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Approximation of SBV functions with possibly infinite jump set

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ABSTRACT

We prove an approximation result for functions $u \in SBV(\Omega;\mathbb{R}^m)$ such that ∇u is p-integrable, $1 \le p < \infty$, and $g_0(|[u]|)$ is integrable over the jump set (whose $\mathcal{H}^{n-1}$ measure is possibly infinite), for some continuous, nondecreasing, subadditive function g_0, with $g_0^{-1}(0) = \{0\}$. The approximating functions u_j are piecewise affine with piecewise affine jump set; the convergence is that of L^1 for u_j and the convergence in energy for $|\nabla u_j|^p$ and $g([u_j], \nu_{u_j})$ for suitable functions g. In particular, u_j converges to u BV-strictly, area-strictly, and strongly in BV after composition with a bilipschitz map. If in addition $\mathcal{H}^{n-1}(J_u) < \infty$, we also have convergence of $\mathcal{H}^{n-1}(J_{u_j})$ to $\mathcal{H}^{n-1}(J_u)$.

Contents

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1. Introduction and main result

Approximation with regular objects is a fundamental tool in many problems in functional analysis and in the Calculus of Variations. For instance, De Giorgi's theory of sets of finite perimeter depends crucially on the approximability with piecewise smooth sets, a key step in the theory of Sobolev spaces is approximation by smooth functions (for example, the proof of the chain rule depends on it), and similarly for functions of Bounded Variation. Indeed, in these cases a possible definition of the relevant function space is via relaxation of a functional defined on smooth maps, and the difficult part is proving that this is equivalent to the intrinsic definition on measurable sets or functions.

More specifically, approximation and density play an important role in relaxation, Γ-convergence, integral representation, semicontinuity and many other aspects of the Calculus of Variations in which the topology of the function space is complemented by a variational functional to be minimized. In these applications it is important to approximate in the relevant topology and in energy. In this respect, the literature contains many approximation results for free discontinuity problems, mainly focused on either linear growth or discontinuity sets with finite measure, as appropriate for example for models of concentration of plastic slip or for the Griffith model of brittle fracture. Our main aim here is approximation in energy without the assumption that the jump set has finite measure. One natural application of our result is the study of superlinear models of cohesive fracture.

The functional framework to settle this kind of problems is provided by (a suitable subspace of) the space of Special functions of Bounded Variation, introduced by De Giorgi and Ambrosio in [23] to model a large class of problems which are described by a volume energy and a surface energy (e.g., mixtures of liquids, liquid crystals, image segmentation, fracture mechanics, ...). Indeed, $SBV(\mathbb{R}^n;\mathbb{R}^m)$ is the set of functions $u \in BV(\mathbb{R}^n;\mathbb{R}^m)$ whose distributional derivative has no Cantor part:

$$Du = \nabla u \mathcal{L}^n + [u] \otimes \nu_u \mathcal{H}^{n-1} \llcorner J_u,$$

where ∇u is the approximate gradient and $J_u, \nu_u, [u] = u^+ - u^-$ are respectively the jump set, its normal, and the amplitude of the jump, see [5] for the definitions.

In these problems, the general form of the energy is

$$F[u,A] := \int_A \Psi(x,\nabla u)dx + \int_{J_u \cap A} g(x,u^+,u^-,\nu_u)d\mathcal{H}^{n-1}, \tag{1.1}$$

for $A \subset \mathbb{R}^n$ open and bounded, Ψ and g satisfying suitable growth and regularity properties, $u \in SBV(A;\mathbb{R}^m)$. If one is interested in the (possibly constrained) global minimization of F, lower semicontinuity and coercivity are further required in order to apply the direct method of the Calculus of Variations and to establish the existence of a solution.

For many applications it is of crucial importance to be able to approximate $u \in SBV(A;\mathbb{R}^m)$ in $L^1(A;\mathbb{R}^m)$ and in the sense of the energy by a sequence u_j of more regular functions (for example piecewise regular), i.e., in a way that $F[u_j,A] \to F[u,A]$ as $j \to \infty$. This was the aim of several works appeared in the recent years. Braides and Chiadò-Piat in [10, Sect. 5] focus on functions $u \in SBV^p \subset SBV$, $p > 1$, i.e. such that $\nabla u \in L^p$ and $\mathcal{H}^{n-1}(J_u) < \infty$. For functions $u \in SBV^p \cap L^\infty$ they provide an approximation $u_j \in SBV^p$, regular out of a closed rectifiable set, satisfying

$$u_j \to u \text{ strongly in } BV, \quad \nabla u_j \to \nabla u \text{ in } L^p, \quad \mathcal{H}^{n-1}(J_{u_j} \triangle J_u) \to 0. \tag{1.2}$$

Cortesani in [19] and Cortesani and Toader in [22], on the positive side, improve this result, by constructing for $u \in SBV^p \cap L^\infty$, $p > 1$, a sequence u_j whose jump set is in addition piecewise regular, and precisely polyhedral. Moreover they get

$$\nabla u_j \to \nabla u \text{ in } L^p,$$
$$\limsup_{j\to\infty} \int_{J_{u_j}\cap\overline{A}} g(x,u_j^+,u_j^-,\nu_{u_j})d\mathcal{H}^{n-1} \le \int_{J_u\cap\overline{A}} g(x,u^+,u^-,\nu_u)d\mathcal{H}^{n-1},$$

on $A \subset\subset \Omega$. On the negative side, they do not obtain strong convergence in SBV.

The strong convergence in SBV holds for the result by De Philippis, Fusco and Pratelli in [24, Theorem C], in which, for $u \in SBV^p$, $p > 1$, the authors construct u_j regular out of the closure of its jump set, which is actually essentially closed being contained in a compact C^1 manifold with C^1 boundary, and differs from it only by an $\mathcal{H}^{n-1}$-negligible set.

The previous four results have been crucial for many applications involving a penalization on the measure of the jump set. The case in which the jump set of u is allowed to have infinite measure is quite different and few approximations are available in the literature. An extension of the result by Cortesani and Toader to BV was obtained in [3] in the setting of BV strict convergence. In [29], the approximation of any BV function is obtained in the area-strict sense through countably piecewise affine functions with the same trace as u at the boundary. A different approximation is provided in SBV in [24, Theorem B]. Precisely, the authors prove that if $u \in SBV$ with $\nabla u \in L^p$, $p > 1$, then it is possible to construct u_j regular out of the closure of its jump set, which is actually essentially closed being (up to $\mathcal{H}^{n-1}$-null sets) a compact C^1 manifold with C^1 boundary, and satisfying

$$u_j \to u \text{ strongly in } BV, \quad \nabla u_j \to \nabla u \text{ in } L^p .$$

In particular, the convergence $\mathcal{H}^{n-1}(J_{u_j} \setminus J_u) \to 0$ is not ensured. Moreover, in case $p = 1$, the jump of u_j can be additionally taken contained in the intersection of a compact C^1 manifold with C^1 boundary and of the jump set of the function to be approximated (see [24, Theorem A]). Related density results, with different functional settings such as $(G)SBD$ or BH, have been obtained in the last years (see for example [17,28,13,26,27,14,20,12] and [1,2], respectively).

