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Digital Experiences of Mathematical Cognitive Functions in Learning the Basic Concepts of General Topology

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Abstract

This paper aims at defining and exploring design principles in a distance technological setting for an educational activity for mathematics undergraduate students, devoted to the construction of basic concepts in general topology, the promotion of problem-solving processes, the development of metacognitive aspects, and, in general, the development of the students' mathematical identity. The design exploits the production of examples and investigation of variations and invariants, exploration of problems and generation of conjectures, and an extension intertwining of the 'inside-out' model from the Digital Interactive Storytelling in Mathematics with the Thinking Classroom model at university education. We present a didactic activity based on the identified design principles and discuss the preliminary results of a pilot carried out with fifty mathematics undergraduate students, attending their second year of the mathematics degree.

Keywords Digital role-play collaboration · Mathematical identity · General topology · Example generation

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Introduction

General topology is an interesting and fundamental field that studies the properties of abstract and general mathematical objects. The abstractness and generality make the basic concepts of topology some powerful theoretical and working tools in many areas, not only in mathematics, and, on the other side, can be a source of difficulties for undergraduate students. General topology involves an exquisitely "mathematical eye", fundamental to undergraduate students' education, that requires particular educational attention. Let's take an example. Think about the property of a function as being continuous. Is the identity function continuous or not? The first answer could seem to be that a function as simple as the identity function is, of course, continuous! However, we know that it depends on the topology at stake! So, once you have defined a topology on the set X (every topology you want), the function f, where f(x) = x for all x in X, is obviously continuous. However, in general, the answer is negative! This is because continuity depends both on the topology of the domain and on the codomain of the function. For example, if the topology of X as the domain of the function f is the trivial topology and we consider the space X with the discrete topology as the codomain, the function is not continuous! Therefore, continuity is not an absolute property of a function! It depends on the topology. Similar arguments hold for other concepts, such as compactness or connectedness.

Moving from considering some familiar properties (i.e., continuity) as absolute to relative requires a change of perspective very deep from the mathematical viewpoint and relevant to mathematical expertise. Thus, general topology is a topic that offers opportunities for such a shift due to a different point of view on definitions and it doesn't only have an instrumental role for the advanced courses in analysis. Moreover, activities with general topology can engage the students in real problem-solving processes involving exploration of problems, generation of conjecture, argumentation, and proof, and can offer the experience of engagement in a construction of examples and counterexamples very different from the usual mathematical objects.

This paper aims at defining and exploring design principles in a distance technological setting for an educational activity for mathematics undergraduate students, devoted to reach the following didactical objectives: construction of basic concepts in general topology; promotion of problem-solving processes; development of metacognitive aspects, and, in general, development of the students' mathematical identity. A technology-enhanced environment is pivotal to the design for promoting a participation of students that allows them to be engaged in problem solving and at the same time to reflect on the on-going process from two points of view. On one hand, students activate, share and exploit their knowledge to solve the problem at stake and to control their process, and on the other hand, each participant looks at that process using a specific lens due to a specific cognitive role personified.

Previous studies (Albano et al., 2021) on the promotion of metacognitive reflections and the development of students' mathematical identity have given rise to a model, so-called 'inside-out' model. In this paper, we extend the model incorporating also epistemological and cognitive elements to design and manage activities for undergraduate students.

We present the design of a didactic activity, based on the identified design principles, and discuss the preliminary results of a pilot carried out with second year mathematics undergraduate students (see Albano et al., 2022).

Conceptual Background

To define the design principles in a distance technological setting for an educational activity for mathematics undergraduate students, the elements to be taken into consideration concern various aspects that are involved in the activity: the methodological choices for the activity management, including relationships with and between students (methodological plane), an epistemological analysis of concepts of general topology and the cognitive aspects related to the initial learning of the concepts (cognitive-epistemological plane). In particular, the cognitive-epistemological principles of design will have the role of guiding the formulation of tasks aimed at promoting cognitive processes considered crucial in relation to mathematical knowledge (general topology) which is the object of the educational objective. The design principles of the methodological plane will guide the management of tasks in the technological environment.

Methodological Plane

The didactical organisation was guided by the main methodological assumption, which is the importance of reflection to promote deep understanding. To this aim, we start from considering writing as pivotal to learning: to construct knowledge, to go back and forward along the constructed knowledge, to reflect on what is emerging or emerged. We exploited two models that are in line with this aim: the 'inside-out' model (Albano et al., 2021) and the 'thinking classroom' (Lilijedhal, 2016), that we are going to describe in the following.

Writing-to-learn

A basic element assumed in the design that we consider pivotal to promote cognition and strictly linked to argumentation is the widely recognised benefits of writing-to-learn (Morgan, 1998). An important attribute of writing is that it produces something that lasts over time, which can be used during and after the writing process. This allows the writer to reread, integrate, and edit at the very moment she is producing the text, as well as to reflect on and review it afterwards (Morgan, 2002). It is worthwhile to note that we assume that mathematical writing should take into account the main characteristics of mathematical discourse: multisemioticity and multivariety (Ferrari, 2020). The former refers to the great variety of semiotic representations used in mathematics: verbal language, symbolic notations, and figural representations. The latter refers to the different registers, intended as a linguistic variety based on use, due to the two-fold function of the verbal language: to represent mathematical knowledge and be able to communicate with each other. The former shares the features of the literate registers, usually used in textbooks, while the latter refers to colloquial registers used by people sharing a context such as in everyday communication. Therefore, it is important to provide students with opportunities to produce texts making use of appropriate representations of mathematical concepts as well as to use colloquial registers, which are needed, for instance, at the beginning of all mathematical activities (Albano & Ferrari, 2013). In this respect, an e-learning platform can offer the chance to use a large variety of both semiotic systems and linguistic registers.

The Inside-out Model

In the following, we detail the so called 'inside-out' model (Albano et al., 2021), aimed at reaching the following two basic goals:

1) to let the students be in the mind of a mathematician as a problem solver and experience the various cognitive functions coming into play during the problem solving process;
2) to foster the students' awareness of the above cognitive functions and of the importance of all of them in a collaborative flavour.

The Cognitive Roles With respect to item 1), the 'inside-out' model identified five cognitive functions, becoming cognitive roles personified by characters of a story-problem, that are:

- Boss, i.e., the *organiser function*, is essential to carry out the whole problem-solving process. She is both task and group oriented, keeping track of what is happening and taking care of the members' participation.
- Blogger, i.e., the *editor function*, required to produce a publishable text. She is in charge of summarising what the group produced (i.e. arguments, observations, doubts, questions, and answers) and presenting it according to literate registers.
- Critical Mind, i.e., the *critical thinking function*, necessary to test and validate the findings. She is in charge of promoting the validation of the group processes and products.
- Promoter, i.e., the *scouting function*, needed to initiate the problem-solving process. She looks for insights, also in cases of impasse, based on the group's knowledge or searching outside (e.g., external resources or asking an expert).
- Guru, i.e., the *knowledge and wisdom function*. She is the expert in the Vygotskian perspective.

The Appropriation of the Cognitive Roles With respect to item 2), the 'inside-out' model assumes that the appropriation of roles can be fostered through social practice. In this respect, it stresses the importance of reflecting more than acting. In order to promote such reflections, the model foresees splitting students into groups of (generally) four people (eventually, Critical Mind can be duplicated), and the groups alternatively engaged as ‘Actors’ or ‘Onlookers’, along the episode of the story(-problem). For each episode, one group is engaged as Actors, that is actively engaged in solving the problem at stake, and the other ones are engaged as Onlookers, that is observers from an external point of view with a reflective lens what happens in the Actors group. Thus, a two-layer structure of engagement can be devised.

