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A set of well-defined operations on succession rules

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Abstract. *In this paper we introduce a system of* well-defined operations *on the set of* succession rules. *These operations allow us to tackle combinatorial enumeration problems simply by using succession rules instead of generating functions. Finally we suggest several open problems the solution of which should lead to an algebraic characterization of the set of succession rules.*

1 Introduction

A *succession rule* Ω is a system having the form:

$$\begin{cases} (b) \\ (k) \rightsquigarrow (e_1(k))(e_2(k))\dots(e_k(k)), \end{cases}$$

where $b, k \in \mathbb{N}^+$, and $e_i : \mathbb{N}^+ \to \mathbb{N}^+$; (b) is the *axiom* and $(k) \rightsquigarrow (e_1(k))(e_2(k))\dots(e_k(k))$ is the *production*; (b), (k), $(e_i(k))$, are called *labels* of Ω. The rule Ω can be represented by means of a *generating tree*, that is a rooted tree whose vertices are the labels of Ω; (b) is the label of the root and each node labelled (k) produces k sons labelled $(e_1(k)), \dots, (e_k(k))$ respectively. We refer to [4] for further details and examples. A succession rule Ω defines such a sequence of positive integers $\{f_n\}_{n\geq 0}$, that f_n is the number of the nodes belonging to the generating tree defined by Ω and lying at level n. By convention the root is at level 0, so $f_0 = 1$. The function $f_\Omega(x) = \sum_{n\geq 0} f_n x^n$ is the *generating function* derived from Ω.

The concept of succession rules was first introduced in [6] by Chung and al. to study reduced Baxter permutations; later, West applied succession rules to the enumeration of permutations with forbidden subsequences [11]. Moreover, they are a fundamental tool used by the ECO method [4], which is a general method for the enumeration of combinatorial objects consisting essentially in the recursive construction of a class of objects. A generating tree is then associated to a certain combinatorial class, according to some enumerative parameter, so that the number of nodes appearing on level n of the tree gives the number of n-sized objects in the class. In [1] the relationships between structural properties of the rules and the rationality, algebraicity or trascendance of the corresponding generating function are studied. We wish to point out that in the present paper we deal with "pure" succession rules [4], instead of generalizations [2], or specializations [7].

Two rules Ω_1 and Ω_2 are said to be *equivalent*, $\Omega_1 \cong \Omega_2$, if they define the same number sequence, that is $f_{\Omega_1}(x) = f_{\Omega_2}(x)$. For example, the following rules are equivalent and define Schröder numbers ([5]):

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$$\left\{\begin{array}{l}(2)\\(2k)\rightsquigarrow(2)(4)^2\dots(2k)^2(2k+2),\end{array}\right. \qquad \left\{\begin{array}{l}(2)\\(k)\rightsquigarrow(3)\dots(k)(k+1)^2,\end{array}\right.$$

where the power notation is used to express repetitions: $(h)^i$ stands for (h) repeated i times.

Starting from classical succession rules we define *coloured rules* in the following way: a rule Ω is coloured when there are at least two labels (k) and $(\bar{k})$ having the same value but different productions. For example, it is easily proved, that the sequence $1, 2, 3, 5, 9, 17, 33, \dots, 2^{n-1}+1$, having $f(x) = \frac{1-x-x^2}{1-3x+2x^2}$ as generating function, can only be described by means of coloured rules, such as:

$$\Omega' : \quad \left\{\begin{array}{l}(2)\\(1)\rightsquigarrow(\bar{2})\\(2)\rightsquigarrow(1)(2)\\(\bar{2})\rightsquigarrow(\bar{2})(\bar{2}).\end{array}\right.$$

A succession rule Ω is *finite* if it has a finite number of different labels. The number sequences $\{a_{n,k}\}_n$, defined by the recurrences:

$$\sum_{j=0}^{k}(-1)^j\binom{k}{j}a_{n-j,k} = 0 \qquad k \in \mathbb{N},$$

having $\frac{1}{(1-x)^k}$ as generating function, have finite succession rules:

$$\Omega(k) : \left\{\begin{array}{l}(k)\\(1)\rightsquigarrow(1)\\(2)\rightsquigarrow(1)(2)\\(3)\rightsquigarrow(1)(2)(3)\\\dots \quad \dots\\(k)\rightsquigarrow(1)(2)(3)\dots(k-1)(k).\end{array}\right.$$