Although all the quoted results are important advances, they are in general not enough for many applications, and in general do not imply convergence of the surface term or of the total energy F, in particular if the measure of the jump set is not finite. An easy example is that of an energy F where $\Psi = \Psi(\nabla u)$ is superlinear for large gradients and $g = g([u], \nu_u)$ is superlinear for small amplitudes, the natural domain of finiteness being (a subset of) SBV. In this case, the only result available in the literature is [9, Sect. 4], which however applies only to $u \in GSBV$ with $\nabla u = 0$ $\mathcal{L}^n$-a.e. on Ω. The approximants satisfy $\nabla u_j = 0$ $\mathcal{L}^n$-a.e. on Ω and have jump sets of finite measure. The convergence is that of L^1 together with the convergence of the energies.

In this paper, we develop an original multiscale technique to approximate functions $u \in SBV$ with jump set of possibly infinite measure and $\nabla u \in L^p$, with $p \geq 1$. We stress that it encompasses at the same time both superlinear, cohesive-type and Griffith, brittle-type surface energies as shown in Section 2.

Theorem 1.1.
Let $\Omega \subseteq \mathbb{R}^n$ be an open bounded Lipschitz set, $u \in SBV(\Omega; \mathbb{R}^m)$ such that $\nabla u \in L^p(\Omega; \mathbb{R}^{m\times n})$ for some $p \in [1, \infty)$, and $g_0(|[u]|) \in L^1(\Omega; \mathcal{H}^{n-1} \llcorner J_u)$, with $g_0 : [0, \infty) \to [0, \infty)$ continuous, nondecreasing, subadditive, and $g_0^{-1}(0) = \{0\}$.

Then there are sequences $u_j \in SBV \cap L^\infty(\Omega; \mathbb{R}^m)$ and $\Phi_j \in \mathrm{Lip}\,(\mathbb{R}^n; \mathbb{R}^n)$ such that

(i) *for each j there is a locally finite decomposition of $\mathbb{R}^n$ in simplexes such that u_j is affine in the interior of each of them;*
(ii) *$u_j \to u$ in $L^1(\Omega; \mathbb{R}^m)$;*
(iii) *$\nabla u_j \to \nabla u$ in $L^p(\Omega; \mathbb{R}^{m\times n})$;*
(iv) *Φ_j is bilipschitz, with $\Phi_j(x) - x \to 0$ in $L^\infty(\mathbb{R}^n; \mathbb{R}^n)$, $D\Phi_j \to \mathrm{Id}$ in $L^\infty(\mathbb{R}^n; \mathbb{R}^{n\times n})$, and $\Phi_j(x) = x$ for $x \in \mathbb{R}^n \setminus \Omega$;*
(v) *one can choose the orientation of the normal ν_j to J_{u_j} so that*

$$\lim_j \int_{J_u \cup \Phi_j^{-1}(J_{u_j})} g_0(|[u] - [u_j] \circ \Phi_j|)\, d\mathcal{H}^{n-1} = 0 \tag{1.3}$$

(with $[u] = 0$ outside J_u, and similarly for u_j), and

$$\lim_j \int\limits_{J_u \cup \Phi_j^{-1}(J_{u_j})} g_0(|[u]| + |[u_j] \circ \Phi_j|) \big|\nu_u - \nu_j \circ \Phi_j\big| \, d\mathcal{H}^{n-1} = 0; \tag{1.4}$$

(vi) if $\mathcal{H}^{n-1}(J_u) < \infty$, then also $\mathcal{H}^{n-1}(J_u \triangle \Phi_j^{-1}(J_{u_j})) \to 0$;
(vii) if $\nabla u = 0$ $\mathcal{L}^n$-almost everywhere on Ω, then $\nabla u_j = 0$ $\mathcal{L}^n$-almost everywhere on Ω for all j. If instead $u \in W^{1,p}(\Omega; \mathbb{R}^m)$ then $u_j \in W^{1,p}(\Omega; \mathbb{R}^m)$ for all j.

A few remarks are in order. First, since $\Phi_j(\Omega) = \Omega$, the integrals in (1.3) and (1.4) are over subsets of Ω. Then, thanks to the subadditivity of g_0 from items (iv) and (v) it follows that

$$\lim_j \int\limits_{J_{u_j}} g_0(|[u_j]|) \, d\mathcal{H}^{n-1} = \int\limits_{J_u} g_0(|[u]|) \, d\mathcal{H}^{n-1}$$

(see the proof of Corollary 2.1 below). Moreover, under suitable assumptions discussed in details in Section 2, we can deduce the convergence of surface energies with density $g : \mathbb{R}^m \times S^{n-1} \to [0, \infty)$ depending suitably on the full jump and the normal. Finally, if $\Psi \in C^0(\mathbb{R}^{m\times n})$ has p-growth (cf. again Section 2) then (iii) implies

$$\lim_j \int\limits_{\Omega} \Psi(\nabla u_j) \, dx = \int\limits_{\Omega} \Psi(\nabla u) \, dx \, .$$

In addition, the sequence $(u_j)_{j\in\mathbb{N}}$ can be chosen such that the convergence to u is stronger, namely strict in BV and in area, see Corollary 2.3 below.

We stress that energies with bulk density Ψ and surface density g as above are in general not L^1 or weakly*-BV lower semicontinuous. Hence, our approximations can be used to prove relaxation formulas in the spirit of [6,8,7]. This will be the object of future work in [16].

The proof of Theorem 1.1 is obtained through an explicit construction in several steps. First, u can be extended to a function defined on a slightly larger set at a small energy cost. This is not achieved by local reflections at the boundary and a partition of unity process as usually done, which would require $u \in L^p$. It is rather pursued through a regularization of the normal vector at the boundary and the definition of a bilipschitz map which swaps an inner neighborhood of the boundary with an outer one. Further details can be found in Section 3.1.

We employ next a multiscale approach. More precisely, we find a suitable scale $\delta > 0$, such that ∇u is close to a constant and J_u is close to a C^1 manifold in each cube of side δ of a partition of $\mathbb{R}^n$. This is the object of Proposition 3.6. At this point, we introduce a second scale $\varepsilon \ll \delta$. In each cube of side δ we consider a finer triangulation with simplexes of diameter less than $c\varepsilon$ and volume larger than $c\varepsilon^n$. The heart of the paper is Proposition 4.1, which, given the values of u on the vertices of a single simplex, and two vectors for each edge, representing the cumulated jump and the average gradient of u on the edge, provides a piecewise affine interpolation, whose gradient and jump can be estimated respectively only through the given gradient vector or the given jump vector (see Fig. 1). Proposition 4.1 is then employed in Proposition 4.3 (see Fig. 2) to define a global projection, with good energy estimates, of any SBV function on the space of piecewise affine functions.

