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Heckman, Kostant, and Steinberg Formulas for Symplectic Manifolds

VICTOR GUILLEMIN* AND ELISA PRATO†

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

Let M be a compact symplectic manifold on which a compact connected Lie group, K, acts in a Hamiltonian fashion. In Part I we derive a formula for $J_*(\beta_M)$ which generalizes the Heckman formula for co-adjoint orbits. (Here J is the moment map associated with the action of K on M and β_M the Liouville measure on M.) In Part II we derive a "quantum" analogue of this formula which extends to the symplectic setting classical multiplicity formulas of Kostant and Steinberg.

1. INTRODUCTION

Let M be a compact symplectic manifold, K a connected compact Lie group with Lie algebra $\mathbf{k}$, and

$$\rho\colon K \to \operatorname{Diff}(M) \tag{1.1}$$

a Hamiltonian action of K on M with moment map

$$J\colon M \to \mathbf{k}^*.$$

The pushforward, $J_*(\beta_M)$, of the Liouville measure on M appears to be an extremely interesting symplectic invariant of (M, ρ) and has been considerably studied in recent years (see [3–5, 9, 12]).

In his thesis Heckman derived an explicit formula for $J_*(\beta_M)$ in the special case K a torus and M a co-adjoint orbit of a compact semi-simple Lie group, G, with K as Cartan subgroup. His formula involved a kind of continuous limit of the partition function occurring in the Kostant multiplicity formula, and using it he was able to give sharp asymptotic results on the distribution of weights for a ladder of irreducible representations of G. This asymptotic result was generalized by one of us [8] to compact

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symplectic manifolds by Toeplitz theoretical methods which did not require writing down an explicit formula for $J_*(\beta_M)$ itself. Recently, however, we realized that such a formula exists for actions of any compact connected Lie group; this will be the topic of Part I of our article.

In Part II we will give a "quantized" version of this formula which generalizes to symplectic manifolds the classical multiplicity formulas of Kostant [16] and Steinberg [18]. Among other things our formula will say that all quantizations of the classical action (1.1) are unitarily equivalent answering affirmatively in the compact case a conjecture posed a long time ago by Kostant.

Results similar to ours for the special case of K abelian were obtained in [9].

One thing which we have not succeded in doing in this paper is to extend the Blattner formula to a symplectic setting; however, recent results of Duflo, Heckman, and Vergne [5] and Duflo and Vergne [6] strongly suggest that such a generalization exists.

PART I: HECKMAN FORMULA FOR SYMPLECTIC MANIFOLDS

2. Statement and Proof of the Formula

Let M be a compact symplectic manifold,

$$K \to \mathrm{Diff}(M)$$

a Hamiltonian action of a compact connected Lie group, K, on M, and

$$J\colon M \to \mathbf{k}^* \tag{2.1}$$

the corresponding moment map. Let T be a Cartan subgroup of K with Lie algebra $\mathbf{t}$; the induced action,

$$T \to \mathrm{Diff}(M), \tag{2.2}$$

of T on M is also Hamiltonian. Denote by $\mathbf{k}^*_{\mathrm{reg}}$ the set of regular elements of $\mathbf{k}^*$ and by $\mathbf{t}^*_{\mathrm{reg}}$ the intersection of $\mathbf{k}^*_{\mathrm{reg}}$ with $\mathbf{t}^*$. We will make the following assumption on the action (2.2):

> There are a finite number of fixed points, $p_1, ..., p_N$, and their images with respect to J lie in $\mathbf{t}^*_{\mathrm{reg}}$.

Remark 2.3. This property implies that the stabilizer group in K of each p_i is T.

Consider now the pushforward, $J_*(\beta_M)$, of the Liouville measure, β_M, on M. This measure has a natural restriction, v, to $\mathbf{t}^*$; v is Weyl group invariant and defined by the property that if f is a K-invariant function on $\mathbf{k}^*$ and g its restriction to $\mathbf{t}^*$, then

$$\int_{\mathbf{k}^*} f(x)\, dJ_*(\beta_M) = \int_{\mathbf{t}^*} g(y)\, dv.$$

The "Heckman formula" that we are about to prove below gives an explicit description of the measure v as a sum of a finite number of terms, one for each fixed point of the action (2.2). Such a description has a rather interesting geometric interpretation which we will discuss in Appendix B.

We will begin by making a few definitions. First of all identify $\mathbf{t}^*$ with the set of T-fixed elements in $\mathbf{k}^*$; let $\mathbf{r}$ be the annihilator of $\mathbf{t}^*$ in $\mathbf{k}$ and $\mathbf{r}^*$ the annihilator of $\mathbf{t}$ in $\mathbf{k}^*$. Then

$$\mathbf{k} = \mathbf{r} \oplus \mathbf{t}$$

and

$$\mathbf{k}^* = \mathbf{r}^* \oplus \mathbf{t}^*.$$

Consider now a set of positive roots, $\alpha_1, \dots, \alpha_d$, for the pair (K, T); let $X_1, Y_1, \dots, X_d, Y_d$ be a basis for $\mathbf{r}$ such that each vector $\xi_i = [X_i, Y_i]$ is dual to the root α_i in the isomorphism of $\mathbf{t}$ with $\mathbf{t}^*$ given by the Killing form. Denote by η the standard volume form on $\mathbf{r}$ relative to this basis. Next define, for each y in $\mathbf{t}^*$, an alternating two-form on $\mathbf{r}$ by

$$\Omega_y(v, w) = \langle y, [v, w] \rangle, \qquad v, w \in \mathbf{r}.$$

Then Ω_y^d is a volume form on $\mathbf{r}$ which depends on y and that can therefore be written as $\pi(y)$ times η. It can be easily verified that $\pi(y) = \prod \langle y, \xi_i \rangle$. Obviously $\pi(y)$ is a homogeneous polynomial of degree $d = \frac{1}{2}(\dim \mathbf{k} - \dim \mathbf{t})$ in y and has the property that $\pi(y) \neq 0$ when y is in $\mathbf{t}^*_{\text{reg}}$ (for more on $\pi(y)$, see [13]).