Moreover, let $\{a_n\}_n$ be the sequence of integers satisfying the recurrence:

$$a_n = ka_{n-1} + ha_{n-2}, \qquad k \in \mathbb{N}^+, h \in \mathbb{Z},$$

subject to the initial conditions $a_0 = 1$, $a_1 = b \in \mathbb{N}^+$; every term of the sequence is a positive number, if $k + h > 0$. In this case, the sequence $\{a_n\}_n$ is defined by the finite succession rule:

$$\Omega_{\mathcal{F}^b_{k,h}} : \left\{\begin{array}{l}(b)\\(b)\rightsquigarrow(k)^{b-1}(k+h)\\(k)\rightsquigarrow(k)^{k-1}(k+h)\\(k+h)\rightsquigarrow(k)^{k+h-1}(k+h).\end{array}\right. \tag{1}$$

Finite succession rules play an important role in enumerative combinatorics, such as in the enumeration of restricted classes of combinatorial objects ([8]). Moreover, we can regard any finite succession rule Ω as a particular PDOL system

([9]), (Σ, P, w_0), where the alphabet Σ is the set of labels of Ω, P is the set of its productions and $w_0 \in \Sigma$. These remarks lead to the solution of two open problems for finite succession rules:

Equivalence. *Let Ω_1 and Ω_2 be two finite succession rules having h_1 and h_2 labels respectively, then $\Omega_1 \cong \Omega_2$, if and only if the first $h_1 + h_2$ terms of the two sequences defined by Ω_1 and Ω_2 coincide.*

For example, the number sequences defined by Ω' and by $\Omega_{\mathcal{F}^2_{1,1}}$ (the rule for Fibonacci numbers) coincide for the first 4 terms, but not for the fifth, so the rules are not equivalent.

Generating functions. *The function $f(x)$ is the generating function of a finite succession rule iff:*

1. $f(x) = \frac{P(x)}{Q(x)}$, with $P(x), Q(x) \in \mathbb{Z}[x]$, and $Q(0) = P(0) = 1$;
2. $\frac{1}{x}(f(x) - 1) - f(x)$ is $\mathbb{N}$-rational.

Roughtly speaking, $\mathbb{N}$-rational functions are the generating functions of regular languages, and their analytic characterization is given by Soittola's Theorem [9] (for further details see [3]). We are so ensured that each generating function of a finite succession rule is the generating function of a regular language, while the converse does not hold. For example, let $g(x) = \frac{1}{1-10x}$ and $h(x) = \frac{1-3x+36x^2}{(1-9x)(1+2x+81x^2)}$; $h(x)$ is a rational function having all positive coefficients (see [3] for the proof) but it is not $\mathbb{N}$-rational, since the poles of minimal modulus are complex numbers. Let

$$f(x) = g(x^2) + x[g(x^2) + h(x^2)] = k_1(x^2) + xk_2(x^2); \qquad (2)$$

$f(x)$ is $\mathbb{N}$-rational, since it is the merge of the two functions $k_1(x)$ and $k_2(x)$, each of them having a real positive dominating root, $x = 10$. This proves the existence of a regular language having $f(x)$ as its generating function. Moreover, it is clear that $f(x)$ defines a strictly increasing sequence of positive numbers. Neverthless $\frac{1}{x}(f(x) - 1) - f(x)$ is not $\mathbb{N}$-rational, since it is a merge of $g(x)$ and $h(x)$, and $h(x)$ is not $\mathbb{N}$-rational. Thus there are no finite succession rules having $f(x)$ as its generating function. The previous problems remain still open in the case of not finite succession rules.

