Periodic perturbations of reducible scalar second order functional differential equations

Electronic Journal of Qualitative Theory of Differential Equations
2023, No. 18, 1–23; https://doi.org/10.14232/ejqtde.2023.1.18 www.math.u-szeged.hu/ejqtde/

Periodic perturbations of reducible scalar second order functional differential equations

Alessandro Calamai ✉ [1], **Maria Patrizia Pera**[2] and **Marco Spadini**[2]

[1]Dipartimento di Ingegneria Civile, Edile e Architettura, Università Politecnica delle Marche, Via Brecce Bianche, I–6013 Ancona, Italy
[2]Dipartimento di Matematica e Informatica "Ulisse Dini", Università degli Studi di Firenze, Via S. Marta 3, I–50139 Florence, Italy

Received 16 September 2022, appeared 11 May 2023
Communicated by Ferenc Hartung

Abstract. Using a topological approach we investigate the structure of the set of forced oscillations of a class of reducible second order functional retarded differential equations subject to periodic forcing. More precisely, we consider a delay-type functional dependence involving a gamma probability distribution and, using a linear chain trick, we formulate a first order system of ODEs whose T-periodic solutions correspond to those of the functional equation.

Keywords: functional differential equations, branches of periodic solutions, linear chain trick, Brouwer degree.

2020 Mathematics Subject Classification: 34K13, 34C25.

1 Introduction and setting of the problem

Integro-differential and functional-delayed equations are often used to describe phenomena whose dynamics depends on the past states of the system of concern. We mention, in particular, scalar second order functional differential equation of the following form:

$$\ddot{x}(t) = g\left(x(t), \dot{x}(t), \int_{-\infty}^{t} \mathcal{K}(t-s)\varphi(x(s), \dot{x}(s))\, ds\right), \tag{1.1}$$

where $g\colon \mathbb{R}^3 \to \mathbb{R}$ is a continuous map, $\varphi\colon \mathbb{R}^2 \to \mathbb{R}$ is continuous and $\mathcal{K}$ is some integral kernel. Such equations, often studied in the first order case, have been considered in many contexts. A comprehensive bibliography is beyond the scope of this paper, we only mention a few relevant papers and books, e.g., [14,22,32,34,35,40,41].

In this paper we focus on the case when the integral kernel in (1.1) is the gamma probability distribution γ_a^b, for $a > 0$ and $b \in \mathbb{N} \setminus \{0\}$ given by

$$\gamma_a^b(s) = \frac{a^b s^{b-1} e^{-as}}{(b-1)!} \ \text{ for } s \geq 0, \qquad \gamma_a^b(s) = 0 \ \text{ for } s < 0, \tag{1.2}$$

✉ Corresponding author. Email: calamai@dipmat.univpm.it

with mean b/a and variance b/a^2. Namely, we study the following equation:

$$\ddot{x}(t) = g\left(x(t), \dot{x}(t), \int_{-\infty}^{t} \gamma_a^b(t-s)\varphi(x(s), \dot{x}(s))\,\mathrm{d}s\right). \tag{1.3}$$

An equation as (1.3) is often called *reducible* because, unlike equation (1.1) whose dynamics is, in general, infinite dimensional (see, e.g., [31]), equation (1.3) can be essentially reduced to a system of ordinary differential equations by the so-called *linear chain trick* (see Section 2). Indeed, backtracking the linear chain trick procedure, one can see how equations of the form (1.3) may arise from the coupling of a nonlinear equation with a linear one (see, e.g., [35, Ch. 10]). We will briefly come back on this last topic in Section 8 on perspectives and further developments. Another of the reasons that motivates the interest in this particular type of kernels is that, at least for some linear functional equations containing a convolution-like term as in equation (1.1) with general kernel, it is possible to construct gamma-like kernels such that the solutions of the corresponding equations approximate those of the equation we started with (see [6, 9, 10]).

The particular dependence on the past of the solution that is considered in equation (1.3) naturally arises in many contexts (not necessarily for second order equations) see, e.g., [5, 10, 21, 22, 34, 35, 40, 41]. In the equation we examine, the delay is spread along the whole history but more concentrated at a given distance in the past. We can interpret this, from a probabilistic point of view, as a delay that follows a gamma-type distribution with a given mean (and variance). This approach seems to be reasonable in contexts when the delay cannot be measured with precision, but its mean value and variance are known, or it is genuinely spread in time as, e.g., in [18, 19]. Notice, in particular, that letting a and b tend to infinity in such a way that the quotient $r := a/b$ remains constant (for instance, put $a = nr$ and $b = n$ and let $n \to \infty$), one concentrates the memory effect close to the delay r. Indeed, at least in the sense of distributions, (1.3) approximates the second order delay differential equation

$$\ddot{x}(t) = g\big(x(t), \dot{x}(t), \varphi\big(x(t-r), \dot{x}(t-r)\big)\big).$$

Observe that the function γ_a^b defined in (1.2) is continuous for $b = 2, 3, \ldots$ but not for $b = 1$. However, also in the latter case, assuming x in C^1, the function

$$t \mapsto \int_{-\infty}^{t} \gamma_a^1(t-s)\varphi(x(s), \dot{x}(s))\,\mathrm{d}s$$

is continuous. Then the right hand side of (1.3) is continuous as well; consequently, any C^1 solution of (1.3) is actually of class C^2. It is also worth noticing (and we will use this fact later) that, since in the integral one has $t - s \geq 0$, then

$$\int_{-\infty}^{t} \gamma_a^1(t-s)\varphi(x(s), \dot{x}(s))\,\mathrm{d}s = \int_{-\infty}^{t} a\mathrm{e}^{-a(t-s)}\varphi(x(s), \dot{x}(s))\,\mathrm{d}s,$$

so that the function $t \mapsto \int_{-\infty}^{t} \gamma_a^1(t-s)\varphi(x(s), \dot{x}(s))\,\mathrm{d}s$ is actually in C^1.

Our main concern will be the structure of periodic solutions of (1.3) when subject to a periodic forcing. Given $T > 0$, we consider the following T-periodic perturbation of (1.3):

$$\ddot{x}(t) = g\left(x(t), \dot{x}(t), \int_{-\infty}^{t} \gamma_a^b(t-s)\varphi(x(s), \dot{x}(s))\,\mathrm{d}s\right) + \lambda f\big(t, x(t), \dot{x}(t)\big), \tag{1.4}$$

in which we assume that the map $f\colon \mathbb{R}^3 \to \mathbb{R}$ is continuous and T-periodic in the first variable, and λ is a nonnegative real parameter. Our main purpose is to investigate the set of T-periodic solutions of (1.4). Here, given $\lambda \geq 0$, by a *T-periodic solution* (on $\mathbb{R}$) of (1.4) we mean a C^2 function $x\colon \mathbb{R} \to \mathbb{R}$ of period T that satisfies identically equality (1.4). In that case, we will call (λ, x) a *T-forced pair* of (1.4). The T-forced pairs of the type $(0, x)$ with x constant will be called *trivial*.

Roughly speaking, we explicitly construct a scalar function Φ whose change of sign implies the existence of a connected set, called a *"branch"*, of nontrivial pairs (λ, x) that emanates out of the set of zeros of Φ and whose closure – in a suitable Banach space – is not compact.

We also give sufficient conditions yielding the multiplicity of T-periodic solutions of (1.4) for $\lambda > 0$ small. Such conditions are essentially based upon the notion of *ejecting set* (see, e.g., [26]).

The methods that we employ are topological in nature and are based on the Brouwer degree. Nevertheless, the use of our main results is accessible even without a familiarity with this notion since the application of the degree theory is restricted to the proofs and some preliminary results proved elsewhere. This is possible since we deal with scalar equations for which degree-theoretic assumptions can be replaced with more elementary ones. It would be not so if the vector case was considered, see Section 8 for a brief discussion. However, for completeness, we provide a short summary of degree theory in Section 4.

Our results are so-to-speak dual to those of [12], where periodic perturbations containing delay terms are applied to scalar, second order ODEs. The study, by means of topological methods, of the branching and multiplicity of periodic solutions of periodically perturbed equations, is now a well-investigated subject in the case of ODEs both in Euclidean spaces and on manifolds (see, e.g., [25, 28]). The case of ODEs perturbed with delayed forcing terms is also studied in the literature, although not so broadly (see, e.g., [12, 13]). However, the presence of delay terms in the *unperturbed* equation (1.3) is peculiar of the problem addressed here. In spite of the apparent similarities, a different approach is called upon in order to manage the unperturbed equation.

In the undelayed case, that is for periodically forced second-order scalar autonomous ODEs, both the problems of existence and multiplicity of periodic solutions are quite classical. However, this is still the subject of active research by the mathematical community. There are many approaches that have been successfully pursued to get multiplicity results: among the others let us mention here, e.g., the recent contributions [3, 4, 11, 23], the survey papers [37, 38] as well as the monograph [24] and the references therein.

The strategy adopted in this paper is inspired to [42] and can be summarized as follows. First we construct a system of $b+2$, T-periodically perturbed first-order ordinary differential equations whose T-periodic solutions correspond to that of (1.4). We then use known results about these perturbed systems to ensure the existence of a branch as sought when the topological degree of the unperturbed field G has nonzero degree and finally, we show, by homotopy techniques, that the change of sign of Φ implies that G has nonzero degree.

It should be noted that the result just described does not guarantee the existence of forced oscillations even for very small values of λ. For this reason, we also provide a nonlocal condition, based on an inequality of J. A. Yorke [43], implying that the branch projects nontrivially onto $[0, \infty)$.

In order to develop a better understanding of the nature of the branch of T-periodic solutions and its relation with multiplicity results we discuss, following [2] and [42], a method to visualize, in finite dimension, a homemorphic set that retains all the relevant properties. We illustrate the procedure with a numeric example.

2 The linear chain trick

As pointed out in the introduction, the purpose of this paper is to investigate the structure of the set of the T-periodic solutions of the parametric ODE

$$\ddot{x}(t) = g\left(x(t), \dot{x}(t), \int_{-\infty}^{t} \gamma_a^b(t-s)\varphi(x(s), \dot{x}(s))\,\mathrm{d}s\right) + \lambda f\big(t, x(t), \dot{x}(t)\big), \tag{2.1}$$

where we make the following set of assumptions:

$$\begin{aligned}
&i.\quad g\colon \mathbb{R}^3 \to \mathbb{R} \text{ is a continuous map;}\\
&ii.\quad \varphi\colon \mathbb{R}^2 \to \mathbb{R} \text{ is a continuous map;}\\
&iii.\quad \gamma_a^b, \text{ for } a>0 \text{ and } b \in \mathbb{N}\setminus\{0\}, \text{ represents the gamma probability distribution (1.2);}\\
&iv.\quad f\colon \mathbb{R}^3 \to \mathbb{R} \text{ is continuous and } T\text{-periodic in the first variable;}\\
&v.\quad \lambda \geq 0 \text{ is a real parameter.}
\end{aligned} \tag{2.2}$$

We start with a crucial remark; namely, the fact that the T-periodic solutions of (2.1) and those of the following system of $b+2$ ordinary differential equations correspond in some sense:

$$\dot{\xi} = G(\xi) + \lambda F(t, \xi), \tag{2.3}$$

where $\xi = (u, v_0, v_1, \ldots, v_b) \in \mathbb{R}^{b+2}$, and the maps $G\colon \mathbb{R}^{b+2} \to \mathbb{R}^{b+2}$ and $F\colon \mathbb{R} \times \mathbb{R}^{b+2} \to \mathbb{R}^{b+2}$ are respectively defined as

$$G(u, v_0, v_1, \ldots, v_b) = \Big(v_0, g(u, v_0, v_b), a\big(\varphi(u, v_0) - v_1\big), a(v_1 - v_2)\ldots, a(v_{b-1} - v_b)\Big) \tag{2.4}$$

and

$$F(t, u, v_0, v_1, \ldots, v_b) = \Big(0, f(t, u, v_0), \underbrace{0, \ldots, 0}_{b \text{ times}}\Big),$$

with g, φ and f as in equation (2.1). Clearly, G and F are continuous maps. By a T-periodic solution (on $\mathbb{R}$) of system (2.3) we mean a C^1 function $\xi\colon \mathbb{R} \to \mathbb{R}^{b+2}$ of period T that satisfies (2.3) identically.