LEMMA 2.4. *Let f be a K-invariant smooth function on $\mathbf{k}^*$ which is compactly supported on $\mathbf{k}^*_{\text{reg}}$; denote by g its restriction to $\mathbf{t}^*$; then, for ξ in $\mathbf{t}$,*

$$\left(\prod_{i=1}^{d} \alpha_i(\xi)\right) \int_{\mathbf{k}^*} f(x)\, e^{\sqrt{-1}\langle x, \xi \rangle}\, dx = \int_{\mathbf{t}^*} \frac{g(y)}{\pi(y)}\, e^{\sqrt{-1}\langle y, \xi \rangle}\, dy, \tag{2.5}$$

dx and dy being, respectively, Lebesgue measure on $\mathbf{k}^*$ and $\mathbf{t}^*$.

Proof. Let $\mathbf{t}_+^*$ be the set of y in $\mathbf{t}^*$ such that $\langle y, \xi_i \rangle > 0$ for each $i = 1, ..., d$. Consider the topological fibration

$$\kappa \colon \mathbf{k}_{\mathrm{reg}}^* \to \mathbf{t}_+^*$$

whose fibers are the co-adjoint orbits in $\mathbf{k}_{\mathrm{reg}}^*$. Denote by β the measure on the fibers of κ whose restriction to each fiber is the Liouville measure associated to the canonical Kostant–Kirillov symplectic form. Notice that, by K-invariance, we obtain

$$dx = \kappa^*(\pi^{-1}\, dy)\, d\beta.$$

Now, since $f = \kappa^* g$, we have that, for ξ a regular element in $\mathbf{t}$,

$$\begin{aligned}
\hat{f}(\xi) &= \int_{\mathbf{k}^*} f(x)\, e^{\sqrt{-1}\langle x, \xi \rangle}\, dx \\
&= \int_{\mathbf{k}^*} (\kappa^* g)(x)\, e^{\sqrt{-1}\langle x, \xi \rangle} (\kappa^*(\pi^{-1}\, dy))\, d\beta \\
&= \int_{\mathbf{t}_+^*} \left(\int_{K \cdot y} e^{\sqrt{-1}\langle x, \xi \rangle}\, d\beta \right) \frac{g(y)}{\pi(y)}\, dy.
\end{aligned}$$

However, the inner integral over the fiber above y can be computed by exact stationary phase (see [3]) and is equal to

$$\left(\prod_{i=1}^{d} \alpha_i(\xi) \right)^{-1} \sum_{w \in W} \varepsilon(w)\, e^{\sqrt{-1}\langle wy, \xi \rangle}, \tag{2.6}$$

W being the Weyl group and $\varepsilon(w)$ being the determinant of w (as a linear transformation of $\mathbf{t}$). Substituting (2.6) into the expression for $\hat{f}$ we get, for ξ in $\mathbf{t}$,

$$\left(\prod_{i=1}^{d} \alpha_i(\xi) \right) \hat{f}(\xi) = \int_{\mathbf{t}_+^*} \sum_{w \in W} \varepsilon(w)\, e^{\sqrt{-1}\langle wy, \xi \rangle} \frac{g(y)}{\pi(y)}\, dy.$$

Finally, using the fact that $\pi(wy) = \varepsilon(w)\, \pi(y)$, we can replace the integral on the right by an integral over all of $\mathbf{t}^*$,

$$\int_{\mathbf{t}^*} e^{\sqrt{-1}\langle y, \xi \rangle} \frac{g(y)}{\pi(y)}\, dy,$$

establishing (2.5). Q.E.D.

As an immediate consequence of this lemma we get:

PROPOSITION 2.7. *Let f and g be as in Lemma 2.4. Then for ξ in* $\mathbf{t}$,

$$\prod_{i=1}^{d} \alpha_i(\xi) \int_{\mathbf{k}^*} f(x)\, e^{\sqrt{-1}\langle x,\xi\rangle}\, dJ_*(\beta_M) = \int_{\mathbf{t}^*} \frac{g(y)}{\pi(y)}\, e^{\sqrt{-1}\langle y,\xi\rangle}\, dv. \tag{2.8}$$

Next, by differentiating under the integral sign, we can rewrite (2.8) as

$$\left(\prod_{i=1}^{d} D_{\xi_i}\right)\left(\left(\prod_{i=1}^{d} \alpha_i(\xi)\right) \int_{\mathbf{k}^*} f(x)\, e^{\sqrt{-1}\langle x,\xi\rangle}\, dJ_*(\beta_M)\right)$$
$$= \int_{\mathbf{t}^*} e^{\sqrt{-1}\langle y,\xi\rangle}\, g(y)\, dv, \tag{2.9}$$

where $\sqrt{-1}\, D_{\xi_i}$ is differentiation with respect to ξ_i. Take now an increasing sequence, f_n, of K-invariant smooth functions on $\mathbf{k}^*$, compactly supported in $\mathbf{k}^*_{\text{reg}}$ and pointwise convergent to 1 on $\mathbf{k}^*$; let g_n be the restriction of f_n to $\mathbf{t}^*$. Then, using (2.9) with f_n and g_n instead of f and g and applying the Lebesgue dominated convergence theorem we get

THEOREM 2.10.

$$\hat{v}(\xi) = \left(\prod_{i=1}^{d} D_{\xi_i}\right)\left(\left(\prod_{i=1}^{d} \alpha_i(\xi)\right) \widehat{J_*(\beta_M)}(\xi)\right), \qquad \xi \in \mathbf{t}.$$

Consider now the representation of T on $T_{p_i}M$, p_i being a fixed point of the action of T on M. Under T, $T_{p_i}M$ breaks up into a direct sum of two-dimensional subspaces indexed by weights

$$\alpha_{i1}, ..., \alpha_{in}, \qquad i = 1, ..., N. \tag{2.11}$$

Moreover, by Remark 2.3, $\mathbf{k}/\mathbf{t}$ sits inside $T_{p_i}M$ as a T-invariant subspace, so the α_{ij}'s contain, for all k, either the root α_k or the root $-\alpha_k$.

