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On the exact controllability of a Galerkin scheme for 3D viscoelastic fluids with fractional Laplacian viscosity and anisotropic filtering

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We study a mathematical model describing 3D viscoelastic fluids with memory, fractional viscosity, and regularized by means of a horizontal anisotropic filter. This regularization is obtained through the action of the inverse of the horizontal Helmholtz operator, and the system is considered in a fully periodic space-domain Ω. After introducing a controlled version of such a model, we take into account for it a suitable Galerkin approximation scheme. Exploiting the Hilbert uniqueness method, we establish the exact controllability of the finite-dimensional Galerkin system.

1 | INTRODUCTION

The mathematical theory of the three-dimensional viscoelastic fluid flow equations, arising from the Kelvin–Voigt model for the non-Newtonian fluids, has been analyzed in many papers in the last decades (see, e.g., Refs. [38–45]). Here, in principle, we are interested in the particular case of Kelvin–Voigt fluids of order $L = 1$ (see, e.g., Mohan [38]) for which we have the following law connecting the stress tensor σ and the strain tensor D, that is,

$$\sigma(x,t) = \mu_1 \frac{\partial \mathrm{D}(x,t)}{\partial t} + \mu_0 \mathrm{D}(x,t) + \beta_1 \int_0^t e^{\alpha_1(t-s)} \mathrm{D}(x,s)\mathrm{d}s, \tag{1.1}$$

where α_1, β_1, μ_0, and μ_1 are suitable positive constants, with $x \in \Omega$ and $t > 0$.

Then, setting $\alpha_1 = \delta$ and $\beta_1 = \gamma$, and inserting the above relation into the equations describing the motion of a continuous incompressible medium in Cauchy form, that is,

$$\begin{aligned} &\partial_t \mathbf{u} + \nabla \cdot (\mathbf{u} \otimes \mathbf{u}) + \nabla p = \nabla \cdot \sigma + \mathbf{f}, \\ &\nabla \cdot \mathbf{u} = 0, \end{aligned}$$

we obtain the following system:

$$\partial_t \mathbf{u} - \mu_1 \Delta \partial_t \mathbf{u} + \nabla \cdot (\mathbf{u} \otimes \mathbf{u}) - \mu_0 \Delta \mathbf{u} - \gamma \int_0^t e^{\delta(t-s)} \Delta \mathbf{u}(s)\mathrm{d}s + \nabla p = \mathbf{f}, \tag{1.2}$$

$$\nabla \cdot \mathbf{u} = 0. \tag{1.3}$$

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The above equations are considered on the periodic space domain

$$\Omega = \left\{ \mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 \ : \ -\pi L < x_1, x_2, x_3 < \pi L \right\},$$

with $L > 0$, and $2\pi L$ periodicity with respect to $\mathbf{x} := (x_1, x_2, x_3)$, that is, we are actually taking into account Equations (1.2)–(1.3) on the 3D torus $\mathbb{T}^3 = \Omega/\partial\Omega$.

We now set $\mu_1 = 0$ in Equation (1.2), to re-introduce later a term with the same regularity but only involving derivatives about ∂_t, $\partial^2_{x_1}$, and $\partial^2_{x_2}$ as a consequence of the use of an anisotropic horizontal filtering regularization (see Equations (1.6) and (1.7) below).

Before introducing the smoothing procedure that will lead us to the main model analyzed in this paper, starting from Equations (1.2) and (1.3) (with $\mu_1 = 0$), we consider a weaker dissipative configuration. We assume the presence, in Equation (1.2), of the fractional-order dissipative term $\mu_0 \Lambda^{2\beta} \mathbf{u}$, $\Lambda = (-\Delta)^{1/2}$, $3/4 \le \beta < 1$ (this range of values for β is used for proving existence and regularity), in place of $-\mu_0 \Delta \mathbf{u}$. The fractional dissipation can model a number of physical phenomena in hydrodynamics, and it appears in describing geophysical fluid flows (see, e.g., Droniou and Imbert [26]), see also Refs. [1, 13, 48, 56, 57] for the case of Boussinesq equations) and, in particular, can be related to the description of anomalous diffusion in viscoelastic polymers [30, 54]; the anomalous diffusion typically occurs when the rates of diffusion and viscoelastic relaxation are comparable (see Hajikarimi and Nejad [30]).

When $\beta \ge 1$, the system becomes more regular and the results that we provide can be achieved even more easily, so we restrict ourselves to the case $3/4 \le \beta < 1$.

As already observed, the presence of fractional diffusion is clearly a situation of reduced regularity, compared to that involving the standard Laplacian. Here, also for the memory, we consider a less regular scheme by substituting $-\gamma \Delta \mathbf{u}$ with the nondifferential term $\gamma \mathbf{u}$, to get

$$\partial_t \mathbf{u} + \nabla \cdot (\mathbf{u} \otimes \mathbf{u}) + \mu_0 \Lambda^{2\beta} \mathbf{u} + \gamma \int_0^t e^{\delta(t-s)} \mathbf{u}(s) \mathrm{d}s + \nabla p = \mathbf{f}, \tag{1.4}$$

$$\nabla \cdot \mathbf{u} = 0. \tag{1.5}$$

This is in line with the derivation of the models considered in Refs. [3, 23]; actually we turn back to the case of a differential memory term after the action of the anisotropic Helmholtz's filter (see Equations (1.9) and (1.10) below).

We highlight that controllability results for linearized versions of Equations (1.4) and (1.5) have been obtained, taking $\beta = 1$ and considering a suitable range of values for μ_0, in the case of viscoelastic fluids of the Maxwell and Jeffreys types (see, e.g., Doubova et al. [24]); indeed the purely Jeffreys system can be seen as a modified Stokes equation with memory (see, e.g., Refs. [31, 50, 51]). Linearizations emerge quite naturally as approximations for the description of viscoelastic fluids behavior, and related models have been analyzed in a number of papers (see, e.g., Refs. [21, 22, 46, 47, 52]) where, depending on the situation, the authors study and deduce various types (null, approximate, exact) of controllability results.

Let us now introduce the horizontal differential filtering procedure that, although coming from situations (and models) typical of turbulence theory, finds application in the present context. Our interest here is mainly of a mathematical nature; however, the choice of an anisotropic filter could be combined with the presence of privileged directions in the considered flow, in the case, for instance, of horizontal stratifications [8] (in this case, anisotropic spatial filtering is used to study turbulent mixing in stratified flows) or in presence of a flow in an unlimited horizontal channel, with a bounded third direction, that is, a strip-like region (see Bisconti and Catania [16]). The anisotropic filter acts in correspondence with the horizontal layer, providing a certain regularization, which then actually affects the entire fluid flow as a consequence of the divergence-free condition $\nabla \cdot \mathbf{u} = 0$.

Consider

$$\begin{gathered} \mathbf{x} = (x_1, x_2, x_3), \qquad \mathbf{x}_h = (x_1, x_2), \\ \partial_j = \partial_{x_j}, \qquad \Delta_h = \partial_1^2 + \partial_2^2, \qquad \nabla_h = (\partial_1, \partial_2), \end{gathered} \tag{1.6}$$

where "h" stays for "horizontal," and we take into account the anisotropic horizontal filter given by the inverse of the following horizontal Helmholtz operator (see e.g., Berselli [9]), that is,

$$A_h = I - \alpha^2 \Delta_h \qquad \text{with } \alpha > 0. \tag{1.7}$$

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From the point of view of the numerical simulations (see, e.g., Refs. [2, 29, 33]), this filter is less memory consuming with respect to the standard isotropic one. Among the others, the large eddy simulation (LES) community has manifested interest in models involving such a kind of filtering procedure (see, e.g., Refs. [2, 7, 12, 14–18, 55]) and, in particular, the connection between anisotropic α-models and turbulence has been investigated by Berselli in Ref. [9] for the 3D Navier–Stokes equations, even if, in fact, the first use of anisotropic filters in turbulence dates back to the approach of Germano [29] (see also Refs. [10, 11] for the case corresponding to the 3D Boussinesq equations with anisotropic filters for turbulent flows). See also Refs. [20, 59] for the use of Helmholtz's filter in contexts not strictly related to turbulence.

Set $\mathbf{w} = \overline{\mathbf{u}}^h = A_h^{-1}\mathbf{u}$ and $q = \overline{p}^h = A_h^{-1}p$, so that $\mathbf{u} = A_h\mathbf{w}$. Then, applying the horizontal filter "$\overline{\,\cdot\,}^h$" component by component to the various fields and tensor fields in Equations (1.4) and (1.5), and solving the interior closure problem by the approximation

$$\overline{\mathbf{u}\otimes\mathbf{u}}^h \approx \overline{\overline{\mathbf{u}}^h\otimes\overline{\mathbf{u}}^h}^h = \overline{\mathbf{w}\otimes\mathbf{w}}^h, \tag{1.8}$$

we finally get the regularized model

$$\partial_t\mathbf{w} + \nabla\cdot\overline{\mathbf{w}\otimes\mathbf{w}}^h + \mu_0\Lambda^{2\beta}\mathbf{w} + \int_0^t \varpi(t-s)\mathbf{w}(s)\mathrm{d}s + \nabla q = \overline{\mathbf{f}}^h, \tag{1.9}$$

$$\nabla\cdot\mathbf{w} = 0, \tag{1.10}$$

where $\varpi(\tau) = \gamma e^{-\delta\tau}$ and we impose a suitable initial condition $\mathbf{w}|_{t=0} = \mathbf{w}_0$.