Let us show now how a "linear chain trick" (see, e.g., [41, 42]) can be used to prove the correspondence between T-periodic solutions of the second order equation (2.1) and of the first order system (2.3).

Let us introduce some notation. By $C_T^n(\mathbb{R}^s)$, $n = 0, 1, 2$, we will denote the Banach space of the T-periodic C^n maps $x\colon \mathbb{R} \to \mathbb{R}^s$ with the the standard norm

$$\|x\|_{C^n} = \sum_{i=0}^{n} \max_{t\in\mathbb{R}} |x^{(i)}(t)|.$$

Here, $x^{(i)}$ denotes the i-th derivative of x, in particular $x^{(0)}$ coincides with x.

Theorem 2.1. *Assume (2.2) and suppose x_0 is a T-periodic solution of (2.1), and let*

$$\begin{cases} y_0(t) := \dot{x}_0(t), \\ y_i(t) := \int_{-\infty}^{t} \gamma_a^i(t-s)\varphi(x_0(s), y_0(s))\,\mathrm{d}s, \quad i = 1, \ldots, b \end{cases}$$

for $t \in \mathbb{R}$. Then, $(x_0, y_0, y_1 \ldots, y_b)$ is a T-periodic solution of (2.3).

Proof. First notice that, since $x_0 \in C^2_T(\mathbb{R})$ and $\varphi\colon \mathbb{R}^2 \to \mathbb{R}$ is continuous, it is not difficult to prove that the functions y_i, $i = 0, \dots, b$, are in $C^1_T(\mathbb{R})$.

Moreover, we claim that, for any $t \in \mathbb{R}$,

$$\begin{cases} \dot{x}_0(t) = y_0(t), \\ \dot{y}_0(t) = g\big(x_0(t), y_0(t), y_b(t)\big) + \lambda f\big(t, x_0(t), y_0(t)\big), \quad \lambda \geq 0, \\ \dot{y}_1(t) = a\big(\varphi(x_0(t), y_0(t)) - y_1(t)\big), \\ \dot{y}_i(t) = a\big(y_{i-1}(t) - y_i(t)\big), \quad i = 2, \dots, b. \end{cases}$$

Indeed, the first equality follows by definition. Since x_0 is a solution of (2.1), for $\lambda \geq 0$ we have

$$\begin{aligned} \dot{y}_0(t) = \ddot{x}_0(t) &= g\left(x_0(t), \dot{x}_0(t), \int_{-\infty}^{t} \gamma_a^b(t-s)\varphi(x_0(s), \dot{x}_0(s))\,\mathrm{d}s\right) + \lambda f\big(t, x_0(t), \dot{x}_0(t)\big) \\ &= g\big(x_0(t), y_0(t), y_b(t)\big) + \lambda f\big(t, x_0(t) y_0(t)\big), \quad \forall t \in \mathbb{R}. \end{aligned}$$

Now observe that, when $i = 1$, we have for any $t \in \mathbb{R}$,

$$y_1(t) := \int_{-\infty}^{t} \gamma_a^1(t-s)\varphi(x_0(s), y_0(s))\,\mathrm{d}s = \int_{-\infty}^{t} a\mathrm{e}^{-a(t-s)}\varphi(x_0(s), y_0(s))\,\mathrm{d}s.$$

Thus, taking the derivative under the integral sign,

$$\frac{\mathrm{d}}{\mathrm{d}t} y_1(t) = a\varphi(x_0(t), y_0(t)) - a^2 \int_{-\infty}^{t} \mathrm{e}^{-a(t-s)}\, \varphi(x_0(s), y_0(s))\,\mathrm{d}s$$

so that

$$\dot{y}_1(t) = a\big(\varphi(x_0(t), y_0(t)) - y_1(t)\big).$$

Also, for $i = 2, \dots, b$, we get for any $t \in \mathbb{R}$,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} y_i(t) &= \gamma_a^i(0)\, \varphi(x_0(t), y_0(t)) + \int_{-\infty}^{t} \frac{\mathrm{d}}{\mathrm{d}t}\gamma_a^i(t-s)\, \varphi(x_0(s), y_0(s))\,\mathrm{d}s \\ &= \int_{-\infty}^{t} a\left(\gamma_a^{i-1}(t-s) - \gamma_a^i(t-s)\right)\, \varphi(x_0(s), y_0(s))\,\mathrm{d}s \\ &= a\left(\int_{-\infty}^{t} \gamma_a^{i-1}(t-s)\, \varphi(x_0(s), y_0(s))\,\mathrm{d}s - \int_{-\infty}^{t} \gamma_a^i(t-s)\, \varphi(x_0(s), y_0(s))\,\mathrm{d}s\right) \\ &= a\,(y_{i-1}(t) - y_i(t)) \end{aligned}$$

proving our claim.

Finally, to see that $(x_0, y_0, y_1 \dots, y_b)$ is T-periodic, observe that so are x_0 and y_0, and with the change of variable $\sigma = s - T$ we get

$$\begin{aligned} y_i(t+T) &= \int_{-\infty}^{t+T} \gamma_a^i(T+t-s)\varphi(x_0(s), y_0(s))\,\mathrm{d}s \\ &= \int_{-\infty}^{t} \gamma_a^i(t-\sigma)\varphi(x_0(\sigma+T), y_0(\sigma+T))\,\mathrm{d}\sigma \\ &= \int_{-\infty}^{t} \gamma_a^i(t-\sigma)\varphi(x_0(\sigma), y_0(\sigma))\,\mathrm{d}\sigma = y_i(t), \end{aligned}$$

for $i = 1, \dots, b$ and any $t \in \mathbb{R}$, and this completes the proof. $\square$

Conversely, we have the following:

Theorem 2.2. *Assume* (2.2) *and suppose that* $(x_0, y_0, y_1 \dots, y_b)$ *is a* T*-periodic solution of* (2.3)*, then* x_0 *is a* T*-periodic solution of* (2.1)*.*

The proof of Theorem 2.2 is based on the following technical lemma on linear systems of ODEs, see [42, Lemma 3.3], cf. also [41, Prop. 7.3]. The proof is omitted.

Lemma 2.3 ([42], Lemma 3.3)**.** *Given any continuous and bounded function* $z_0 \colon \mathbb{R} \to \mathbb{R}$ *and any* $a > 0$*, there exists a unique* C^1 *solution* $z = (z_1, \dots, z_b)$*,* $z_i \colon \mathbb{R} \to \mathbb{R}$*,* $i = 1, \dots, b$*, of the system in* $\mathbb{R}^b$

$$\dot{z}_i(t) = a\big(z_{i-1}(t) - z_i(t)\big),$$

which is bounded in the C^1 *norm. This solution is given by*

$$z_i(t) = \int_{-\infty}^{t} \gamma_a^i(t-s) z_0(s)\,\mathrm{d}s, \quad i = 1, \dots, b.$$

Remark 2.4. Observe in particular that, for $i = 1$ in the previous lemma, one has

$$z_1(t) = \int_{-\infty}^{t} \gamma_a^1(t-s) z_0(s)\,\mathrm{d}s = a \int_{-\infty}^{t} \mathrm{e}^{-a(t-s)} z_0(s)\,\mathrm{d}s,$$

so that z_1 is actually a C^1 function, and thus so are all the z_i's for $i = 2, \dots, b$.

Proof of Theorem 2.2. Let $(x_0, y_0, y_1 \dots, y_b)$ be a T-periodic solution of (2.3), and define $z_0(t) = \varphi(x_0(t), y_0(t))$ for $t \in \mathbb{R}$. Observe that z_0 is bounded and continuous. Thus, by Lemma 2.3,

$$y_i(t) = \int_{-\infty}^{t} \gamma_a^i(t-s) \varphi(x_0(s), y_0(s))\,\mathrm{d}s, \quad i = 1, \dots, b,$$

is the unique solution of class C^1 of

$$\begin{cases} \dot{y}_1(t) = a\big(\varphi(x_0(t), y_0(t)) - y_1(t)\big), \\ \dot{y}_i(t) = a\big(y_{i-1}(t) - y_i(t)\big), \quad i = 2, \dots, b. \end{cases}$$

In particular, we have

$$y_b(t) = \int_{-\infty}^{t} \gamma_a^b(t-s) \varphi(x_0(s), y_0(s))\,\mathrm{d}s.$$

Thus, from (2.3),

$$\begin{aligned} \ddot{x}_0(t) = \dot{y}_0(t) &= g\big(x_0(t), y_0(t), y_b(t)\big) + \lambda f\big(t, x_0(t) y_0(t)\big) \\ &= g\left(x_0(t), \dot{x}_0(t), \int_{-\infty}^{t} \gamma_a^b(t-s)\varphi(x_0(s), \dot{x}_0(s))\,\mathrm{d}s\right) + \lambda f\big(t, x_0(t), \dot{x}_0(t)\big), \quad \lambda \geq 0, \end{aligned}$$

for all $t \in \mathbb{R}$, whence the assertion. □

3 Branches of T-pairs

This section investigates the structure of the set of T-periodic solutions of (2.1). We begin by recalling some notation and basic facts.

Consider the following first-order parameterized ODE on $\mathbb{R}^k$:

$$\dot{x}(t) = \mathfrak{g}\big(x(t)\big) + \lambda\mathfrak{f}\big(t, x(t)\big), \tag{3.1}$$

where $\lambda \geq 0$, the maps $\mathfrak{g}\colon \mathbb{R}^k \to \mathbb{R}^k$ and $\mathfrak{f}\colon \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}^k$ are continuous and $\mathfrak{f}$ is T-periodic in the first variable.