2 Operations on succession rules

A n-ary operation $\circ$ on the set $\mathcal{S}$ of all succession rules is said to be *well-defined* if the equivalences $\Omega_1 \cong \Omega'_1, \ldots \Omega_n \cong \Omega'_n$ imply $\circ(\Omega_1, \ldots, \Omega_n) \cong \circ(\Omega'_1, \ldots, \Omega'_n)$. Our aim is to determine a set of well-defined operations on $\mathcal{S}$, in order to build an algebraic system on $\mathcal{S}$.

Let Ω and Ω' be two succession rules, defining the sequences $\{f_n\}_n$ and $\{g_n\}_n$, and having $f(x)$ and $g(x)$ as generating functions, respectively. In the sequel we deal with Ω and Ω' having the following general forms:

$$\Omega : \begin{cases} (a) \\ (h) \rightsquigarrow (e_1(h))(e_2(h))\dots(e_h(h)), \end{cases} \qquad \Omega' : \begin{cases} (b) \\ (k) \rightsquigarrow (c_1(k))(c_2(k))\dots(c_k(k)). \end{cases}$$

2.1 Sum of succession rules

Given two succession rules Ω and Ω', their *sum*, $\Omega \oplus \Omega'$, is the rule defining the sequence $\{h_n\}_n$ so that $h_0 = 1$ and $h_n = f_n + g_n$, if $n > 0$, and having $f(x) + g(x) - 1$ as generating function. Let $(a) \rightsquigarrow (A_1)\dots(A_a)$ and $(b) \rightsquigarrow (B_1)\dots(B_b)$ be the productions for the axiom (a) in Ω and (b) in Ω', then the following succession rule:

$$\Omega \oplus \Omega' : \qquad \begin{cases} (a+b) \\ (a+b) \rightsquigarrow (A_1)\dots(A_a)(\overline{B}_1)\dots(\overline{B}_b) \\ (h) \rightsquigarrow (e_1(h))(e_2(h))\dots(e_h(h)) \\ (\overline{k}) \rightsquigarrow (\overline{c_1(k)})(\overline{c_2(k)})\dots(\overline{c_k(k)}), \end{cases}$$

gives the sequence $\{h_n\}_n$.

2.2 Bisection of succession rules

Given a succession rule Ω, its *bisection*, denoted as $\frac{\Omega}{2}$, is the rule defining the sequence $\{f_{2n}\}_n$, and having $\frac{f(\sqrt{x})+f(-\sqrt{x})}{2}$ as generating function. Let $(a) \rightsquigarrow (A_1)\dots(A_a)$ be the production for the axiom (a), and $s = A_1 + \dots + A_a$, then (s) is the axiom for $\frac{\Omega}{2}$. Let $e_i(h) \rightsquigarrow e^i_1(h)\dots e^i_{e_i(h)}(h)$ be the production for $e_i(h)$, $i = 1, \dots, h$, and $g(h) = e_1(h) + \dots + e_h(h)$, then the rule for $\frac{\Omega}{2}$ is:

$$\begin{cases} (s) \\ (g(h)) \rightsquigarrow (g(e^1_1(h)))\dots(g(e^1_{e_1(h)}(h)))\dots(g(e^h_1(h)))\dots(g(e^h_{e_h(h)}(h))). \end{cases}$$

Example 2.1

i) Bisection of the rule for the Fibonacci numbers. By applying the previous definition we obtain:

$$\Omega_{\mathcal{F}} : \qquad \begin{cases} (2) \\ (1) \rightsquigarrow (2) \\ (2) \rightsquigarrow (1)(2), \end{cases} \qquad \frac{\Omega_{\mathcal{F}}}{2} : \qquad \begin{cases} (3) \\ (2) \rightsquigarrow (2)(3) \\ (3) \rightsquigarrow (2)(3)(3). \end{cases}$$

ii) Bisection of the rule for Catalan numbers. We start from the rule Ω_C,

$$\begin{cases} (2) \\ (h) \rightsquigarrow (2)(3)\dots(h)(h+1); \end{cases}$$

the axiom for $\frac{\Omega_C}{2}$ is (5); moreover, $g(h) = 2 + 3 + \dots h + h + 1 = \frac{h^2+3h}{2}$, so the rule for Catalan numbers of even index, $\left(1, 5, 42, 429, \dots, \frac{1}{2n+2}\binom{4n+2}{2n+1}\right)$ is:

$$\frac{\Omega_C}{2}: \begin{cases} (5) \\ \left(\frac{h^2+3h}{2}\right) \rightsquigarrow (5)^h(9)^h(14)^{h-1}(20)^{h-2}\dots\left(\frac{(h+1)^2+3(h+1)}{2}\right)^2\left(\frac{(h+2)^2+3(h+2)}{2}\right). \end{cases}$$