The proof of Theorem 1.1 contains a few additional steps, since the direct application of Proposition 4.3 to the given u would provide a piecewise affine approximation with surface energy controlled only up to a multiplicative factor by the surface energy of u. To avoid this problem, we first consider the extensions $U^{\pm}$ of u with respect to the C^1 manifold approximating J_u in each cube of side δ. We then apply the previous projection to $U^{\pm}$. We finally introduce a piecewise affine interpolation of the C^1 manifold and define the approximation of u as the projections of $U^{\pm}$ on the two sides of it. This is performed in the proof of Theorem 1.1 in Section 4.3, see also Fig. 3.

The structure of the paper is the following. In Section 2 we provide several consequences of Theorem 1.1, in particular we show that the approximating sequence can be constructed such that it converges also BV-strictly, area-strictly and BV-strongly after composition with a bilipschitz map. Section 3 addresses two key technical issues: the extension tool in Section 3.1 and the regularization at scale δ in Section 3.2. Section 4 is devoted to the proof of Theorem 1.1. Precisely, Section 4.1 contains the construction of a relevant piecewise affine interpolation on a single simplex. Section 4.2 applies such construction to produce a piecewise affine approximation of a given SBV function. Finally, Section 4.3 provides the full proof of Theorem 1.1 by applying the projection of Section 4.2 to the extensions of u on the two sides of the regularized jump set and by defining the approximation of u as such projections on the two sides of a suitable perturbation of a piecewise interpolation of the regularized jump set.

2. Consequences of the approximation theorem

We discuss here some consequences of Theorem 1.1. To this aim we fix $p \in [1, \infty)$ and consider $\Psi \in C^0(\mathbb{R}^{m\times n})$ obeying for some $C > 0$,

$$|\Psi(\xi)| \le C(|\xi|^p + 1). \tag{2.1}$$

Throughout the paper C will denote a constant, possibly depending on the dimension (if not otherwise specified) and changing from line to line. Next we select a function $g_0 : [0, \infty) \to [0, \infty)$ which represents a modulus of continuity of the surface energy g introduced below (see in particular (2.4)) satisfying:

($\mathrm{H}_1^{g_0}$) g_0 is continuous, nondecreasing, and $g_0^{-1}(0) = \{0\}$,

($\mathrm{H}_2^{g_0}$) g_0 is subadditive, namely for every $(t,t') \in [0,\infty) \times [0,\infty)$

$$g_0(t+t') \leq g_0(t) + g_0(t') \,.$$

For example either $g_0(t) = 1 \wedge t^q$ or $g_0(t) = t^q$, for $q \in (0,1]$, will do. Note that by subadditivity and continuity of g_0 in zero, for every $\lambda > 0$ there is $C_\lambda > 0$ such that for all $t \in [0,\infty)$

$$g_0(t) \leq \lambda + C_\lambda t. \tag{2.2}$$

Then we consider any function $g \in C^0(\mathbb{R}^m \times S^{n-1}; [0,\infty))$, such that

(H_1^g) $g(-s,-\nu) = g(s,\nu)$ for all $(s,\nu) \in \mathbb{R}^m \times S^{n-1}$;
(H_2^g) for all $(s,s',\nu) \in \mathbb{R}^m \times \mathbb{R}^m \times S^{n-1}$

$$g(s+s',\nu) \leq g(s,\nu) + C\, g_0(|s'|); \tag{2.3}$$

and either

(H_3^g) $g(0,\nu) = 0$ for all $\nu \in S^{n-1}$

or

($\mathrm{H}_{3'}^g$) there is $\alpha > 0$ such that $g(0,\nu) \geq \alpha$ for all $\nu \in S^{n-1}$.

Thanks to assumption (H_1^g), the surface energy with density g is well defined as it does not depend on the chosen orientation of the normal to the jump set. Assumption (H_3^g) is useful to model cohesive-type energies, such as for example the one of the Barenblatt model. Assumption ($\mathrm{H}_{3'}^g$) is instead useful for surface energies typical of brittle fracture, such as the one of the Griffith model (or, in the scalar case, of the Mumford-Shah model) for which g is constant.

Exchanging the roles of s and $s+s'$ in (2.3) yields that

$$|g(s+s',\nu) - g(s,\nu)| \leq C\, g_0(|s'|). \tag{2.4}$$

Moreover, if (H_3^g) holds, the latter estimate with $s' = -s$ implies that for all $(s,\nu) \in \mathbb{R}^m \times S^{n-1}$

$$g(s,\nu) \leq C\, g_0(|s|) \,. \tag{2.5}$$

If instead ($\mathrm{H}_{3'}^g$) holds, then by continuity there is also $\beta > 0$ such that $g(0,\nu) \leq \beta$ for all ν, and in particular for all $(s,\nu) \in \mathbb{R}^m \times S^{n-1}$

$$g(s,\nu) \leq \beta + C\, g_0(|s|). \tag{2.6}$$

For $u \in SBV(\Omega;\mathbb{R}^m)$ and for a Borel set $A \subseteq \Omega$, we define the energy

$$E_{\Psi,g}[u,A] := \int_A \Psi(\nabla u)dx + \int_{J_u \cap A} g([u],\nu_u)d\mathcal{H}^{n-1}\,,$$

where for any $u \in SBV(\Omega;\mathbb{R}^m)$ we denote by $[u]$ the function which is the usual jump of u on J_u and 0 on $\Omega \setminus J_u$.

Corollary 2.1. *Under the assumptions of Theorem 1.1, the sequence $(u_j)_{j\in\mathbb{N}}$ introduced there satisfies*

$$\lim_j \int_\Omega \Psi(\nabla u_j)dx = \int_\Omega \Psi(\nabla u)dx\,, \tag{2.7}$$

$$\lim_j \int_{J_{u_j}} g([u_j],\nu_{u_j})d\mathcal{H}^{n-1} = \int_{J_u} g([u],\nu_u)d\mathcal{H}^{n-1} \tag{2.8}$$

for all functions $\Psi \in C^0(\mathbb{R}^{m\times n})$ satisfying (2.1), and all $g \in C^0(\mathbb{R}^m \times S^{n-1};[0,\infty))$ satisfying (H_1^g), (H_2^g), and (H_3^g). In particular,

$$\lim_j E_{\Psi,g}[u_j,\Omega] = E_{\Psi,g}[u,\Omega]\,.$$

We stress that the assumptions of Theorem 1.1 include in particular integrability of $g_0(|[u]|)$ and ensure via (2.1) and (2.5) that $E_{\Psi,g}[u,\Omega]$ is finite.