In order to get the considered model, we apply the operator $A_h = I - \alpha^2\Delta_h$, term by term, to Equations (1.9) and (1.10) and set $\nu = \mu_0$ to reach

$$\partial_t A_h\mathbf{w} + (\mathbf{w}\cdot\nabla)\mathbf{w} + \nu\Lambda^{2\beta}A_h\mathbf{w} + \int_0^t \varpi(t-s)A_h\mathbf{w}(s)\mathrm{d}s + \nabla p = \mathbf{f}, \tag{1.11}$$

$$\nabla\cdot\mathbf{w} = 0, \tag{1.12}$$

with $\mathbf{w}|_{t=0} = \mathbf{w}_0$.

Existence and uniqueness of strong solutions for the model above can be established by using the same methods developed in Annese et al. [3], Sections 2 and 3] with minor modifications, regarding the treatment of the memory term, based on what has been done in Mohan [38].

Remark 1.1. A natural domain for the horizontal filter A_h^{-1} would be

$$\Omega_h = \left\{\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 \ : \ -\pi L < x_1, x_2 < \pi L, -d < x_3 < d\right\},$$

$L > 0$, $d > 0$, with $2\pi L$ periodicity with respect to $\mathbf{x}_h := (x_1, x_2)$, and homogeneous Dirichlet boundary conditions on

$$\Gamma = \left\{\mathbf{x} \in \mathbb{R}^3 \ : \ -\pi L < x_1, x_2 < \pi L,\ x_3 = \pm d\right\}.$$

This choice, although highlights the privileged directions associated with the filter, makes problematic to properly define the fractional-order operator $\Lambda^{2\beta}$ on Ω_h. A possible alternative approach, which is well suited to the choice of this domain, with homogeneous boundary conditions, is the one about the use of the fractional anisotropic filter $A_h^\beta = I + \alpha^{2\beta}{\Lambda_h}^{2\beta}$, $\Lambda_h = (-\Delta_h)^{1/2}$, with $3/4 \le \beta < 1$ (see, e.g., Berselli [9]), coupled with the standard dissipative term $-\nu\Delta\mathbf{w}$, with $\nu = \mu_0$, instead of $\nu\Lambda^{2\beta}\mathbf{w}$, in Equation (1.9). Proceeding as before, this yields the equations

$$\partial_t\mathbf{w} + \nabla\cdot\overline{\mathbf{w}\otimes\mathbf{w}}^h - \nu\Delta\mathbf{w} + \int_0^t \varpi(t-s)\mathbf{w}(s)\mathrm{d}s + \nabla q = \overline{\mathbf{f}}^h, \tag{1.13}$$

$$\nabla\cdot\mathbf{w} = 0, \tag{1.14}$$

where now $\overline{(\,\cdot\,)}^h = (A_h^\beta)^{-1}(\,\cdot\,)$.

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It is possible to verify that the results that we obtain in this paper can be achieved, following the same approach, also for the above system. In order to give a more precise idea about this point, apply A_h^β term by term to Equation (1.13), to get

$$\partial_t A_h^\beta \mathbf{w} + (\mathbf{w}\cdot\nabla)\mathbf{w} + \nu\Delta A_h^\beta \mathbf{w} + \int_0^t \varpi(t-s)A_h^\beta \mathbf{w}(s)\mathrm{d}s + \nabla p = \mathbf{f}, \tag{1.15}$$

$$\nabla\cdot\mathbf{w} = 0. \tag{1.16}$$

Assuming to have at disposal sufficient regularity to test and manipulate Equation (1.15) against $\mathbf{w}$ in $L^2(\Omega_h)$, after an integration by parts we reach

$$\frac{1}{2}\frac{d}{dt}\left(\|\mathbf{w}\|_{L^2}^2 + \alpha^{2\beta}\|\Lambda_h^\beta \mathbf{w}\|_{L^2}^2\right) + \nu\left(\|\Lambda_h^\beta \mathbf{w}\|_{L^2}^2 + \alpha^{2\beta}\|\Lambda_h^\beta \nabla\mathbf{w}\|_{L^2}^2\right)$$
$$+\int_0^t \varpi(t-s)(\mathbf{w}(s),\mathbf{w}(t))_{L^2}\mathrm{d}s + \alpha^{2\beta}\int_0^t \varpi(t-s)(\Lambda_h^\beta \mathbf{w}(s),\Lambda_h^\beta \mathbf{w}(t))_{L^2}\mathrm{d}s = (\mathbf{f},\mathbf{w})_{L^2},$$

with $\|\cdot\|_{L^2}$ and $(\cdot,\cdot)_{L^2}$, respectively, the norm and the scalar product in $L^2(\Omega_h)$. The presence of the term $\nu\alpha^{2\beta}\|\Lambda_h^\beta\nabla\mathbf{w}\|_{L^2}^2$ in the left-hand side of the above relation (which is better than $\nu\alpha^2\|\Lambda^\beta\nabla_h\mathbf{w}\|_{L^2}^2$, obtained after applying A_h to Equation (1.11) and performing the L^2-test), allows us to recover existence, uniqueness, well-posedness, and regularity results adapting, to this case, the calculations in Section 3.1 (see also Annese et al. [3, § 3–to–5]) performed in the fully periodic setting. Thus, from the mathematical point of view, we consider more interesting to deal with the case presenting the weaker dissipative term, that is, $\nu\Lambda^{2\beta}\mathbf{w}$ in Equation (1.11) and, also for this reason, in what follows, we will consider the Equations (1.11) and (1.12) in the fully periodic context.

Because of the nature of the considered system, as in the cases treated in Refs. [4, 37, 38], it is not evident how to achieve the exact controllability for the Equations (1.11) and (1.12) with arbitrary target functions (see also Bisconti and Mariano [19] for further comments). In this paper, we restrict ourselves to analyzing the exact controllability of a suitable Galerkin's approximation scheme $\{\mathbf{w}^n\}$ for Equations (1.11) and (1.12) on $\mathbb{T}^3$, and for $t>0$, that is

$$\left((I-\alpha^2\Delta_h)\partial_t\mathbf{w}^n,\mathbf{e}\right) + ((\mathbf{w}^n\cdot\nabla)\mathbf{w}^n,\mathbf{e}) + \nu\left(\Lambda^\beta A_h^{1/2}\mathbf{w}^n,\Lambda^\beta A_h^{1/2}\mathbf{e}\right)$$
$$+\left(\int_0^t \varpi(t-s)A_h\mathbf{w}^n(s)\mathrm{d}s,\mathbf{e}\right) = (U\chi_{\mathcal{O}},\mathbf{e}),\quad \forall\mathbf{e}\in E, \tag{1.17}$$

$$\mathbf{w}^n(0)=\mathbf{w}_0^n=\mathbf{P}(\mathbf{w}_0),\quad n\le \dim E = d_N,$$

where $A_h^{1/2}=(I-\alpha^2\Delta_h)^{1/2}$, $E=\mathrm{span}\{\mathbf{e}_1,\dots,\mathbf{e}_{d_N}\}$ is a suitable finite-dimensional space, and $\mathbf{w}^n(x,t)=\sum_{j=1}^{d_N}\varrho_{n,j}(t)\mathbf{e}_j(x)$, with $\varrho_{n,j}(t)$ coefficient functions (see, e.g., Refs. [4, 19, 38]). Also, $\mathbf{P}(\mathbf{w}_0)=\sum_{j=1}^{d_N}(\mathbf{w}_0,\mathbf{e}_j)\mathbf{e}_j$. The function $U\chi_{\mathcal{O}}$ is given as follows: $\mathcal{O}$ is the control domain (the domain where the control is acting upon), U is the distributed control function acting on the system, and $\chi_{\mathcal{O}}$ denotes the characteristic function of the control domain $\mathcal{O}$.

In the main result of this work, that is, Theorem 4.1, we prove the exact controllability of the above Galerkin system.

There is a large literature about control results (also of the type considered here) for many different fluid models. In addition to the cases already mentioned above, let us recall, for instance, the problem of optimal control for certain families of polymer solutions that has been considered in Refs. [58, 59], while in Doubova and Fernández-Cara [25], viscoelastic Jeffreys (Oldroyd) fluids are handled and controllability results are obtained. In Refs. [35, 37], the authors proved that the considered Galerkin's approximations, for the Navier–Stokes system, are exactly controllable. Also, in Araruna et al. [4], the authors consider the problem of the exact controllability of Galerkin's approximations of micropolar fluids. A similar analysis, in the case of polymer fluids, is addressed in Bisconti and Mariano [19]. The recent paper [32] addresses the problem of controllability of an ideal MHD system with a control prescribed on the boundary.

In Araruna et al. [5], a Leray-α turbulence model is considered and it is proved, among other things, that the equations are locally null controllable with controls independent of α, so that, if the initial data are sufficiently small, the controls converge as $\alpha\to 0^+$ to a null control of the Navier–Stokes equations. In this paper, we do not address such an issue, and we postpone a similar study to a future work.

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As a possible further point of interest, we emphasize that the analysis provided here does not seem suitable for addressing the problem of taking the limit for $d_N \to +\infty$ in Equation (1.17) (see Remark 4.2 for further details), essentially because the estimates used to obtain the exact controllability (at the finite-dimensional level d_N) are not preserved when d_N becomes arbitrarily large, and, as a consequence, they cannot be used directly for the original infinite-dimensional system. This appears to be standard for similar problems (see also Lions and Zuazua [37]).

We now give, in the following sections, a number of preliminaries and the results obtained in this paper.