2.2.1 Product of succession rules

Given the rules Ω and Ω', their *product* $\Omega \otimes \Omega'$, is the succession rule defining the sequence $\{\sum_{k\leq n} f_{n-k}g_k\}_n$, and having $f(x)\cdot g(x)$ as generating function. Let $(b) \rightsquigarrow B_1 \dots B_b$ be the production of (b), then:

$$\Omega\otimes\Omega': \quad \begin{cases} (a+b) \\ (h+b) \rightsquigarrow (e_1(h)+b)(e_2(h)+b)\dots(e_h(h)+b)\overline{B}_1\dots\overline{B}_b \\ \overline{(k)} \rightsquigarrow \overline{c_1(k)}\dots\overline{c_k(k)}. \end{cases}$$

This statement is easily proved as follows: let $t(x) = f(x)\cdot g(x)$, and t_n the number of nodes at level n in the generating tree of $\Omega\otimes\Omega'$. The statement clearly holds for $n=0$, $t_0 = 1 = f_0g_0$, and for $n=1$, $t_1 = f_1g_0 + f_0g_1 = a+b$. Figure 1 shows the generating tree for the rule $\Omega\otimes\Omega'$; the number of nodes at level n of the tree can be considered as a sum of $n+1$ terms, which are as follows:
- $g_n = f_0g_n$; indeed this is the number of nodes at level n in Ω' generating tree, which is a proper subtree of $\Omega\otimes\Omega'$ having the axiom as its root.
- f_1g_{n-1}; indeed, by construction, the generating tree of $\Omega\otimes\Omega'$ has f_1 subtrees, isomorphic to Ω', the roots of which lie at level 1; each of them has g_{n-1} nodes at level n of $\Omega\otimes\Omega'$ generating tree. Generally, at level n of $\Omega\otimes\Omega'$ generating tree, there are f_i times the nodes at level $n-i$ of Ω' generating tree (that is g_{n-i} nodes), $0\leq i\leq n$.

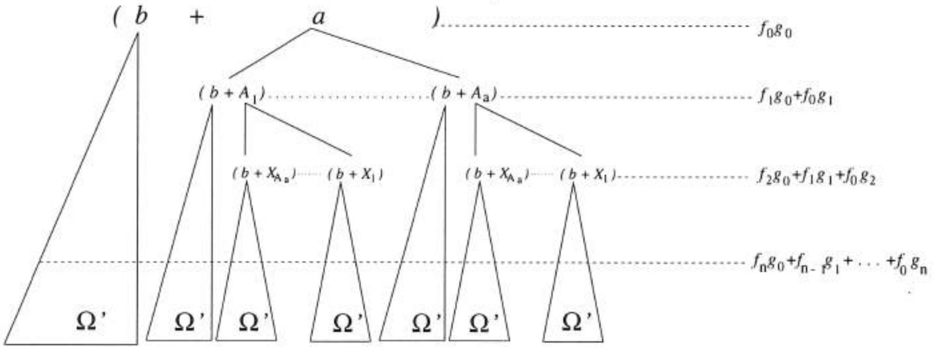

Figure 1: The generating tree for $\Omega\otimes\Omega'$.

Thus the total number of nodes at level n of $\Omega\otimes\Omega'$ generating tree is $t_n = f_0g_n + \dots + f_ng_0$. As the product is commutative, $\Omega\otimes\Omega'$ and $\Omega'\otimes\Omega$ are equivalent rules but of a different shape.