In this proof and in the rest of the paper we shall use repeatedly the Area formula in the following form. For $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ bilipschitz, and $u \in BV(\mathbb{R}^n;\mathbb{R}^m)$, one has

$$\int_{J_u} g([u],\nu_u)d\mathcal{H}^{n-1} = \int_{\Phi^{-1}(J_u)} g([u]\circ\Phi,\nu_u\circ\Phi)\mathbf{J}_{n-1}d^{\Phi^{-1}(J_u)}\Phi\, d\mathcal{H}^{n-1} \tag{2.9}$$

(cf. [5, Theorem 2.91], with $f = \Phi$ and $E = \Phi^{-1}(J_u)$, or [25, Theorem 3.2.22]). In (2.9) we write $\mathbf{J}_{n-1}d^{\Phi^{-1}(J_u)}\Phi$ for the tangential Jacobian and remark that if Φ is differentiable then $\mathbf{J}_{n-1}d^{\Phi^{-1}(J_u)}\Phi = |\mathrm{cof}(D\Phi)(\nu_u\circ\Phi)|$. For $\mathcal{H}^{n-1}$-almost every $x \in \Phi^{-1}(J_u)$, the map Φ is differentiable in x in the directions of the tangent space. The same holds for $y \mapsto \Phi(y) - y$, which is Lipschitz with Lipschitz constant bounded by $\|D\Phi - \mathrm{Id}\|_{L^\infty(\mathbb{R}^n)}$. Therefore for $\mathcal{H}^{n-1}$-almost every $x \in \Phi^{-1}(J_u)$ we have

$$|\mathbf{J}_{n-1}d^{\Phi^{-1}(J_u)}\Phi - 1|(x) \leq C\|D\Phi - \mathrm{Id}\|_{L^\infty(\mathbb{R}^n)} \tag{2.10}$$

and, in particular, $|\mathbf{J}_{n-1}d^{\Phi^{-1}(J_u)}\Phi| \leq C$ (with constants that may depend on n and on the Lipschitz constants of Φ and Φ^{-1}).

Proof. Given the L^p convergence of $(\nabla u_j)_{j\in\mathbb{N}}$ to ∇u, we may consider a subsequence, which we do not relabel, such that

$$\limsup_j \left|\int_\Omega (\Psi(\nabla u_j) - \Psi(\nabla u))dx\right|$$

is actually a limit, and $(\nabla u_j)_{j\in\mathbb{N}}$ converges to ∇u $\mathcal{L}^n$-almost everywhere on Ω. Thanks to Egorov's theorem, for every $\varepsilon > 0$ there is E with $|\Omega \setminus E| \le \varepsilon$ such that $\nabla u \in L^\infty(E;\mathbb{R}^{m\times n})$ and $(\nabla u_j)_{j\in\mathbb{N}}$ converges to ∇u uniformly on E. Therefore, we may use (2.1) and item (iii) in Theorem 1.1 to get

$$\begin{aligned}\limsup_j \left|\int_\Omega (\Psi(\nabla u_j) - \Psi(\nabla u))dx\right| &= \limsup_j \left|\int_{\Omega\setminus E} (\Psi(\nabla u_j) - \Psi(\nabla u))dx\right| \\ &\le C\Big(\int_{\Omega\setminus E} |\nabla u|^p dx + |\Omega \setminus E|\Big).\end{aligned}$$

The conclusion then follows as $\varepsilon \downarrow 0$ by absolute continuity. As the limit is unique, convergence holds for the entire sequence.

We next deal with (2.8). To this aim we first use the Area formula in (2.9) to obtain

$$\int_{J_{u_j}} g([u_j], \nu_{u_j})d\mathcal{H}^{n-1} = \int_{\Phi_j^{-1}(J_{u_j})} g([u_j]\circ\Phi_j, \nu_{u_j}\circ\Phi_j)\mathbf{J}_{n-1}d^{\Phi_j^{-1}(J_{u_j})}\Phi_j d\mathcal{H}^{n-1}. \tag{2.11}$$

By (2.10) and (iv), the tangential Jacobian converges in L^∞ to 1. We observe that by subadditivity of g_0

$$\begin{aligned}&\int_{\Phi_j^{-1}(J_{u_j})} g_0(|[u_j]\circ\Phi_j|)d\mathcal{H}^{n-1} \\ &\le \int_{J_u\cup\Phi_j^{-1}(J_{u_j})} g_0(|[u] - [u_j]\circ\Phi_j|)\, d\mathcal{H}^{n-1} + \int_{J_u} g_0(|[u]|)\, d\mathcal{H}^{n-1}.\end{aligned}$$

Using (1.3) and the assumption that $g_0(|[u]|) \in L^1(\Omega, \mathcal{H}^{n-1} \llcorner J_u)$ we obtain that

$$\int_{\Phi_j^{-1}(J_{u_j})} g_0(|[u_j]\circ\Phi_j|)d\mathcal{H}^{n-1} \le C < \infty \tag{2.12}$$

for all j. By (2.11), the growth condition in (2.5), and the last step (2.12), we obtain

$$
\begin{aligned}
&\left|\int_{J_{u_j}} g([u_j],\nu_{u_j})d\mathcal{H}^{n-1} - \int_{\Phi_j^{-1}(J_{u_j})} g([u_j]\circ\Phi_j,\nu_{u_j}\circ\Phi_j)d\mathcal{H}^{n-1}\right| \\
&\le \|1-\mathbf{J}_{n-1}d^{\Phi_j^{-1}(J_{u_j})}\Phi_j\|_{L^\infty(\Phi_j^{-1}(J_{u_j});\mathcal{H}^{n-1})} \int_{\Phi_j^{-1}(J_{u_j})} g([u_j]\circ\Phi_j,\nu_{u_j}\circ\Phi_j)d\mathcal{H}^{n-1} \\
&\le o(1)\int_{\Phi_j^{-1}(J_{u_j})} g_0(|[u_j]\circ\Phi_j|)d\mathcal{H}^{n-1} = o(1)\,. \qquad (2.13)
\end{aligned}
$$

Using (1.3) and (1.4) in Theorem 1.1 and the fact that g_0 is nondecreasing with $g_0^{-1}(0) = \{0\}$ we deduce $\chi_{\Phi_j^{-1}(J_{u_j})} \to 1$ and $\nu_{u_j}\circ\Phi_j \to \nu_u$, $\mathcal{H}^{n-1}$-almost everywhere on J_u. Thus, $\chi_{\Phi_j^{-1}(J_{u_j})}\nu_{u_j}\circ\Phi_j \to \nu_u$, $\mathcal{H}^{n-1}$-almost everywhere on J_u. Dominated convergence, which we can use by (2.5) and integrability of $g_0(|[u]|)$, then yields

$$
\limsup_j \int_{J_u\cap\Phi_j^{-1}(J_{u_j})} \left|g([u],\nu_{u_j}\circ\Phi_j) - g([u],\nu_u)\right| d\mathcal{H}^{n-1} = 0. \qquad (2.14)
$$