Regardless of the modality of the group involvement, each member personifies one cognitive role, except Guru (played by the teacher or generally by an expert). Depending on whether the member is an Actor or an Onlooker, she is respectively required to contribute or reflect on the observed character paying attention to the point of view of a specific cognitive role personified, with respect to both the task and the role played.

Moreover, the model foresees that both the groups of Actors and Onlookers rotate, as well as the roles within each group. This means that: a) each group plays as Onlookers most often; b) for each episode, each student plays a cognitive role different from the ones already played (if any). So, each student experiences (as Actor or Onlooker) all cognitive functions (as the story lasts more than four episodes), and this can favour the appropriation of these functions by each student.

Roles and Writing The model also assumes the importance of writing-to-learn with respect to the cognitive functions identified. Thus the students are asked to:

- interact among themselves exclusively in written form (e.g., in the pilot they are required to communicate using a chat) for solving the given problem;
- at the end of the episodes of the story-problem, produce a shared (for each group) mathematical narrative reporting entirely how they reach the solution of the problem;
- along the episodes of the story-problem, write a personal logbook reporting reflections on the role played or observed, with respect to how the role has been played, which contribution have been given, how useful and why the interventions have been, and what a posteriori changes she would make in playing the role or what she would have done in the shoes of the observed role.

The Thinking Classroom Model

The notion of a Thinking Classroom (Liljedahl, 2016) stems from the realisation that what was missing in a lot of observed classroom practices was that the students were not thinking when engaged in problem solving packaged enough to get stuck and then persist through being stuck enough to experience an illumination moment about a key idea to unlock the problem solving process. Such realisation motivated the necessity to understand how to transform a non-thinking classroom into a thinking one (Liljedahl, 2022) and to find a way to build a culture of thinking, both for the students and the teachers, which led to the definition of a thinking classroom:

> A thinking classroom is a classroom that is not only conducive to thinking but also occasions thinking, a space that is inhabited by thinking individuals as well as individuals thinking collectively, learning together, and construct-

> ing knowledge and understanding through activity and discussion. It is a space wherein the teacher not only fosters thinking but also expects it, both implicitly and explicitly (Liljedahl, 2016, p. 364).

This definition is in tune with studies on mathematical thinking (Dahlberg & Housman, 1997; Watson & Mason, 2005). Random grouping on a regular basis is the most effective technique to create a thinking classroom. Students will require a work surface once they are in groups so they may display their group thinking. A cloud-based digital whiteboard is used to accomplish this, and students can use a mouse or stylus to doodle, add text boxes or sticky notes, add images, and erase (Liljedahl, 2022). Ultimately, it facilitates group collaboration by enabling simultaneous work inside the same representational area, thereby simulating the traditional model's characteristic of working on a vertical, non-permanent surface. Just as in a traditional thinking classroom, when students are working on a task in a virtual synchronous setting, "proximity questions", asked when the teacher is close, and "stop thinking" questions, most often of the form "is this right?" should not be answered by the instructor. Only keep thinking questions, questions that students ask so they can get back to work should be answered (Liljedahl, 2016).

Methodological Design Principles

In summary, the assumption of the above section brought us to define the following methodological principles:

> M1 **-** to plan activities that force students' writing and require the use of various semiotic systems;
> M2 **-** to provide students with environments, such as non-permanent digital or non-digital whiteboards, that allow them to interact in written form and to take meaningful notes, giving them the opportunity to use colloquial as well as literate registers on their choice;
> M3 **-** to split students into random working groups of 4 (if possible) students for solving the problems, engaging them according to the modalities of Actors and Onlookers and to ensure that each group works at least one time as Actors; and engaging students in a group according to the various cognitive roles;
> M4 **-** to promote reflection on their writings at two levels: a) on the cognitive roles by means of personal logbooks; b) on mathematics at stake requiring each group to produce a publishable mathematical text concerning their own problem solving process;
> M5 **-** to give thinking tasks, that is, highly engaging, non-standard tasks to think about, requiring the construction of new mathematical objects different from the conventional ones discussed during the lectures;
> M6 **-** to answer only keep-thinking questions, only hints, when students are working on tasks; so, as an example, if students ask, "Is this right?" the expert could respond with, "I don't know. Is there a way we can check if it is?", as a groupmate (i.e. Guru);
> M7 **-** to ask check-your-understanding questions to help students see where they are and where they are going. The goal is to give questions to check groups'

understanding to begin triggering the transfer of collective knowing and doing to individual knowing and doing.

Technology as Pivotal It is worthwhile to note that the above methodological design principles cannot disregard a technological component that is itself part of the methodological plane. Indeed, a digital environment should be foreseen that allows the participants to:

– be engaged in solving a problem according to different engagement layers modalities, Actors or Onlookers, and to different roles: to do that, students log into a digital environment where Actors can interact among themselves in written form (e.g. chat or forum or whiteboard) and Onlookers can follow the problem-solving process without intervening and discussing in their own chat room about what the Actors are doing with respect to the problem and the roles assigned;
– discuss the problem at stake using various linguistic registers as well as various semiotic systems, that can be supported by a digital whiteboard allowing students to put notes on a stick, to attach pieces of mathematical knowledge like excerpts from a textbook, to sketch figures, to draw conjectures or reasoning, to connect ideas on the whiteboard, and so on;
– reflect on the cognitive role played or observed: to do that, students log into the digital environment as characters personified a cognitive role and along the educational path they are required to fulfil a personal logbook;
– reflect on the mathematical problem at stake and on the solution produced: to do that, at any moment students can go back and forward along the end of the solving process, they can look back at the whole process accessing the previous chats, or whiteboard, or any material produced, and work on a shared document to produce a collective mathematical narrative.

Epistemological-cognitive Plane

The literature on teaching and learning topology is very limited, but in recent years, research in mathematics education, recognising topology as a crucial component of the basic language of mathematics, has devolved increasing attention to the power of topology for the development of advanced mathematical thinking (Gallagher & Infante, 2019a, b, 2022; Kontorovich & Greenwood, 2023; Miranda, 2023a, b).

The epistemological-cognitive plane develops from the epistemological analysis of the mathematical knowledge that is the goal of the teaching; on the basis of this analysis, specific research in mathematics education was identified to plan the formulation of the tasks.

Epistemological Analysis

In his seminal book on the fundamental concepts of general topology, Willard (1970) explains his view on this field, identifying two areas in general topology:

> The first, which could be called “continuous topology”, centres on the results about compactness and metrization which are indispensable tools of the modern analyst. This is what Kelley has labelled “what every young analyst should know ” [...] The second area, which might be called “geometric topology”, is primarily concerned with the connectivity properties of topological spaces and provides the cores of results from general topology which are necessary preparation for later courses in geometry and algebraic topology. (p. 5)

To design the didactical activity, we identified four key concepts, presented in every general topology book: boundary, continuity, compactness, and connectedness. The notion of boundary, based on the abstract notions of open and closed sets, allows students to apply the first steps to abstraction through the concepts of closure and interior, which are very far from the Euclidean ones. The notions of boundary, closure, and interior completely describe the topology. The concept of continuous function is central to the study of analysis, and the need for a notion of general continuity comes naturally after the introduction of metric spaces (Frechet, 1906) and the definition of topological space (Hausdorff, 1914) and is necessary to treat the maps identifying two topological spaces, the homeomorphisms. Compactness and connectedness, both properties useful to topologically distinguish spaces, originate both from Euclidean intervals’ properties and give students the possibility to individuate classes of spaces satisfying or not that property, according to the definition and the topology (structure) to be considered, and to reflect on how a space living within a fixed structure must be made to possess or not possess a certain property.