2 | SOME BASIC FACTS

2.1 | Function spaces

For the sake of simplicity, in the sequel, we set $\Omega = \mathbb{T}^3$. We introduce the following function spaces:

$$L^2(\Omega) = \left\{ \phi : \Omega \to \mathbb{R} \text{ measurable, } 2\pi L \text{ periodic in } \mathbf{x},\ \int_\Omega |\phi|^2 d\mathbf{x} < \infty \right\}$$

$$L_0^2(\Omega) = \left\{ \phi \in L^2(\Omega) \text{ with zero mean with respect to } \mathbf{x} \right\},$$

$$H = \left\{ \mathbf{v} \in (L_0^2(\Omega))^3 \ :\ \nabla \cdot \mathbf{v} = 0 \text{ in } \Omega \right\},$$

all with L^2 norm denoted by $\|\cdot\|$, and scalar product $(\cdot\,,\cdot)$ in L^2. Also, we set

$$V = \left\{ \mathbf{v} \in H \ :\ \nabla \mathbf{v} \in (L^2(\Omega))^9 \right\},$$

$$V_h = \left\{ \mathbf{v} \in H \ :\ \nabla_h \mathbf{v} \in (L^2(\Omega))^6 \right\}.$$

The space V_h is endowed with the inner product

$$\langle \mathbf{u}, \mathbf{v} \rangle_{V_h} = (\mathbf{u}, \mathbf{v}) + \alpha^2 (\nabla_h \mathbf{u}, \nabla_h \mathbf{v}),$$

and $\|\mathbf{u}\|_{V_h}^2 = \|\mathbf{u}\|^2 + \alpha^2 \|\nabla_h \mathbf{u}\|^2$.

Further, for any $0 < \beta < 1$, we define

$$H^\beta = \left\{ \mathbf{v} \in H \ :\ \Lambda^\beta \mathbf{v} \in L^2(\Omega)^3 \right\},$$

$$H_h^{1+\beta} = \left\{ \mathbf{v} \in V_h \ :\ \Lambda^\beta \mathbf{v} \in V_h \right\},$$

with norms, respectively,

$$\|\mathbf{v}\|_{H^\beta}^2 = \|\mathbf{v}\|^2 + \|\Lambda^\beta \mathbf{v}\|^2,$$

$$\|\mathbf{v}\|_{H_h^{1+\beta}}^2 = \|\mathbf{v}\|_{V_h}^2 + \|\Lambda^\beta \mathbf{v}\|_{V_h}^2.$$

Let us consider the Sobolev spaces $W^{s,2}(\Omega)$, $s > 0$. Also, let us recall the homogeneous Sobolev spaces $\dot{H}^s := \dot{H}^s(\Omega)$ defined for $s \geq 0$ (see, e.g., Refs. [27, 49]), as

$$\dot{H}^s = (W^{s,2}(\Omega))^3 \cap H,$$

that, in our case, coincide with the zero-mean spaces H^s introduced above. Actually, for the elements in $\dot{H}^s$, we drop the L^2 part of the H^s-norm: In fact, making use of the Fourier series expansion, for any $u \in \dot{H}^s$, the well-defined quantity

$$\|u\|_{\dot{H}^s}^2 = \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} |k|^{2s} |\hat{u}_k|^2, \quad (\text{here } \hat{u}_{-k} = \bar{\hat{u}}_k, \text{ and } k \cdot \hat{u}_k = 0, \forall k \in \mathbb{Z}^3)$$

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introduces a norm on $\dot{H}^s$. Here, $\bar{\hat{u}}_k$ denotes the complex conjugate of the Fourier coefficient $\hat{u}_k$. Observe that the above expression does not define a norm on $(W^{s,2}(\Omega))^3$ since it is equal to zero for the constant vector fields. We denote the dual of $\dot{H}^s$ by $H^{-s} := H^{-s}(\Omega)$, where $s > 0$ can be any positive number.

In the sequel, in order to keep the notation compact, we omit the superscript indicating the dimension of the considered spaces, reintroducing it only when it is strictly required by the context.

Remark 2.1. For $\mathbf{v} \in V_h$, we have that

$$\|A_h^{1/2}\mathbf{v}\|^2 = \|(I - \alpha^2\Delta_h)^{1/2}\mathbf{v}\|^2 = \big((I - \alpha^2\Delta_h)\mathbf{v}, \mathbf{v}\big) = \|\mathbf{v}\|^2 + \alpha^2\|\nabla_h\mathbf{v}\|^2. \tag{2.1}$$

Similarly, for $\mathbf{w} \in H_h^{1+\beta}$, it holds true that

$$\|\Lambda^\beta A_h^{1/2}\mathbf{w}\|^2 = \|\Lambda^\beta\mathbf{w}\|^2 + \alpha^2\|\Lambda^\beta\nabla_h\mathbf{w}\|^2. \tag{2.2}$$

In the following, we indicate the partial derivatives in time, with $\mathbf{v}_t = \partial_t\mathbf{v}$. Also, in what follows, we always use the compact notation A_h (or $A_h^{1/2}$) to indicate $I - \alpha^2\Delta_h$ (respectively $(I - \alpha^2\Delta_h)^{1/2}$) in the considered formulas, but not when this operator acts on $\mathbf{w}_t$, and in this case, we leave $I - \alpha^2\Delta_h$ written explicitly.

We denote by c or $\bar{C}$ generic positive constants, which may change from line to line; explicit dependencies will be subscribed to the constant or placed in parentheses.

2.2 | Properties of the kernel

Let us now introduce

$$(L\mathbf{w})(t) := (\varpi * \mathbf{w})(t) = \int_0^t \varpi(t-s)\mathbf{w}(s)\mathrm{d}s. \tag{2.3}$$

A function $\varpi(\cdot)$ is called positive kernel if the operator L is positive on $L^2(0,T;H)$ for all $T > 0$. That is, we have

$$\int_0^T (L\mathbf{w}(t), \mathbf{w}(t))\mathrm{d}t = \int_0^T\int_0^t \varpi(t-s)(\mathbf{w}(s), \mathbf{w}(t))\mathrm{d}s\mathrm{d}t \geq 0 \tag{2.4}$$

for all $\mathbf{w} \in H$ and every $T > 0$.

We have the following results on positive kernels.

Lemma 2.1 Lemma 4.1, [6]. *Let $\varpi \in L^\infty(0,\infty)$, and let $\widehat{\varpi}(\theta)$ be the* Laplace transform *of $\varpi(t)$ such that*

$$\operatorname{Re}\widehat{\varpi}(\theta) > 0 \;\; \textit{for} \;\; \operatorname{Re}\theta > 0.$$

Then, $\varpi(t)$ defines a positive kernel.

Also, $\varpi(\cdot)$ is said to be a strongly positive kernel if there exist constants $\varepsilon > 0$ and $\alpha > 0$ such that $\varpi(t) - \varepsilon e^{-\alpha t}$ is a positive kernel, that is,

$$\int_0^T\int_0^t \varpi(t-s)(\mathbf{w}(s), \mathbf{w}(t))\mathrm{d}s\mathrm{d}t \geq \varepsilon\int_0^T\int_0^t e^{-\alpha(t-s)}(\mathbf{w}(s), \mathbf{w}(t))\mathrm{d}s\mathrm{d}t \geq 0, \tag{2.5}$$

for all $T > 0$ and for all $\mathbf{w} \in L^2(0,T;H)$.

Lemma 2.2 Proposition 4.1, [6]. *Let $\varpi(t)$ satisfies the following conditions:*

(i) $\varpi \in C[0,\infty) \cap C^2(0,\infty)$,
(ii) $(-1)^k \dfrac{d^k}{dt^k}\varpi(t) \geq 0$ *for* $t > 0$, $k = 0, 1, 2,$

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(iii) $\varpi(t) \neq const.$

Then $\varpi(t)$ is a strongly positive kernel.

Lemma 2.3 Lemma 2.6, [38]. *Let $\varpi \in L^1(0,T)$, $f, g \in L^2(0,T)$ for some $T > 0$. Then, we have*

$$\left[\int_0^T g^2(t)\left(\int_0^t \varpi(t-s)f(s)\mathrm{d}s\right)^2 \mathrm{d}t\right]^{1/2} \leq \left(\int_0^T |\varpi(t)|\mathrm{d}t\right)\left(\int_0^T g^2(t)f^2(t)\mathrm{d}t\right)^{1/2}.$$

Remark 2.2. The following facts hold true:

(I) If we take $g(t) = 1$ and $f = \|\mathbf{w}\|$ with $\mathbf{w} \in L^2(0,T;H)$ in the above formula, we obtain

$$\left[\int_0^T \left(\int_0^t \varpi(t-s)\|\mathbf{w}(s)\|\mathrm{d}s\right)^2 \mathrm{d}t\right]^{1/2} \leq \left(\int_0^T |\varpi(t)|\mathrm{d}t\right)\left(\int_0^T \|\mathbf{w}(t)\|^2\mathrm{d}t\right)^{1/2}. \tag{2.6}$$

(II) Moreover, for $\varpi(t) = \gamma e^{-\delta t}$, we know that

$$\int_0^\infty \varpi(t)\mathrm{d}t = \frac{\gamma}{\delta} \quad \text{and} \quad \hat{\varpi}(\theta) = \frac{\gamma}{\theta+\delta} \quad \text{for} \quad Re\,\theta > 0. \tag{2.7}$$

and so, by Lemma 2.1, $\varpi(t)$ is a positive kernel.

(III) As a consequence of Lemma 2.2, we also get that $\varpi(t)$, introduced in *(II)*, is a strongly positive kernel.