Moreover, (1.3) in Theorem 1.1(v) yields (with (2.5)) that

$$
\limsup_j \left(\int_{J_u\setminus\Phi_j^{-1}(J_{u_j})} g([u],\nu_u)d\mathcal{H}^{n-1} + \int_{\Phi_j^{-1}(J_{u_j})\setminus J_u} g([u_j]\circ\Phi_j,\nu_{u_j}\circ\Phi_j)d\mathcal{H}^{n-1}\right) = 0. \qquad (2.15)
$$

Therefore, we conclude that

$$
\begin{aligned}
&\limsup_j \left|\int_{J_{u_j}} g([u_j],\nu_{u_j})d\mathcal{H}^{n-1} - \int_{J_u} g([u],\nu_u)d\mathcal{H}^{n-1}\right| \\
&\le \limsup_j \left|\int_{\Phi_j^{-1}(J_{u_j})} g([u_j]\circ\Phi_j,\nu_{u_j}\circ\Phi_j)d\mathcal{H}^{n-1} - \int_{J_u} g([u],\nu_u)d\mathcal{H}^{n-1}\right| \\
&\le \limsup_j \int_{\Phi_j^{-1}(J_{u_j})\cap J_u} \left|g([u_j]\circ\Phi_j,\nu_{u_j}\circ\Phi_j) - g([u],\nu_u)\right| d\mathcal{H}^{n-1} \\
&\le \limsup_j \int_{\Phi_j^{-1}(J_{u_j})\cap J_u} \left|g([u_j]\circ\Phi_j,\nu_{u_j}\circ\Phi_j) - g([u],\nu_{u_j}\circ\Phi_j)\right| d\mathcal{H}^{n-1}
\end{aligned}
$$

$$\leq C \limsup_j \int_{\Phi_j^{-1}(J_{u_j})\cap J_u} g_0(|[u]-[u_j]\circ\Phi_j|)\,d\mathcal{H}^{n-1} = 0\,,$$

where we have used (2.13) in the first inequality, (2.15) in the second one, (2.14) in the third one, (2.4) in the fourth one, and (1.3) in Theorem 1.1(v) in the last equality. □

We next show how to treat the case that g is bounded from below, in which ($\mathrm{H}^g_{3'}$) holds. We stress that the case of the Mumford-Shah energy functional corresponds to the choices $\Psi = |\cdot|^2$ and $g \equiv 1$ as $|[u]| > 0$ on J_u for $u \in SBV$.

Corollary 2.2. *Under the assumptions of Theorem 1.1, if $\mathcal{H}^{n-1}(J_u) < \infty$ the sequence $(u_j)_{j\in\mathbb{N}}$ introduced there satisfies*

$$\lim_j \int_\Omega \Psi(\nabla u_j)dx = \int_\Omega \Psi(\nabla u)dx\,, \tag{2.16}$$

$$\lim_j \int_{J_{u_j}} g([u_j],\nu_{u_j})d\mathcal{H}^{n-1} = \int_{J_u} g([u],\nu_u)d\mathcal{H}^{n-1} \tag{2.17}$$

for all functions $\Psi \in C^0(\mathbb{R}^{m\times n})$ satisfying (2.1), and $g \in C^0(\mathbb{R}^m \times S^{n-1};[0,\infty))$ satisfying (H^g_1), (H^g_2), and ($H^g_{3'}$). In particular,

$$\lim_j E_{\Psi,g}[u_j,\Omega] = E_{\Psi,g}[u,\Omega]\,.$$

Proof. The proof is very similar to the one of Corollary 2.1. The first part, until (2.12), is identical. Using (2.6), $\mathcal{H}^{n-1}(J_u) < \infty$, (vi) in Theorem 1.1, and (2.12) we have

$$\begin{aligned}
&\int_{\Phi_j^{-1}(J_{u_j})} g([u_j]\circ\Phi_j,\nu_{u_j}\circ\Phi_j)d\mathcal{H}^{n-1}\\
&\leq \beta(\mathcal{H}^{n-1}(J_u)+\mathcal{H}^{n-1}(\Phi_j^{-1}(J_{u_j})\setminus J_u)) + C\int_{\Phi_j^{-1}(J_{u_j})} g_0(|[u_j]\circ\Phi_j|)d\mathcal{H}^{n-1}\\
&\leq C < \infty
\end{aligned} \tag{2.18}$$

for all j. We use the latter and (2.11) to conclude that

$$\begin{aligned}
&\left|\int_{J_{u_j}} g([u_j],\nu_{u_j})d\mathcal{H}^{n-1} - \int_{\Phi_j^{-1}(J_{u_j})} g([u_j]\circ\Phi_j,\nu_{u_j}\circ\Phi_j)d\mathcal{H}^{n-1}\right|\\
&\leq \|1-\mathbf{J}_{n-1}d^{\Phi_j^{-1}(J_{u_j})}\Phi_j\|_{L^\infty(\Phi_j^{-1}(J_{u_j});\mathcal{H}^{n-1})}\int_{\Phi_j^{-1}(J_{u_j})} g([u_j]\circ\Phi_j,\nu_{u_j}\circ\Phi_j)d\mathcal{H}^{n-1}
\end{aligned}$$

$$= o(1) \tag{2.19}$$

which replaces (2.13). Using (vi) in Theorem 1.1, $\chi_{\Phi_j^{-1}(J_{u_j})} \to 1$ pointwise $\mathcal{H}^{n-1}$-almost everywhere on J_u. As above, $\chi_{\Phi_j^{-1}(J_{u_j})}\nu_{u_j} \circ \Phi_j \to \nu_u$ $\mathcal{H}^{n-1}$-almost everywhere on J_u. From $\mathcal{H}^{n-1}(J_u) < \infty$ and integrability of $g_0(|[u]|)$ we obtain that $\beta + g_0(|[u]|) \in L^1(\Omega; \mathcal{H}^{n-1} \llcorner J_u)$. Dominated convergence, which we can use by (2.6), then yields

$$\limsup_j \int_{J_u \cap \Phi_j^{-1}(J_{u_j})} \left| g([u], \nu_{u_j} \circ \Phi_j) - g([u], \nu_u) \right| d\mathcal{H}^{n-1} = 0. \tag{2.20}$$

Moreover, items (v) and (vi) in Theorem 1.1 and (2.6), yield that

$$\limsup_j \left(\int_{J_u \setminus \Phi_j^{-1}(J_{u_j})} g([u], \nu_u) d\mathcal{H}^{n-1} + \int_{\Phi_j^{-1}(J_{u_j}) \setminus J_u} g([u_j] \circ \Phi_j, \nu_{u_j} \circ \Phi_j) d\mathcal{H}^{n-1} \right) = 0. \tag{2.21}$$

The rest of the proof is unchanged. □

We can actually strengthen the conclusions of Corollary 2.1 and Corollary 2.2 by constructing an approximating sequence converging in a stronger sense. We recall that the push-forward of a measure μ through a map $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ is defined by $(\Phi_\# \mu)(E) := \mu(\Phi^{-1}(E))$ for any measurable set E, and that this implies $\int f d(\Phi_\# \mu) = \int (f \circ \Phi) d\mu$ for any measurable function f.