DEFINITION 2.12. *Let k_i be the number of k's for which $-\alpha_k$ belongs to the set of weights* (2.11).

Deleting the $\pm\alpha_k$'s from the set (2.11) we get a subset

$$\beta_{i1}, ..., \beta_{im}, \qquad i = 1, ..., N.$$

We now want to "renormalize" these weights as follows: consider in $\mathbf{t}$ the family of hyperplanes

$$\beta_{ij}(\xi) = 0, \qquad i = 1, ..., N, j = 1, ..., m. \tag{2.13}$$

The complement in $\mathbf{t}$ of the union of these hyperplanes is highly disconnected. We will define a *positive Weyl chamber* to be a choice of a connected component of this set. Now let ξ_0 be a fixed vector in the positive Weyl chamber and define

$$\beta_{ij}^{w}(\xi)=\begin{cases}\beta_{ij} & \text{if } \ \beta_{ij}(\sqrt{-1}\,\xi_0)>0\\ -\beta_{ij} & \text{if } \ \beta_{ij}(\sqrt{-1}\,\xi_0)<0.\end{cases}$$

Finally, for fixed i, let w_i be the number of β_{ij}'s for which $\beta_{ij}(\sqrt{-1}\,\xi_0)<0$.

Remark 2.14. The renormalization $\beta_{ij}\to\beta_{ij}^{w}$ does not depend on the choice of ξ_0 but only on the choice of the positive Weyl chamber in which ξ_0 sits. The same is true of the w_i's.

Now let us return to our formula for $\hat{v}(\xi)$ (cf. Theorem 2.10). If $\alpha_{ij}(\xi)\neq 0$ for all $i=1, ..., N$ and all $j=1, ..., n$, we can evaluate the right-hand side of this formula by exact stationary phase (see [4]), obtaining

$$\hat{v}(\xi)=\left(\prod_l D_{\xi_l}\right)\left(\left(\prod_l \alpha_l(\xi)\right)\sum_i\left(\prod_j \alpha_{ij}(\xi)\right)^{-1} e^{\sqrt{-1}\langle J(p_i),\xi\rangle}\right),$$

which we can rewrite (using the above definitions) as

$$\hat{v}(\xi)=\left(\prod_l D_{\xi_l}\right)\sum_i(-1)^{k_i}\left(\prod_j \beta_{ij}(\xi)\right)^{-1} e^{\sqrt{-1}\langle J(p_i),\xi\rangle}.$$

Note that, by continuity, this formula is true for all ξ in $\mathbf{t}$ *not* lying on the union of the hyperplanes (2.13). Finally, we want to renormalize this formula and write it as

$$\hat{v}(\xi)=\left(\prod_l D_{\xi_l}\right)\left(\sum_i(-1)^{k_i+w_i}\left(\prod_j \beta_{ij}^{w}(\xi)\right)^{-1} e^{\sqrt{-1}\langle J(p_i),\xi\rangle}\right).$$

Consider now the distribution on $\mathbf{t}^*$ defined by the sum of convolutions

$$v'=\pi(y)\sum_i(-1)^{k_i+w_i}\,\delta_{J(p_i)} * H_{\beta_{i1}^{w}} * \cdots * H_{\beta_{im}^{w}}, \tag{2.15}$$

$\delta_{J(p_i)}$ being the delta distribution at $J(p_i)$, and H_α (α in $\mathbf{t}^*$) being the distribution

$$\langle H_\alpha, f\rangle=\int_0^\infty f(t\alpha)\,dt, \qquad f\in C_0^\infty(\mathbf{t}^*).$$

Notice that in each summand of (2.15) the distributions occurring in the convolution product are tempered, hence so is (2.15) itself. Notice also that (2.15) is supported in a half-space.

THEOREM 2.16. *The distribution* (2.15) *is identical with* ν.

Proof. By construction $\hat{\nu}-\hat{\nu}'$ is supported on the union of the hyperplanes (2.13) and, since ν and ν' are tempered, $\hat{\nu}-\hat{\nu}'$ is tempered. Thus for some large integer N_0 (see the Appendices, Corollary A.5)

$$\left(\prod_{ij} \beta_{ij}(\xi)\right)^{N_0} (\hat{\nu}-\hat{\nu}')=0.$$

On the Fourier transform side this says that

$$\left(\prod_{ij} D_{\beta_{ij}}\right)^{N_0} (\nu-\nu')=0.$$

However, $\nu-\nu'$ is supported on a half-space so a very simple inductive argument based on Theorem A.1 shows that $\nu-\nu'=0$. Q.E.D.

We conclude this section with a few remarks on formula (2.15).

Remark 2.17. The k_i's in this formula can be computed as follows. If $J(p_i)$ is in $\mathbf{t}_+^*$, then $k_i=0$. In general k_i is congruent modulo two to the number of Weyl group involutions (wall crossings) required to get $J(p_i)$ from the open Weyl chamber in which it sits into $\mathbf{t}_+^*$.

Remark 2.18. Consider the set $X=J^{-1}(\mathbf{t}_{\text{reg}}^*)$; X is a symplectic submanifold of M, T acts on M in a Hamiltonian fashion, and the corresponding moment map is simply the restriction of (2.1) to X, which we will still denote by J (see [10]). Consider the pushforward, $J_*(\beta_X)$, of the Liouville measure, β_X, on X. $J_*(\beta_X)$ is not defined on all of $\mathbf{t}^*$ but only on $\mathbf{t}_{\text{reg}}^*$; however, by a formula of Duflo, Heckman, and Vergne (see [5, (I.1.8)]) one can extend $|\pi(y)|\, J_*(\beta_X)$ to all of $\mathbf{t}^*$ and

$$|\pi(y)|\, J_*(\beta_X)=\nu.$$

Hence from Theorem 2.16 we get an explicit formula for the measure $J_*(\beta_X)$ in terms of the distributions $H_{\beta_{ij}^w}$; notice finally that the β_{ij}'s are nothing but the weights of the representation of T on $T_{p_i}X$.

