider the change $\mathcal{O}' \longrightarrow \mathcal{O}$, we have now different (in principle) infinitesimal generators. Then, in the pull-back change $\mathcal{O}' \longrightarrow \mathcal{O}$, the Clausius–Duhem inequality reads $B' + B^\ddagger \leq 0$. Above all, $B^\ddagger$ *is in principle different from* $B^\dagger$.

We impose the following **covariance principle in dissipative setting**: it states, essentially, that both $B^\dagger$ and $B^\ddagger$ are *always* non-positive.

Axiom 6.1. *In any change of observer, the additional term arising after pulling-back the Clausius–Duhem inequality evaluated by the second observer in a frame defining the first one is always non-positive.*

By accepting such a principle, which I introduced in reference [20] for a discussion concerning multiplicative elastic-perfectly plastic behaviour under large strains, we proceed as follows:

— We write a version of the Clausius–Duhem inequality that includes the relative power, namely

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathfrak{b}} \psi \,\mathrm{d}x - \mathcal{P}^{\mathrm{rel}}_{\mathfrak{b}}(\dot{y}, w) \leq 0, \tag{6.1}$$

and we presume that it holds for any choice of the rates involved and the part $\mathfrak{b}$.

— We presume that contact actions admit an additive decomposition into energetic t^e_∂ and dissipative t^d_∂ components, as it is customary in viscoelasticity. The former component is called an energetic one because it is considered to be fully determined by the free energy; the latter by assumption does not depend necessarily on any potential.

— We assume that the free energy varies tensorially under changes of observers. We replace ψ with $\rho\tilde{\psi}$ in the inequality (6.1), with ρ the referential mass density and $\tilde{\psi}$ the free energy per unit mass. By indicating once again with the symbol $\diamond$ in superscript position values measured after changes of observers and pulled back in a frame defining the first observer, we write

$$\rho\frac{\mathrm{d}\tilde{\psi}^{\diamond}}{\mathrm{d}t} = \rho\frac{\mathrm{d}\tilde{\psi}}{\mathrm{d}t} + \rho\frac{\partial\tilde{\psi}}{\partial g}\cdot\mathfrak{L}_{w'}g + \rho\frac{\partial\tilde{\psi}}{\partial\tilde{g}}\cdot\mathfrak{L}_{v'}\tilde{g}, \tag{6.2}$$

which is a version of (4.4) where we just substitute $\tilde{e}^{\diamond}$ with $\tilde{\psi}^{\diamond}$; also, w' and v' are the same infinitesimal generators for the changes of observers already introduced in proving theorem 4.1.

Axiom 6.1 and identity (6.2) imply

$$\begin{aligned}\int_{\mathfrak{b}}\left(\rho\frac{\partial\tilde{\psi}}{\partial g}\cdot\mathfrak{L}_{w'}g + \rho\frac{\partial\tilde{\psi}}{\partial\tilde{g}}\cdot\mathfrak{L}_{v'}\tilde{g}\right)\mathrm{d}x - \int_{\mathfrak{b}} b^{\ddagger}\cdot v'\,\mathrm{d}x \\ - \int_{\partial\mathfrak{b}}(\mathrm{t}_\partial\cdot v' - (\rho\tilde{e}n - F^*\mathrm{t}_\partial)\cdot w')\,\mathrm{d}\mathcal{H}^{k-1}(x) \\ + \int_{\mathfrak{b}}\left(\rho\frac{\partial\tilde{\psi}}{\partial x} + f\right)\cdot w'\,\mathrm{d}x - \int_{\mathfrak{b}}\mu\cdot\mathrm{curl}w'\,\mathrm{d}x \leq 0.\end{aligned}$$

Set $w' = 0$. The inequality reduces to

$$\int_{\mathfrak{b}}\rho\frac{\partial\tilde{\psi}}{\partial\tilde{g}}\cdot\mathfrak{L}_{v'}\tilde{g}\,\mathrm{d}x - \int_{\mathfrak{b}} b^{\ddagger}\cdot v'\,\mathrm{d}x - \int_{\partial\mathfrak{b}}\mathrm{t}_\partial\cdot v'\,\mathrm{d}\mathcal{H}^{k-1}(x) \leq 0.$$

If the assumptions of Cauchy's theorem hold true, the inequality notwithstanding, by following the classical proof of Cauchy's theorem, where a key role is played by an estimate of the contact actions, we recover the action-reaction relation and $\mathrm{t}_\partial = \mathrm{t}(x,t,n) = P(x,t)n$, with n the normal to $\partial\mathfrak{b}$ at all points where it is well-defined. Then, if $x \longmapsto P(x,t)$ is a $PC^1(\mathcal{B})$ map at every t, with a fixed discontinuity surface Σ, we get

$$\begin{aligned}\int_{\mathfrak{b}}\left(\rho\frac{\partial\tilde{\psi}}{\partial g}\cdot\mathfrak{L}_{w'}g + \rho\frac{\partial\tilde{\psi}}{\partial\tilde{g}}\cdot\mathfrak{L}_{v'}\tilde{g}\right)\mathrm{d}x - \int_{\mathfrak{b}}(b^{\ddagger} + \mathrm{Div}P)\cdot v'\,\mathrm{d}x \\ - \int_{\mathfrak{b}}(\mathrm{Div}\mathbb{P} - \partial_x\psi - f)\cdot w'\mathrm{d}x \\ - \int_{\mathfrak{b}}(P\cdot Dv' + \mathbb{P}\cdot Dw' + 2(\mu\times)\cdot\mathrm{skw}Dw')\,\mathrm{d}x \\ - \int_{\mathfrak{b}\cap\Sigma}([P]n\cdot v' + [\mathbb{P}]n\cdot w')\,\mathrm{d}x \leq 0\,,\end{aligned} \tag{6.3}$$