Example 2.2 i) *Product of Catalan and Fibonacci numbers.* The succession rule obtained by applying the operation $\otimes$ to the rules for Catalan and Fibonacci numbers is as follows:

$$\Omega_C \otimes \Omega_{\mathcal{F}} \qquad \begin{cases} (4) \\ (k+2) \rightsquigarrow (1)(2)(4)(5)...(k)(k+1) \\ (1) \rightsquigarrow (2) \\ (2) \rightsquigarrow (1)(2), \end{cases}$$

and it defines the number sequence $1, 4, 12, 35, 95, ...$

ii) *The rule for the square Catalan numbers.* The rule obtained is:

$$\Omega_C^2: \qquad \begin{cases} (4) \\ (k+2) \rightsquigarrow (\bar{2})(\bar{3})(4)(5)...(k)(k+1)(k+2)(k+3) \\ (\bar{k}) \rightsquigarrow (\bar{2})(\bar{3})...(\bar{k})(\overline{k+1}). \end{cases}$$

which can be easily simplified as the following nice rule:

$$\Omega_C^2: \qquad \begin{cases} (4) \\ (k) \rightsquigarrow (2)(3)\dots(k)(k+1). \end{cases}$$

2.3 The Star of a succession rule

The *star* of the succession rule Ω, denoted as Ω^*, is the rule defining the number sequence having $\frac{1}{1-f_0(x)} = 1 + f_0(x) + f_0^2(x) + \dots + f^n(x) + \dots = \sum_{n\geq 0} f_0^n(x)$ as generating function, where $f_0(x) = f(x) - 1$. Let (a) be the axiom and $(a) \rightsquigarrow A_1 \dots A_a$ the production of (a). The rule Ω^* is:

$$\Omega^*: \begin{cases} (a) \\ (a) \rightsquigarrow (A_1 + a)\dots(A_a + a) \\ (h+a) \rightsquigarrow (A_1 + a)\dots(A_a + a)(e_1(h) + a)(e_2(h) + a)\dots(e_h(h) + a). \end{cases}$$

The proof is similar to the one given for the product of two succession rules.

Example 2.3 *The star of Schröder numbers.* We start from the rule Ω_S:

$$\begin{cases} (2) \\ (2k) \rightsquigarrow (2)(4)^2\dots(2k)^2(2k+2), \text{and obtain } \Omega_S^*: \end{cases}$$

$$\begin{cases} (2) \\ (2) \rightsquigarrow (4)(6) \\ (2k+2) \rightsquigarrow (4)^2(6)^3\dots(2k+2)^2(2k+4). \end{cases}$$

2.4 An application of rule operations to enumerative combinatorics

A Grand Dyck path is a a sequence of rise and fall steps ($(1,1)$ and $(1,-1)$ respectively) in the plane $\mathbb{N} \times \mathbb{Z}$, running from $(0,0)$ to $(2n,0)$.
Let us determine a succession rule that enumerates Grand Dyck paths according to their semilenght, by applying some operations to succession rules. Grand Dyck paths are in bijection with Grand Dyck words, which are generated by the following non-ambiguous grammar:

$$\begin{cases} S \to aAbS|bBaS|\epsilon \\ A \to aAbA|\epsilon \\ B \to bBaB|\epsilon, \end{cases}$$

where a encodes a rise step, and b a fall step; its generating function is $f(x) = \frac{1}{\sqrt{1-4x}}$. Let $f_D(x)$ be the generating function for Dyck words, enumerated by Catalan numbers. We can write $f(x)$ as:

$$f(x) = \frac{1}{1 - [(f_D(x) - 1) + (f_D(x) - 1)]}.$$

Thus the rule Ω for Grand Dyck paths can be obtained as $\Omega = (\Omega_C \oplus \Omega_C)^*$, where Ω_C represents the rule for Catalan numbers:

$$\Omega_C : \quad \begin{cases} (1) \\ (h) \rightsquigarrow (2)(3)\dots(h)(h+1). \end{cases} \tag{3}$$

By applying sum and star operations, we obtain the following rule:

$$\Omega : \quad \begin{cases} (2) \\ (h) \rightsquigarrow (3)(3)(4)\dots(h)(h+1). \end{cases} \tag{4}$$

It should be noticed that this is the same rule found in [8] and which recursively constructs the class of Grand Dyck paths according to the ECO method.