3. THE "STEINBERG FORMULA"

Let O be a generic orbit of K, that is, an orbit of the co-adjoint action of K on $\mathbf{k}^*$ through a point, μ, in $\mathbf{t}_+^*$. Consider the space $O\times M$ with M

as in the previous section; $O \times M$ is a Hamiltonian K-space with moment map, $\Phi: O \times M \to \mathbf{k}^*$, given by

$$\Phi(\lambda, p) = \lambda + J(p), \qquad (\lambda, p) \in O \times M.$$

Let $\beta_{O \times M}$ be the Liouville measure on $O \times M$ and denote by γ the restriction to $\mathbf{t}^*$ of the measure $\Phi_*(\beta_{O \times M})$.

Using the results of Section 2 we will derive the following

THEOREM 3.1.

$$\gamma = \pi(y) \sum_{u \in W} \varepsilon(u)\, \delta_{u\mu} * (J_T)_*(\beta_M), \tag{3.2}$$

J_T: $M \to \mathbf{t}^*$ being the moment map associated to the action of T on M.

Proof. Notice first of all that the fixed points of the action of T on $O \times M$ are those of the type $q_i = (u\mu, p_i)$ with u in the Weyl group, and with p_i a fixed point of the action of T on M. Moreover the weights of the isotropy representation of T on $T_{q_i}(O \times M) = T_{u\mu} O \oplus T_{p_i} M$ are

$$\alpha_1^u, \ldots, \alpha_d^u, \beta_{i1}, \ldots, \beta_{im},$$

where the α_i^u's are the roots with respect to which $u\mu$ is dominant, and where the β_{ij}'s are the weights of the representation of T on $T_{p_i} M$. Since $\prod \alpha_i^u = \varepsilon(u) \prod \alpha_i$, we get, from formula (2.15),

$$\gamma = \pi(y) \sum_{u \in W} \varepsilon(u) \sum_i (-1)^{w_i}\, \delta_{u\mu + J(p_i)} * H_{\beta_{i1}^w} * \cdots * H_{\beta_{im}^w}.$$

Notice now that $\delta_{u\mu + J(p_i)} = \delta_{u\mu} * \delta_{J(p_i)}$ so that the right-hand side can be rewritten as

$$\pi(y) \sum_{u \in W} \varepsilon(u)\, \delta_{u\mu} * \left(\sum_i (-1)^{w_i}\, \delta_{J(p_i)} * H_{\beta_{i1}^w} * \cdots * H_{\beta_{im}^w} \right).$$

However, by Theorem 2.16 (in the special case $K = T$), the second sum is the measure $(J_T)_*(\beta_M)$, therefore

$$\gamma = \pi(y) \sum_{u \in W} \varepsilon(u)\, \delta_{u\mu} * (J_T)_*(\beta_M). \qquad \text{Q.E.D.}$$

PART II: KOSTANT AND STEINBERG FORMULAS FOR SYMPLECTIC MANIFOLDS

4. Statement and Proof of the Formulas

As in Part I, let M be a compact symplectic manifold and

$$K \to \operatorname{Diff}(M) \tag{4.1}$$

a Hamiltonian action of K on M with moment map

$$J: M \to \mathbf{k}^*. \tag{4.2}$$

We will make the same assumptions about (4.1) that we made in Part I, namely, we will assume that the induced action of T on M has a finite number of fixed points, $p_1, ..., p_N$, and that their images, $J(p_i)$, $i = 1, ..., N$, are in $\mathbf{t}^*_{\text{reg}}$.

Suppose now that, in Kostant's language, the action (4.1) can be quantized; i.e., suppose, to begin with, that the symplectic form on M is *integral*. This means, in Kostant's language, that (4.1) can be prequantized: there exists a Hermitian line bundle

$$L \to M,$$

a connection, ∇, on L, and a lifting of the action (4.1) to L which preserves ∇. Next suppose that, in Kostant's language, there exists a K-invariant positive-definite polarization of M. In other words, suppose there exists a positive-definite Kaehler structure on L compatible with its given symplectic structure. Consider the alternating sum

$$\sum (-1)^r H^r(M, \mathscr{L}), \tag{4.3}$$

$\mathscr{L}$ being the sheaf of holomorphic sections of L. There is a natural representation of K on this "virtual vector space," and this representation is, in Kostant's language, the *quantization* of the action (4.1). We will henceforth assume that all terms except the $r = 0$ term in the sum (4.3) are zero, so that we can think of the quantization of (4.1) as being an honest representation

$$\rho: K \to \mathscr{U}(V), \tag{4.4}$$

where $V = H^0(M, \mathscr{L})$ is equipped with its intrinsic K-invariant Hermitian structure (as a finite-dimensional subspace of the space of square-integrable sections of L).

For every integer lattice point, $\mu \in \mathbf{t}_+^*$, we will denote by $\#(\mu, \rho)$ the multiplicity with which the irreducible representation of K with maximal weight μ occurs in ρ. The "Kostant–Steinberg formula" alluded to above is a recipe for computing $\#(\mu, \rho)$ in terms of symplectic invariants of the K-action (4.1). It bears a very close resemblance to the "Heckman formula" we derived in Part I and, in fact, the Heckman formula can be viewed as a continuous limit of the formula, (4.42), we are about to prove below (see Appendix B).