The topological properties of mathematical objects as sets and functions are not intrinsic properties of these objects but are properties of objects once a topology is fixed. The same set X can be thought of as different topological spaces depending on the topology with which it is equipped. This implies that the same subset A of X, depending on the topological space (X, T) in which it is considered (T is a topology), may or may not verify a topological property.

The teaching activity we wanted to design focused on both of these elements - mathematical sets or functions, and topology. As part of the theoretical background, we refer to studies of two strands of mathematics teaching which we are now going to describe: exemplification and variation.

Exemplification and Variation

In a study on initial approach to new concepts, Dahlberg and Housman (1997) identified four learning strategies used by students in studying new definitions - example generation, reformulation, decomposition and synthesis, memorization - and they observed that.

> Students […] who employed an example generation learning strategy were more effective in attaining an initial understanding of the new concept than those who primarily employed other learning strategies… (p. 283)

This result has important consequences for teaching and leads to value the generation of examples by students:

> …it may be beneficial to introduce students to new concepts by requiring them to generate their own examples or have them verify and work with instances of a concept before providing them with examples and commentary (p. 297-298)

The importance of having students generate their own examples, rather than providing them with teachers' examples, is shared in several works in mathematics education and has been studied from different perspectives (see Watson & Mason, 2005). In previous studies on processes of example generation, Antonini (2011) and Furinghetti et al. (2011) focused on the role of the production of own examples in the construction of concept image and in harmonisation between concept image and concept definition (in terms of Tall & Vinner, 1981).

Beyond the use of examples in building basic concepts, Zaslavsky and Peled (1996) observe that ''the state of generating examples can be seen as a problem solving situation, for which different people employ different strategies'' (p. 76). As a problem-solving activity, generating examples can require processes of exploring, of looking for regularities, producing conjectures, argumentation, and proof, and involves metacognitive processes for managing the cognitive resources (Antonini, 2011). Watson and Mason (2005) describe a "personal example space" as " what is accessible in response to a particular situation, to particular prompts or propensities" (p. 51) and state that expanding the personal example spaces should be one of the goals in mathematics learning.

The experience of exemplification is related to the experience of variation, and both assume a central role in the acquisition of mathematical concepts. In mathematics education, in the last decades, various research has supported how variation and invariants can be crucial in the design of teaching activities (see, for example, Marton & Booth, 1997). As Watson (2017) wrote:

> Mathematical concepts are often encountered by learners through examples, and the variation they experience through examples that have some similarity in structure leads them to generalise either about the properties of mathematical objects or about relations between them (Michener, 1978). [...] Task design always has with it, either explicitly or implicitly, assumptions about pedagogy, a fundamental belief being that learners will notice and generalize from patterns and relationships between what aspects vary and what aspects are invariant (p. 85)

Coherently with the above epistemological analysis, the aspects we are interested in that vary and that are invariant are: a) the topologies; b) the mathematical objects as sets, subsets, and functions.

In the hypothesis that this double variation also has a central role for the cognitive construction of topological concepts, instructions for building examples will be formulated to promote the investigation about variations and relationships between topology and the topological properties of mathematical objects.

Cognitive-epistemological Design Principles

In summary, in the design of the activities, we assume the following principles, that are ask students to:

CE.1 - construct their own examples of topologies and mathematical objects as sets, subsets, and functions, varying both the topologies and the mathematical objects;
CE.2 - explore topological properties of sets and functions in a particular topology, and the topological properties of a particular set or function varying the topologies;
CE.3 - describe how to construct objects with particular properties;
CE.4 - explain the relationships between the topological properties and the variation of the topologies.

Methodology

Participants

The problem-solving activities were carried out in the Geometry III course within a Bachelor of Mathematics at the University of Salerno, in Italy. The third author was the course's teacher, too. The course involved fifty second-year students, who had already attended the following first-year courses: calculus, linear algebra, and algebra. The course provides a bridge between intuition and formalism and teaches students how to handle abstraction and construct examples, counterexamples, and rigorous proofs. It aims at developing the fundamental concepts of general topology (e.g., continuity in the setting of topological spaces, new spaces from old, separation and countability, compactness and connectedness) and providing topological strategies to prove fundamental theorems. Students usually experience difficulties facing problems in topology that seem to depend on the transition between two distant approaches to understanding: procedural and conceptual. In order to bridge the gap between these approaches, the course foresees cycles of lecture, cooperative working, homeworking, and discussion for the main concepts. Each biweekly problem-solving activity about a given concept alternated with biweekly individual homework dealing with the same concept, both followed by a discussion with the teacher (in the order lecture-group activity-homework-discussion).

The Experimental Design

The set of all the participants, referred to as Working Group (WG), was split into three working groups, i.e. WG1, WG2, and WG3, each of which consisted of 16–20 students. Each WGi (i = 1,2,3) was engaged in the same didactical path consisting of four problem-solving cooperative working activities (CW) referred to as CW_k (k = 1,2,3,4), implemented 15 days apart during the second term of 2020–21 academic year, in distance setting due to the pandemic. Each WGi (i = 1, 2, 3) was split into four subgroups: WSGi.1, WSGi.2, WSGi.3, and WSGi.4, according to the methodological design principle M3-M4 and engaged as Solvers (corresponding to Actors) or Onlookers. Within each WSGi.j, each student was required to be in the shoes of a given cognitive role of the inside-out model. The Solver group S was committed to collectively facing the problem-solving activity, each member contributing from the viewpoint of the cognitive function she played. The three Onlooker groups O1, O2, and O3 observed how the Solver group worked, focusing on how each single member acted in relation to both the mathematical issue and her function. In addition to the reflective level of the Onlookers, a new degree of reflection was introduced: as the problem-solving activity consisted of three problems, each one aimed to achieve specific sub-goals, ranging from the use of mathematical ideas, facts, methods, and reasoning to the creation of new mathematical knowledge, each Onlooker group was required to primarily focused on one problem (i.e. group O1 on Problem 1, group O2 on Problem 2, group O3 on Problem 3). As the activity CW_k (k = 1,2,3,4) varied, the groups WSGi.j (j = 1,…,4) changed roles (S, O1, O2, O3) and the students within the group also changed their cognitive roles.

The groups were thought to be inhabited by students cooperating to solve 'thinking tasks', according to the methodological design principle M5, who collaborated for the development and creation of mathematical meanings by using non-permanent spacious digital interactive boards, according to principle M2.

The intertwining between the Thinking classroom model and the role assignments gave rise to the notion of Thinking Group (Miranda, 2023a):

> A Thinking Group (hereafter TG) can be defined as a group composed of thinking individuals equipped with a shared space to think individually according to an assigned cognitive role as well as to think collectively, learning together and constructing knowledge through activity and discussion. (p. 720)

According to the methodological design principle M6 while students were working on a thinking task, an expert, playing the role of Guru, could answer only keep-thinking questions that is to say questions so they could get back to work and think enough to have an illumination moment. In addition, to help students see where they are and where they are going, teachers can give check-your-understanding questions and encourage, eventually by keep-thinking hints, the transition to the next moment (principle M7). Only the Promoter had a communication link with an expert (a Guru) who may intervene to unlock a barrier if necessary.

Each student was expected to create a personal logbook with certain guidelines to reflect on their individual roles as Solver and Onlooker (principle M5). Furthermore, at the end of each problem-solving activity, each TG, was required to create a collective logbook detailing the problem solution and the process used to reach it (detailing the experience, how the answers were constructed, and paying special attention to the arguments they adduced).