We say that a pair $(\lambda, x) \in [0, \infty) \times C_T(\mathbb{R}^k)$ is a *T-pair* (for (3.1)) if x is a T-periodic solution of (3.1) corresponding to λ. If $\lambda = 0$ and x is constant then the T-pair is said *trivial*. It is not hard to see that the trivial T-periodic pairs of (3.1) correspond to the zeros of $\mathfrak{g}$. From now on, given any $p \in \mathbb{R}^k$, we will denote by $\overline{p}$ the constant map $t \mapsto p$, $t \in \mathbb{R}$. Given an open subset $\mathcal{O}$ of $[0, \infty) \times C_T(\mathbb{R}^k)$, we will denote by $\widetilde{\mathcal{O}} \subseteq \mathbb{R}^k$ the open set $\widetilde{\mathcal{O}} := \big\{p : (0, \overline{p}) \in \mathcal{O}\big\}$.

We have the following fact concerning the T-pairs of (3.1), see [27, Theorem 3.3]. The statement of Theorem 3.1 involves the Brouwer degree of the map $\mathfrak{g}$, see Section 4 for a definition and related notions.

Theorem 3.1 ([27]). *Let $\mathcal{O}$ be open in $[0, \infty) \times C_T(\mathbb{R}^k)$, and assume that $\deg(\mathfrak{g}, \widetilde{\mathcal{O}})$ is well defined and nonzero. Then there exists a connected set $\Gamma \subseteq \mathcal{O}$ of nontrivial T-pairs whose closure in $\mathcal{O}$ is not compact and meets the set of trivial T-pairs contained in $\mathcal{O}$, namely the set: $\big\{(0, \overline{p}) \in \mathcal{O} : \mathfrak{g}(p) = 0\big\}$.*

Let us now go back to the second-order equation (2.1). Roughly speaking we will state a global bifurcation result, analogous to Theorem 3.1, whose assumptions do not involve the topological degree, but rely only on sign-changing properties of the real-valued function $\Phi\colon \mathbb{R} \to \mathbb{R}$, given by

$$\Phi(u) = g(u, 0, \varphi(u, 0)). \tag{3.2}$$

To make a precise statement we need some further notation. The pairs $(\lambda, x) \in [0, +\infty) \times C^1_T(\mathbb{R})$, with $x\colon \mathbb{R} \to \mathbb{R}$ a T-periodic solution of (2.1) corresponding to λ, will be called *T-forced pairs* (of (2.1)), and we denote by X the set of these pairs. Among the T-forced pairs we shall consider as *trivial* those of the type $(0, x)$ with x constant. Given an open subset Ω of $[0, \infty) \times C^1_T(\mathbb{R})$, we will denote by $\widetilde{\Omega} := \big\{u \in \mathbb{R} : (0, \overline{u}) \in \Omega\big\}$, where by $\overline{u}$ we mean the constant map $t \mapsto u$, $t \in \mathbb{R}$.

As pointed out in Section 2, there is a correspondence between T-periodic solutions of the second order equation (2.1) and of the first order system in (2.3). A similar correspondence holds between their "T-periodic pairs" in a sense that we are going to specify. We point out that, since equation (2.3) can be seen as a special case of (3.1), in accordance with the notation introduced above, a pair $(\lambda, \xi) \in [0, \infty) \times C_T(\mathbb{R}^{b+2})$ is called a *T-pair* if ξ is a T-periodic solution of (2.3) corresponding to λ. We also recall that the set of T-forced pairs of (2.1) is regarded as a subset of $[0, +\infty) \times C^1_T(\mathbb{R})$.

As a first remark we relate the corresponding "trivial T-periodic pairs". This observation will be deduced from the following link between the zeros of the map G, defined in (2.4), and those of Φ, introduced in (3.2).

Remark 3.2. Observe that if $(\bar{u}, v_0, v_1, \ldots, v_b) \in G^{-1}(0)$, then $\Phi(\bar{u}) = 0$, $v_0 = 0$ and $v_1 = v_2 = \ldots = v_b = \varphi(\bar{u}, 0)$. Conversely, for any $\bar{u} \in \Phi^{-1}(0)$, then $G(\bar{u}, 0, \varphi(\bar{u}, 0) \ldots, \varphi(\bar{u}, 0)) = 0$.

Remark 3.3. Observe that if $(0, \overline{q})$, with $q \in \mathbb{R}$, is a trivial T-forced pair of (2.1), then $(0, \overline{\xi_0})$ is a trivial T-pair for equation (2.3), where $\xi_0 \in \mathbb{R}^{b+2}$ is given by

$$\xi_0 := (q, 0, \varphi(q, 0) \ldots, \varphi(q, 0)),$$

that is, by Remark 3.2, $\xi_0 \in G^{-1}(0)$. Conversely, any trivial T-pair of (2.3) must be of the form $(0, \overline{\xi_0})$, where $\xi_0 \in \mathbb{R}^{b+2}$ is such that $G(\xi_0) = 0$. Thus, again by Remark 3.2, we have $\xi_0 = (q, 0, \varphi(q,0) \ldots, \varphi(q,0))$ for some $q \in \mathbb{R}$, and consequently $(0, \overline{q})$ is a trivial T-forced pair for (2.3).

Let us now establish a general correspondence between the sets of T-forced pairs and of T-pairs, that preserves the notion of triviality. Let $\mathcal{J} \colon [0, +\infty) \times C^1_T(\mathbb{R}) \to [0, \infty) \times C_T(\mathbb{R}^{b+2})$ be the map defined as follows:

$$\mathcal{J} : (\lambda, x_0) \mapsto (\lambda, \xi)$$

where $\xi := (x_0, y_0, y_1 \ldots, y_b)$ is given by

$$\begin{cases} y_0(t) := \dot{x}_0(t), \\ y_i(t) := \int_{-\infty}^{t} \gamma_a^i(t-s)\varphi(x_0(s), y_0(s))\,\mathrm{d}s, \quad i = 1, \ldots, b. \end{cases}$$

As above, denote by X the set of the T-forced pairs of (2.1), and let $Y \subseteq [0, \infty) \times C_T(\mathbb{R}^{b+2})$ be the set of the T-pairs of (2.3).

Lemma 3.4. *Let $\mathcal{J}|_X : X \to Y$ be the restriction to X of $\mathcal{J}$. Then $\mathcal{J}|_X$ is a homeomorphism of X onto Y, which establishes a bijective correspondence between the trivial T-forced pairs in X and the trivial T-pairs in Y.*

Proof. Since φ is continuous, the continuity of $\mathcal{J}$ is obtained by construction. Whence we get the continuity of the restriction $\mathcal{J}|_X$. Injectivity and surjectivity of $\mathcal{J}|_X : X \to Y$ follow from Theorems 2.1 and 2.2. The inverse map $(\mathcal{J}|_X)^{-1}$ can be seen merely as the projection onto the first two components, so it is continuous. Finally the last part of the assertion follows by Remark 3.3. □

The following fact will be crucial for the proof of our main result. Its proof heavily relies on the properties of the topological degree, it is therefore postponed to Section 4 where this concept and its features are discussed. Given a bounded open interval $(\alpha, \beta) \subseteq \mathbb{R}$, define the open subset $W^* = (\alpha, \beta) \times \mathbb{R}^{b+1}$ of $\mathbb{R}^{b+2}$.

Theorem 3.5. *Assume (2.2), let G be as in (2.4) and let Φ be the real-valued function defined in (3.2). Suppose that $\Phi(\alpha) \cdot \Phi(\beta) < 0$. Then G is admissible for the degree in W^* and we have $\deg(G, W^*) \neq 0$.*

We are now in a position to state and prove our main result concerning the set of T-forced pairs of (2.1).

Theorem 3.6. *Consider equation (2.1) and assume that (2.2) hold. Denote by $X \subseteq [0, \infty) \times C^1_T(\mathbb{R})$ the set of its T-forced pairs. Let Φ be the real-valued function defined in (3.2), and suppose that $(\alpha, \beta) \subseteq \mathbb{R}$ is such that $\Phi(\alpha) \cdot \Phi(\beta) < 0$. Let $\Omega \subseteq [0, \infty) \times C^1_T(\mathbb{R})$ be open and such that $\widetilde{\Omega} = (\alpha, \beta)$. Then, in $X \cap \Omega$ there is a connected subset Γ of nontrivial T-forced pairs of (2.1) whose closure relative to Ω is not compact and intersects the set*

$$\big\{(0, \overline{u}) \in \Omega : u \in (\alpha, \beta) \cap \Phi^{-1}(0)\big\}. \tag{3.3}$$

Proof. Let $\mathcal{O} = \Omega \times C_T(\mathbb{R} \times \mathbb{R}^b)$. Clearly, $\widetilde{\mathcal{O}} = \widetilde{\Omega} \times \mathbb{R} \times \mathbb{R}^b = (\alpha, \beta) \times \mathbb{R} \times \mathbb{R}^b$. Thus, Theorem 3.5 implies that $\deg(G, \widetilde{\mathcal{O}})$ is well-defined and nonzero. So, there exists a connected set Y of nontrivial T-pairs for (2.3) as in Theorem 3.1. Let $\Gamma := \mathcal{J}^{-1}(\mathrm{Y})$; by Lemma 3.4, this is a connected set, it is made up of nontrivial T-forced pairs of (2.1), and its closure relative to Ω must intersect the set (3.3) of the trivial T-forced pairs contained in Ω. We claim that Γ is not contained in any compact subset of Ω. By contradiction, assume that there exists a compact set $K \subseteq \Omega$ containing Γ. Since X is closed, $K \cap X$ is compact. Thus, its image $\mathcal{J}(K \cap X)$ is a compact subset of $\mathcal{O}$ containing Υ, which is a contradiction. The assertion follows. □

We remark that, as in the well-known case of the resonant harmonic oscillator

$$\ddot{x} = -x + \lambda \sin t,$$

this unbounded branch is possibly contained in the slice $\{0\} \times C^1_T(\mathbb{R})$. We will discuss some conditions preventing this "pathological" situation in Section 5.

4 Computation of the degree

4.1 Brouwer degree in Euclidean spaces

We will make use of the Brouwer degree in $\mathbb{R}^k$ in a slightly extended version (see e.g. [1,20,39]). Let U be an open subset of $\mathbb{R}^k$, f a continuous $\mathbb{R}^k$-valued map whose domain contains the closure $\overline{U}$ of U, and $q \in \mathbb{R}^k$. We say that the triple (f, U, q) is *admissible (for the Brouwer degree)* if $f^{-1}(q) \cap U$ is compact.

The Brouwer degree is a function that to any admissible triple (f, U, q) assigns an integer, denoted by $\deg(f, U, q)$ and called the *Brouwer degree of f in U with target q*. Roughly speaking, $\deg(f, U, q)$ is an algebraic count of the solutions in U of the equation $f(p) = q$. In fact, one of the properties of this integer-valued function is given by the following *computation formula.* Recall that, if $f \colon U \to \mathbb{R}^k$ is a C^1 map, an element $p \in U$ is said to be a *regular point* (of f) if the differential of f at p, df_p, is surjective. Non-regular points are called *critical (points)*. The *critical values* of f are those points of $\mathbb{R}^k$ which lie in the image $f(C)$ of the set C of critical points. Any $q \in \mathbb{R}^k$ which is not in $f(C)$ is a *regular value*. Therefore, q is a regular value for f in U if and only if $\det(df_p) \neq 0$, $\forall p \in f^{-1}(q) \cap U$. Observe, in particular, that any element of $\mathbb{R}^k$ which is not in the image of f is a regular value.