Our strategy for getting hold of $\#(\mu, \rho)$ will be studying the measure

$$\sum \varepsilon(w)\ \#(\mu, \rho)\, \mathbf{m}(w(\mu+\delta)-\delta) \tag{4.5}$$

on $\mathbf{t}$. (In this expression, $\mathbf{m}(\alpha)$ is the Dirac delta measure concentrated at the point α, δ is half the sum of the positive roots of the group K, and the summation is over the lattice points, μ, of $\mathbf{t}_+^*$ and the elements, w, of the Weyl group.) The first step involved in computing this measure will be to compute its Fourier transform at generic points of $\mathbf{t}$, and this in turn will involve two ingredients:

(i) A modified version of the Lefschetz formula for compact groups described in Section 3 of Atiyah and Bott [1]. This will play the same role here as the exact stationary phase formula did in Part I.

(ii) A technical lemma of Kostant which will play the same role here as did Lemma 2.5 in Part I.

We will begin by describing (ii). Let G be the complexification of the group K and let $\mathbf{g}$ be its Lie algebra. Then $\mathbf{g}$ has a decomposition into subalgebras

$$\mathbf{g} = \mathbf{g}_+ \oplus \mathbf{t}_{\mathbf{C}} \oplus \mathbf{g}_-, \tag{4.6}$$

where

$$\mathbf{g}_+ = \sum \mathbf{g}_\alpha, \qquad \alpha > 0 \tag{4.7}$$

(the sum of the root spaces corresponding to the positive roots of $\mathbf{g}$) and $\mathbf{g}_-$ is the analogous sum over the negative roots. Now let ρ_μ be the irreducible representation of G of maximal weight μ and let V_μ be the finite-dimensional vector space on which this representation lives. This representation can be viewed infinitesimally as a representation of $\mathbf{g}$ on V_μ, and by restriction, as a representation of $\mathbf{g}_+$ on V_μ. In particular, thinking of V_μ as a $\mathbf{g}_+$-module we can consider its cohomology groups

$$H^i(\mathbf{g}_+, V_\mu), \qquad i = 0, ..., \dim \mathbf{g}_+. \tag{4.8}$$

(See, for instance, Jacobson [15, Chap. III]). Moreover, since T acts on V_μ as a group of automorphisms of V_μ (qua $\mathbf{g}_+$-module) there is a natural representation of T on the spaces (4.8).

PROPOSITION 4.9. (See Kostant [17] or Bott [2]). *The weights of the representation of T on the space* (4.8) *are exactly all the weights of the form*

$$w(\mu+\delta)-\delta,$$

w being an element of the Weyl group of length i. Moreover, each of these weights occurs in (4.8) *with multipicity one.* (*Here $\delta=\frac{1}{2}\sum\alpha$, the sum taken over the positive roots.*)

Next let us turn to item (i). The Lefschetz formula in Section 3 of Atiyah and Bott [1] describes the action of T on the cohomology groups of the Dolbeault complex

$$0 \longrightarrow L \xrightarrow{\bar\partial} L\otimes \Lambda^{0,1} \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} L\otimes\Lambda^{0,n} \longrightarrow 0. \tag{4.10}$$

What we will need is a variant of their result for a bi-complex of differential operators of which (4.10) is the "horizontal part." To manufacture this bi-complex we first observe that the action of K on M extends to an action of G on M which is no longer Hamiltonian but is still holomorphic. Moreover, this holomorphic action of G on M lifts to a holomorphic action of G on L (see, for instance, [11]). The infinitesimal version of this action is a rule which assigns to each element ξ of $\mathbf{g}$ a first-order differential operator

$$D_\xi : \Gamma(L) \to \Gamma(L),$$

and, more generally, a first-order differential operator

$$D_\xi : \Gamma(L\otimes\Lambda^{0,i}) \to \Gamma(L\otimes\Lambda^{0,i}), \tag{4.11}$$

which commutes with $\bar\partial$. Now set

$$C^{i,j}=\Gamma(L\otimes\Lambda^{0,i}\otimes\Lambda^j(\mathbf{g}_+^*)). \tag{4.12}$$

We will define a first-order differential operator

$$d: C^{i,j}\to C^{i,j+1} \tag{4.13}$$

by the formula

$$d(f\otimes\beta)=\sum D_{\xi_\alpha} f\otimes\xi_\alpha^*\wedge\beta+f\otimes C^\alpha_{\gamma\delta}\xi_\gamma^*\wedge\xi_\delta^*\,\iota(\xi_\alpha)\,\beta. \tag{4.14}$$

(Here f is a section of $L\otimes\Lambda^{0,i}$, β an element of $\Lambda^j(\mathbf{g}_+^*)$, ξ_α a basis vector of $\mathbf{g}_\alpha$, ξ_α^* the dual basis vector of $\mathbf{g}_\alpha^*$, and $C_{\gamma\delta}^\alpha$ the structure constants of $\mathbf{g}_+$ with respect to the ξ_α's.) It is easy to check that $d^2=d\bar\partial+\bar\partial d=0$, so $\{C^{i,j},\bar\partial,d\}$ is a bi-complex in the sense of Godement [7]; moreover there is a natural action of T on this bi-complex which commutes with $\bar\partial$ and d. We will be interested in the cohomology of the associated complex

$$\{C^*,\bar\partial+d\},\tag{4.15}$$

where

$$(C^*)^i=\sum_{j+k=i}C^{j,k}.\tag{4.16}$$

Ignoring the d-cohomology of this complex, we can compute its $\bar\partial$-cohomology just by tensoring all the terms of (4.10) by $\Lambda^j(\mathbf{g}_+^*)$ for $j=0,1,\ldots$. Since (4.10) computes the cohomology groups, $H^i(M,\mathscr{L})$, we have, by assumption, $E_1^{i,j}=0$ for $i>0$ and

$$E_1^{0,j}=V\otimes\Lambda^j(\mathbf{g}_+^*),$$

where V is, as above, $H^0(M,\mathscr{L})$. Hence by Theorem 4.4.1 of Godement [7] the ith cohomology group of the complex (4.15) is just