2.5 Partial sum of a succession rule

Let Ω be a succession rule, defining the sequence $\{f_n\}_n$ and having $f(x)$ as generating function. The *partial sum* $\Sigma\Omega$, is the rule defining the sequence $\{F_n\}_n = \left\{\sum_{j\leq n} f_j\right\}_n$. We can obtain $\Sigma\Omega$ by means of the product operation, since $F(x) = \sum_n F_n x^n = \frac{1}{1-x} \cdot f(x)$. Thus:

$$\Sigma\Omega = \Omega_1 \otimes \Omega,$$

where $\Omega_1 \begin{cases} (1) \\ (1) \rightsquigarrow (1) \end{cases}$ is the natural rule for the sequence $f_n = 1, \forall n$. By applying the product operation we get:

$$\Sigma\Omega : \begin{cases} (a+1) \\ (1) \rightsquigarrow (1) \\ (h+1) \rightsquigarrow (1)(e_1(h)+1)(e_2(h)+1)\dots(e_h(h)+1). \end{cases}$$

For example, the rule Ω_C for Catalan numbers leads to the rule:

$$\Sigma\Omega_C : \begin{cases} (3) \\ (1) \rightsquigarrow (1)(h+1) \rightsquigarrow (1)(3)(4)\dots(h+1)(h+2), \end{cases}$$

giving the sequence $1, 3, 8, 22, 64, \dots$

3 Other operations

Let Ω and Ω' be succession rules, and as usual, $\{f_n\}_n$ and $\{g_n\}_n$ their sequences, with their respective generating functions $f(x)$ and $g(x)$. The *Hadamard product* of Ω and Ω', denoted as $\Omega \odot \Omega'$, is the rule defining the sequence $\{f_n g_n\}_n$. Generally, it is not so simple to determine the generating function $f(x) \odot g(x)$, but the Hadamard product of two $\mathbb{N}$-rational series was proved to be $\mathbb{N}$-rational ([9]).

We start by giving an example of how to construct the rule $\Omega \odot \Omega'$ in the case of finite rules. Let Ω be the rule for the Pell numbers, $\{1, 2, 5, 12, 29, \dots\}$, and Ω' the rule for the Fibonacci numbers having an odd index, $\{1, 2, 5, 13, 34, \dots\}$,

$$\Omega : \quad \begin{cases} (2) \\ (2) \rightsquigarrow (2)(3) \\ (3) \rightsquigarrow (2)(2)(3), \end{cases} \qquad \Omega' : \quad \begin{cases} (\bar{2}) \\ (\bar{2}) \rightsquigarrow (\bar{2})(\bar{3}) \\ (\bar{3}) \rightsquigarrow (\bar{2})(\bar{3})(\bar{3}). \end{cases}$$

For each label (h) of Ω and $(\bar{k})$ of Ω', $(h \cdot k)$ is a label of the rule $\Omega \odot \Omega'$, and it is coloured only if there is already another label having the same value; the axiom is $(a \cdot b)$, where (a) and (b) are the axioms of the rules; if the productions of (h) and $(\bar{k})$ are:

$$(h) \rightsquigarrow (c_1)\dots(c_h)$$

$$(\bar{k}) \rightsquigarrow \overline{(e_1)}\dots\overline{(e_k)},$$

then the production of $(h \cdot k)$ is:

$$(h \cdot k) \rightsquigarrow (c_1 \cdot e_1)\dots(c_1 \cdot e_k)\dots(c_h \cdot e_1)\dots(c_h \cdot e_k).$$

Going back to our example, the labels of $\Omega \odot \Omega'$ are $(2 \cdot \bar{2}) = (4)$, $(2 \cdot \bar{3}) = (6)$, $(3 \cdot \bar{2}) = (\bar{6})$, $(3 \cdot \bar{3}) = (9)$. For instance, the production for the label (4) is:

$$(4) = (2 \cdot \bar{2}) \rightsquigarrow (2 \cdot \bar{2})(2 \cdot \bar{3})(3 \cdot \bar{2})(3 \cdot \bar{3}) = (4)(6)(\bar{6})(9).$$