The Technology-enhanced Environment

Each WGi was provided with a technology-enhanced environment consisting of:

- a digital whiteboard, that is Mirò, one for each Solver group: in tune with principles M2-M3 this choice allows to provide students with: a) space and tools to interact in written form and to take notes; b) the chance to use various semiotic systems, such as figures, text, tables and so on; c) the chance to reason, communicating among them by means of colloquial as well as literate registers; indeed, Mirò allows them to write freely, to attach notesticks, to paste pieces of book, etc.; differently from the Solver group, the Onlookers can access read only;
- a virtual room to meet each other, that is Teams Room, one for each subgroup Solver or Onlooker: this choice allows the students belonging to a group to add a further synchronous interaction modality, that is voice, they are used to;
- private documents, such as a private Google Doc, one for each student, in order to allow them to write a personal logbook (principle M4,a);
- shared documents, such as shared Google Docs, one for each subgroup in order to allow the students to co-construct a publishable mathematical text (principle M4,b).

The whole technology-enhanced environment can be accessed along the whole didactical path, allowing students to go back and forward on their writings, both on the whiteboard and on their documents.

The Problem-solving Activities

The assignments are intended to encourage the creation of mathematical meanings associated with a learned definition by constructing examples and conjectures.

The structure of the problem-solving activity CW_k consisted of three problems, each aiming to fulfil a specific subgoal. Let us see Problem 1 (Figs. 1, 2, 3). The first item of Problem 1, according to the cognitive-epistemological design principle CE1, aimed at activating a process of example generation that takes place in the choice of an unconventional topological space and, within it, of two topological subspaces that verify or do not verify the property under study. There is a constraint between the choice of ambient space and the construction of subspaces. The second item of Problem 1, according to the principles CE2-CE4, focuses on the behaviour of the subsets previously generated, with respect to the property under study, as the topology changes in the ambient space, aiming to activate a verification process.

Problem 2 (Figs. 1, 2, 3), according to the principles CE1-CE4, having ascertained the behaviour of the previous subsets, requires students to construct three subsets that verify the investigated property with respect to all topologies, not

ACTIVITY CW1

PROBLEM n.1

1.a Construct a topological space (S, τ), where S is a non-empty set and τ a topology on S, different from those studied in class (invent it !!), in such that S contains two proper non-empty subsets X_1 e X_2 such that $Fr(X_1) = \emptyset$ e $Fr(X_2) \neq \emptyset$ in (S, τ), where $Fr(X_1)$ e $Fr(X_2)$ are the boundaries of X_1 and X_2 in (S, τ).

1.b Consider the set S introduced in point 1.a. Denoted by τ_1 the topology τ and by τ_2, τ_3, τ_4, respectively, the trivial topology, the discrete topology, a topology of your choice distinct from the previous ones on S, determine

1.b.1 the frontier of X_1 in (S, τ_i), $Fr(X_1)^{\tau_i}$, for $i = 2, 3, 4$.

1.b.2 the frontier of X_2 in (S, τ_i), $Fr(X_2)^{\tau_i}$, for $i = 2, 3, 4$.

The boundary of a set can vary as the topology varies. Explain why.

PROBLEM n.2 Consider the topological space you defined in the previous problem (point 1.a). Construct, if they exist, three proper non-empty subsets of S, X_3, X_4, X_5, each distinct from X_1 and X_2, so that

2.a X_3 has an empty boundary with respect to all topologies.

2.b X_4 has a non-empty boundary with respect to all topologies.

2.c X_5 has empty boundary only with respect to some topology.

PROBLEM n.3

3.a Choose one of the topological spaces (S, τ_i) with $i \neq 2$ among those considered in problems 1 and 2 and investigate how a subset with an empty boundary can be constructed. In other words, characterize, if possible, the subsets with empty boundaries.

3.b Do the considerations made in the previous point or the statements you have reached also apply to the other topologies? Consider at least one of the two topologies τ_j on S with $j \notin \{i, 2\}$ and determine if the results found in (S, τ_i) continue to hold in (S, τ_j).

Fig. 1 First problem-solving activity

verify it with respect to all topologies, and verify it with respect to some topologies, respectively.

Finally, Problem 3 (Figs. 1, 2, 3), according to the principle CE3, encourages a reflection on how a space (or a function) verifying the involved property (continuity, compactness) is structured, specifically on whether what has been observed to be true in a particular case is true for the other topologies handled or always true, stimulating the activation of a process of generalisation.

Data Collection and Research Method Analysis

As argued by the previous sections, this research was born in the frame of educational design-based research, with the aim of innovating university mathematics

ACTIVITY CW2 - CONTINUITY

PROBLEM n.1

1.a Construct two topological spaces (S, τ), (S', τ') where S and S' are non-empty sets, τ and τ' are topologies on S and S', respectively, different from those studied during lessons (invent them !!), such that there exist a function $f_1 : (S, \tau) \to (S', \tau')$ which is continuous and a function $f_2 : (S, \tau) \to (S', \tau')$ which is not continuous.

1.b Consider the topological spaces you have defined in the previous problem (at point 1.a). Denote with τ_1 the topology τ, with τ_2 the topology τ', with τ_3, τ_4 the *trivial topology* on S and the *trivial topology* on S', respectively, and with τ_5, τ_6 the *discrete topology* on S and the *discrete topology* on S', respectively. Determine if $f : (S, \tau_i) \to (S', \tau_j)$ is continuous varying $i \in \{1, 3, 5\}$ and $j \in \{2, 4, 6\}$.

The continuity of a function essentially depends on the topologies with which its domain and its range are endowed. Explain why.

PROBLEM n.2

Consider the topological spaces you have defined in the problem above (in point 1.a). Construct, if they exist, three different functions f_3, f_4, f_5 defined on S and with values in S', each distinct from f_1 f_2, satisfying the following conditions:

2.a $f_3 : (S, \tau_i) \to (S', \tau_j)$ is continuous in (S, τ_i), $\forall i \in \{1, 3, 5\}$ and $\forall j \in \{2, 4, 6\}$

2.b $f_4 : (S, \tau_i) \to (S', \tau_j)$ is not continuous in (S, τ_i), $\forall i \in \{1, 3, 5\}$ and $\forall j \in \{2, 4, 6\}$

2.c $\exists i \in \{1, 3, 5\}$ e $\exists j \in \{2, 4, 6\}$: $f_5 : (S, \tau_i) \to (S', \tau_j)$ is continuous and $\exists h \in \{1, 3, 5\}$ and $\exists k \in \{2, 4, 6\}$: $f_5 : (S, \tau_h) \to (S', \tau_k)$ is not continuous.

PROBLEM n.3

3.a Select two topological spaces (S, τ_i) and (S', τ_j) with $i \neq 5$ among those considered in problems 1 and 2 and investigate how this can be done: to define a continuous function $f : (S, \tau_i) \to (S', \tau_j)$. In other words, characterize, if possible, the continuous functions defined on (S, τ_i) and valued in (S', τ_j).

3.b Do the considerations made in the previous point or the statements you have reached also apply to the other topologies?
Consider at least two of the other topologies, (S, τ_k) on S and (S, τ_h) on S', with $h, k \notin \{i, j\}$, and establish if the results found for $f : (S, \tau_i) \to (S', \tau_j)$ still hold for $f : (S, \tau_k) \to (S', \tau_h)$.

Fig. 2 Second problem-solving activity

education practices. The design of a technology-enhanced learning environment is strictly interwoven with the development of a theory based on our hypothesis of suitably combining the development of cognitive roles in mathematics and of pivotal mathematical processes such as exemplification and variation.

ACTIVITY CW3 - COMPACTNESS

PROBLEM n.1

1.a Construct a topological space (S, τ), where S is a non-empty set and τ a topology on S, different from those studied in class (invent it !!), such that S contains two proper non-empty subsets X_1 and X_2 such that X_1 is compact and X_2 is not compact in (S, τ).

1.b Consider the set S introduced in point 1.a. Denoted by τ_1 the topology τ and by τ_2, τ_3, τ_4, respectively, the trivial topology, the discrete topology, a topology of your choice distinct from the previous ones on S, determine if

1.b.1 X_1 is compact in (S, τ_i), for $i = 2, 3, 4$.