$$\sum\#(\mu,\rho)\,H^i(\mathbf{g}_+,V_\mu)\tag{4.17}$$

and, by Proposition 4.9, the trace of T on (4.17) is

$$\sum\#(\mu,\rho)\,e^{\sqrt{-1}(w\mu+w\delta-\delta)},\tag{4.18}$$

the sum being over all lattice points μ in $\mathbf{t}_+$ and all Weyl group elements, w, of length i (i.e., for every element $\xi\in\mathbf{t}$ the trace of $\exp\xi\in T$ is just (4.18) evaluated on ξ). Thus the alternating sum of the traces is

$$\sum\varepsilon(w)\;\#(\mu,\rho)\,e^{\sqrt{-1}(w\mu+w\delta-\delta)}\tag{4.19}$$

summed over all lattice points $\mu\in\mathbf{t}_+^*$ and *all* elements of the Weyl group. Notice, however, that (4.19) is just the Fourier transform of the measure (4.5). We will evaluate the Fourier transform of (4.5) on generic elements of $\mathbf{t}$ by applying the Atiyah and Bott fixed point formula to the complex (4.15). However, to justify this application of Atiyah and Bott we must verify

LEMMA 4.20. *The complex* (4.15) *is elliptic.*

Proof. We must show that for every p in M and every non-zero covector, ξ, in the cotangent space to p at M, the symbol complex

$$\{C_p^*, \sigma(\bar{\partial}+d)(\xi)\}$$

is exact. This, however, follows from the theorem of Godement cited above. The cohomology of this symbol complex is the E_∞ term in a spectral sequence whose E_1 term is the cohomology of the complex

$$\{C_p^*, \sigma(\bar{\partial})(\xi)\},$$

which is zero since $\sigma(\bar{\partial})(\xi)$ is exact. Q.E.D.

For generic elements of $\mathbf{t}$ the fixed points of $\exp \xi$ are $p_1, ..., p_N$; so the local contributions to the fixed point formula come from these points. To compute these contributions, let us, as in Part I, denote by

$$\alpha_{ij}, \qquad j=1, ..., n \tag{4.21}$$

the weights of the isotropy representation of T on $T_{p_i}M$. We will say that $\xi \in \mathbf{t}$ is *generic* if

$$\alpha_{ij}(\xi) \neq 0 \qquad \text{mod } 2\pi\mathbf{Z}. \tag{4.22}$$

If this condition holds then, according to [1, Sect. 3], the local Lefschetz number of $\exp(\xi)$ at p_i is the product of the three terms:

(a) The *Dolbeault* contribution: this is just the product

$$\prod_j (1-e^{\sqrt{-1}\alpha_{ij}(\xi)})^{-1}. \tag{4.23}$$

(b) The trace of $\exp(\xi)$ on L_{p_i}: this is just

$$e^{\sqrt{-1}\langle J(p_i), \xi\rangle}. \tag{4.24}$$

(See, for instance, Guillemin and Sternberg [10].)

(c) The alternating sum of the traces of $\exp(\xi)$ on the spaces $\Lambda^i(\mathbf{g}_+^*)$. By elementary linear algebra this is the product

$$\prod_l (1-e^{\sqrt{-1}\alpha_l(\xi)}). \tag{4.25}$$

Multiplying (a), (b), and (c) together and summing over i we get the value of the Fourier transform of (4.5) on generic elements of $\mathbf{t}$:

$$\prod_l (1-e^{\sqrt{(-1)}\alpha_l(\xi)}) \left(\sum_i \left(\prod_j (1-e^{\sqrt{-1}\alpha_{ij}(\xi)})^{-1}\right) e^{\sqrt{-1}\langle J(p_i), \xi\rangle}\right). \tag{4.26}$$

We will now *renormalize* this formula as we did in Part I. To begin with we can, as in Part I, get rid of the first factor in (4.26). Since $J(p_i)$ is in $\mathfrak{t}^*_{\mathrm{reg}}$ there exists a unique element, σ_i, in the Weyl group such that

$$\sigma_i^{-1} J(p_i) \subseteq \mathrm{Int}(\mathfrak{t}^*_+).$$

Moreover, the root vectors $\sigma_i \alpha_l$ are contained among the α_{ij}'s; so we can factor out of the ith item in the sum (4.26) the expression

$$\prod_l (1 - e^{\sqrt{-1}\alpha_l(\xi)})(1 - e^{\sqrt{-1}\langle \sigma_i \alpha_l, \xi \rangle})^{-1}$$

and after a short computation rewrite this as

$$(-1)^{\sigma_i} e^{\sqrt{-1}(\delta - \sigma_i \delta)},$$

where

$$\delta = \tfrac{1}{2} \sum \alpha_l.$$

Thus (4.26) takes the much simpler form

$$\sum_i (-1)^{\sigma_i} e^{\sqrt{-1}\langle J(p_i) + \delta - \sigma_i \delta, \xi \rangle} \left(\prod_j (1 - e^{\sqrt{-1}\beta_{ij}(\xi)})^{-1} \right), \tag{4.27}$$

where

$$\beta_{i1}, ..., \beta_{im} \tag{4.28}$$

is the set of weights obtained by deleting $\sigma_i \alpha_1, ..., \sigma_i \alpha_d$ from the set (4.21).