In the same way we obtain:

$$\Omega \odot \Omega' : \quad \begin{cases} (4) \\ (4) \rightsquigarrow (4)(6)(\bar{6})(9) \\ (6) \rightsquigarrow (4)(6)(6)(\bar{6})(9)(9) \\ (\bar{6}) \rightsquigarrow (4)(4)(6)(6)(\bar{6})(9) \\ (9) \rightsquigarrow (4)(4)(6)(6)(6)(6)(\bar{6})(9)(9). \end{cases}$$

The rule $\Omega \odot \Omega'$ has ij labels, being i and j the numbers of labels of Ω and Ω' respectively. The method we described above enables us to obtain the product $\Omega \odot \Omega'$ for finite rules, and it also proves that *the Hadamard product of two finite rules is a finite rule.* Some problems arise when attempting to find a general formula for the Hadamard product of two rules Ω and Ω' having each an infinite number of labels. Generally speaking, the best we can do is to write a finite rule Ω^k which *approximates* the rule $\Omega \odot \Omega'$ with the required precision, depending on the parameter $k \in \mathbb{N}^+$ (see [8]).

Neverthless there are some cases when the application of the previously defined operation on succession rules becomes particularly easy and helpful.

(1) Let $k \in \mathbb{N}$; the rule Ω_k is a rule for the sequence $\{F_n\}_n = \{k^n f_n\}_n$; since the generating function $F(x) = \sum_n F_n x^n = f(x) \odot \frac{1}{1-kx}$, we have:

$$\Omega_k = \Omega \odot \Omega_1,$$

where Ω_1 is the rule for $\{k^n\}_n$, that is:

$$\begin{cases} (k) \\ (k) \rightsquigarrow (k)^k; \end{cases}$$

by applying the operation $\odot$ we get:

$$\Omega_k : \quad \begin{cases} (ka) \\ (kh) \rightsquigarrow (e_1(h))^k (e_2(h))^k \dots (e_h(h))^k. \end{cases}$$

For example, let us take into consideration the rule Ω for Motzkin numbers $\{1, 1, 2, 4, 9, 21, \dots, M_n, \dots\}$:

$$\Omega : \quad \begin{cases} (1) \\ (h) \rightsquigarrow (1)(2)\dots(h-1)(h+1); \end{cases} \tag{5}$$

for $k = 2$ we get the rule Ω_2 giving the sequence $\{1, 2, 8, 32, 144, \dots, 2^n M_n, \dots\}$:

$$\Omega_2 : \quad \begin{cases} (2) \\ (2h) \rightsquigarrow (2)^2 (4)^2 \dots (2h-2)^2 (2h+2)^2. \end{cases}$$

(2) The rule $[n+1]\Omega$ defines the sequence $F_n = (n+1)f_n$, $n \in \mathbb{N}$. As the generating function is $F(x) = \sum_n F_n x^n = f(x) \cdot \frac{1}{(1-x)^2}$, we have:

$$[n+1]\Omega = \Omega \odot \Omega',$$

where Ω' is the rule for the sequence $n+1$, that is:

$$\Omega': \quad \begin{cases} (2) \\ (2) \rightsquigarrow (1)(2) \\ (1) \rightsquigarrow (1), \end{cases}$$

so we obtain:

$$[n+1]\Omega: \quad \begin{cases} (\overline{2a}) \\ (h) \rightsquigarrow (e_1(h))(e_2(h))\dots(e_h(h)) \\ (\overline{2h}) \rightsquigarrow (e_1(h))(e_2(h))\dots(e_h(h))(\overline{2e_1(h)})\dots(\overline{2e_h(h)}). \end{cases}$$