Computation formula. If (f, U, q) is admissible, f is smooth, and q is a regular value for f in U, then

$$\deg(f, U, q) = \sum_{p \in f^{-1}(q) \cap U} \operatorname{sign} \det(df_p). \tag{4.1}$$

This formula is actually the basic definition of the Brouwer degree, and the integer associated to any admissible triple (g, U, r) is defined by

$$\deg(g, U, r) := \deg(f, U, q),$$

where f and q satisfy the assumptions of the Computation Formula and are, respectively, "sufficiently close" to g and r. It is known that this is a well-posed definition.

The more classical and well-known definition of Brouwer degree is usually given in the subclass of triples (f, U, q) such that $f \colon \overline{U} \to \mathbb{R}^k$ is continuous, U is bounded and $q \notin f(\partial U)$. However, all the standard properties of the degree, such as homotopy invariance, excision, additivity, existence, are still valid in this more general context. For a detailed list of such properties we refer, e.g., to [30,33,39].

Since in this paper the target point q will always be the origin, for the sake of simplicity, we will simply write $\deg(f, U)$ instead of $\deg(f, U, 0)$. In this context, we will say that an element $p \in f^{-1}(0)$ is a *nondegenerate zero* (of f) if $\det(df_p) \neq 0$; this means, equivalently, that p is a regular point. Accordingly, we will also say that (f, U) is an *admissible pair* (or that the map f is admissible in U) if so is the triple $(f, U, 0)$. Observe that $\deg(f, U)$ can be regarded also as the degree (or characteristic, or rotation) of the map f seen as a tangent vector field on $\mathbb{R}^k$.

In what follows we will make use of the following elementary fact whose proof we include for the sake of completeness.

Lemma 4.1. *Let $\psi\colon [a,b] \to \mathbb{R}$ be a continuous function with $\psi(a)\psi(b) < 0$. Then the pair $\big(\psi, (a,b)\big)$ is admissible and*

$$\deg\big(\psi, (a,b)\big) = \operatorname{sign}\psi(b).$$

In particular $\deg\big(\psi, (a,b)\big) \neq 0$*.*

Proof. It is sufficient to observe that if $\psi(b) > 0$ the map $H\colon [0,1] \times [a,b] \to \mathbb{R}$ given by $H(\lambda, s) = \lambda\big(s - \frac{b+a}{2}\big) + (1-\lambda)\psi(s)$ is an admissible homotopy with the identity map translated by $\frac{b+a}{2}$. Hence

$$\deg\big(\psi, (a,b)\big) = \deg\big(H(0,\cdot), (a,b)\big) = \deg\big(H(1,\cdot), (a,b)\big) = +1.$$

If, otherwise, one has $\psi(b) < 0$, it is possible to define an admissible homotopy H by setting $H(\lambda, s) = -\lambda\left(s - \frac{b+a}{2}\right) + (1-\lambda)\psi(s)$. Thus, in this case, $\deg\big(\psi, (a,b)\big) = -1$. $\square$

4.2 The degree of the map G

The last part of this section is devoted to the proof of Theorem 3.5. Roughly speaking we will relate the degree of the map G, defined in (2.4) by

$$G(u, v_0, v_1, \ldots, v_b) = \big(v_0, g(u, v_0, v_b), a(\varphi(u, v_0) - v_1), a(v_1 - v_2) \ldots, a(v_{b-1} - v_b)\big),$$

with that of the function $\Phi\colon \mathbb{R} \to \mathbb{R}$ in (3.2), given by

$$\Phi(u) = g(u, 0, \varphi(u, 0)).$$

To simplify the computation of the degree, it is convenient to introduce the next map on $\mathbb{R}^{b+2}$:

$$\mathcal{G}(u, v_0, v_1, \ldots, v_b) := \big(v_0, g(u, v_0, v_b), a(\varphi(u, v_0) - v_b), a(v_1 - v_2) \ldots, a(v_{b-1} - v_b)\big).$$

Remark 4.2. As in Remark 3.2, we have that the zeros of $\mathcal{G}$ and those of Φ correspond. In fact, if $\mathcal{G}(\bar{u}, v_0, v_1, \ldots, v_b) = 0$ then $\Phi(\bar{u}) = 0$, $v_0 = 0$ and $v_1 = v_2 = \ldots = v_b = \varphi(\bar{u}, 0) =: \bar{w}$. Conversely, for any $\bar{u} \in \Phi^{-1}(0)$, then $\mathcal{G}(\bar{u}, 0, \bar{w} \ldots, \bar{w}) = 0$. Indeed, $G^{-1}(0) = \mathcal{G}^{-1}(0)$.

Lemma 4.3. *Let G and $\mathcal{G}$ be as above, and let $V \subseteq \mathbb{R}^{b+2}$ be open. Suppose that one of the maps G or $\mathcal{G}$ is admissible for the Brouwer degree in V; then, so is the other and they are admissibly homotopic in V.*

Proof. For $(\lambda, u, v_0, v_1, \ldots, v_b) \in [0,1] \times V$, consider the map

$$\begin{aligned}\mathcal{H}(&\lambda, u, v_0, v_1, \ldots, v_b) \\ &= \big(v_0, g(u, v_0, v_b), a(\varphi(u, v_0) - [\lambda v_b + (1-\lambda)v_1]), a(v_1 - v_2) \ldots, a(v_{b-1} - v_b)\big),\end{aligned}$$

and observe that $\mathcal{H}(\lambda, u, v_0, v_1, \ldots, v_b) = 0$ if and only if $(u, v_0, v_1, \ldots, v_b) \in V \cap G^{-1}(0) = V \cap \mathcal{G}^{-1}(0)$. Hence $\mathcal{H}$ is an admissible homotopy. $\square$

Now we prove that if Φ is smooth and the zeros of Φ are nondegenerate, so are those of $\mathcal{G}$.

Lemma 4.4. *Assume that g and φ are of class C^1. Then, all zeros of Φ are nondegenerate if and only if all zeros of $\mathcal{G}$ are nondegenerate.*

Proof. First notice that, by Remark 4.2 the zeros of $\mathcal{G}$ and those of Φ correspond. Assume now that $\bar{\xi} := (\bar{u}, 0, \bar{w} \dots, \bar{w})$ is a nondegenerate zero of $\mathcal{G}$. We represent the differential $d\mathcal{G}_{\bar{\xi}}$ of $\mathcal{G}$ at $\bar{\xi}$ as the following Jacobian matrix:

$$\left(\begin{array}{cc|cccccc|c}
0 & 1 & 0 & 0 & \dots & \dots & 0 & & 0 \\
\partial_1 g(\bar{u},0,\bar{w}) & \partial_2 g(\bar{u},0,\bar{w}) & 0 & 0 & \dots & \dots & 0 & & \partial_3 g(\bar{u},0,\bar{w}) \\
a\,\partial_1\varphi(\bar{u},0) & a\,\partial_2\varphi(\bar{u},0) & 0 & 0 & \dots & \dots & 0 & & -a \\
\hline
0 & 0 & a & -a & 0 & \dots & 0 & & 0 \\
0 & 0 & 0 & a & -a & \ddots & 0 & & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\
0 & 0 & 0 & 0 & \dots & 0 & a & & -a
\end{array}\right),$$

where "∂_i", $i = 1,2,3$, represents the partial derivation with respect to the i-th variable. By direct computation based on Laplace expansion we obtain

$$\begin{aligned}
\det d\mathcal{G}_{\bar{\xi}} &= (-a)^{b-1} \det \begin{pmatrix} 0 & 1 & 0 \\ \partial_1 g(\bar{u},0,\bar{w}) & \partial_2 g(\bar{u},0,\bar{w}) & \partial_3 g(\bar{u},0,\bar{w}) \\ a\partial_1\varphi(\bar{u},0) & a\partial_2\varphi(\bar{u},0) & -a \end{pmatrix} \\
&= -(-a)^{b-1} \det \begin{pmatrix} \partial_1 g(\bar{u},0,\bar{w}) & \partial_3 g(\bar{u},0,\bar{w}) \\ a\partial_1\varphi(\bar{u},0) & -a \end{pmatrix} \\
&= (-1)^{b-1} a^b \cdot [\partial_1 g(\bar{u},0,\bar{w}) + \partial_3 g(\bar{u},0,\bar{w})\,\partial_1\varphi(\bar{u},0)] = (-1)^{b-1} a^b \cdot \Phi'(\bar{u}).
\end{aligned}$$

Thus $\bar{u}$ is a nondegenerate zero of the function Φ. The proof of the converse implication is analogous. □

Let us now compute the degree of the map $\mathcal{G}$ in the smooth case.

Lemma 4.5. *Assume that g and φ are of class C^1. Assume further that Φ is admissible on an open set $U \subseteq \mathbb{R}$ and all its zeros are nondegenerate. Then, $\mathcal{G}$ is admissible in $U^* = U \times \mathbb{R}^{b+1}$ and*

$$\deg(\mathcal{G}, U^*) = (-1)^{b-1} \deg(\Phi, U).$$

Proof. By Remark 4.2 we have that all zeros of $\mathcal{G}$ are of the form $\bar{\xi} := (\bar{u}, 0, \bar{w} \dots, \bar{w})$ with $\Phi(\bar{u}) = 0$. Moreover, they are all nondegenerate. In particular, if Φ is admissible in U then $\Phi^{-1}(0) \cap U$ is compact and so is $\mathcal{G}^{-1}(0) \cap U^*$, whence the admissibility of $\mathcal{G}$ in U^*. Let now $\bar{u} \in \Phi^{-1}(0) \cap U$ be a nondegenerate zero of Φ. As in the proof of Lemma 4.4 we have

$$\det d\mathcal{G}_{\bar{\xi}} = (-1)^{b-1} a^b \cdot \Phi'(\bar{u})$$

and, consequently,

$$\operatorname{sign}\det\left(d\mathcal{G}_{\bar{\xi}}\right) = (-1)^{b-1} \operatorname{sign}\det \Phi'(\bar{u})$$

Thus, by formula (4.1),

$$\begin{aligned}\deg(\mathcal{G},U^*) &= \sum_{\bar{\xi}\in\mathcal{G}^{-1}(0)\cap U^*} \operatorname{sign}\det\left(d\mathcal{G}_{\bar{\xi}}\right)\\ &= \sum_{\bar{u}\in\Phi^{-1}(0)\cap U} (-1)^{b-1}\operatorname{sign}\det\Phi'(\bar{u}) = (-1)^{b-1}\deg(\Phi,U),\end{aligned}$$

whence the assertion. □

The above lemmas imply now the assertion of Theorem 3.5.

Proof of Theorem 3.5. First notice that, by Lemma 4.1, assumption $\Phi(\alpha)\cdot\Phi(\beta)<0$ implies that Φ is admissible in (α,β) and $\deg(\Phi,(\alpha,\beta))\neq 0$. Thus, if Φ is of class C^1 and all zeros of Φ in (α,β) are nondegenerate, the assertion follows from Lemmas 4.3 and 4.5.