1.b.2 X_2 is compact (S, τ_i), for $i = 2, 3, 4$.

The compactness of a set can vary as the topology varies. Explain why.

PROBLEM n.2

Consider the topological space you defined in the previous problem (point 1.a). Construct, if they exist, three proper non-empty subsets of S, X_3, X_4, X_5, each distinct from X_1 and X_2, so that

2.a X_3 is compact with respect to all topologies.

2.b X_4 is not compact with respect to all topologies.

2.c X_5 is compact with respect to some topologies and not compact with respect to the others.

PROBLEM n.3

3.a Choose one of the topological spaces (S, τ_i) with $i \neq 2$ among those considered in problems 1 and 2 and investigate how a compact subset can be constructed. In other words, characterize, if possible, the compact subsets.

3.b Do the considerations made in the previous point or the statements you have reached also apply to the other topologies? Consider at least one of the two topologies τ_j on S with $j \notin \{i, 2\}$ and determine if the results found in (S, τ_i) continue to hold in (S, τ_j).

Fig. 3 Third problem-solving activity

The data to be analysed consist of the whiteboards generated by Mirò and used by each group of students to interact and discuss on how to solve the posed problems and the shared Google docs (i.e. groups' final reports) that each group composed in order to report their solutions as a publishable mathematical text.

We conducted a qualitative and theory-guided analysis of the collected data. Indeed, we analysed them according to two main lines linked to the described methodological and epistemological-cognitive planes. Both planes are strictly connected, on the one hand, to the didactical objectives of the educational activity and, on the other hand, to the conceptual background underlying the design principles. The epistemological-cognitive plane concerns aspects related to the construction of basic concepts of general topology. The methodological plane, strictly dependent on the technology-enhanced environment, concerns metacognitive aspects, linked to the mathematical identity experienced by the students by means of cognitive roles. As our data consists of written texts, we carried out a content analysis according to the theory-guided categories on the methodological plane, corresponding to the cognitive roles, and on the epistemological-cognitive plane, corresponding to the construction's processes of topological concepts. After the three authors of this paper independently read the final reports, they compared their own analysis and identified some excerpts from final reports considered by all the authors to be more representative, because they allow to illustrate in a broad and exhaustive way the richness of the dynamics of the individual and group roles (i.e. methodological plane) and of the processes involved in the construction of examples and investigation of variation and invariants (i.e. cognitive-epistemological plane).

Data Analysis

In this section, we present some data and a preliminary analysis of the pilot's outcomes consisting in a first analysis of the students' reports according to the research method described in the previous subsection.

Analysis on the Metacognitive Plane

We discuss some excerpts from the collective final report of the Thinking Groups.

We note that the narratives attribute sense to the actions of the various participants in relation to the cognitive roles played by each one and emphasise the two-layers engagement structure (Solver and Onlookers), enabled by the technology-enhanced environment.

Let us see some excerpts from WG2.1's report concerning the problem-solving activity CW_1. It is worthwhile to note that unexpectedly, the Solvers refer to help received from the Onlookers that unlocks them. This seems to confirm that the layer Onlookers favours reflection in action.

In the excerpt from Fig. 4, we can identify some significant elements that we are going to describe:

> Critical Mind's role seems to have been played effectively in the sense that her questions or remarks acted as a driving force for reasoning among the participants. In fact:
>
> – her request for clarification of what was said by the solving group prompted the Boss to recall some useful definitions (see (1), (2));
> – her suggestion (at the onlooker level) fostered the activation of the same role of the solving group that led to a new remark (see (3), (4));
> – her acute remark triggered the Promoter to conjecture that what the Critical Mind said was a real and proper characterization and not just a sufficient condition (see (4), (5));

Dopo aver iniziato a ragionare sulla risoluzione dell'esercizio 1.a., la Critical Mind del gruppo osservatore 02 chiede al gruppo risolutore che cosa volesse dire quando afferma che la chiusura è uguale all'interno.
Il Boss del gruppo risolutore ha, quindi, scritto le definizioni di Chiusura, Interno e Frontiera per facilitare gli altri membri del gruppo.
Dopo aver scritto tutte le definizioni utili, è stato suggerito dall'Osservatore-Critical Mind-GO2 di osservare quando la frontiera di un insieme S risulta essere uguale all'insieme vuoto.
Quindi, la Critical Mind del gruppo risolutore è intervenuta osservando con l'aiuto di alcuni membri dei gruppi osservatori che i Clopen hanno frontiera vuota.
Dopo questa acuta osservazione, il promoter del gruppo risolutore ipotizza che di questa affermazione, fatta dalla Critical Mind del gruppo risolutore, potrebbe valere anche il contrario, ovvero che: gli insiemi con frontiera vuota sono Clopen.
Gli osservatori aiutano quindi a dimostrare questa tesi, ed infatti sia il Critical Mind del gruppo osservatore O3, che il boss che gruppo risolutore avanzano l'ipotesi di prendere gli aperti di un insieme S e per verificare che esso sia un Clopen, mettere tra gli aperti l'insieme stesso ed il suo complemento.
Grazie a queste osservazione il gruppo risolutore, con l'aiuto dei gruppi osservatori, è giunta alla seguente osservazione:

<u>OSSERVAZIONE: Dopo aver scelto un insieme S, abbiamo osservato che affinché la frontiera sia vuota, la chiusura deve essere uguale all'interno. Inoltre, ciò avviene se e solo sé l'insieme è un clopen.</u>
E grazie a questa, il gruppo risolutore è riuscito facilmente a dare un esempio di topologia che andasse a soddisfare le richieste della traccia.

(1) After starting to reason how to solve sub-task 1.a., the Onlooker-Critical Mind-G02 asked the solving group what they meant when they said that closure equals interior.
(2) The Boss of the solving group then wrote down the definitions of Closure, Interior and Boundary to facilitate the other group members.
(3) After writing down all the useful definitions, it was suggested by the Onlooker-Critical Mind-G02 to note when the boundary of a set S is equal to the empty set.
(4) Then, the Critical Mind of the solving group intervened by observing with the help of some members of the onlookers groups that clopen has an empty boundary.
(5) After this acute remark, the Promoter of the solving group hypothesised that this statement, made by the Critical Mind of the solving group, could also be valid for the opposite, i.e. that: sets with empty boundary are Clopen.
(6) The onlookers then help to prove this thesis, and in fact both the Onlooker-Critical Mind-G03 and the Boss of the solver group put forward the hypothesis of taking the openings of a set S and, in order to verify that it is a Clopen, putting the set itself and its complement among the openings.
(7) With the help of the onlookers groups, the solving group arrived at the following:
<u>REMARK: After choosing a set S, we noted that for the boundary to be empty, the closure must be equal on the inside. Moreover, this happens if and only if the set is a clopen.</u>
(8) And thanks to this, the solving group was easily able to give an example of a topology that would meet the demands of the task.

Fig. 4 First excerpt and English translation from the WG2.1's report on CW_1

– together with the Boss made a new hypothesis which became crucial to solving the task (see (6), (7), and (8)).

Also the other roles appeared to be consistent:

– The Boss acted as organiser and facilitator of the group's work (see (2), (6));
– The Promoter provides the group with a new path to follow by suggesting a new hypothesis to work on (see (5)).

The Blogger, who wrote the narrative from which the excerpts in Fig. 4 were taken, produced a text that reported not only the mathematical solutions to the proposed tasks but also the solving process seen as an interconnected evolution of mathematical facts and participation of the various cognitive roles.