We will next renormalize the β_{ij}'s as in Part I, i.e., we will select a connected component of the set

$$\beta_{ij}(\xi) \neq 0 \qquad \text{for all } i \text{ and } j \tag{4.29}$$

and designate it to be our positive Weyl chamber; and having made this arbitrary choice, define w_i and β^w_{ij} as in Part I. We will also need the notation

$$\beta_i = \tfrac{1}{2} \sum_j \beta_{ij} \tag{4.30}$$

and

$$\beta^w_i = \tfrac{1}{2} \sum \beta^w_{ij}. \tag{4.31}$$

Then after a little fiddling we can rewrite the factor involving the β_{ij}'s in (4.27) as

$$(-1)^{w_i} e^{\sqrt{-1}(\beta_i - \beta_i^w)} \prod_j (1 - e^{\sqrt{-1}\beta_{ij}^w})^{-1}$$

evaluated on ξ. Thus (4.27) itself can be rewritten as the sum over i:

$$\sum (-1)^{\sigma_i + w_i} e^{\sqrt{-1}(J(p_i) + \delta - \sigma_i\delta + \beta_i - \beta_i^w)} \prod (1 - e^{\sqrt{-1}\beta_{ij}^w})^{-1} \tag{4.32}$$

evaluated on ξ. To summarize what we have proved so far: *the expression* (4.32) *is equal, by the Atiyah–Bott Lefschetz theorem, to the Fourier transform of the measure* (4.5) *evaluated at* ξ, *providing* ξ *satisfies the inequalities* (4.22). We will now get rid of the denominators in (4.32) by means of a classical "trick" that goes back to Euler. Let us introduce the counting function, $P_i(\mu)$, which for every integer lattice point, μ, in $\mathbf{t}$ counts the number of ways that μ can be written as a sum

$$\sum_j k_j \beta_{ij}^w,$$

the k_j's being non-negative integers. Then a little manipulation shows that formally

$$\prod_j (1 - e^{\sqrt{-1}\beta_{ij}^w})^{-1} = \sum P_i(\mu) e^{\sqrt{-1}\mu}. \tag{4.33}$$

(See, for instance, Humphreys [14, p. 136].) Thus we can formally rewrite (4.32) as a sum over i:

$$\sum (-1)^{\sigma_i + w_i} e^{\sqrt{-1}(J(p_i) + \delta - \sigma_i\delta + \beta_i - \beta_i^w)} \left(\sum P_i(\mu) e^{\sqrt{-1}\mu}\right)$$

or, alternatively, as a sum over i:

$$\sum (-1)^{\sigma_i + w_i} P_i(\lambda + \sigma_i\delta - \delta + \beta_i^w - \beta_i - J(p_i)) e^{\sqrt{-1}\lambda}. \tag{4.34}$$

This expression is formally the Fourier transform of the measure

$$\sum (-1)^{\sigma_i + w_i} P_i(\lambda + \sigma_i\delta - \delta + \beta_i^w - \beta_i - J(p_i)) \mathbf{m}(\lambda) \tag{4.35}$$

summed over the integer lattice points in $\mathbf{t}^*$ and over $i = 1, \ldots, N$. However, simple estimates on $P_i(\lambda)$ show that $P_i(\lambda)$ has polynomial growth in λ as $|\lambda| \to \infty$. Hence the measure (4.35) is a *tempered* measure on $\mathbf{t}^*$, and its

Fourier transform is therefore a tempered distribution on $\mathbf{t}$ which is equal, on the set (4.22), to the function (4.32).

Finally, notice that (4.35) is supported in a half-space and hence so is the difference between (4.5) and (4.35). By the main result of this section, the difference between their Fourier transforms is supported on the union of the hyperplanes

$$\beta_{ij} = 0 \qquad \bmod 2\pi\mathbf{Z}. \tag{4.36}$$

Thus (to repeat the punch line in our proof of the Heckman formula in Part I) this difference has to be identically zero on *all* of $\mathbf{t}$. In other words, (4.5) and (4.35) have to be identical. Making the substitution

$$\lambda = w(\mu + \delta) - \delta.$$

in (4.5) (and defining $\#(\mu, \rho)$ to be zero if μ is not in $\mathbf{t}_+^*$) we can rewrite (4.5) in the form

$$\sum \varepsilon(w)\ \#(w(\lambda + \delta) - \delta, \rho)\, \mathbf{m}(\lambda);$$

and hence, comparing (4.5) and (4.35), we obtain the following

LEMMA 4.37. *For all integer lattice points, $\lambda \in \mathbf{t}^*$, the alternating sums*

$$\sum \varepsilon(w)\ \#(w(\lambda + \delta) - \delta, \rho) \tag{4.38}$$

and

$$\sum (-1)^{\sigma_i + w_i} P_i(\lambda + \sigma_i\delta - \delta + \beta_i^w - \beta_i - J(p_i)) \tag{4.39}$$

are identical.

Suppose now that λ is in $\mathbf{t}_+^*$. For the following, see Humphreys [14, p. 72, Exercise 9].

LEMMA 4.20. *The weight $w(\lambda + \delta) - \delta$ is in $\mathbf{t}_+^*$ if and only if w is the identity element.*

Thus if λ is in $\mathbf{t}_+^*$, all terms in the alternating sum (4.38) are zero except the term corresponding to the identity, and we obtain the following symplectic version of the Steinberg formula:

THEOREM 4.41. *Let ρ be a representation of K which "quantizes" (4.1), and let λ be an integer lattice point in $\mathbf{t}_+$. Then the multiplicity, $\#(\lambda, \rho)$,* *with which the irreducible representation of K with maximal weight λ occurs in ρ is given by the alternating sum*

$$\sum (-1)^{\sigma_i + w_i} P_i(\lambda + \sigma_i \delta - \delta + \beta_i^w - \beta_i - J(p_i)). \tag{4.42}$$

(*In particular, all quantizations of* (4.1) *are unitarily equivalent*!)

PART III: APPENDICES

A. Two Elementary Theorems about Distributions

In our proof of the Heckman formula we needed the following two elementary results.

THEOREM A.1. *Given v in $\mathbf{R}^n$ and l in $(\mathbf{R}^n)^*$ with $\langle l, v \rangle \neq 0$ let ϕ be a distribution on $\mathbf{R}^n$ which satisfies*

(i) $D_v \phi = 0$,

(ii) $\phi \equiv 0$ *on the half-space* $l(x) < 0$.

Then $\phi = 0$ everywhere.