Corollary 2.3. *In Corollary 2.1 and Corollary 2.2 the sequence $(u_j)_{j \in \mathbb{N}}$ can be chosen to additionally satisfy*

$$\lim_j |(\Phi_j)_\# Du_j - Du|(\Omega) = 0 \tag{2.22}$$

and

$$\lim_j \|u_j \circ \Phi_j - u\|_{BV(\Omega)} = 0\,. \tag{2.23}$$

In particular,

$$\lim_j \int_\Omega |\nabla u_j| dx = \int_\Omega |\nabla u| dx\,, \tag{2.24}$$

$$\lim_j \int_\Omega \sqrt{1 + |\nabla u_j|^2} dx = \int_\Omega \sqrt{1 + |\nabla u|^2} dx\,, \tag{2.25}$$

$$\lim_j \int_{J_{u_j}} |[u_j]| d\mathcal{H}^{n-1} = \int_{J_u} |[u]| d\mathcal{H}^{n-1} , \tag{2.26}$$

so that $(u_j)_{j\in\mathbb{N}}$ converges to u strictly in $BV(\Omega;\mathbb{R}^m)$ and in area.

Proof. The proof is based on the fact that the construction of the sequence in Theorem 1.1 does not depend on the details of the energy considered. We define the auxiliary functions $\widetilde{g}_0 : [0,\infty) \to [0,\infty)$, $\widetilde{g} : \mathbb{R}^m \to [0,\infty)$, by

$$\widetilde{g}_0(t) := g_0(t) + t , \qquad t \in [0,\infty) ,$$

and

$$\widetilde{g}(s) := |s| , \qquad s \in \mathbb{R}^m .$$

It is easy to check that $\widetilde{g}_0$ satisfies $(\mathrm{H}_1^{g_0})$-$(\mathrm{H}_2^{g_0})$, and moreover that both g and $\widetilde{g}$ satisfy (H_1^{g})-(H_2^{g}) with respect to $\widetilde{g}_0$. Further, $\widetilde{g}$ satisfies (H_3^{g}). In addition, if $u \in SBV(\Omega;\mathbb{R}^m)$, having assumed that $g_0(|[u]|) \in L^1(\Omega;\mathcal{H}^{n-1} \llcorner J_u)$, we infer that $\widetilde{g}_0(|[u]|) \in L^1(\Omega;\mathcal{H}^{n-1} \llcorner J_u)$. Therefore, we may consider the sequence $(u_j)_{j\in\mathbb{N}}$ provided by Theorem 1.1 with surface density $\widetilde{g}_0$. Thus, to get (2.24)–(2.26) it is sufficient to apply Corollary 2.1 with $\Psi_1(\xi) := |\xi|$ and $\widetilde{g}$, and then $\Psi_2(\xi) := \sqrt{1+|\xi|^2}$ and $\widetilde{g}$. One applies either Corollary 2.1 or Corollary 2.2 with densities Ψ and g to obtain convergence of the energy. From Theorem 1.1(v) for $\tilde{g}_0$ we obtain

$$\lim_j \int_{J_u \cup \Phi_j^{-1}(J_{u_j})} |[u_j] \circ \Phi_j - [u]| d\mathcal{H}^{n-1} = 0 ,$$

and with (iii) and (iv) we conclude $|(\Phi_j)_\# Du_j - Du|(\Omega) \to 0$. It remains to prove (2.23). Recalling Theorem 1.1 (ii) and (iv) and (2.22), it is enough to check that

$$\lim_j \int_\Omega |\nabla(u_j \circ \Phi_j) - \nabla u| dx = 0.$$

Let $\mathcal{S}_j$ be the decomposition of Theorem 1.1 (i), then u_j is affine in T, for all $T \in \mathcal{S}_j$, and the chain rule gives

$$\begin{aligned}\int_\Omega |\nabla(u_j \circ \Phi_j) - \nabla u| dx &= \sum_{T\in\mathcal{S}_j} \int_{\Omega\cap\Phi_j^{-1}(T)} |\nabla(u_j \circ \Phi_j) - \nabla u| dx \\ &= \sum_{T\in\mathcal{S}_j} \int_{\Omega\cap\Phi_j^{-1}(T)} |(\nabla u_j \circ \Phi_j) D\Phi_j - \nabla u| dx.\end{aligned}$$

The triangular inequality yields

$$\int_{\Omega\cap\Phi_j^{-1}(T)} |(\nabla u_j \circ \Phi_j)D\Phi_j - \nabla u|dx \le \int_{\Omega\cap\Phi_j^{-1}(T)} |\nabla u_j \circ \Phi_j||D\Phi_j - \mathrm{Id}|dx$$
$$+ \int_{\Omega\cap\Phi_j^{-1}(T)} |(\nabla u_j - \nabla u)\circ \Phi_j|dx + \int_{\Omega\cap\Phi_j^{-1}(T)} |\nabla u \circ \Phi_j - \nabla u|dx$$

and all terms tend to zero by Theorem 1.1 (iv) and (2.22). This gives the conclusion. □

Finally, we extend the above approximation results to functions belonging to $(GSBV(\Omega))^m$ with energy $E_{|\cdot|^p,g_0}$ finite (we refer to [5, Section 4.5] for the basic notation and theory).

Corollary 2.4. *Let $\Omega \subseteq \mathbb{R}^n$ be an open bounded Lipschitz set, $u \in L^1(\Omega;\mathbb{R}^m) \cap (GSBV(\Omega))^m$ be such that $\nabla u \in L^p(\Omega;\mathbb{R}^{m\times n})$ for some $p \in [1,\infty)$, and $g_0(|[u]|) \in L^1(\Omega;\mathcal{H}^{n-1}\llcorner J_u)$, with $g_0 : [0,\infty) \to [0,\infty)$ continuous, nondecreasing, subadditive, and $g_0^{-1}(0) = \{0\}$.*

Then, there exists a sequence $(u_j)_{j\in\mathbb{N}} \subseteq SBV \cap L^\infty(\Omega;\mathbb{R}^m)$ such that all the conclusions in Theorem 1.1 hold, and in addition

$$\lim_j \int_\Omega \Psi(\nabla u_j)dx = \int_\Omega \Psi(\nabla u)dx\,, \tag{2.27}$$

$$\lim_j \int_{J_{u_j}} g([u_j],\nu_{u_j})d\mathcal{H}^{n-1} = \int_{J_u} g([u],\nu_u)d\mathcal{H}^{n-1} \tag{2.28}$$

for all functions $\Psi \in C^0(\mathbb{R}^{m\times n})$ satisfying (2.1), and all $g \in C^0(\mathbb{R}^m \times S^{n-1};[0,\infty))$ satisfying (H_1^g), (H_2^g), and (H_3^g). In addition, if $\mathcal{H}^{n-1}(J_u) < \infty$, (2.28) holds for all $g \in C^0(\mathbb{R}^m \times S^{n-1};[0,\infty))$ satisfying (H_1^g), (H_2^g), and ($H_{3'}^g$).