(IV) Assume that $\varpi(t)$ is a strongly positive kernel, in particular, of the type in *(II)*. Using an integration by parts, we have

$$-\int_0^T \langle(\varpi * (\Delta\mathbf{w})(t)), \mathbf{w}(t)\rangle\mathrm{d}t = \int_0^T ((\varpi * (\nabla\mathbf{w})(t)), \nabla\mathbf{w}(t))\mathrm{d}t \geq 0, \tag{2.8}$$

and also

$$-\int_0^T \langle(\varpi * (\Delta_h\mathbf{w})(t)), \mathbf{w}(t)\rangle\mathrm{d}t = \int_0^T ((\varpi * (\nabla_h\mathbf{w})(t)), \nabla_h\mathbf{w}(t))\mathrm{d}t \geq 0. \tag{2.9}$$

3 | WEAK SOLUTIONS

We have the following definition:

Definition 3.1. Let $T > 0$. A map $\mathbf{w} \in C([0,T];V_h) \cap L^2(0,T;H_h^{1+\beta})$ with $A_h\partial_t\mathbf{w} \in L^2(0,T;H^{-\beta})$, is a weak solution to the system (1.11)–(1.12), if for $\mathbf{f} \in L^2(0,T;H^{-\beta})$, and $\mathbf{w}_0 \in V_h$, then $\mathbf{w}$ satisfies the following weak formulation for all $\mathbf{v} \in H_h^{1+\beta}$, that is,

$$\Big(\big((I-\alpha^2\Delta_h)\mathbf{w}_t(t)\big) + \nu\Lambda^{2\beta}A_h\mathbf{w}(t) + (\varpi * A_h)\mathbf{w}(t) + (\mathbf{w}\cdot\nabla)\mathbf{w}, \mathbf{v}\Big) \tag{3.1}$$

$$= (\mathbf{f}(t), \mathbf{v}),$$

$$\lim_{t\to 0}(\mathbf{w}(t), \mathbf{v}) = (\mathbf{w}_0, \mathbf{v}), \tag{3.2}$$

and the energy equality

$$\frac{d}{dt}\left(\|\mathbf{w}(t)\|^2 + \alpha^2\|\nabla_h\mathbf{w}(t)\|^2\right) + \nu\left(\|\Lambda^\beta\mathbf{w}(t)\|^2 + \alpha^2\|\Lambda^\beta\nabla_h\mathbf{w}(t)\|^2\right)$$

$$+ (\varpi * \mathbf{w}(t), \mathbf{w}(t)) = (\mathbf{f}(t), \mathbf{w}(t)).$$

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Remark 3.1. In proving the above energy equality, we use a suitable regularization procedure for the considered equations and the related solutions (see Annese et al. [3]): Let $\mathbf{w}_\varepsilon$ denote the standard convolution regularization, in time, of $\mathbf{w}$, with $0 < t_0 < t_1 < T$ fixed, and $0 < \varepsilon < t_0$, $\varepsilon < T - t_1$, $\varepsilon < t_1 - t_0$ (see, e.g., Berselli [9], see also Annese et al. [3]). For each $t \in [t_0, t_1]$, we have

$$\mathbf{w}_\varepsilon(t) = (j_\varepsilon * \mathbf{w})(t) = \int_{t_0}^{t_1} j_\varepsilon(t-\tau)\mathbf{w}(\tau)\,\mathrm{d}\,\tau,$$

where the smooth function j_ε is even, positive, supported in $(-\varepsilon, \varepsilon)$, and $\int_{-\varepsilon}^{\varepsilon} j_\varepsilon(s)\,\mathrm{d}\,s = 1$.

For the existence, we take initial data in $\mathbf{w}_0 \in V_h$ (see Section 3.1 below), and consider an appropriate Galerkin approximating scheme $\{\mathbf{w}^n\}$ such that, up to pass to subsequences, satisfies

$$\mathbf{w}^n \to \mathbf{w} \quad \text{in} \qquad L^\infty(0,T;V_h) \cap L^2(0,T;V \cap H_h^{1+\beta}), \tag{3.3}$$

with the limiting vector fields $\mathbf{w}$ verifying Equations (3.1) and (3.2) in Definition 3.1. Then, setting $\mathbf{w}_\varepsilon^n(t) = (j_\varepsilon * \mathbf{w}^n)(t)$, we have, for $1 \le q < +\infty$, that (see, e.g., Galdi [28])

$$\lim_{n\to+\infty} \|\mathbf{w}_\varepsilon^n - \mathbf{w}_\varepsilon\|_{L^q(t_0,t_1;X)} = 0 \ \text{ for } \ \mathbf{w}^n \in L^q(t_0,t_1;X) \text{ s.t. } \mathbf{w}^n \to \mathbf{w} \in L^q(t_0,t_1;X), \tag{3.4}$$

and X is any of the Hilbert spaces, that is, V_h, V, or $H_h^{1+\beta}$, in Equation (3.3). Then, testing Equation (1.11) against $\mathbf{w}_\varepsilon^n$ in L^2, and integrating on $[t_0, t_1]$ in $\mathrm{d}\,\tau$, we get

$$\begin{aligned}
&\int_{t_0}^{t_1} \big[(\mathbf{w}, \partial_t \mathbf{w}_\varepsilon^n) + \alpha^2(\nabla_h \mathbf{w}, \partial_t \nabla_h \mathbf{w}_\varepsilon^n) - \nu\big(\Lambda^\beta \mathbf{w}, \Lambda^\beta \mathbf{w}_\varepsilon^n\big) - \nu\alpha^2\big(\Lambda^\beta \nabla_h \mathbf{w}, \Lambda^\beta \nabla_h \mathbf{w}_\varepsilon^n\big)\\
&\quad +(\mathbf{w}\cdot\nabla)\mathbf{w}_\varepsilon^n, \mathbf{w})\big](\tau)\,\mathrm{d}\tau - \int_{t_0}^{t_1}\int_0^\tau \varpi(\tau - s)\langle \mathbf{w}(s), \mathbf{w}_\varepsilon^n(\tau)\rangle_{V_h}\,\mathrm{d}s\,\mathrm{d}\tau \\
&= -\int_{t_0}^{t_1} (\mathbf{f}, \mathbf{w}_\varepsilon^n)(s)\,\mathrm{d}s + (\mathbf{w}(t_1), A_h \mathbf{w}_\varepsilon^n(t_1)) - (\mathbf{w}(t_0), A_h \mathbf{w}_\varepsilon^n(t_0)),
\end{aligned} \tag{3.5}$$

where $\langle \mathbf{w}_\varepsilon^n, \mathbf{w}\rangle_{V_h} = (\mathbf{w}, \mathbf{w}_\varepsilon^n) + \alpha^2(\nabla_h \mathbf{w}, \nabla_h \mathbf{w}_\varepsilon^n)$. To gain the energy equality, in integral form, we use the same argument as in Annese et al. [3] (see also Refs. [10, 28]), by passing to the limit as $n \to +\infty$, and using the convergence types listed in Equations (3.3) and (3.4).

We now provide a few facts about the existence of such a type of weak solutions.

3.1 | Global existence and uniqueness

Let us now show that the system (1.11)–(1.12) has a unique weak solution using Galerkin approximations and energy estimates.

Theorem 3.1. *Let $\mathbf{w}_0 \in V_h$ and $\mathbf{f} \in L^2(0,T;H^{-\beta})$ be given. Assume $3/4 \le \beta < 1$. Then, there exists a unique weak solution $\mathbf{w}$ to the system (1.11)–(1.12), in the sense of Definition 3.1.*

This result is based on a compactness argument in the style of Aubin–Lions (see Annese et al. [3]) combined with proper a priori estimates.

Let us assume $\mathbf{w}_0 \in V_h$ and $\mathbf{f} \in L^2(0,T;V_h)$. We proceed formally to get the needed a priori estimates, but the procedure actually goes through the use of the Galerkin approximation scheme (1.17).

Let us test Equation (1.11) against $\mathbf{w}$ in L^2, to get

$$\frac{1}{2}\frac{d}{dt}\big(\|\mathbf{w}\|^2 + \alpha^2\|\nabla_h \mathbf{w}\|^2\big) + \nu\big(\|\Lambda^\beta \mathbf{w}\|^2 + \alpha^2\|\Lambda^\beta \nabla_h \mathbf{w}\|^2\big) + (\varpi * \mathbf{w}, \mathbf{w}) = (\mathbf{f}, \mathbf{w}).$$

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Then, using Equation (2.5) along with Equation (2.9), we get

$$\frac{1}{2}\frac{d}{dt}\left(\|\mathbf{w}\|^2+\alpha^2\|\nabla_h\mathbf{w}\|^2\right)+(\nu-\varepsilon)\|\Lambda^\beta\mathbf{w}\|^2+\nu\|\Lambda^\beta\nabla_h\mathbf{w}\|^2\le C_\varepsilon\|\mathbf{f}\|_{H^{-\beta}}. \tag{3.6}$$

Testing Equation (1.11) against $\phi\in H^\beta$, we infer

$$\begin{aligned}\left|\left((I-\alpha^2\Delta_h)\mathbf{w}_t,\phi\right)\right| &\le c\left(\nu(\|\Lambda^\beta\mathbf{w}\|+\alpha^2\|\Lambda^\beta\nabla_h\mathbf{w}\|)+\|\varpi*\mathbf{w}\|+\|\mathbf{f}\|_{H^{-\beta}}\right)\|\phi\|_{H^\beta}\\ &\quad+\left|((\mathbf{w}\cdot\nabla)\mathbf{w},\phi)\right|.\end{aligned}$$

In particular, it holds that (see [3])

$$\begin{aligned}\left|((\mathbf{w}\cdot\nabla)\mathbf{w},\phi)\right| &\le \left|((\mathbf{w}_h\cdot\nabla_h)\mathbf{w},\phi)\right|+\left|(w_3\partial_3\mathbf{w},\phi)\right|\\ &\le c\left(\|(\mathbf{w}_h\cdot\nabla_h)\mathbf{w}\|_{H^{-\beta}}+\|w_3\partial_3\mathbf{w}\|_{H^{-\beta}}\right)\|\phi\|_{H^\beta}\\ &\le c\left(\|\Lambda^\beta\mathbf{w}_h\|\|\nabla_h\mathbf{w}\|+\|w_3\|_{H^1}\|\Lambda^\beta\mathbf{w}\|\right)\|\phi\|_{H^\beta}\\ &\le c\|\nabla_h\mathbf{w}\|\left(\|\Lambda^\beta\mathbf{w}_h\|+\|\Lambda^\beta\mathbf{w}\|\right)\|\phi\|_{H^\beta},\end{aligned} \tag{3.7}$$

and so, using (2.6), we infer

$$\begin{aligned}&\int_0^T\|(I-\alpha^2\Delta_h)\mathbf{w}_t\|^2_{H^{-\beta}}\mathrm{d}s\\ &\quad\le c\int_0^T\left(\nu(\|\Lambda^\beta\mathbf{w}\|+\alpha^2\|\Lambda^\beta\nabla_h\mathbf{w}\|)+\|\mathbf{f}\|_{H^{-\beta}}\right)^2\mathrm{d}s\\ &\quad+\left(\int_0^T|\varpi(s)|\mathrm{d}s\right)^2\int_0^T\|\mathbf{w}\|^2\mathrm{d}s+c\int_0^T\|\Lambda^\beta\mathbf{w}\|^2\mathrm{d}s.\end{aligned} \tag{3.8}$$

Hence, it follows that $\|(I-\alpha^2\Delta_h)\mathbf{w}_t\|^2_{H^{-\beta}}\in L^2_{\mathrm{loc}}(0,+\infty)$.