Consider now the case in which Φ is merely continuous, or the zeros of Φ in (α,β) may not be nondegenerate. Clearly the above Lemmas 4.4 and 4.5 are not directly applicable, nevertheless it is possible to prove the assertion by applying some well-known approximation theorems and the properties of the Brouwer degree. More precisely, using the so-called smooth Urysohn Theorem and Sard's Theorem, we will construct a real-valued function $\widetilde{\Phi}_r\colon\mathbb{R}\to\mathbb{R}$, depending on a suitable small parameter r, satisfying the assumptions of Lemma 4.5 in (α,β).

For this purpose, observe first that, as a consequence of Remark 4.2, the map $\mathcal{G}$ is admissible in W^* so that $\deg(\mathcal{G},W^*)$ is well-defined. Let $\mathcal{W}\subseteq W^*$ be an open relatively compact neighborhood of the compact set $\mathcal{G}^{-1}(0)\cap W^*$. Without loss of generality, one can choose $\mathcal{W}$ with the following properties:

$$\begin{aligned}&i.\quad \mathcal{W}=(\alpha,\beta)\times\mathcal{U} \text{ where } \mathcal{U}\subseteq\mathbb{R}^{b+1} \text{ is open and bounded;}\\ &ii.\quad (u,0,\varphi(u,0)\dots,\varphi(u,0))\in\overline{\mathcal{W}} \text{ for all } u\in[\alpha,\beta].\end{aligned}\tag{4.2}$$

Let $\alpha'<\alpha$ and $\beta'>\beta$ be such that Φ does not change sign in the intervals $[\alpha',\alpha]$ and $[\beta,\beta']$. Let also $\mathcal{U}'\subseteq\mathbb{R}^{b+1}$ be an open and bounded set such that $\overline{\mathcal{U}}\subseteq\mathcal{U}'$. Consider the open set

$$\mathcal{V}=(\alpha',\beta')\times\mathcal{U}'.$$

Note that $\mathcal{V}$ is bounded and contains $\overline{\mathcal{W}}$; in particular, $\mathcal{G}^{-1}(0)\cap W^*\subseteq\mathcal{W}\subseteq\mathcal{V}$.

By Lemma 4.3 and the excision property of the degree,

$$\deg(G,W^*)=\deg(\mathcal{G},W^*)=\deg(\mathcal{G},\mathcal{W}).\tag{4.3}$$

Now, take $\delta>0$ such that

$$\begin{aligned}&i.\quad \delta<\min\{|\Phi(\alpha)|,|\Phi(\beta)|\};\\ &ii.\quad \delta<\min_{(u,v_0,\dots,v_b)\in\overline{\mathcal{V}}\setminus\mathcal{W}}\|\mathcal{G}(u,v_0,\dots,v_b)\|.\end{aligned}\tag{4.4}$$

By a smooth approximation argument similar to the one used in the definition of the degree (cfr. [30, Ch. 5, §1], see in particular [30, Theorem. 2.6]), we can choose smooth approximations $\widetilde{g}$ and $\widetilde{\varphi}$ of g and φ, respectively, such that, setting

$$\widetilde{\mathcal{G}}(u,v_0,v_1,\dots,v_b)=\Big(v_0,\widetilde{g}(u,v_0,v_b),a\big(\widetilde{\varphi}(u,v_0)-v_b\big),a(v_1-v_2)\dots,a(v_{b-1}-v_b)\Big),$$

the map $\widetilde{\mathcal{G}}$ has the following properties: $\big(\widetilde{\mathcal{G}}\big)^{-1}(0) \cap W^* \subseteq \mathcal{W}$, all the zeros of $\widetilde{\mathcal{G}}$ are nondegenerate, we have $\deg(\mathcal{G}, W^*) = \deg(\widetilde{\mathcal{G}}, W^*)$ and

$$\max_{(u,v_0,\dots,v_b)\in\overline{\mathcal{V}}} \|\mathcal{G}(u,v_0,v_1,\dots,v_b) - \widetilde{\mathcal{G}}(u,v_0,v_1,\dots,v_b)\| < \delta/4. \tag{4.5}$$

Let now $\widetilde{\Phi}(u) = \widetilde{g}\big(u,0,\widetilde{\varphi}(u,0)\big)$. By (4.2)–*(ii)* and (4.5) it follows $|\widetilde{\Phi}(\alpha) - \Phi(\alpha)| < \delta/4$ and $|\widetilde{\Phi}(\beta) - \Phi(\beta)| < \delta/4$, so that from $\Phi(\alpha)\Phi(\beta) < 0$ we get

$$\widetilde{\Phi}(\alpha)\widetilde{\Phi}(\beta) < 0. \tag{4.6}$$

Now, recall that by Sard's Theorem (see, e.g., [30, Ch. 3, §1] or [29, Ch. 1, §7]) the set of the critical values of a smooth function has Lebesgue measure zero. Thus, one can choose some $r \in (-\delta/4, \delta/4)$ such that 0 is a regular value of the map

$$u \mapsto \widetilde{g}\big(u,0,\widetilde{\varphi}(u,0)\big) - r. \tag{4.7}$$

In particular, for such an r, all the zeros of this map are nondegenerate. Let $\sigma\colon \mathbb{R}^{b+2} \to [0,1]$ be a smooth function that vanishes identically on $\mathbb{R}^{b+2} \setminus \mathcal{V}$ and is identically equal to 1 in $\overline{\mathcal{W}}$. The existence of such a function follows from the so-called smooth Urysohn Theorem; see, e.g., [29, Ch. 1, §8, Exercise 15]. Define

$$\widetilde{\Phi}_r(u) = \widetilde{g}\big(u,0,\widetilde{\varphi}(u,0)\big) - \sigma\big(u,0,\widetilde{\varphi}(u,0),\dots,\widetilde{\varphi}(u,0)\big)r.$$

Observe that (4.2)-*(ii)* implies that $\widetilde{\Phi}_r$ coincides with the map in (4.7) in $[\alpha,\beta]$; consequently, all of its zeros in this interval are nondegenerate. From (4.6) it follows $\widetilde{\Phi}_r(\alpha)\widetilde{\Phi}_r(\beta) < 0$, which implies that $\widetilde{\Phi}_r$ is admissible in (α,β). Let now

$$\begin{aligned}&\widetilde{\mathcal{G}}_r(u,v_0,v_1,\dots,v_b)\\ &\quad = \Big(v_0, \widetilde{g}(u,v_0,v_b) - \sigma(u,v_0,\dots,v_b)r, a\big(\widetilde{\varphi}(u,v_0) - v_b\big), a(v_1 - v_2)\dots, a(v_{b-1} - v_b)\Big),\end{aligned}$$

where σ is the scalar function introduced above. Applying Lemma 4.1, Lemma 4.4 and Lemma 4.5 to $\widetilde{\Phi}_r$ and $\widetilde{\mathcal{G}}_r$ we obtain that $\widetilde{\mathcal{G}}_r$ is admissible in $W^* = (\alpha,\beta) \times \mathbb{R}^{b+1}$, meaning that $\big(\widetilde{\mathcal{G}}_r\big)^{-1}(0) \cap W^*$ is compact, and

$$\deg(\widetilde{\mathcal{G}}_r, W^*) = \deg\Big(\widetilde{\Phi}_r, (\alpha,\beta)\Big) \neq 0. \tag{4.8}$$

Furthermore, by construction, the map $\widetilde{\mathcal{G}}_r$ coincides with $\widetilde{\mathcal{G}}$ on $\mathbb{R}^{b+2} \setminus \mathcal{V}$, hence the compact set $\big(\widetilde{\mathcal{G}}_r\big)^{-1}(0) \cap W^*$ is contained in $\mathcal{V} \cap W^*$. Actually, as a consequence of (4.4)-*(ii)*, (4.5) and the choice of r, $\big(\widetilde{\mathcal{G}}_r\big)^{-1}(0) \cap W^* \subseteq \mathcal{W}$. Thus, from (4.8) and the excision property of the degree we get

$$\deg(\widetilde{\mathcal{G}}_r, \mathcal{W}) \neq 0. \tag{4.9}$$

Moreover, note that

$$\max_{(u,v_0,\dots,v_b)\in\overline{\mathcal{W}}} \|\mathcal{G}(u,v_0,v_1,\dots,v_b) - \widetilde{\mathcal{G}}_r(u,v_0,v_1,\dots,v_b)\| < \delta/2,$$

and consider the homotopy $H\colon [0,1] \times \overline{\mathcal{W}} \to \mathbb{R}$ defined by

$$H(\lambda,u,v_0,v_1,\dots,v_b) = \lambda\mathcal{G}(u,v_0,v_1,\dots,v_b) + (1-\lambda)\widetilde{\mathcal{G}}_r(u,v_0,v_1,\dots,v_b).$$

Observe that, by construction,

$$\min\{\|H(\lambda,u,v_0,v_1,\dots,v_b)\| : \lambda\in[0,1],(u,v_0,\dots,v_b)\in\partial\mathcal{W}\}>\delta/2>0;$$

hence, the compact set

$$\big\{(\lambda,u,v_0,\dots,v_b)\in[0,1]\times\overline{\mathcal{W}} : H((\lambda,u,v_0,\dots,v_b)=0\big\},$$

is contained in $[0,1]\times\mathcal{W}$. Thus, H is an admissible homotopy, so that

$$\deg(\mathcal{G},\mathcal{W})=\deg(\widetilde{\mathcal{G}}_r,\mathcal{W}). \tag{4.10}$$

Finally, from (4.3) and (4.10) we get

$$\deg(G,W^*)=\deg(\mathcal{G},\mathcal{W})=\deg(\widetilde{\mathcal{G}}_r,\mathcal{W})$$

and the assertion follows from inequality (4.9). □

5 Ejecting points and small perturbations

As we have seen by the simple example at the end of Section 3, it may happen that the branch Γ of T-forced pairs of equation (2.1), as in the assertion of Theorem 3.6, is completely contained in the slice $\{0\}\times C^1_T(\mathbb{R})$. In this section we show simple conditions ensuring that this is not the case. Such conditions are based on the key notion of *ejecting set* (see, e.g., [26]).

Let us first introduce the following notation: let Y be a subset of $[0,\infty)\times C^1_T(\mathbb{R})$. Given $\lambda\geq 0$, let Y_λ be the slice $\big\{x\in C^1_T(\mathbb{R}) : (\lambda,x)\in \mathrm{Y}\big\}$. Below, we adapt to our context the definition of ejecting set of [26]:

Definition 5.1. Let $X\subseteq[0,\infty)\times C^1_T(\mathbb{R})$ be the set of T-forced pairs of (2.1), and let X_0 be the slice of X at $\lambda=0$. We say that $A\subset X_0$ is an *ejecting set* (for X) if it is relatively open in X_0 and there exists a connected subset of X which meets A and is not contained in X_0. In particular, when $A=\{p_0\}$ is a singleton we say that p_0 is an *ejecting point*.

We now discuss a sufficient condition for an isolated point of Φ to be ejecting for the set X of T-forced pairs of (2.1). This condition is based on a result by J. Yorke ([43], see also [7]) concerning the period of solutions of an autonomous ODE with Lipschitz continuous right-hand side.