Moreover, the sequence $(u_j)_{j\in\mathbb{N}}$ can be chosen such that (2.24) and (2.25) hold.

Proof. We argue by density by constructing a sequence $(\tilde u_k)_{k\in\mathbb{N}} \subset SBV \cap L^\infty(\Omega;\mathbb{R}^m)$ converging in $L^1(\Omega;\mathbb{R}^m)$ and in energy to u. This is well-known nowadays, in any case for the readers' convenience we recall the definition. To this aim we fix a sequence $(a_k)_{k\in\mathbb{N}} \subset (0,\infty)$ such that $a_k < a_{k+1}$, $a_k \uparrow \infty$, and such that there are functions $\mathcal{T}_k \in C_c^1(\mathbb{R}^n;\mathbb{R}^m)$ satisfying $\mathcal{T}_k(z)= z$ if $|z| \le a_k$, $\mathcal{T}_k(z) = 0$ if $|z| \ge a_{k+1}$, and $\|D\mathcal{T}_k\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^m)} \le 1$. Then, the sequence $\tilde u_k := \mathcal{T}_k(u)\in SBV(\Omega;\mathbb{R}^m)$ converges to u in $L^1(\Omega;\mathbb{R}^m)$, $\nabla \tilde u_k = \nabla u$ $\mathcal{L}^n$-almost everywhere on $\Omega_k := \{x \in \Omega : |u(x)| \le a_k\}$, $J_{\tilde u_k} \subseteq J_u$, $\nu_{\tilde u_k} = \nu_u$ $\mathcal{H}^{n-1}$-almost everywhere on $J_{\tilde u_k}$. Moreover, as

$$\mathcal{H}^{n-1}(\{x \in J_u : |u^\pm(x)| = \infty\}) = 0$$

(see [4, Proposition 2.12, Remark 2.13]) we infer that $\chi_{J_{\tilde{u}_k}} \to \chi_{J_u}$, $\tilde{u}_k^{\pm}(x) = u^{\pm}(x)$ $\mathcal{H}^{n-1}$-almost everywhere on $J_{\tilde{u}_k} \cap \Omega_k$, $|[\tilde{u}_k]| \leq |[u]|$ and $\tilde{u}_k^{\pm} \to u^{\pm}$ $\mathcal{H}^{n-1}$-almost everywhere on J_u. Therefore, we get

$$\lim_k \int_\Omega |\nabla \tilde{u}_k|^p dx = \int_\Omega |\nabla u|^p dx \,,$$

and thanks to the subadditivity and monotonicity properties of g_0 also that

$$\lim_k \int_{J_u} g_0(|[u] - [\tilde{u}_k]|) d\mathcal{H}^{n-1} = 0$$

(for detailed proofs of similar properties see, for instance, [4, Lemma 6.1] and [15, Proposition 4.8]).

Next, for every $k \in \mathbb{N}$ we apply Theorem 1.1 in order to get a sequence $(\tilde{u}_{k,j})_{j\in\mathbb{N}}$ approximating $\tilde{u}_k$ and satisfying all the conditions in that statement. Eventually, we conclude thanks to a diagonalization argument in view of the properties of g_0, Ψ and g and the arguments in Corollaries 2.1 or 2.2, and in Corollary 2.3. $\square$

3. Technical results

In this section we collect the key technical tools we use to prove Theorem 1.1. For SBV functions having finite energy according to Theorem 1.1, we establish first an extension result, and then some measure theoretic properties crucial for our constructions.

3.1. Extension

In this section we prove an extension result for SBV functions. A standard local reflection argument would work for $SBV \cap L^p$ functions. However, in the present setting it is not clear that having finite energy implies finiteness of the L^p norm. In particular, to the aim of applications, both for approximation via Γ-convergence and for the determination of relaxation of variational integrals, it is not natural to assume additionally $u \in L^p$ (see for example the forthcoming paper [16]), therefore we avoid the extra L^p integrability condition. To this aim, we introduce a global reflection argument based on a bilipschitz map reflecting a neighborhood of $\partial\Omega$ in Ω, outside of Ω itself.

The general strategy is standard, but to the best of our knowledge the details are new. For example, a similar result was obtained in [21, Th. 3.1] with a more complex construction using the solution of an ODE (see [21, Eq. (3.5)]) instead of the specific formula (3.21) below for the construction of the reflection.

Theorem 3.1. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded Lipschitz set. Then there are an open set $\omega \subseteq \mathbb{R}^n$ with $\partial\Omega \subset \omega$ and a bilipschitz map $\Phi : \omega \to \omega$ such that $\Phi(x) = x$ for $x \in \partial\Omega$ and $\Phi(\omega \cap \Omega) = \omega \setminus \overline{\Omega}$.*

We recall that a map $f : E \to f(E) \subseteq \mathbb{R}^n$, for $E \subseteq \mathbb{R}^n$, is bilipschitz if there is $L > 0$ such that

$$\frac{1}{L}|x-y| \le |f(x)-f(y)| \le L|x-y| \tag{3.1}$$

for all $x, y \in E$. This is the same as saying that f is injective, Lipschitz, with a Lipschitz inverse $f^{-1} : f(E) \to E$.

Moreover, a set Ω is Lipschitz if for every $x \in \partial\Omega$ there are $\varepsilon_x > 0$, $G_x \in \mathrm{Lip}\,(\mathbb{R}^{n-1})$ and an isometry $A_x : \mathbb{R}^n \to \mathbb{R}^n$ such that $A_x 0 = x$ and

$$B_{\varepsilon_x}(x) \cap \Omega = B_{\varepsilon_x}(x) \cap A_x\{(y', y_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : y_n < G_x(y')\}. \tag{3.2}$$

Obviously $G_x(0) = 0$; if $|y'| < \varepsilon_x/(\mathrm{Lip}\,(G_x)+1)$ then $A_x(y', G_x(y')) \in B_{\varepsilon_x}(x) \cap \partial\Omega$. If Ω is bounded, there are ε_0 and L_0 such that one can choose $\varepsilon_x \ge \varepsilon_0$ and $\mathrm{Lip}\,(G_x) \le L_0$ for all $x \in \partial\Omega$.