for any choice of the rate fields involved, with $\mathbb{P} := \psi I - F^*P$, and I the second-rank unit tensor. The arbitrariness of w' and v' implies the balance equations (3.9), (3.11), and (3.13) in the bulk and along the unstructured discontinuity surface Σ, after the identification of the inertial term obtained accepting the identity (3.8) of theorem 3.1. Moreover, the assumption on t_∂ requires an obvious additive decomposition of P into energetic (P^e) and dissipative (P^d) components, namely

$P = P^e + P^d$. For such components, we assume that P^e depends on the same entries of ψ while $P^d = \tilde{P}^d(x, t, \ldots, \dot{F})$, where the dots indicate possible state variables and their derivatives, according to *Truesdell's equipresence principle*. We also compute

$$P \cdot Dv' = \frac{1}{2} P^e F^* \cdot \mathfrak{L}_{v'} \tilde{g} + P^d F^* \cdot \mathrm{sym} D_y v' + P F^* \cdot \mathrm{skw} D_y v', \tag{6.4}$$

with D_y the derivative with respect to coordinates y on $\mathfrak{S}$. By substituting the identity (6.4) into inequality (6.3), from the arbitrariness of $\mathrm{skw} D_y v'$ and $\mathrm{skw} Dw'$ we get (3.10) and (3.12). The inequality then reduces to

$$\int_{\mathfrak{b}} \left(\left(\rho \frac{\partial \tilde{\psi}}{\partial g} - \frac{1}{2} \mathrm{sym} \mathbb{P} \right) \cdot \mathfrak{L}_{w'} g + \left(\rho \frac{\partial \tilde{\psi}}{\partial \tilde{g}} - \frac{1}{2} P^e F^* \right) \cdot \mathfrak{L}_{v'} \tilde{g} \right) \mathrm{d}x$$
$$- \int_{\mathfrak{b}} P^d F^* \cdot \mathrm{sym} D_y v' \, \mathrm{d}x \leq 0.$$

Also, the arbitrariness of $\mathfrak{L}_{w'} g$ and $\mathfrak{L}_{v'} \tilde{g}$ implies

$$\mathrm{sym} \mathbb{P} = 2\rho \frac{\partial \tilde{\psi}}{\partial g} \quad \text{and} \quad P^e = 2\rho \frac{\partial \tilde{\psi}}{\partial \tilde{g}} F^{-*}. \tag{6.5}$$

The second equation implies also that $P^e F^*$ is symmetric, due to the symmetry of $\tilde{g}$. Moreover, since $P F^* = P^e F^* + P^d F^*$, and (3.10) holds, symmetry of the first two terms in the identity implies also that $P^d F^*$ is symmetric too. So, we get

$$P^d F^* \cdot \mathrm{sym} D_y v' = P^d F^* \cdot D_y v' = P^d \cdot Dv',$$

and the inequality above reduces to

$$\int_{\mathfrak{b}} P^d \cdot Dv' \, \mathrm{d}x \geq 0.$$

The arbitrariness of $\mathfrak{b}$ implies

$$P^d \cdot Dv' \geq 0 \tag{6.6}$$

for any choice of Dv'. Such an arbitrariness allows us to identify Dv' with $\dot{F}$ so that an expression of the type

$$P^d = a(\cdot) \dot{F}, \tag{6.7}$$

with $a(\cdot)$ a positive-definite scalar function of state variables and, possibly, their gradients, is compatible with the inequality (6.6).

These results can be summarized in the theorem below.

Theorem 6.2. *Consider valid for a mutant body the modified Clausius–Duhem inequality (6.1). Accept the covariance principle in dissipative setting, under the assumptions stated above, and presume that the identity (3.8) holds. Then,*

- *(i) contact actions admit the standard representation in terms of stress under the assumptions of Cauchy's theorem 2.1;*
- *(ii) under PC^1 regularity for the stress field and continuity for the bulk actions, all balance equations in theorem 3.1 hold true;*
- *(iii) the energetic part of the first Piola–Kirchhoff stress is given by equation $(6.5)_2$;*
- *(iv) the Hamilton–Eshelby stress satisfies relation $(6.5)_1$, and*
- *(v) the dissipative part of P is given by (6.7) or similar relations with $a(\cdot)$ replaced by a positive-definite second-rank tensor valued function of state variables, not necessarily only those entering the free energy.*

7. Additional remarks

— The second law of thermodynamics, written in the form of Clausius–Duhem's inequality is not only a source of *a priori* constitutive restrictions on stresses and a criterion of stability and admissibility for the propagation of discontinuities like shocks: indeed, it is also a source of the representation of contact actions in terms of stresses and implies the local forms of the balance equations under appropriate regularity. In other words, it is intrinsically a source of all basic ingredients of a mechanical model.

— For the reasons above, asking as in the previous section covariance of the second law, written as (thermo)mechanical dissipation inequality, is a safe tool to build up models when more refined representations of the body morphology come into play. In this sense, such a covariance requirement is a counterpart in dissipative setting of the invariance for the Lagrangian required in Noether's theorem.

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