We point out that an analysis of (2.3), for $\lambda=0$, linearized at its zeros leads to a different, not entirely comparable, approach. A discussion of the latter technique, which is based on the notion of T-resonance and requires the knowledge of the spectrum of the linearized equation is outside the scope of the present paper (see, e.g., [17, Ch. 7] and [16, Ch. 2], a similar idea can be traced back to Poincaré see, e.g., [38]; see also [2] for an application of this idea to multiplicity results).

The aforementioned result of Yorke is the following:

Theorem 5.2 ([43])**.** *Let ξ be a nonconstant τ-periodic solution of*

$$\dot{x}=\mathcal{F}(x)$$

where $\mathcal{F}\colon W\subseteq\mathbb{R}^k\to\mathbb{R}^k$ is Lipschitz of constant L. Then the period τ of ξ satisfies $\tau\geq 2\pi/L$.

Inequalities on the period like that of Theorem 5.2 have been object of study in different contexts, see e.g., the introduction of [8] for an interesting discussion.

Since we are going to use Theorem 5.2, besides our standard assumptions (2.2), we need to assume the following fact on the maps g and φ throughout this and the following sections:

$$g \text{ and } \varphi \text{ are such that } G \text{, defined in (2.4), is (locally) Lipschitz.} \tag{5.1}$$

Observe that taking g and φ locally Lipschitz allows us to apply Theorem 5.2 in a suitable set $W \subseteq \mathbb{R}^{b+2}$. In particular, since we are going to apply such a result only to unperturbed part of (2.3), no further hypothesis on f is necessary.

From Theorem 5.2, we immediately deduce the following facts concerning the branch of T-forced pairs of (2.1) given by Theorem 3.6:

Corollary 5.3. *Consider equation* (2.1)*, assume that* (2.2) *hold and let* (α, β)*,* Ω *and* Φ *be as in Theorem 3.6. Suppose further that* g *and* φ *are such that* G *is Lipschitz with constant* L_G*. If the period* T *of the forcing term* f *satisfies* $T < 2\pi/L_G$*, then the set* Γ *of Theorem 3.6 cannot intersect the slice* $\{0\} \times C^1_T(\mathbb{R})$*.*

Proof. Assume by contradiction that there exists a nontrivial T-forced pair $(0, x_0) \in \Gamma$. Then x_0 corresponds (in the sense of Theorem 2.1) to a T-periodic solution of $\dot{\xi} = G(\xi)$ with $T < 2\pi/L_G$, violating Theorem 5.2. □

A similar argument yields the following result:

Corollary 5.4. *Consider equation* (2.1)*, suppose that* (2.2) *hold and let* (α, β)*,* Ω *and* Φ *be as in Theorem 3.6. Let* $p \in (\alpha, \beta)$ *be an isolated zero of* Φ *and assume that* g *and* φ *are such that* G *is locally Lipschitz in a neighborhood* W *of the corresponding zero* $P = \big(p, 0, \varphi(p, 0), \dots, \varphi(p, 0)\big)$ *of* G*. Suppose that a set* Υ *of* T*-pairs for* (2.3) *contains* $\big(0, \overline{P}\big)$*. Let* $\Lambda = \mathcal{J}^{-1}(\Upsilon)$ *be the corresponding set of* T*-forced pairs of* (2.1)*. As above, call* L_G *the Lipschitz constant of* G *in* W*. Then, the trivial* T*-forced pair* $(0, \bar{p})$ *is isolated in the slice* Λ_0*. In particular, if* $T < 2\pi/L_G$*, the set* Λ *cannot be contained in the slice* $\{0\} \times C^1_T(\mathbb{R})$*.*

Observe that, unlike Γ in Corollary 5.3, the set Υ in Corollary 5.4 does not necessarily consist of nontrivial T-pairs. Consequently, Λ does not necessarily consist of nontrivial T-forced pairs as well. Notice also that since p is an isolated zero of Φ then $\big(0, \overline{P}\big)$ is isolated in the set of trivial T-pairs of (2.3) and, similarly, $(0, \bar{p})$ is isolated in the set of trivial T-forced pairs of (2.1).

Proof of Corollary 5.4. By construction of the map $\mathcal{J}$, the T-forced pairs of (2.1) that lie in the slice $\{0\} \times C^1_T(\mathbb{R})$ corresponds bijectively to the T-pairs of (2.3) contained in $\{0\} \times C_T(\mathbb{R}^{b+2})$. Theorem 5.2 implies that there are not nonconstant T-periodic solutions of (2.3) contained in W. Thus Υ cannot be contained in the slice $\{0\} \times C_T(\mathbb{R}^{b+2})$. Hence, the same is true for $\Lambda = \mathcal{J}^{-1}(\Upsilon)$. □

Remark 5.5. Suppose that g and φ are such that G is C^1 in a neighborhood of a nondegenerate zero, then it is locally Lipschitz in this neighborhood. Then by Corollary 5.4 we have that *if the frequency of the forcing term is sufficiently high, then a nondegenerate zero of the unperturbed vector field is necessarily an ejecting point of nontrivial* T*-forced pairs of* (2.1).

6 Ejecting sets and multiplicity

This section is devoted to the illustration of sufficient conditions on f, g, and φ yielding the multiplicity of T-periodic solutions of (2.1) for $\lambda > 0$ small.

As above, let G be as in (2.4), and let $X \subseteq [0, \infty) \times C^1_T(\mathbb{R})$ be the set of nontrivial T-forced pairs of (2.1).

For brevity, we will say that a continuous function $\psi : (\alpha, \beta) \to \mathbb{R}$ *changes sign at the isolated zero* $p \in (\alpha, \beta)$ if there exists $\delta > 0$ such that $\forall x \in (p - \delta, p)$, $\forall y \in (p, p + \delta)$ we have $\psi(x) \cdot \psi(y) < 0$.

Corollary 5.4 and Theorem 3.6 yield the following result about multiple periodic solutions of (2.1).

Theorem 6.1. *Consider equation* (2.1) *and suppose that* (2.2) *hold. Let* $(\alpha, \beta) \subseteq \mathbb{R}$ *be an open interval and assume that* Φ *changes sign at the isolated zeros* $p_1, \dots, p_n \in (\alpha, \beta)$. *For* $i = 1, \dots, n$, *let* $P_i = \big(p_i, 0, \varphi(p_i, 0) \dots, \varphi(p_i, 0)\big)$. *Assume that* g *and* φ *are such that* G *is Lipschitz with constant* L *on a neighborhood of* $\{P_1, \dots, P_n\}$ *and the period* T *of* f *satisfies* $T < 2\pi/L$. *Then:*

1. *for* $i = 1, \dots, n$, *there exist open subsets* Ω_i *of* $[0, \infty) \times C^1_T(\mathbb{R})$ *and connected sets* $\Gamma_i \subseteq \Omega_i$ *of nontrivial* T*-pairs of* (2.1) *whose closure* Ξ_i *in* Ω_i *contains* $(0, \bar{p}_i)$ *and is not compact;*
2. *there exists* $\lambda_* > 0$ *such that the projection of each* Ξ_i, $i = 1, \dots, n$, *on the first component contains* $[0, \lambda_*]$. *Thus,* (2.1) *has at least* n *solutions* $x_1^\lambda, \dots, x_n^\lambda$ *of period* T *for* $\lambda \in [0, \lambda_*]$;
3. *the images of* $x_1^\lambda, \dots, x_n^\lambda$ *are pairwise disjoint.*

Proof. Let I_i, $i = 1, \dots, n$, be open intervals with pairwise disjoint closures each isolating the zero p_i. Let $U_i = C^1_T(\mathbb{R}, I_i)$ and $\Omega_i = [0, \infty) \times U_i$. Then by Theorem 3.6, for each $i \in \{1, \dots, n\}$, there is a connected set Γ_i in $X \cap \Omega_i$ whose closure, $\Xi_i := \mathrm{cl}_{\Omega_i}(\Gamma_i)$, in Ω_i is not compact and intersect the set

$$\big\{(0, \overline{u}) \in \Omega_i : u \in I_i \cap \Phi^{-1}(0)\big\} = \{(0, \bar{p}_i)\}.$$

This proves assertion (1).

To prove assertion (2) let, for $i = 1, \dots, n$, $K_i = \{(0, \bar{p}_i)\}$. By Corollary 5.4, K_i is isolated in the slice $\{0\} \times U_i$. Since Ξ_i is not compact, it does not consist only of the singleton $\{(0, \overline{p}_i)\}$. Also, since Ξ_i is connected it cannot be completely contained in $\{0\} \times U_i$ (otherwise $(0, \overline{p}_i)$ would not be an isolated point). So, the projection π_1 of Ξ_i onto the first factor of $[0, \infty) \times C^1_T(\mathbb{R})$ cannot reduce to $\{0\}$. Thus, π_1 being continuous, $\pi_1\big(\Xi_i\big)$ is a nontrivial interval J_i with $0 \in J_i$. Let $\delta_i > 0$ be such that $[0, \delta_i] \subseteq J_i$. The proof is completed by taking $\lambda_* = \min\{\delta_1, \dots, \delta_n\}$.

To prove the last assertion, observe that by part (2) there exists $\lambda_* > 0$ such that each $\{\lambda\} \times U_i$, contains at least one T-forced pair, say (λ, x_i^λ) for any $\lambda \in [0, \lambda_*]$. Hence, for $j, k = 1, \dots, n$ and $j \neq k$, the images of x_j^λ and x_k^λ are confined to the disjoint sets I_i and I_k. □

Restricting to a neighborhood of the set of zeros of G corresponding to $p_1, \dots, p_n$, we can give a somewhat less technical and perhaps more elegant formulation of our multiplicity result.

Corollary 6.2. *Consider equation* (2.1) *and suppose that* (2.2) *and* (5.1) *hold. Assume that* Φ *changes sign at the isolated zeros* $p_1, \dots, p_n \in (\alpha, \beta)$. *Then, for sufficiently high frequency of the perturbing term* f *and sufficiently small* $\lambda > 0$, *equation* (2.1) *has at least* n *solutions of period* T *whose images are pairwise disjoint.*

7 Visual representation of branches

We briefly discuss a method allowing to represent graphically the infinite dimensional set of T-forced pairs of (2.1). In other words, as in [2], we create a homeomorphic finite dimensional image of the set Γ yielded by Theorem 3.6 and show a graph of some relevant functions of the point of Γ as, for instance, the sup-norm or the diameter of the orbit of the solution x in any T-forced pair (λ, x).

In this section we assume g and φ as well as the perturbing term f to be at least Lipschitz continuous, so that continuous dependence on initial data of (2.3) holds.