Thanks to the above estimates, we can proceed as in Annese et al. [3] to prove, first, existence and then uniqueness.

4 | EXACT CONTROLLABILITY OF GALERKIN APPROXIMATIONS

In order to establish the exact controllability of the Galerkin approximation scheme for the considered model, we start by introducing the following controlled system, for $x\in\Omega=\mathbb{T}^3$, and $t>0$, that is,

$$\begin{aligned}&\left(I-\alpha^2\Delta_h\right)\mathbf{w}_t+\nu\Lambda^{2\beta}A_h\mathbf{w}+(\varpi*A_h\mathbf{w})+(\mathbf{w}\cdot\nabla)\mathbf{w}+\nabla p=U\chi_{\mathcal{O}},\\ &(\nabla\cdot\mathbf{w})(t,x)=0,\\ &\mathbf{w}(0,x)=\mathbf{w}_0(x),\end{aligned} \tag{4.1}$$

where the control domain $\mathcal{O}$ (i.e., the domain on which the control is acting) is supposed to be as small as needed, U is the distributed control function acting over the system, and $\chi_{\mathcal{O}}$ denotes the characteristic function of $\mathcal{O}$.

If we take the inner product of the first equation in Equation (4.1) with $\mathbf{v}\in H_h^{1+\beta}$, we arrive at

$$\begin{aligned}&\left((I-\alpha^2\Delta_h)\mathbf{w}_t,\mathbf{v}\right)+\nu(\Lambda^\beta A_h^{1/2}\mathbf{w},\Lambda^\beta A_h^{1/2}\mathbf{v})+((\varpi*\mathbf{w}),\mathbf{v})\\ &\quad+\alpha^2((\varpi*\nabla_h\mathbf{w}),\nabla_h\mathbf{v})+((\mathbf{w}\cdot\nabla)\mathbf{w},\mathbf{v})=(U\chi_{\mathcal{O}},\mathbf{v}),\\ &\mathbf{w}(0)=\mathbf{w}_0\in H_h^{1+\beta}.\end{aligned}$$

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Now, consider $\{\mathbf{e}_j\}_{j=1}^{\infty}$ basis of $H_h^{1+\beta}$ made of elements that are linearly independent in H. The existence of such a basis is guaranteed by the following result due to Lions and Zuazua (see Theorem 3.1, [36] and Proposition 2.1, [37]) when the immersion $J\ :\ H_h^{1+\beta} \to H$ is utilized.

Proposition 4.1 Theorem 3.1, [36]. *Let H_1 and H_2 be Hilbert spaces. Let $J\ :\ H_1 \to H_2$ be a bounded linear operator with an infinite-dimensional range. Then, there exists a Riesz basis $\{\mathbf{e}_j\}_{j=1}^{\infty}$ of H_1 such that $\{J\mathbf{e}_j\}_{j=1}^{\infty}$ are linearly independent in H_2.*

Next, as standard, we consider the finite-dimensional space

$$E = \operatorname{span}\{\mathbf{e}_1, \dots, \mathbf{e}_{d_N}\} \tag{4.2}$$

and take into account a suitable Galerkin approximation scheme $\{\mathbf{w}^n\}$ for the system (1.11)–(1.12) with $\mathbf{f} = 0$.

We are going to use the following Galerkin approximation (already introduced in Equation (1.17)), where we set $\mathbf{w} = \mathbf{w}^n$ in order to keep the notation concise, so that

$$\begin{aligned}
&((I - \alpha^2\Delta_h)\mathbf{w}_t, \mathbf{e}) + \nu(\Lambda^\beta A_h^{1/2}\mathbf{w}, \Lambda^\beta A_h^{1/2}\mathbf{e}) + ((\varpi * \mathbf{w}), \mathbf{e}) \\
&\qquad + \alpha^2((\varpi * \nabla_h \mathbf{w}), \nabla_h \mathbf{e}) + ((\mathbf{w}\cdot\nabla)\mathbf{w}, \mathbf{e}) = (U\chi_{\mathcal{O}}, \mathbf{e}), \quad \forall \mathbf{e} \in E, \\
&\mathbf{w}(0,x) = \mathbf{w}_0 \in E,
\end{aligned} \tag{4.3}$$

Here, we allow a slight abuse of notation having set

$$\mathbf{w}_0 = \mathbf{P}(\mathbf{w}_0) = \sum_{i=1}^{d_N} (\mathbf{w}_0, e_i)e_i.$$

This system admits a unique solution $\mathbf{w} \in C([0,T]; E)$, as a consequence of the Carathéodory's existence theorem and uniqueness follows from the local Lipschitz property (see Annese et al. [3], see also Mohan [38]) and this is one of the basic points behind the proof of Theorem 3.1.

As a further matter of notation, in what follows, we will use the notations $\|\cdot\|$ and $(\cdot\,,\cdot)$ to denote the norm and inner product, both on $L^2(\mathbb{T}^3)$ and on the finite-dimensional space E.

4.1 | Exact controllability

Let us start by defining what we mean by exact controllability for system (4.3).

Definition 4.1. The Galerkin approximated system (4.3) is said to be exactly controllable at time $T > 0$ if, for given $\mathbf{w}_0$ and $\mathbf{w}_T \in E$, there exists a control $U \in L^2(0,T; L^2(\mathcal{O}))$ such that the solution $\mathbf{w}$ of Equation (4.3) satisfies:

$$\mathbf{w}(\cdot\,, T; U) = \mathbf{w}_T. \tag{4.4}$$

In order to achieve this goal, it will be useful to consider the following cost functional:

$$\mathcal{J}(U) = \frac{1}{2}\int_0^T \|U(t)\|_{L^2(\mathcal{O})}\mathrm{d}t = \frac{1}{2}\int_0^T \int_{\mathcal{O}} |U(t)|^2 \mathrm{d}x\mathrm{d}t. \tag{4.5}$$

Now we are ready to formulate and demonstrate the main result.

Theorem 4.1. *Let $T > 0$. Then the Galerkin approximation scheme (4.3) is exactly controllable in the sense of Definition 4.1. Moreover, the cost functional described in Equation (4.5) is bounded independently of the nonlinearity.*

In proving this result, we show that the cost functional (4.5) is bounded independently of the magnitude of nonlinear term in Equation (4.3) (see Equation (4.20) below) so, for convenience, we modify the nonlinearity in the considered model

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by introducing a multiplicative scalar quantity $\mu \in \mathbb{R}$, to get $\mu(\mathbf{w}\cdot\nabla)\mathbf{w}$, and after proving our controls in this situation, we turn back, as a special subcase, to the actual system.

Proof. Let us introduce a coefficient $\mu \in \mathbb{R}$ and consider the following system:

$$\begin{aligned}
&(I-\alpha^2\Delta_h)\mathbf{w}_t + \nu\Lambda^{2\beta}A_h\mathbf{w} + (\varpi * A_h\mathbf{w}) + \mu(\mathbf{w}\cdot\nabla)\mathbf{w} + \nabla p = U\chi_{\mathcal{O}},\\
&(\nabla\cdot\mathbf{w})(t,x) = 0,\ \ x\in\mathbb{T}^3,\ t>0,\\
&\mathbf{w}(0,x) = \mathbf{w}_0(x),\ \ x\in\mathbb{T}^3.
\end{aligned} \tag{4.6}$$

In step 2, we will show some estimates for this system that are uniform with respect to μ. Then, we will establish the exact controllability for system (4.6), which easily gives the required result for system (4.1) by setting $\mu = 1$.