For example, let Ω be the rule for Pell numbers, then:

$$[n+1]\Omega: \quad \begin{cases} (4) \\ (2) \rightsquigarrow (2)(3) \\ (3) \rightsquigarrow (2)(2)(3) \\ (4) \rightsquigarrow (2)(3)(4)(6) \\ (6) \rightsquigarrow (2)(2)(3)(4)(4)(6). \end{cases}$$

Let G_n be the number of Grand Dyck paths having semilength n and C_n the nth Catalan number. As usual, let Ω_C be the rule (3) for Catalan numbers, and Ω_G the rule defining the sequence $\{G_n\}$. From the combinatorial identity $G_n = (n+1)C_n$, we have:

$$\Omega_G = [n+1]\Omega_C,$$

and thus

$$\Omega_G: \quad \begin{cases} (\overline{2}) \\ (h) \rightsquigarrow (2)(3)\dots(h)(h+1) \\ (\overline{2h}) \rightsquigarrow (2)(3)\dots(h)(h+1)(\overline{4})(\overline{6})\dots(\overline{2h})(\overline{2h+2}), \end{cases}$$

is a rule counting $\{G_n\}$, equivalent to (4).

Moreover it is easy to prove the following property.

Proposition 3.1 *Let Ω be a rule defining the sequence $\{f_n\}_n$. Then a rule Ω' defining a sequence $\{g_n\}_n$, such that $f_n = g_n - rg_{n-1}$, for $n > 1$, exists:*

$$\Omega': \quad \begin{cases} (a+r) \\ (r) \rightsquigarrow (r)^r \\ (h+r) \rightsquigarrow (r)^r(e_1(h)+r)(e_2(h)+r)\dots(e_h(h)+r). \end{cases}$$

3.1 Open problems

There are several open problems related to the definition of an algebra on succession rules and arising from the operations we have introduced. Below the most interesting problems are mentioned:

- *Equivalence.* Is there a criterion allowing us to establish whether two given succession rules are equivalent simply by working on their labels, that is, with no need to determine the corresponding generating functions?

- *Subtraction.* Given two rules Ω and Ω', defining the sequences $\{f_n\}$ and $\{g_n\}$ respectively, such that $f_n > g_n$ for each $n > 0$, let $\Omega \ominus \Omega'$ be the rule defining the sequence $\{h_n\}_n$ such that $h_n = \begin{cases} 1 & \text{if } n = 0 \\ f_n - g_n & \text{otherwise.} \end{cases}$

  The construction of the rule $\Omega_1 \ominus \Omega_2$ constitutes an open problem.

- *Inversion.* Let $\{f_n\}_n$ be a non decreasing sequence of positive integers. Is there a method allowing us to decide whether a succession rule defining the sequence $\{f_n\}_n$ exists and, in this case, to find it? We remark that this problem can be solved for finite rules.

3.2 A Conjecture

Conjecture: *if a succession rule has a rational generating function, then it is equivalent to a finite succession rule.* It is sufficient to prove that each rational generating function of a succession rule satisfies the properties of the generating functions of finite rules established in Section 1. If the conjecture proves true, rational functions such as (2) cannot be the generating functions of any succession rule. For example, let Ω be the rule, studied in [1], whose set of labels is the whole set of prime numbers:

$$\Omega : \quad \begin{cases} (2) \\ (p_n) \rightsquigarrow (p_{n+1})(q_n)(r_n)(2)^{p_n-3}, \end{cases}$$

where p_n denotes the nth prime number, and q_n and r_n are two primes such that $2p_n - p_{n+1} + 3 = q_n + r_n$; its generating function is rational, $f(x) = \frac{1-2x}{1-4x+3x^2}$, thus, according to our conjecture, a finite succession rule Ω' equivalent to Ω can be found:

$$\Omega' : \quad \begin{cases} (2) \\ (2) \rightsquigarrow (2)(3) \\ (3) \rightsquigarrow (2)(3)(4) \\ (4) \rightsquigarrow (2)(3)(4)(4). \end{cases}$$

It should be noticed that the rule Ω' was further exploited in [8], being the 4-approximating rule for Catalan numbers, and it describes a recursive construction for Dyck paths whose maximal ordinate is 4.

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