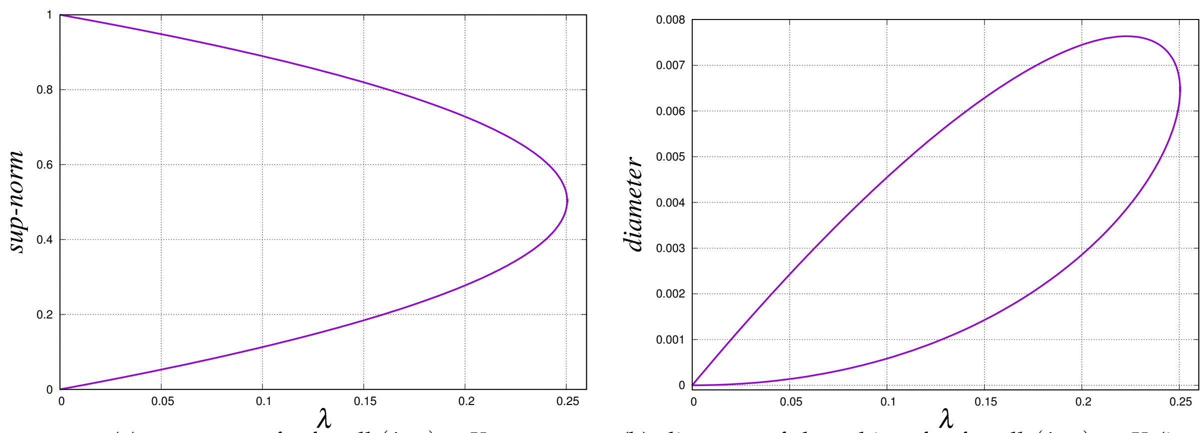
Let us consider the set

$$S = \left\{ (\lambda, q, p_0, \dots, p_b) \in [0,\infty) \times \mathbb{R}^{b+2} \;\middle|\; \begin{array}{l} (q, p_0, \dots, p_b) \text{ is an initial condition at } t = 0 \\ \text{for a } T\text{-periodic solution of (2.3)} \end{array} \right\}.$$

The elements of S are called *starting points* for (2.3). A starting point $(\lambda, q, p_0, \dots, p_b)$ is *trivial* when $\lambda = 0$ and the solution of (2.3) starting a time $t = 0$ from $(q, p_0, \dots, p_b)$ is constant. By uniqueness and continuous dependence on initial data the map $\mathfrak{p}\colon Y \to S$ given by

$$(\lambda, x_0, y_0, \dots . y_b) \mapsto \big(\lambda, x_0(0), y_0(0), \dots . y_b(0)\big)$$

is a homeomorphism that establishes a correspondence between trivial T-pairs and trivial starting points. Thus, the composition $\mathfrak{h} = \mathfrak{p} \circ \mathcal{J}^{-1}\colon X \to S$ is as well a homeomorphism that establishes a correspondence between trivial T-forced pairs and trivial starting points. In other words, $\Sigma := \mathfrak{h}(\Gamma) \subseteq [0,\infty) \times \mathbb{R}^{b+2}$ is the desired homemorphic image of Γ.



(a) sup-norm of x for all $(\lambda, x) \in X$

(b) diameter of the orbits of x for all $(\lambda, x) \in X$ (i.e., $\max_{t\in[0,T]} x(t) - \min_{t\in[0,T]} x(t)$)

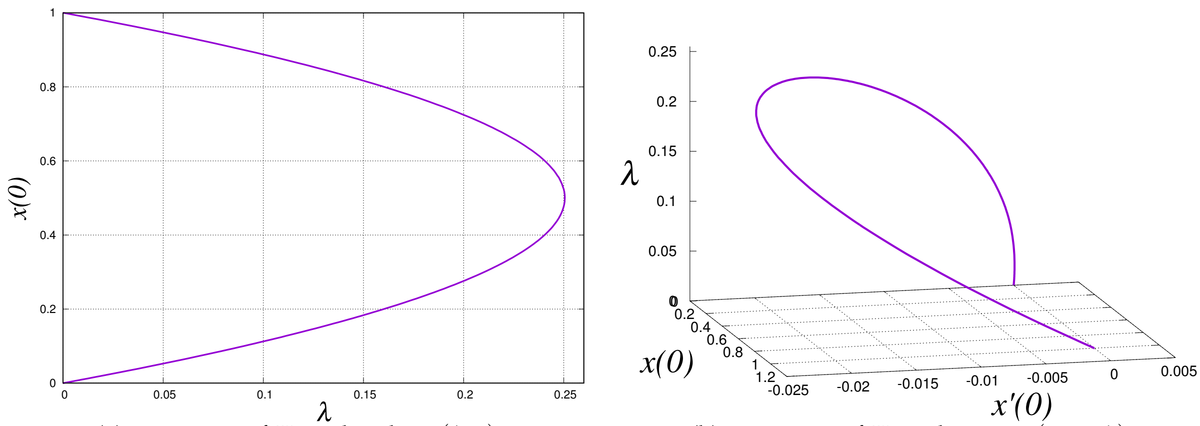
Figure 7.1: Sup-norm and diameter of points in Γ

Example 7.1. Consider the following scalar equation:

$$\ddot{x}(t) = -x(t)\left(1 + \int_{-\infty}^{t} \gamma_2^2(t-s)\big(\dot{x}(s) - x(s)\big)\,\mathrm{d}s\right) + \lambda\big(1 + x(t)\sin(2\pi t)\big), \tag{7.1}$$

with $\lambda \geq 0$. Here, $T = 1$, $g(\xi, \eta, \zeta) = -\xi(1+\zeta)$, $\varphi(p, q) = q - p$, so that

$$\Phi(u) = g\big(u, 0, \varphi(u, 0)\big) = -u(1-u).$$



(a) projection of X on the plane (λ, q) (b) projection of X on the space (q, p_0, λ)

Figure 7.2: Projections of Σ: Initial values and speed of x for $(\lambda, x) \in \Gamma$

One has

$$G(u, v_0, v_1, v_2) = \big(v_0, -u(1+v_2), 2(v_0 - u - v_1), 2(v_1 - v_2)\big).$$

Note that Φ changes sign at the zeros 0 and 1, the corresponding zeros of G being $P_0 := (0,0,0,0)$ and $P_1 := (1,0,-1,-1)$. Observe that G is locally Lipschitz. In particular, sufficiently small neighborhoods of P_0 and P_1 can be chosen where the Lipschitz constant L of G is smaller than 2π. The validity of this statement can be checked by computing the operator norms of the Jacobian matrices of G at P_0 and P_1 that actually turn out to be strictly smaller than 4. Thus, by Theorem 6.1 there are connected sets Γ_0 and Γ_1 of nontrivial T-forced pairs of (7.1) emanating, respectively, from $(0, \bar{0})$ and $(0, \bar{1})$. In particular, by Theorem 6.1 it follows that there exists $\lambda_* > 0$ such that for $\lambda \in [0, \lambda_*)$ there are two periodic solutions of (7.1) with disjoint images. Corollary 6.2 shows that the same is true (with possibly different values of λ_*) for all perturbations with sufficiently high frequency.

Figure 7.1 shows sup-norm and diameter of the solutions of the T-forced pairs of (7.1). Figure 7.2, instead, shows the projections of Σ on the plane (λ, q) and on the 3-dimensional space (q, p_0, λ). Indeed, Figures 7.1 and 7.2 suggest that, for Γ_0 and Γ_1 maximal, $\Gamma_1 = \Gamma_0$. The figures suggest that the value of λ_* in Theorem 6.1 is about 0.25.

8 Perspectives and future developments

The results of this paper can be naturally extended along two lines. The first and more direct one is to consider systems of equations, say in $\mathbb{R}^n$, allowing φ to take vector values, say in $\mathbb{R}^k$. In this case one considers (1.3) with $g\colon \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$ and $\varphi\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^k$. This generalization that, for example, finds application to coupled nonlinear oscillators with memory, has been considered in [36]. Here we only mention that the analogous of the function Φ that has to be constructed in this more general environment is a map of an open set of $\mathbb{R}^n$ in $\mathbb{R}^n$, hence the sign change hypothesis has to be replaced with an assumption about the degree.

A second extension that can be considered is the case when more than one distributed delay is allowed. This situation is considered in some models (see, e.g., [35, 41]), indeed it arises naturally in some situations where a nonlinear equation is coupled with a linear "subsystem". To illustrate this point we consider the following example:

Example 8.1. Let $f\colon \mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and $\varphi\colon \mathbb{R}\to\mathbb{R}$ be given continuous functions. We are interested in the bounded (in C^0) solutions, if any, of the following system of coupled scalar differential equations

$$\begin{cases}\ddot{x}(t) = f\big(t, x(t), y(t)\big),\\ \ddot{y}(t) + (a+b)\,\dot{y}(t) + ab\,y(t) = \varphi\big(x(t)\big),\end{cases} \tag{8.1}$$

with $a \neq b$ positive constants. It is well known that if x is a bounded function, the second equation in (8.1) admits a unique bounded solution which, as it is easy to check, is given by (see, e.g., [15, Lemma 4.1])

$$\begin{aligned} y(t) &= \frac{1}{b-a}\left(\int_{-\infty}^{t} \mathrm{e}^{-a(t-s)}\varphi\big(x(s)\big)\,\mathrm{d}s - \int_{-\infty}^{t} \mathrm{e}^{-b(t-s)}\varphi\big(x(s)\big)\,\mathrm{d}s\right)\\ &= \frac{1}{b-a}\left(\frac{1}{a}\int_{-\infty}^{t} \gamma_a^1(t-s)\varphi\big(x(s)\big)\,\mathrm{d}s - \frac{1}{b}\int_{-\infty}^{t} \gamma_b^1(t-s)\varphi\big(x(s)\big)\,\mathrm{d}s\right). \end{aligned}$$

Using this expression for y, one sees that the bounded solutions of (8.1) are determined by the bounded solutions of the following equation with two distributed delays:

$$\ddot{x}(t) = F\left(t, x(t), \frac{1}{a}\int_{-\infty}^{t} \gamma_a^1(t-s)\varphi\big(x(s)\big)\,\mathrm{d}s - \frac{1}{b}\int_{-\infty}^{t} \gamma_b^1(t-s)\varphi\big(x(s)\big)\,\mathrm{d}s\right),$$

with $F(t,\xi,\eta) = f\big(t,\xi,\eta/(b-a)\big)$. Notice that when $a = b$ one obtains the unique bounded solution of the second equation in (8.1) as

$$y(t) = \int_{-\infty}^{t} (t-s)\mathrm{e}^{-a(t-s)}\varphi\big(x(s)\big)\,\mathrm{d}s = \frac{1}{a^2}\int_{-\infty}^{t} \gamma_a^2(t-s)\varphi\big(x(s)\big)\,\mathrm{d}s.$$

Hence, in this case, the bounded solutions of (8.1) are determined by the following differential equation with a single distributed delay:

$$\ddot{x}(t) = f\left(t, x(t), \frac{1}{a^2}\int_{-\infty}^{t} \gamma_a^2(t-s)\varphi\big(x(s)\big)\,\mathrm{d}s\right).$$

We observe that the above perspective extensions can be combined and further expanded to the case where the ambient space is a differentiable manifold. Such generalization will be investigated elsewhere.