We are going to exploit the following variational formulation of system (4.6):

$$\begin{aligned}
&\big((I-\alpha^2\Delta_h)\mathbf{w}_t,\mathbf{e}\big) + \nu(\Lambda^\beta A_h^{1/2}\mathbf{w},\Lambda^\beta A_h^{1/2}\mathbf{e}) + (\varpi * \mathbf{w},\mathbf{e})\\
&\quad + \alpha^2((\varpi * \nabla_h\mathbf{w}),\nabla_h\mathbf{e}) + \mu((\mathbf{w}\cdot\nabla)\mathbf{w},\mathbf{e}) = (U\chi_{\mathcal{O}},\mathbf{e}),\ \ \forall\mathbf{e}\in E,\\
&\mathbf{w}(0) = \mathbf{w}_0 \in E.
\end{aligned} \tag{4.7}$$

– Step 1: Linear system. Take a function $\mathbf{h} \in L^2(0,T;E)$ and consider the following linearized system:

$$\begin{aligned}
&\big((I-\alpha^2\Delta_h)\mathbf{w}_t,\mathbf{e}\big) + \nu(\Lambda^\beta A_h^{1/2}\mathbf{w},\Lambda^\beta A_h^{1/2}\mathbf{e}) + (\varpi * \mathbf{w},\mathbf{e})\\
&\quad + \alpha^2((\varpi * \nabla_h\mathbf{w}),\nabla_h\mathbf{e}) + \mu((\mathbf{h}\cdot\nabla)\mathbf{w},\mathbf{e}) = (U\chi_{\mathcal{O}},\mathbf{e}),\\
&\mathbf{w}(0) = \mathbf{0}.
\end{aligned} \tag{4.8}$$

Since system (4.8) is linear, it is immediate that it has a unique solution $\mathbf{w} \in C([0,T];E)$. Let us notice that we can work with null initial data because of the linearity of system (4.8) and obtain a result, which is still valid for non-null initial data, that is to say, if we take $\mathbf{w}(0) = \mathbf{w}_0 \in E$.

Now, let us show that system (4.8) is exactly controllable for any time $T > 0$, in the sense of Definition 4.1. In order to establish this, it is sufficient to prove that (see Lions [34] for details)

$$\begin{aligned}
&\text{if } \mathbf{g}\in E \text{ satisfies } (\mathbf{w}(\,\cdot\,,T;U),\mathbf{g}) = 0 \ \text{ for all } U \in L^2(0,T;L^2(\mathcal{O})),\\
&\quad\text{then } \mathbf{g} = \mathbf{0}.
\end{aligned} \tag{4.9}$$

Observe that this same approach is used in a number of papers to study finite-dimensional systems coming as approximations of the Navier–Stokes equations (see Refs. [36, 37]) or other similar systems for the description of fluid models (e.g., besides some families viscoelastic fluids, also micropolar fluids and polymer fluids, see Refs. [4, 19, 38]).

To prove Equation (4.9), we take into account the following adjoint system (here $\mathbf{p}_t = \partial_t\mathbf{p}$):

$$\begin{aligned}
&-(I-\alpha^2\Delta_h)\mathbf{p}_t + \nu\Lambda^{2\beta}A_h\mathbf{p} + \varpi\circ A_h\mathbf{p} - \mu(\mathbf{h}\cdot\nabla)\mathbf{p} + \nabla q(t,x) = \mathbf{0},\\
&\quad x\in\mathbb{T}^3,\ t>0,\\
&(\nabla\cdot\mathbf{p})(t,x) = 0,\ \ x\in\mathbb{T}^3,\ t>0,\\
&\mathbf{p}(T,x) = (I-\alpha^2\Delta_h)^{-1}\mathbf{g},\ \ x\in\mathbb{T}^3,
\end{aligned} \tag{4.10}$$

where $\mathbf{g} \in E$ and

$$(\varpi\circ A_h\mathbf{p})(t) = \int_t^T \varpi(s-t)A_h\mathbf{p}(s)\mathrm{d}s.$$

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Remark 4.1. From $\int_0^T ((\varpi * f)(t), g(t))\mathrm{d}t$, with f, g sufficiently regular, a change in the order of integration yields:

$$\begin{aligned}\int_0^T ((\varpi * f)(t), g(t))\mathrm{d}t &= \int_0^T \int_0^t \varpi(t-s)(f(s), g(t))\mathrm{d}s\mathrm{d}t \\ &= \int_0^T (f(t), (\varpi \circ g)(t))\mathrm{d}t,\end{aligned} \tag{4.11}$$

where, as defined above, $(\varpi \circ g)(t) = \int_t^T \varpi(s-t)g(s)\mathrm{d}s$.

The variational formulation of the last system is

$$\begin{gathered}-((I-\alpha^2\Delta_h)\mathbf{p}_t, \mathbf{e}) + \nu(\Lambda^\beta A_h^{1/2}\mathbf{p}, \Lambda^\beta A_h^{1/2}\mathbf{e}) + ((\varpi \circ \mathbf{p}), \mathbf{e}) + \alpha^2((\varpi \circ \nabla_h \mathbf{p}), \nabla_h \mathbf{e}) - \mu((\mathbf{h}\cdot\nabla)\mathbf{p}, \mathbf{e}) = 0 \\ \mathbf{p}(T) = (I-\alpha^2\Delta_h)^{-1}\mathbf{g},\end{gathered} \tag{4.12}$$

again, the system being linear, it has a unique solution $\mathbf{p} \in C([0,T]; E)$. Setting $\mathbf{e} = \mathbf{w}$ in Equation (4.12), we obtain

$$\begin{gathered}-((I-\alpha^2\Delta_h)\mathbf{p}_t, \mathbf{w}) + \nu(\Lambda^\beta A_h^{1/2}\mathbf{p}, \Lambda^\beta A_h^{1/2}\mathbf{w}) + ((\varpi \circ \mathbf{p}), \mathbf{w}) + \alpha^2((\varpi \circ \nabla_h \mathbf{p}), \nabla_h \mathbf{w}) - \mu((\mathbf{h}\cdot\nabla)\mathbf{p}, \mathbf{w}) = 0 \\ \mathbf{p}(T) = (I-\alpha^2\Delta_h)^{-1}\mathbf{g},\end{gathered} \tag{4.13}$$

and integrating in time from 0 to T, we find

$$\begin{aligned}-\big(\mathbf{w}(T), (I-\alpha^2\Delta_h)\mathbf{p}(T)\big) + \int_0^T &\big[((I-\alpha^2\Delta_h)\mathbf{w}_t, \mathbf{p}) + \nu(\Lambda^{2\beta}A_h\mathbf{w}, \mathbf{p}) + ((\varpi * \mathbf{w}), \mathbf{p}) \\ &+\alpha^2((\varpi * \nabla_h \mathbf{w}), \nabla_h \mathbf{p}) + \mu((\mathbf{h}\cdot\nabla)\mathbf{w}, \mathbf{p})\big]\mathrm{d}t = 0.\end{aligned} \tag{4.14}$$

Now we exploit the fact that $\mathbf{p}(T) = (I-\alpha^2\Delta_h)^{-1}\mathbf{g}$ in Equation (4.14) along with Equation (4.8), so that

$$-(\mathbf{w}(T), \mathbf{g}) = \int_0^T (U(t)\chi_{\mathcal{O}}, \mathbf{p}(t))\mathrm{d}t. \tag{4.15}$$

If the assumption given in Equation (4.9) holds true, then, from Equation (4.15), we can easily obtain that

$$\int_0^T (U(t)\chi_{\mathcal{O}}, \mathbf{p}(t))\mathrm{d}t = 0, \text{ for all } U \in L^2(0,T; L^2(\mathcal{O})). \tag{4.16}$$

The above equality also shows that

$$\mathbf{p} = \mathbf{0} \text{ in} \mathcal{O} \times (0,T). \tag{4.17}$$

Finally, from $\mathbf{p} = \sum_{i=1}^n \mathbf{p}_i(t)e_i$ and the fact that the basis $\{e_j\}_{j=1}^\infty$ is linearly independent in $L^2(\mathcal{O})$ (see, e.g., Araruna et al. [4]) and from Equation (4.17), we can conclude that $\mathbf{p}_i = \mathbf{0}$ for all $i = 1, \dots, n$, that is to say that $\mathbf{p} = \mathbf{0}$, and thus $\mathbf{g} = \mathbf{0}$, and the statement (4.9) is proved. Therefore, we have shown that the linear system (4.8), and hence also Equation (4.6), is exactly controllable.

– Step 2: Estimates based on the duality argument. The results shown in Step 1 allow us to define the operator $\mathcal{M} : L^2(0,T;E) \to \mathbb{R}$ given by

$$\mathcal{M}(\mathbf{h}) = \inf_{U \in \mathcal{U}_{ad}} \frac{1}{2}\int_0^T \int_{\mathcal{O}} |U|^2 \mathrm{d}x\mathrm{d}t, \tag{4.18}$$

where $\mathcal{U}_{ad}$ denotes the set of all admissible controls, that is to say

$$\mathcal{U}_{ad} = \big\{U \in L^2(\mathcal{O} \times (0,T)) \ : \ \mathbf{w} \text{ solution of (4.8) satisfying (4.4)}\big\}. \tag{4.19}$$

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We will obtain that

$$\mathcal{M}(\mathbf{h}) \leq C, \tag{4.20}$$

where the positive constant C is independent of $\mathbf{h}$ and μ.