As noted, the excerpt in Fig. 4 reported that the onlooker groups, contrary to the design, actively intervened in task solving by interacting with the solving group. However, in some cases, it seems that although they intervene explicitly, their interventions are a sign of their being onlookers, that is, of being able to look, think, and reason with greater detachment from those more directly involved in the solving process (according to the principles M3-M4). It would therefore seem that their point of view, i.e., their role as onlookers, favours interventions that take the form of "observations" concerning what the solving group is doing and that may be useful to them in moving forward in the resolution (according to the principle M7). In addition to the excerpt (3) in Fig. 4, what is reported in Fig. 5 also appears to be of the same type:

Note that in (9) when students say “because every Clopen has a non-empty boundary," even if they wrote "non-empty," they would have liked to write empty. This is interpreted as a careless error; otherwise, the preceding discussion would not make sense.

Let us now take a look at some excerpts from the final report produced by the onlooker group WG3 (Fig. 6) in which students are required to generate a topology under some constraints in line with the principle CE1.

Note that in the onlooker group’s proposal, axiom ST3 does not hold. This created the occasion for a class discussion in which a counterexample was found and students choose K equal to the union of subsets each to the most countable, in such a way that all three axioms were satisfied.

Unlike the onlooker groups we saw in the previous case, here we see that the onlooker group sticks to the onlooker role envisaged by the design and does not interact with the solving group.

The first observation worth making is to note that the onlooker group is actually fully involved in the mathematical problem at hand. The observation of the mathematics that brings the solving group into play becomes an ’additional task’ for the onlooker group. The problem to be solved for the onlooker group is not merely the mathematical task assigned in the activity but rather the review of the solution and solving process that the actor group is producing. This means, on the one hand working with the mathematics needed to solve the task, but on the other hand, working with the mathematics needed to corroborate or refute what is produced by the solving group.

This is shown in Fig. 6, where the onlooker group reported the topological space proposed by the actor group and justified its incorrectness as a solution proposing counterexamples (see (10), (11), (12)). Then, the onlooker group solved the task (see (15)).

Per la risoluzione del punto 2.b i gruppi osservatori, acutamente, osservano che nella topologia discreta Non Esistono X4£S tale che Fr(X4) è non vuota nella topologia discreta (perché ogni Clopen ha frontiera non vuota).

(9) For solving sub-task 2.b, the onlookers groups keenly observe that there is no X4 belonging to S such that Fr(X4) is non-empty in the discrete topology (because every Clopen has a non-empty boundary).

Fig. 5 Second excerpt and English translation from the WG2.1’s report on CW_1

Il gruppo risolutore ha affrontato l'attività partendo dal problema 1 che richiedeva, inizialmente, di creare uno spazio topologico a sostegno infinito rispetto al quale individuare due sottoinsiemi propri, uno compatto e l'altro non.
È stato proposto il seguente spazio topologico:

(R,τ) , $\tau = \{\emptyset , R , K \cup \{1\}\}$ **dove** K **è un sottoinsieme numerabile di** R

uno spazio topologico:
La mancanza di dimostrazioni degli assiomi di spazio topologico ha impedito agli studenti risolutori di osservare che la struttura così definita non è uno spazio topologico, infatti, non si sono accorti che non è soddisfatto il secondo assioma di spazio topologico. Si pensi al controesempio:

Z^- **ed** N **sono sottoinsiemi numerabili di** R **, dunque** $Z^- \cup \{1\} , N \cup \{1\} = N \in \tau$ **.**
Ma $\{Z^- \cup \{1\}\} \cap N = \{1\} = \emptyset \cup \{1\} \notin \tau$ **perché** $\emptyset$ **non è numerabile.**

Si sarebbe potuto risolvere facilmente il problema pensando ai sottoinsiemi al più numerabili di R**, infatti, la seguente struttura è**

$$(R,\tau) , \ \tau = \{\emptyset , R , K \cup \{1\}\} \ \textbf{con} \ K \subset R , |K| \leq \aleph_0$$

ST1) $\emptyset , R \in \tau$
ST2) Siano $K \subset R , |K| \leq \aleph_0 , K' \subset R , |K'| \leq \aleph_0$ **, allora,** $K \cup \{1\} , K' \cup \{1\} \in \tau$
$\{K \cup \{1\}\} \cap \{ K' \cup \{1\}\} = \{K \cap K'\} \cup \{1\} \in \tau$ **perché l'intersezione di insiemi al più numerabili è al più numerabile.**
ST3) L'unione infinita di insiemi del tipo $K \cup \{1\}$ **con** $K \subset R , |K| \leq \aleph_0$ **è un aperto di** τ **perché contiene {1} e l'unione infinita di insiemi al più numerabili è al più numerabile.**

Non è stato soddisfatto il punto del problema 1 in cui veniva chiesto di spiegare perché uno stesso insieme può risultare compatto o meno al variare della topologia; ciò si verifica perché dipende da come sono fatti gli aperti della topologia visto che per un insieme si ha compattezza o meno in base alla possibilità di ricoprirlo con un numero finito di aperti dello spazio ambiente.

Anche i successivi problemi sono stati trattati senza argomentare molto e questo ha influito negativamente sull'esposizione del lavoro oltre ad essere stato invalidante per rintracciare gli errori commessi.

(10) The solving group approached the activity CW3 (compactness property))starting with task 1, which initially required the creation of a topological space with infinite support from which two proper subsets, one compact and the other not, could be identified. The following topological space was proposed:

(R,τ) ,τ={∅,R ,K∪{1}} where K is a countable subset of R

(11) The lack of proving the axioms of topological space prevented the solving students from observing that the structure thus defined is not a topological space, as the intersection axiom of topological space is not satisfied.

(12) Consider the counterexample:

Z - and N are numerable subsets of R , therefore Z -∪{1},N∪{1}=N ∈τ .

But {Z ∪{1} ∩N={1}= ∅ ∪{1} ∉τ because ∅ is not countable.

(13) The problem could have been easily solved by thinking of subsets of R at most countable, in fact, the following structure is

$$(R,\tau) , \ \tau = \{\emptyset , R , K \cup \{1\}\} \ \textbf{\textit{con}} \ K \subset R , |K| \leq \aleph_0$$

ST1) $\emptyset , R \in \tau$

ST2) Let $K \subset R , |K| \leq \aleph_0 , K' \subset R , |K'| \leq \aleph_0$ *, then* $K \cup \{1\} , K' \cup \{1\} \in \tau$ *and*

$\{K \cup \{1\}\} \cap \{ K' \cup \{1\}\} = \{K \cap K'\} \cup \{1\} \in \tau$ *because the intersection of sets at most countable is at most countable.*

ST3) The infinite union of sets of the type $K \cup \{1\}$ *with* $K \subset R , |K| \leq \aleph_0$ *is an open of* τ

since it contains {1}, and the infinite union of sets at most countable is at most countable.

(14) The point of task 1 in which it was asked to explain why the same set may or may not be compact when the topology varies was not satisfied; this is because it depends on how the openings of the topology are made since for a set one has compactness or not depending on the possibility of covering it with a finite number of openings of the ambient space.

(15) The subsequent problems were also dealt with without much argumentation and this adversely affected the exposition of the work as well as being crippling in terms of tracing the errors made.

Fig. 6 First excerpt and English translation from the WG3.2's report on CW_3

Avrei sicuramente cambiato le modalità argomentative, prediligendo commenti più articolati e dimostrazioni formali degli enunciati proposti perché, a mio avviso, questi strumenti permettono di documentare ogni passaggio e danno la possibilità di rilevare errori la cui percezione non è immediata.

(16) I would certainly have changed the argumentative modalities, preferring more articulated comments and formal proofs of the proposed statements because, in my opinion, these tools make it possible to document each step and give the possibility of detecting errors whose perception is not immediate.