Proof. One can take $\phi = x_n$ and $v = (0, ..., 1)$. The theorem is an easy consequence of the

LEMMA A.2. *Given any function f in $C_0^\infty(\mathbf{R}^n)$, there exist g and h in $C_0^\infty(\mathbf{R}^n)$ such that h is supported in the half-space, $\{x_n < 0\}$, and $f = \partial g / \partial x_n + h$.*

THEOREM A.3. *Let ϕ be a tempered distribution on $\mathbf{R}^n$. Suppose ϕ is supported on the hyperplane $\{x_n = 0\}$. Then for some integer N, $x_n^N \phi = 0$.*

Proof. To say that ϕ is tempered implies that it is continuous with respect to the topology on the Schwartz space $\mathscr{S}(\mathbf{R}^n)$ defined by the semi-norm

$$\|f\|_{N-1} = \sup \sum_{|\alpha| + |\beta| \leqslant N-1} |x^\alpha D^\beta f|$$

for some sufficiently large N. In particular it extends to a continuous linear functional on the space, $C_0^{N-1}(\mathbf{R}^n)$, of compactly supported functions which are $(N-1)$-times differentiable. Now let f_ε be the following function of one variable:

$$f_\varepsilon(t)=\begin{cases}(t-\varepsilon)^N, & t\geqslant\varepsilon\\ 0, & -\varepsilon<t<\varepsilon\\ (t+\varepsilon)^N, & t\leqslant-\varepsilon.\end{cases}$$

LEMMA A.4. *$f_\varepsilon(t)$ converges uniformly to t^N together with all derivatives of order less or equal to $N-1$ on any compact subinterval of the real line as ε tends to zero.*

To prove Theorem A.3 let g be in $C_0^\infty(\mathbf{R}^n)$. Since $f_\varepsilon(x_n)\,g$ is in C_0^{N-1} and is supported on the set $\{x_n\neq 0\}$, $\phi(f_\varepsilon(x_n)\,g)=0$. However, $f_\varepsilon(x_n)\,g$ tends to $x_n^N g$ in C_0^{N-1} by the lemma, so

$$0=\phi(f_\varepsilon(x_n)\,g)\to\phi(x_n^N g)=(x_n^N\phi)(g).$$

as ε tends to zero, proving that $x_n^N\equiv 0$.

COROLLARY A.5. *Let ϕ be a tempered distribution on $\mathbf{R}^n$. Suppose ϕ is supported on a finite union of hyperplanes, $l_i(x)=0$, $i=1,\ldots,r$; then for some N*

$$\left(\prod l_i(x)\right)^N\phi(x)\equiv 0.$$

B. THE MEASURE $H_{\alpha_1}*\cdots*H_{\alpha_N}$

Let V be a vector space and α a non-zero vector in V. The right-hand side of the Heckman formula is a sum of convolution products of the form

$$H_{\alpha_1}*\cdots*H_{\alpha_N}, \tag{B.1}$$

where the α_i's are a spanning set of vectors in V (the vector space, V, in the Heckman formula being $\mathbf{t}^*$). Here is another description of this measure: let

$$\mathbf{R}_+^N=\{(s_1,\ldots,s_N),\ s_i\geqslant 0,\ i=1,\ldots,N\}$$

be the positive orthant in $\mathbf{R}^N$, and let

$$L\colon\mathbf{R}_+^N\to V \tag{B.2}$$

be the mapping which sends $(s_1, \ldots, s_N)$ onto $s_1\alpha_1 + \cdots + s_N\alpha_N$. Suppose that for some ξ_0 in V^*

$$\langle \alpha_i, \xi_0 \rangle > 0, \qquad \text{for all } i.$$

Then the mapping L is *proper*, so the measure

$$(L)_*(ds_1 \cdots ds_N) \tag{B.3}$$

is well defined.

THEOREM B.4. *The measures* (B.1) *and* (B.3) *are identical.*

Proof. In the special case of $V = \mathbf{R}^N$ and α_i equal to the ith standard basis vector of $\mathbf{R}^N$, e_i, this is just the Fubini theorem. Thus one can write the Lebesgue measure on $\mathbf{R}^N_+$ as a convolution product

$$H_{e_1} * \cdots * H_{e_N}.$$

The theorem follows from the fact that $L_* H_{e_i} = H_{\alpha_i}$ and the fact that for any pair of measures, μ_1 and μ_2, with support in $\mathbf{R}^N_+$, $L_*(\mu_1 * \mu_2) = L_*\mu_1 * L_*\mu_2$. Q.E.D.

Let $(x_1, \ldots, x_n)$ be a system of coordinates in V. By the Radon–Nikodym theorem there exists a non-negative locally summable function $P = P(x_1, \ldots, x_n)$ such that

$$(L_*)(ds_1 \cdots ds_N) = P(x_1, \ldots, x_n)\, dx_1 \cdots dx_n, \tag{B.5}$$

and in fact it is not hard to see that, *up to a multiplicative factor*, $P(\alpha)$ *is the Euclidean volume of the set of points* $(s_1, \ldots, s_N)$ *in* $\mathbf{R}^N_+$ *for which*

$$\alpha = s_1\alpha_1 + \cdots + s_N\alpha_N. \tag{B.6}$$

Let us identify V with R^n by means of the coordinates $(x_1, \ldots, x_n)$ and let us suppose that the α_i's belong to the integer lattice, $\mathbf{Z}^n$. Then if α belongs to $\mathbf{Z}^n$ (and is very large) the volume of the set of s's satisfying (2.6) is approximately equal to the number, $N(\alpha)$, of integer solutions of the equation

$$\alpha = k_1\alpha_1 + \cdots + k_N\alpha_N,$$

the k_i's being non-negative integers. Thus the $P(\alpha)$ on the right-hand side of (2.5) satisfies

$$P(\alpha) \sim N(\alpha)$$

for α large. Using this fact one can deduce the "Heckman formula" in Part I from the "Kostant–Steinberg formula" in Part II just as Heckman deduced his original formula from the standard Kostant formula (see [12]).

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