The proof is based on a suitable duality argument (see, e.g., Refs. [4, 38]). Let us consider the continuous linear operator $L : L^2(\mathcal{O} \times (0,T)) \to E$ given by

$$L(U) := \mathbf{w}(\cdot, T; U), \tag{4.21}$$

and also the functionals

$$F_1(U) = \frac{1}{2}\int_0^T \int_{\mathcal{O}} |U|^2 \mathrm{d}x\mathrm{d}t, \tag{4.22}$$

and

$$F_2(\mathbf{g}) = \begin{cases} 0, & \text{if } \mathbf{g} = \mathbf{w}_T, \\ \infty, & \text{otherwise.} \end{cases} \tag{4.23}$$

This allows us to recast the functional $\mathcal{M}$ as

$$\mathcal{M}(\mathbf{h}) = \inf_{U \in L^2(\mathcal{O}\times(0,T))} [F_1(U) + F_2(L(U))]. \tag{4.24}$$

Thanks to the duality theorem due to Fenchel and Rockafellar (see Rockafellar [53], Theorem 31.1], see also Refs. [4, 38]), we derive

$$-\mathcal{M}(\mathbf{h}) = \inf_{\mathbf{g}\in E} \left[F_1^*(L^*(\mathbf{g})) + F_2^*(-\mathbf{g})\right], \tag{4.25}$$

where $L^* : E \to L^2(\mathcal{O} \times (0,T))$ is the adjoint operator of L. A use of relation (4.15) yields

$$L^*(\mathbf{g}) = \mathbf{p} \ \text{in} \mathcal{O} \times (0,T). \tag{4.26}$$

From the fact that

$$F_1^*(\mathbf{p}) = \frac{1}{2}\int_0^T \int_{\mathcal{O}} |\mathbf{p}|^2 \mathrm{d}x\mathrm{d}t \ \text{ and } \ F_2^*(-\mathbf{g}) = -(\mathbf{g}, \mathbf{w}_T), \tag{4.27}$$

where, to keep the notation coincise, we write $|\mathbf{p}|^2 = |\mathbf{p}(x,t)|^2$ in the above integrals. Then, we deduce

$$-\mathcal{M}(\mathbf{h}) = \inf_{\mathbf{g}\in E} \left[\frac{1}{2}\int_0^T \int_{\mathcal{O}} |\mathbf{p}|^2 \mathrm{d}x\mathrm{d}t - (\mathbf{g}_1, \mathbf{w}_T)\right]. \tag{4.28}$$

The hypotheses on the basis of E imply that $\|\mathbf{e}\|_{\mathcal{O}} = \int_{\mathcal{O}} |\mathbf{e}|^2 \mathrm{d}x$ is indeed a norm on E, that is a finite-dimensional space, so that we also have the equivalence of norms given by

$$c\|\mathbf{e}\|^2 \leq \|\mathbf{e}\|_{\mathcal{O}}^2 \leq C\|\mathbf{e}\|^2, \ \forall \, \mathbf{e} \in E, \tag{4.29}$$

with c and C positive constants only depending on E, provided

$$\|\mathbf{e}\|^2 = \int_{\mathbb{T}^2} |\mathbf{e}(x)|^2 \mathrm{d}x, \ \forall \, \mathbf{e} \in E.$$

Hence, we obtain that

$$-\mathcal{M}(\mathbf{h}) \geq \inf_{\mathbf{g}\in E} \left[\frac{c}{2}\int_0^T \int_{\mathbb{T}^2} |\mathbf{p}|^2 \mathrm{d}x\mathrm{d}t - (\mathbf{g}, \mathbf{w}_T)\right]. \tag{4.30}$$

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We set $\mathbf{e} = \mathbf{p}(t)$ in Equation (4.12) and integrate in time from t to T, so that we obtain

$$\frac{1}{2}\left(\|\mathbf{p}\|^2 + \alpha^2\|\nabla_h\mathbf{p}\|^2\right) + \nu\int_t^T \|\Lambda^\beta A_h^{1/2}\mathbf{p}\|^2\,\mathrm{d}s + \int_t^T (\varpi\circ\mathbf{p},\mathbf{p})\mathrm{d}s + \alpha^2\int_t^T (\varpi\circ\nabla_h\mathbf{p},\nabla_h\mathbf{p})\mathrm{d}s = \frac{1}{2}\|\mathbf{g}\|^2. \tag{4.31}$$

An integration from 0 to T and a change in the order of integration produce

$$\frac{1}{2}\int_0^T \left(\|\mathbf{p}\|^2 + \alpha^2\|\nabla_h\mathbf{p}\|^2\right)\mathrm{d}t + \nu\int_0^T t\|\Lambda^\beta A_h^{1/2}\mathbf{p}\|^2\mathrm{d}t + \int_0^T t\left[(\varpi\circ\mathbf{p},\mathbf{p}) + \alpha^2(\varpi\circ\nabla_h\mathbf{p},\nabla_h\mathbf{p})\right]\mathrm{d}t = \frac{T}{2}\|\mathbf{g}\|^2. \tag{4.32}$$

Exploiting the fact that the space E is finite-dimensional, and using Equations (2.1) and (2.2), we have

$$\begin{gathered}
\|A_h^{1/2}\mathbf{p}\|^2 = \|\mathbf{p}\|^2 + \alpha^2\|\nabla_h\mathbf{p}\|^2,\\
\|\Lambda^\beta A_h^{1/2}\mathbf{p}\|^2 = \|\Lambda^\beta\mathbf{p}\|^2 + \alpha^2\|\Lambda^\beta\nabla_h\mathbf{p}\|^2,\\
\|\mathbf{p}\| \le \|\mathbf{p}\| + \|\nabla_h\mathbf{p}\| \le \bar{K}\|\mathbf{p}\|, \quad \|\nabla_h\mathbf{p}\| \le C\|\mathbf{p}\|,\\
\|\mathbf{p}\| \le \|\mathbf{p}\| + \|\nabla_h\mathbf{p}\| + \|\Lambda^\beta\mathbf{p}\| + \|\Lambda^\beta\nabla_h\mathbf{p}\| \le \bar{K}\|\mathbf{p}\|,\\
\|\Lambda^\beta\nabla_h\mathbf{p}\| \le C\|\mathbf{p}\|
\end{gathered} \tag{4.33}$$

for some $C, \bar{K} > 0$ only dependent on E. In particular, we have that $\|A_h^{1/2}\mathbf{p}\| \le \bar{K}\|\mathbf{p}\|$ as well as $\|\Lambda^\beta A_h^{1/2}\mathbf{p}\| \le \bar{K}\|\mathbf{p}\|$, with the same constant $\bar{K}$ as above.

So we can apply the same method shown in Mohan [38] and, using Cauchy–Schwarz's and Hölder's inequalities, we get

$$\begin{aligned}
&\int_t^T t\left[(\varpi\circ\mathbf{p}(t),\mathbf{p}(t)) + \alpha^2(\varpi\circ\nabla_h\mathbf{p}(t),\nabla_h\mathbf{p}(t))\right]\mathrm{d}t\\
&\quad= \int_0^T\int_t^T s\varpi(t-s)\left[(\mathbf{p}(s),\mathbf{p}(t)) + \alpha^2(\nabla_h\mathbf{p}(s),\nabla_h\mathbf{p}(t))\right]\mathrm{d}s\mathrm{d}t\\
&\quad\le T\left(\int_0^T \|\mathbf{p}(t)\|^2\mathrm{d}t\right)^{1/2}\left(\int_0^T\left(\int_0^t \varpi(t-s)\|\mathbf{p}(s)\|\mathrm{d}s\right)^2\mathrm{d}t\right)^{1/2}\\
&\qquad+ \alpha^2 T\left(\int_0^T \|\nabla_h\mathbf{p}(t)\|^2\mathrm{d}t\right)^{1/2}\left(\int_0^T\left(\int_0^t \varpi(t-s)\|\nabla_h\mathbf{p}(s)\|\mathrm{d}s\right)^2\mathrm{d}t\right)^{1/2}\\
&\quad\le CT\int_0^T \|\mathbf{p}(t)\|^2\mathrm{d}t + C\alpha^2 T\int_0^T \|\nabla_h\mathbf{p}(t)\|^2\mathrm{d}t\\
&\quad\le \frac{C\gamma(1+\alpha^2)T}{\delta}\int_0^T \|\mathbf{p}(t)\|^2\mathrm{d}t,
\end{aligned} \tag{4.34}$$

where we utilized Equation (4.33), and γ and δ are positive constants suitably obtained.

Finally, using Equation (4.32) in combination with Equations (4.33) and (4.34), we get

$$\frac{T}{2}\|\mathbf{g}\|^2 \le \left(\frac{1}{2}(1+\alpha^2 C) + CT\left(\nu + \frac{\gamma(1+\alpha^2)}{\delta}\right)\right)\int_0^T \|\mathbf{p}\|^2\mathrm{d}t.$$

If we set

$$\varkappa = \frac{cT}{(1+\alpha^2 C) + 2\left(\nu + \frac{\gamma(1+\alpha^2)}{\delta}\right)CT},$$

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as a consequence of Equation (4.30), we conclude that

$$-\mathcal{M}(\mathbf{h}) \geq \inf_{\mathbf{g}\in E}\left[\frac{\kappa}{2}\|\mathbf{g}\|^2 - (\mathbf{g}, \mathbf{w}_T)\right] \geq \inf_{\mathbf{g}\in E}\left[\frac{\kappa}{2}\|\mathbf{g}\|^2 - \frac{\kappa}{2}\|\mathbf{g}\|^2 - \frac{1}{2\kappa}\|\mathbf{w}_T\|^2\right]$$
$$= -\frac{1}{2\kappa}\|\mathbf{w}_T\|^2,$$

and therefore,

$$\mathcal{M}(\mathbf{h}) \leq \frac{1}{2\kappa}\|\mathbf{w}_T\|^2, \tag{4.35}$$

which is Equation (4.20).