Fig. 7 Second excerpt and English translation from the WG3.2's report on CW_3

We also note that the Onlooker group found a recurring methodological error of the Solver group, consisting in 'the lack of proving' (see (11)) and of explanation, even explicitly required (see (16)). They underlined that the lack of argumentation impacted the clarity of communication and prevented the solving group from tracing the errors.

This erroneous methodological approach has been considered pivotal for a good solving strategy, as shown in the final comment shown in Fig. 7:

Analysis on the Epistemological-cognitive Plane

The exploratory processes activated by the problem-solving activities seem to have produced meaningful outcomes in terms of concept understanding and mathematical thinking, as made evident by some students' reflections we are going to analyse in the following.

Based on the cognitive-epistemological analysis proposed above, we discuss some excerpts from the Thinking Groups' collective final reports on the concepts of boundary, continuity, and compactness that concern the problem-solving activities.

The excerpt in Fig. 8, from the group WSG2.1, concerning problem-solving activity CW_1 (boundary), highlights that the first construction of examples, the exploration of examples (Problem 1 in Fig. 1), and of variations and invariants lead students to generate a conjecture about the characterization required in Problem 3 (Fig. 1).

A different path on this task was proposed by the solver group WSG2.3. The board used by the group (Fig. 9) displays the elaboration of their answers. It is worthwhile to note that the group shows a certain disposition for creativity in the initial phase. In solving sub-task 1.a (Fig. 1), the subgroup WSG2.3 first recalls the two definitions of interior and closure (the boxes indicated by the arrows in Fig. 9) needed to define the boundary. Then, the group produces two different examples of required topological spaces:the first one with infinite support, the Euclidean plane, i.e., S = R2 equipped by the Euclidean topology, choosing the open unit square as a subspace with a non-empty boundary, and a singleton contained in it as a subspace with an empty boundary (see the orange box in Fig. 9), a choice made after a first unsuccessful attempt to consider a strictly coarser topology (they say "we could consider" but as that sets both have non-empty boundary they reasonably change topology); the second example very similar to the one discussed in the WSG2.1's excerpt in Fig. 8, as they consider the support of finite cardinality and as topology some sets of a partition (i.e., S = {1,2,3,4} and what follows, together with the yellow and pink boxes in Fig. 9). In this act, it can be seen a disposition to generate

OSSERVAZIONE: Dopo aver scelto un insieme S, abbiamo osservato che affinché la frontiera sia vuota, la chiusura deve essere uguale all'interno. Inoltre, ciò avviene se e solo sé l'insieme è un clopen.
E grazie a questa, il gruppo risolutore è riuscito facilmente a dare un esempio di topologia che andasse a soddisfare le richieste della traccia. Definiscono t=[insieme vuoto, S [1,2,3,4] e [5]] e scelgono come X1=[5] e X2=[1]
Notano che X1 è un Clopen, quindi la frontiera di X1 è insieme vuoto.

[…] Si osserva che nella topologia discreta abbiamo tutti Clopen, quindi noi possiamo dedurre dalla teoria che la frontiera è sempre vuota (ciò vale per la caratterizzazione).

[…] Le considerazioni fatte al punto precedente o gli enunciati al quale siamo giunti valgono anche per le altre topologie, perché l'OSSERVAZIONE caratterizza tutti gli insiemi quando si ha frontiera vuota in ogni spazio topologico.

REMARK: After choosing a set S contained in the support space, we observed that, for the boundary to be empty, the closure should be equal to the interior. Moreover, this happens if and only if the set is a clopen.

And thanks to this, the solver group was able to easily give an example of a topology that would satisfy the demands of the task. They define t=[empty set, S [1,2,3,4] and [5]], and choose as X1=[5] and X2=[1]. Note that X1 is a clopen, so the boundary of X1 is an empty set.

[…]It is observed that in the discrete topology we have all clopen subsets, so we can deduce from the above definitions that the boundary is always empty and conversely (this leads to a characterization).

[…]The considerations made in the previous point or the statements to which we have arrived also apply to the other topologies, because the REMARK characterises all sets when there is an empty boundary in every topological space.

Fig. 8 Third excerpt and English translation from the WG2.1's report on CW_1

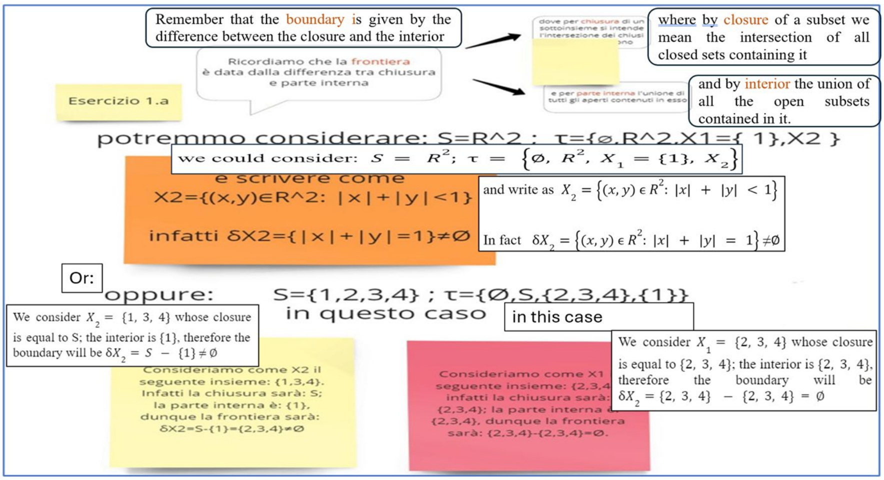

Fig. 9 Board of the group WSG2.3 (For students, $X_1 = \{1\}$ denotes any point subspace of X_2)

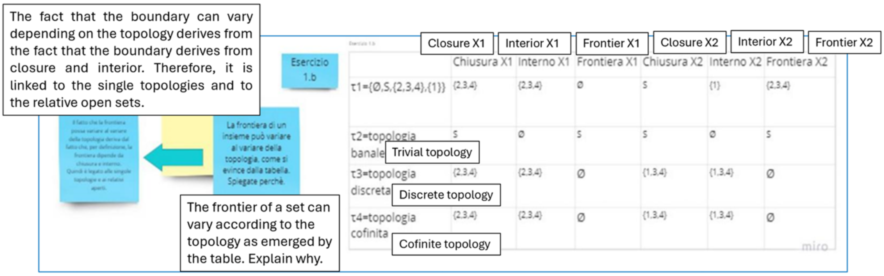

Fig. 10 Board of the group WSG2.3 used for solving sub-task 2.a

several solutions and solution paths to change direction, producing novel and original solutions, and this is a form of creativity.

The exploratory processes activated by the problem-solving activities through the construction of examples, investigation of variations and invariants seem to have produced meaningful outcomes in terms of concept understanding and mathematical thinking.

The following Fig. 10 shows the table where students consider some topologies and how students explain why the boundary of a set can vary as the topology varies, which is a crucial reflection to develop conceptual understanding in topology, as previously highlighted in the epistemological analysis.

It is worthwhile to note that Fig. 9 and 10 show multisemioticiy and multivariety in the students' mathematical discourse, enabled by the technology-enhanced environment they are required to interact with each other.

Let us now see an excerpt concerning the problem-solving activity CW_2 (continuity). The group WG1 reasoned that changing topologies affects continuity, regardless of the correspondence expressing the function (Fig. 11).

nei casi in cui la topologia del dominio è T5=discreta, abbiamo osservato che:
" ogni F(funzione) definita su uno spazio discreto e continua in ogni punto" ,
quindi possiamo affermare che la funzione è continua.

Quando la topologia al codominio è quella banale qualsiasi applicazione è continua,

Se escludiamo il caso in cui la topologia del dominio è quella discreta e il caso in cui la funzione sia quella costante, allora la nostra osservazione non è più valida.