Acknowledgements

The authors would like to thank the referee for the careful reading of the manuscript and the constructive comments. The authors are members of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

References

[1] P. Benevieri, M. Furi, M. P. Pera, M. Spadini, *An introduction to topological degree in Euclidean spaces*, arXiv:2304.06463. https://doi.org/10.48550/arxiv.2304.06463

[2] L. Bisconti, M. Spadini, About the notion of non-T-resonance and applications to topological multiplicity results for ODEs on differentiable manifolds, *Math. Methods Appl. Sci.* **38**(2015), No. 18, 4760–4773. https://doi.org/10.1002/mma.3390; MR3449631; Zbl 1342.34058

[3] A. Boscaggin, A. Fonda, M. Garrione, A multiplicity result for periodic solutions of second-order differential equations with a singularity, *Nonlinear Anal.* **75**(2012), No. 12, 4457–4470. https://doi.org/10.1016/j.na.2011.10.025; MR2927114; Zbl 1278.34038

[4] A. Boscaggin, F. Zanolin, Subharmonic solutions for nonlinear second order equations in presence of lower and upper solutions, *Discrete Contin. Dyn. Syst.* **33**(2013), No. 1, 89–110. https://doi.org/10.3934/dcds.2013.33.89; MR2972947; Zbl 1275.34058

[5] T. A. Burton, *Stability and periodic solutions of ordinary and functional differential equations*, Dover Publications, Inc., Mineola, NY, 2005. Corrected version of the 1985 original. MR2761514; Zbl 1209.34001

[6] S. Busenberg, L. T. Hill, Construction of differential equation approximations to delay differential equations, *Appl. Anal.* **31**(1988), 35–56. https://doi.org/10.1080/00036818808839814; MR1017501; Zbl 0635.34049

[7] Stavros Busenberg, Mario Martelli, Bounds for the period of periodic orbits of dynamical systems, *J. Differential Equations* **67**(1987), No. 3, 359–371. https://doi.org/10.1016/0022-0396(87)90132-X; MR884275; Zbl 0616.34041

[8] S. Busenberg, M. Martelli, Geometric inequalities in normed spaces, *Rocky Mountain J. Math.* **22**(1992), No. 2, 477–491. https://doi.org/10.1216/rmjm/1181072742; MR1180713; Zbl 0788.46017

[9] S. Busenberg, C. Travis, Approximation of functional differential equations by differential systems, in: K. B. Hanngsen, T. L. Herdman, H. W. Stech, R. L. Wheeler (Eds.), *Volterra and functional differential equations*, Marcel Dekker, New York, 1981, pp. 197–205. Zbl 0519.34051

[10] S. Busenberg, C. Travis, On the use of reducible-functional-differential equations in biological models. *J. Math. Anal. Appl.* **89**(1982), No. 1, 46–66. https://doi.org/10.1016/0022-247X(82)90090-7; MR672188; Zbl 0513.34075

[11] A. Calamai, M. P. Pera, M. Spadini, Multiplicity of forced oscillations for the spherical pendulum acted on by a retarded periodic force, *Nonlinear Anal.* **151**(2017), 252–264. https://doi.org/10.1016/j.na.2016.12.006; MR3596680; Zbl 1381.34087

[12] A. Calamai, M. P. Pera, M. Spadini, Multiplicity of forced oscillations for scalar retarded functional differential equations, *Math. Methods Appl. Sci.* **41**(2018), No. 5, 1944–1953. https://doi.org/10.1002/mma.4720; MR3778098; Zbl 1395.34074

[13] A. Calamai, M. P. Pera, M. Spadini, Branches of forced oscillations induced by a delayed periodic force, *Adv. Nonlinear Stud.* **19**(2019), No. 1, 149–163. https://doi.org/10.1515/ans-2018-2028; MR3912426; Zbl 1412.70025

[14] B. D. Coleman, G. H. Renninger, Periodic solutions of certain nonlinear integral equations with a time-lag, *SIAM J. Appl. Math.* **31**(1976), 111–120. https://doi.org/10.1137/0131012; MR405035; Zbl 0328.45008

[15] F. Colonius, M. Spadini, Uniqueness of local control sets, *J. Dynam. Control Systems* **9**(2003), No. 4, 513–530. https://doi.org/10.1023/A:1025692218217; MR2001957; Zbl 1129.93321

[16] J. Cronin, *Fixed points and topological degree in nonlinear analysis*, Mathematical Surveys, No. 11 American Mathematical Society, Providence, R.I. 1964. MR0164101; Zbl 0117.34803

[17] J. Cronin, *Differential equations. Introduction and qualitative theory*, 2nd Ed., Pure and Appl. Math. Vol. 180, Marcel Dekker Inc., New York, 1994. MR1275827; Zbl 0798.34001

[18] R. V. Culshaw, S. Ruan, A delay-differential equation model of HIV infection of CD4$^+$ T-cells, *Math. Biosci.* **165**(2000), No. 1, 27–39. https://doi.org/10.1016/S0025-5564(00)00006-7; Zbl 0981.92009

[19] R. V. Culshaw, S. Ruan, G. Webb, A mathematical model of cell-to-cell spread of HIV-1 that includes a time delay, *J. Math. Biol.* **46**(2003), No. 5, 425–444. https://doi.org/10.1007/s00285-002-0191-5; MR1998996; Zbl 1023.92011

[20] G. Dinca, J. Mawhin, *Brouwer degree. The core of nonlinear analysis*, Progress in Nonlinear Differential Equations and Their Applications, Vol. 95, Birkhäuser, Cham, 2021. https://doi.org/10.1007/978-3-030-63230-4; MR4390178

[21] D. Fargue, Réductibilité des systèmes héréditaires à des systèmes dynamiques (régis par des équations différentielles ou aux dérivées partielles), *C. R. Acad. Sci. Paris Sér. A-B* **277**(1973), B471–B473. MR333630

[22] D. Fargue, Réductibilité des systèmes héréditaires, *Internat. J. Non-Linear Mech.* **9**(1974), 331–338. https://doi.org/10.1016/0020-7462(74)90018-3; MR370100; Zbl 0312.73049

[23] G. Feltrin, F. Zanolin, Multiplicity of positive periodic solutions in the superlinear indefinite case via coincidence degree, *J. Differential Equations* **262**(2017), No. 8, 4255–4291. https://doi.org/10.1016/j.jde.2017.01.009; MR3603271; Zbl 1417.34063

[24] A. Fonda, *Playing around resonance. An invitation to the search of periodic solutions for second order ordinary differential equations*, Birkhäuser, Basel, 2016. https://doi.org/10.1007/978-3-319-47090-0; MR3585909; Zbl 1364.34004

[25] M. Furi, M. P. Pera, M. Spadini, Periodic solutions of retarded functional perturbations of autonomous differential equations on manifolds, *Commun. Appl. Anal.* **15**(2011), No. 2–4, 381–394. MR2867360; Zbl 1235.34188; http://www.acadsol.eu/en/articles/15/3/10.pdf

[26] M. Furi, M. P. Pera, M. Spadini, Forced oscillations on manifolds and multiplicity results for periodically perturbed autonomous systems, *J. Comput. Appl. Math.* **113**(2000), No. 1–2, 241–254. https://doi.org/10.1016/S0377-0427(99)00259-9; MR1735827; Zbl 0936.37038

[27] M. Furi, M. Spadini, On the set of harmonic solutions of periodically perturbed autonomous differential equations on manifolds, *Nonlinear Anal.* **29**(1997), No. 8, 963–970. https://doi.org/10.1016/S0362-546X(96)00296-9; MR1454821; Zbl 0885.34041

[28] M. Furi, M. Spadini, Periodic perturbations with delay of autonomous differential equations on manifolds, *Adv. Nonlinear Stud.* **9**(2009), No. 2, 263–276. https://doi.org/10.1515/ans-2009-0203; MR2503829; Zbl 1193.34141

[29] V. Guillemin, A. Pollack, *Differential topology*, Prentice-Hall Inc., Englewood Cliffs, New Jersey, 1974. MR0348781; Zbl 0361.57001

[30] M. W. Hirsch, *Differential topology*, Vol. 33, Graduate Texts in Mathematics, Springer-Verlag, New York, 1994. Corrected reprint of the 1976 original. MR1336822; Zbl 0356.57001

[31] J. K. Hale, L. T. Magalhães, W. M. Oliva, *An introduction to infinite-dimensional dynamical systems – geometric theory*, Applied Mathematical Sciences, Vol. 47, Springer-Verlag, New York, 1984. https://doi.org/10.1007/0-387-22896-9_9; MR725501; Zbl 0533.58001

[32] M. J. Leitman, V. J. Mizel, On fading memory spaces and hereditary integral equations, *Arch. Rational Mech. Anal.* **55**(1974), 18–51. https://doi.org/10.1007/BF00282432; MR367734; Zbl 0297.45001

[33] N. G. Lloyd, *Degree theory*, Cambridge University Press, Cambridge, 1978. MR0493564; Zbl 0367.47001

[34] N. MacDonald, *Time lags in biological models*, Lecture Notes in Biomathematics, Vol. 27, Springer-Verlag, Berlin–New York, 1978. MR521439; Zbl 0403.92020

[35] N. MacDonald, *Biological delay systems: linear stability theory*, Cambridge Studies in Mathematical Biology, Vol. 8, Cambridge University Press, Cambridge, 1989. MR996637; Zbl 0669.92001

[36] P. M. Mariano, M. Spadini, Periodic solutions to perturbed nonlinear oscillators with memory, *Phys. D* **445**(2023), 133635. https://doi.org/10.1016/j.physd.2022.133635; MR4530994

[37] J. Mawhin, Can the drinking bird explain economic cycles? A history of auto-oscillations and limit cycles, *Bull. Cl. Sci. Acad. R. Belg.* **20**(2009), 49–94. https://doi.org/10.3406/barb.2009.28678

[38] J. Mawhin, The implicit function theorem and its substitutes in Poincaré's qualitative theory of differential equations, *Stud. Hist. Philos. Sci. B Stud. Hist. Philos. Modern Phys.* **47**(2014), 124–130. https://doi.org/10.1016/j.shpsb.2014.01.006; MR3244605; Zbl 1294.01020

[39] L. Nirenberg, *Topics in nonlinear functional analysis*, Courant Institute of Mathematical Sciences, New York, 1974. MR0488102; Zbl 0286.47037

[40] S. Ruan, Delay differential equations in single species dynamics, in: *Delay differential equations and applications*, NATO Sci. Ser. II Math. Phys. Chem., Vol. 205, Springer, Dordrecht, 2006, pp. 477–517. https://doi.org/10.1007/1-4020-3647-7_11; MR2337823; Zbl 1130.34059

[41] H. Smith, *An introduction to delay differential equations with applications to the life sciences*, Texts in Applied Mathematics, Vol. 57, Springer, New York, 2011. https://doi.org/10.1007/978-1-4419-7646-8; MR2724792; Zbl 1227.34001

[42] M. Spadini, Periodic perturbations of a class of functional differential equations, *J. Dynam. Differential Equations* **34**(2022), No. 1, 535–553. https://doi.org/10.1007/s10884-020-09928-2; MR4383297; Zbl 1500.34059

[43] J. A. Yorke, Periods of periodic solutions and the Lipschitz constant, *Proc. Amer. Math. Soc.* **22**(1969), 509–512. https://doi.org/10.2307/2037090; MR245916; Zbl 0184.12103