– Step 3: Nonlinear system. This part is dedicated to the proof of the exact controllability of the nonlinear system (4.6). Assume that $\mathbf{h} \in L^2(0, T; E)$ is given and select the unique element $U \in L^2(0, T; L^2(\mathcal{O}))$ such that

$$\frac{1}{2}\int_0^T \int_{\mathcal{O}} |U(x,t)|^2 \mathrm{d}x\mathrm{d}t = \mathcal{M}(\mathbf{h}), \tag{4.36}$$

thus defining a continuous mapping $\mathbf{h} \mapsto U$ from $L^2(0, T; E)$ to $L^2(0, T; L^2(\mathcal{O}))$ (the continuity and the other properties of this operator are related to those of the bounded functional $\mathcal{M}$, in function of which it is defined). For more details, see the analogous construction in Araruna et al. [4] and, in particular, Rockafellar [53, §31]). Denote by $\mathbf{w}(\mathbf{h})$ the solution of Equation (4.6) with the control $U = U(\mathbf{h})$ and set $\mathbf{e} = \mathbf{w}(t)$ in Equation (4.8) to get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\|\mathbf{w}(t)\|^2 + \alpha^2\|\nabla_h\mathbf{w}(t)\|^2\right) + \nu\|\Lambda^\beta A_h^{1/2}\mathbf{w}(t)\|^2 \\ = -((\varpi * \mathbf{w}(t)), \mathbf{w}(t)) - \alpha^2((\varpi * \nabla_h\mathbf{w}(t)), \nabla_h\mathbf{w}(t)) + (U(t)\chi_{\mathcal{O}}, \mathbf{w}(t)).\end{aligned} \tag{4.37}$$

Integrating from 0 to t, we obtain

$$\begin{aligned}\|\mathbf{w}(t)\|^2 &+ \alpha^2\|\nabla_h\mathbf{w}(t)\|^2 + \nu\int_0^t \|\Lambda^\beta A_h^{1/2}\mathbf{w}(s)\|^2\mathrm{d}s \\ &= -\int_0^t ((\varpi * \mathbf{w}(s)), \mathbf{w}(s))\mathrm{d}s - \alpha^2\int_0^t ((\varpi * \nabla_h\mathbf{w}(s)), \nabla_h\mathbf{w}(s))\mathrm{d}s \\ &\quad + \int_0^t (U(s)\chi_{\mathcal{O}}, \mathbf{w}(s))\mathrm{d}s \\ &\leq \left(\int_0^t \int_{\mathcal{O}} |U(x,s)|^2\mathrm{d}x\mathrm{d}s\right)^{1/2}\left(\int_0^t \int_{\mathcal{O}} |\mathbf{w}(x,s)|^2\mathrm{d}x\mathrm{d}s\right)^{1/2} \\ &\leq \frac{1}{2}\int_0^t \int_{\mathcal{O}} |U(x,s)|^2\mathrm{d}x\mathrm{d}s + \frac{1}{2}\int_0^t \int_{\mathcal{O}} |\mathbf{w}(x,s)|^2\mathrm{d}x\mathrm{d}s,\end{aligned} \tag{4.38}$$

where we took into account Equation (2.9). This implies

$$\|\mathbf{w}(t)\|^2 \leq \|U\|_{L^2(0,T;L^2(\mathcal{O}))} + C\int_0^t \|\mathbf{w}(s)\|^2\mathrm{d}s \tag{4.39}$$

and, thanks to Gronwall's inequality, we get

$$\|\mathbf{w}(t)\| \leq e^{CT}\|U\|_{L^2(0,T;L^2(\mathcal{O}))} \tag{4.40}$$

for all $t \in [0, T]$. Thanks to Equation (4.20) (which in turn depends on Equation (4.35)), as $\mathbf{h}$ varies in $L^2(0, T; E)$, we can conclude that $\mathbf{w}$ is contained in a bounded subset $K \subset L^2(0, T; E)$.

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In order to show that the map $\mathbf{h} \mapsto \mathbf{w}(\mathbf{h})$ admits a fixed point in K, we use directly Schauder's fixed point theorem. To this aim, it is sufficient to prove that the range of $\mathbf{w}(\mathbf{h})$, when $\mathbf{h}$ varies through K, is relatively compact in K. This is an easy consequence of the following statement:

$$\begin{gathered}\mathbf{w}_t \text{ stays bounded in a bounded subset of } L^2(0,T;E)\\ \text{when } \mathbf{h} \text{ spans over } K.\end{gathered} \tag{4.41}$$

Looking for such a control, we start with the estimate (see Equation (4.8), see also Equation (3.8)):

$$\begin{aligned}|((I-\alpha^2\Delta_h)\mathbf{w}_t,\mathbf{e})| &\le \nu|(\Lambda^\beta A_h^{1/2}\mathbf{w},\Lambda^\beta A_h^{1/2}\mathbf{e})| + |((\varpi * \mathbf{w}),\mathbf{e})|\\
&\quad + |((\varpi * \nabla_h\mathbf{w}),\nabla_h\mathbf{e})| + \mu|((\mathbf{h}\cdot\nabla)\mathbf{w},\mathbf{e})| + |(U\chi_{\mathcal{O}},\mathbf{e})|\\
&\le \nu\|\Lambda^\beta A_h^{1/2}\mathbf{w}\|\|\Lambda^\beta A_h^{1/2}\mathbf{e}\| + \|\varpi * \mathbf{w}\|\|\mathbf{e}\| + \|\varpi * \nabla_h\mathbf{w}\|\|\nabla_h\mathbf{e}\|\\
&\quad + \mu\|\mathbf{h}\|\|\nabla\mathbf{w}\|\|\mathbf{e}\|_{L^4} + \|U\|_{L^2(\mathcal{O})}\|\mathbf{e}\|_{L^2(\mathcal{O})}\\
&\le \nu\|\Lambda^\beta A_h^{1/2}\mathbf{w}\|\|\Lambda^\beta A_h^{1/2}\mathbf{e}\| + \|\varpi * \mathbf{w}\|\|\mathbf{e}\| + \|\varpi * \nabla_h\mathbf{w}\|\|\nabla_h\mathbf{e}\|\\
&\quad + \mu\|\mathbf{h}\|\|\nabla\mathbf{w}\|\|\Lambda^\beta\mathbf{e}\| + \|U\|_{L^2(\mathcal{O})}\|\mathbf{e}\|_{L^2(\mathcal{O})}\\
&\le C\Big(\nu\|\Lambda^\beta A_h^{1/2}\mathbf{w}\| + \|\varpi * \mathbf{w}\| + \|\varpi * \nabla_h\mathbf{w}\| + \mu\|\mathbf{h}\|\|\nabla\mathbf{w}\|\\
&\quad + \|U\|_{L^2(\mathcal{O})}\Big)\|\mathbf{e}\|\end{aligned} \tag{4.42}$$

for all $\mathbf{e} \in E$, where we used the embedding $H^\beta \hookrightarrow L^4$ and relations (4.33), and hence

$$\|(I-\alpha^2\Delta_h)\mathbf{w}_t\| \le C\Big(\nu\|\Lambda^\beta A_h^{1/2}\mathbf{w}\| + \|\varpi * \mathbf{w}\| + \|\varpi * \nabla_h\mathbf{w}\| + \mu\|\mathbf{h}\|\|\nabla_h\mathbf{w}\| + \|U\|_{L^2(\mathcal{O})}\Big). \tag{4.43}$$

The memory terms $\|\varpi * \mathbf{w}(t)\|$ and $\|\varpi * \nabla_h\mathbf{w}(t)\|$ can be controlled by Hölder's inequality as

$$\begin{aligned}\|\varpi * \nabla_h\mathbf{w}(t)\| &\le \int_0^t \varpi(t-s)\|\nabla_h\mathbf{w}(s)\|\mathrm{d}s\\
&\le \left(\int_0^t \varpi(t-s)^2\mathrm{d}s\right)^{1/2}\left(\int_0^t \|\nabla_h\mathbf{w}(s)\|^2\mathrm{d}s\right)^{1/2}\\
&\le \frac{C\gamma}{\sqrt{2\delta}}\left(\int_0^t \|\nabla_h\mathbf{w}(s)\|^2\mathrm{d}s\right)^{1/2}\end{aligned}$$

and

$$\|\varpi * \nabla_h\mathbf{w}(t)\| \le \frac{C\gamma}{\sqrt{2\delta}}\left(\int_0^t \|\mathbf{w}(s)\|^2\mathrm{d}s\right)^{1/2}.$$

Exploiting once again the fact that E is finite-dimensional, from Equation (4.43), we also deduce

$$\begin{aligned}c\|\mathbf{w}_t(t)\| &\le \|(I-\alpha^2\Delta_h)\mathbf{w}_t(t)\|\\
&\le C\Bigg[\nu\|\Lambda^\beta A_h^{1/2}\mathbf{w}\| + \frac{C\gamma}{\sqrt{2\delta}}\big(\|\mathbf{w}\|_{L^2(0,T;E)} + \alpha^2\|\nabla_h\mathbf{w}\|_{L^2(0,T;E)}\big)\\
&\quad + \mu\|\mathbf{h}\|\|\nabla_h\mathbf{w}\| + \|U\|_{L^2(\mathcal{O})}\Bigg],\end{aligned} \tag{4.44}$$

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which yields the statement (4.41). Thanks to Schauder's fixed point theorem, we have that $\mathbf{h} \mapsto \mathbf{w}(\mathbf{h})$ admits a fixed point in K. Hence, selected a fixed point $\mathbf{h}$, we conclude that the system (4.7) is exactly controllable since the system (4.8) is exactly controllable for all times $T > 0$. We notice that the system (4.7) is exactly controllable for any $\mu \in \mathbb{R}$, and in particular, if we choose $\mu = 1$, which easily yields the exact controllability of the system (4.3). Further, for any $\mathbf{h}$, we have the uniform estimate (4.20), which establishes that the cost functional introduced in Equation (4.5) is bounded independently of the nonlinearity. □

Remark 4.2. The estimates used to prove the exact controllability of the approximating system (4.3) strongly depend on the dimensions of the space E. Indeed, the inequalities (4.33) (later used in Equations (4.34)–(4.44)) hold true on these finite-dimensional spaces, but they are not uniform with respect to $\dim E$. Thus, at this level, due to the lack of a uniform control, we do not have at disposal an argument to pass to the limit as $\dim E \to +\infty$, and the previous results do not allow us to conclude about the exact controllability of the system (4.1). The same kind of issue is discussed in Lions and Zuazua [37].

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CONFLICT OF INTEREST STATEMENT

The authors declare no conflicts of interest.

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