Se fisso una topologia al codominio diversa da quella banale, non posso dire a priori che la funzione sia continua. Infatti posso sempre trovare un controesempio operando sulla topologia del dominio o su come agisce la funzione.

In cases where the domain topology is T5=discrete, we have observed that:
"every F (function) defined on a discrete and continuous space at every point", so we can say that the function is continuous.
[...] When the codomain topology is the trivial one, any application is continuous,
[...] If we exclude the case where the domain topology is the discrete one and the case where the function is the constant one, then our implication is no longer valid.
If I fix a codomain topology different from the trivial one, I cannot say a priori that the function is continuous. In fact, I can always find a counterexample by operating on the topology of the domain or on how the function acts.

Fig. 11 Excerpt and English translation from the WSG1.1's report on CW_2

[…] La seconda parte del primo esercizio è stata svolta tenendo conto delle topologie studiate quali cofinita, discreta e banale e da lì abbiamo determinato quale insieme fosse compatto e quale no in queste varie situazioni, riportando tutto nella tabella e mostrando accanto i piccoli ragionamenti che sono stati fatti. [...] Se lo spazio è finito allora è compatto con la topologia discreta [...] anche se la topologia è quella cofinita [...] Se lo spazio non è finito allora la topologia banale lo rende compatto [...] grazie al fatto che possiede solo un insieme aperto non vuoto.

[...]The second part of the first exercise was carried out taking into account the studied topologies such as cofinite, discrete and trivial and from there we determined which set was compact and which was not in these various situations, reporting everything in the table and showing alongside the small arguments that were produced. If the space is finite then it is compact with respect to the discrete topology[...] also if the topology is the cofinite one [..] If the space is not finite then the trivial topology makes it compact [...] due to the fact that it possesses only one nonempty open set.

Fig. 12 Excerpt and English translation from the WG3.3's report on CW_3

The group appears to have a deep understanding of what occurs as the topologies evaluated on the domain and codomain change, and therefore an aware control of the concept definition well balanced with the related concept image. In fact, the group bases their reasoning on extreme circumstances taking into consideration the least fine conceivable topology, trivial topology, and the finest possible topology, discrete topology. But the discourse is not limited to extreme cases. In fact, their experience manipulating examples and observing variations and invariants has probably promoted their awareness about the possibility of constructing a counterexample in the case of non-continuity, *operating* on the topological structure.

Finally, we discuss an excerpt concerning the problem-solving activity CW_3 (compactness). The group reflects on the reasons for which a given subset of a topological space should possess or not possess the compactness property, highlighting that it depends on how the open sets defining the topological structure are made. In particular, subgroup WSG3.3, during the solution process, comes across an interesting reflection on how the two components of a topological space should be made, namely its support and the topological structure, so that it satisfies the compactness property. A meaningful aspect seems to emerge: the 'finiteness', a property from which the definition of compactness originated, that is, the compactness property is the topological version of finiteness, as shown in Fig. 12.

In summary, different excerpts show that students have generated their examples - both topologies and mathematical objects as sets, subsets, and functions - managing their variations and invariants and producing some written texts where they show an awareness of the relationships between topological properties and the topologies.

Discussion and Conclusions

The analysis carried out in the previous section led us to formulate the hypothesis that the intertwining of different design elements from different conceptual planes (cognitive-epistemological and methodological) supported the students in achieving the general overall didactic objectives of the designed educational activity.

Starting with an incrementally structured problem-solving activity, the students were involved in a path that led them from the exploration of the link between the validity of a property and the reference topology to the construction of examples of sets satisfying a certain property in a certain topology, and finally to the production of conjectures generated by previous investigations. In cognitive analysis, the report of the group WSG2.1 concerning CW_1 shows that the exploration phase fostered the emergence of conjectures interesting for continuing the path sketched by the problem-solving activity. Our hypotheses, which require further researches, is that this path is strongly influenced by the way in which students are involved: on the one hand, it is a collaborative problem-solving path, engaging them in topological keep-thinking tasks, according to the Liljedahl model (2016), on the other hand, the contribution of each participant is not neutral but depends on the lens provided by the played cognitive role (Albano et al., 2021). The metacognitive analysis shows the key function of the interventions through the lens of Critical Mind, which is actually one of the most important features of mathematical thinking. These interventions opened the way not only to validate some claims, but also to find news results. Further research is also needed to investigate how, by playing their role well and through the interconnection among them, students have promoted the success of the group activity, as shown in the case of WG2.1. As different problem-solving activities progress from CW_1 to CW_3, the cognitive analysis shows elements indicating that the groups become engaged in a more and more expert mathematical discourse. In fact, the discourse moves from managing standard cases (even extreme cases) towards the manipulation of topological structure (as shown by group WG2 in the report concerning CW_2), to the reflections on the link between compactness and topology structure with its support (as shown by group WSG3.3 in the report concerning CW_3).

Concerning the two-level structure of students' engagement, the metacognitive analysis shows how the final reports highlighted the privileged viewpoint provided by being an onlooker, regardless of whether or not the onlookers directly interacted with the Solver group. Although the Onlooker groups are fully involved in the mathematical problem, the fact that they are not expected to produce a solution could allow them to look at the problem and how the Solver group is solving it with that proper detachment that becomes an added value. This sometimes could have brought the Onlooker groups to provide the Solver group with remarks useful to let them move forward (as shown by group WSG2.1 in the report concerning CW_1). Some other times, this could have allowed the Onlooker groups to focus not only on the mathematical content managed by the Solver group but also to reflect on the solving process from a methodological point of view, to the point of being able to detect a recurring methodological error and identify its impact on the managed mathematical content (as shown by group WG3 in the report concerning CW_3).

The analysis presented above highlights how the design principles based on the conceptual background can act synergically to engage undergraduate students in problem-solving activities and to support them in achieving didactical objectives such as the construction of basic concepts in general topology and the development of metacognitive skills, as well as their mathematical identity. The preliminary findings make us confident enough to move forward. Finally, although the analysis has been performed along the two lines described above, some elements testifying to the fostering of problem-solving processes and the construction of proof competency can be devised. Further investigation can be done in this respect.

What emerges from the data analysis is in line with other research. The technological organisation enabled students to be engaged in a way that we can call 'reflection in action'. The Onlookers can reflect on the problem solving process of their peers while it happens, but also go back and forth the writings produced, and in doing so, the Onlookers themselves are engaged in solving the problem. Moreover, their reflections and actions are guided by the roles played. The reflective aspect seems to be greatly favoured by the technology-enhanced environment, which allows the student to navigate the solving process along the dimensions of space (blackboard, chat room, personal logbook, collective narrative) and time (while the mathematical facts are happening or returning later to those same facts, in a back-and-forth). The students' production of examples and counterexamples, the extensions of their personal example spaces, and their investigation of variation and invariants confirm the findings of research on exemplification (Antonini, 2011; Dahlberg & Housman, 1997; Miranda, 2023b; Watson & Mason, 2005). Moreover, a fruitful and rich students' mathematical discourse emerged, characterized by the mobilisation and the coordination of more semiotic systems as well as more linguistic registers, which seems to be fostered by the features of the technology-enhanced environment.

Further research is needed to extend this study in different fields of mathematics. In fact, we believe that the analysis reported in this article goes beyond general topology and can be extended to the teaching and learning of other fields of university mathematics in a distance technological setting. The epistemological-cognitive analysis must be specific to the knowledge that is the object of teaching and will allow, case by case, to identify the crucial epistemological nodes and the cognitive aspects that one consequently chooses to take into consideration. This analysis can guide the formulation of epistemological-cognitive design principles that can be integrated with the methodological plan presented in this article.

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