We start with defining a smooth vector field playing the role of the normal field to $\partial\Omega$, which under our hypotheses is only a function in $L^\infty(\partial\Omega; S^{n-1})$.

Lemma 3.2. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded Lipschitz set. Then there are $\gamma > 0$ and a map $\psi \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n)$ such that $\psi(x) \cdot \nu(x) \ge \gamma$ for $\mathcal{H}^{n-1}$-almost every $x \in \partial\Omega$ and $|\psi| = 1$ on $\partial\Omega$.*

This is well-known (see, for example, [18, Lemma 4.1]), for completeness we include the short proof.

Proof. The compact set $\partial\Omega$ can be covered by a finite family of balls $B_i := B_{r_i}(z_i)$, such that in each of the larger balls $B_i^* := B_{2r_i}(z_i)$ (3.2) reads

$$B_i^* \cap \Omega = B_i^* \cap A_i\{(x', x_n) : x_n < G_i(x')\} \tag{3.3}$$

for some $G_i \in \mathrm{Lip}\,(\mathbb{R}^{n-1})$ and isometry A_i. If x' is such that $y := A_i(x', G_i(x')) \in \partial\Omega \cap B_i^*$ and G_i is differentiable at x', then the outer normal obeys $\nu(y) = R_i(-DG_i(x'), 1)/\sqrt{1 + |DG_i|^2(x')}$, where $R_i := DA_i \in \mathrm{O}(n)$, so that

$$\nu \cdot R_i e_n \ge \gamma^* := \frac{1}{\sqrt{1 + \max_i(\mathrm{Lip}\,(G_i))^2}} > 0 \tag{3.4}$$

$\mathcal{H}^{n-1}$-almost everywhere on $B_i^* \cap \partial\Omega$. We fix cutoff functions $\theta_i \in C_c^\infty(B_i^*; [0, 1])$ with $\theta_i = 1$ on B_i, and set $\psi^*(x) := \sum_i \theta_i(x) R_i e_n$. Then for $\mathcal{H}^{n-1}$-almost every point $x \in \partial\Omega$ we have

$$\psi^*(x) \cdot \nu(x) = \sum_{i : x \in B_i^*} \theta_i(x)(R_i e_n) \cdot \nu(x) \ge \sum_{i : x \in B_i^*} \theta_i(x)\gamma^* \ge \gamma^*, \tag{3.5}$$

since at least one of the $\theta_i(x)$ equals 1.

It only remains to normalize. Condition (3.5) implies $|\psi^*| \geq \gamma^*$ on $\partial\Omega$, and therefore $|\psi^*| > \gamma^*/2$ in a neighborhood of $\partial\Omega$. We select $\varphi \in C^\infty(\mathbb{R}^n; [0,1])$ such that $\varphi = 0$ on $\partial\Omega$, and $\varphi > 0$ on the set $|\psi^*| \leq \gamma^*/2$. Then $\psi := \psi^*/\sqrt{|\psi^*|^2 + \varphi}$ has the desired properties with $\gamma := \gamma^*/\max|\psi^*|(\partial\Omega)$. $\square$

Lemma 3.3. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded Lipschitz set, ψ as in Lemma 3.2. There are $\rho > 0$ and $c > 0$ such that for any $x, y \in \partial\Omega$ with $|x - y| < \rho$ one has*

$$|x - y| \leq c|(\mathrm{Id} - \psi(y) \otimes \psi(y))(x - y)|. \tag{3.6}$$

Proof. We can assume $x \neq y$. After a change of coordinates, and choosing ρ sufficiently small, we can assume that $y = 0$, and that

$$\partial\Omega \cap B_{(1+L)\rho}(0) = B_{(1+L)\rho}(0) \cap \{(z', G(z')) : z' \in \mathbb{R}^{n-1}\} \tag{3.7}$$

for some L-Lipschitz function $G : \mathbb{R}^{n-1} \to \mathbb{R}$. The values of ρ and L have bounds that depend only on Ω. Let $m := \psi(0)$, so that $P := \mathrm{Id} - m \otimes m$ is the projection onto the space orthogonal to m. Condition (3.6) then translates into

$$|x| \leq c|Px| \tag{3.8}$$

for any $x \in \partial\Omega \cap B_\rho(0)$. As both sides of (3.8) are continuous, it suffices to prove it for $\mathcal{H}^{n-1}$-almost every x. By (3.7), we have $x = (x', G(x'))$ for some $x' \in \mathbb{R}^{n-1} \setminus \{0\}$. Define $\hat{x} := (\hat{x}', 0) := (\frac{x'}{|x'|}, 0) \in S^{n-2} \times \{0\} \subset \mathbb{R}^{n-1} \times \{0\}$. Let $\Pi := \mathrm{span}\{\hat{x}, e_n\}$, we remark that $\hat{x} \cdot e_n = 0$ and that $x \in \Pi$. Let m_Π be the orthogonal projection of m on Π, namely

$$m_\Pi := (\hat{x} \otimes \hat{x} + e_n \otimes e_n)m = (\hat{x} \cdot m)\hat{x} + (e_n \cdot m)e_n \in \Pi, \tag{3.9}$$

and $m_\perp := m - m_\Pi$, so that $m = m_\Pi + m_\perp$. Then, as $x \in \Pi$ and $m_\perp \in \Pi^\perp$ we have

$$\begin{aligned} |Px|^2 =& |(\mathrm{Id} - (m_\Pi + m_\perp) \otimes (m_\Pi + m_\perp))x|^2 \\ =& |x - (m_\Pi \cdot x)m_\Pi|^2 + |m_\perp|^2 |m_\Pi \cdot x|^2. \end{aligned} \tag{3.10}$$

We distinguish two cases. If $|m_\perp| \geq \frac{1}{2}\gamma$, with γ the constant from Lemma 3.2, the first term leads to

$$|Px| \geq |x - (m_\Pi \cdot x)m_\Pi| \geq |x| - |m_\Pi|^2|x| = |m_\perp|^2|x| \geq \frac{\gamma^2}{4}|x| \tag{3.11}$$

which concludes the proof of (3.8) in this case.

Assume now that $|m_\perp| \leq \frac{1}{2}\gamma$. For any $z' \in \mathbb{R}^{n-1}$ with $|z'| < \rho$ we have $z := (z', G(z')) \in \partial\Omega \cap B_{\rho(1+L)}(0)$, and if G is differentiable in z' the outer normal is

$$\nu(z) = \frac{1}{\sqrt{1+|DG|^2(z')}} \begin{pmatrix} -DG(z') \\ 1 \end{pmatrix}. \tag{3.12}$$

Recalling $\psi(z) \cdot \nu(z) \geq \gamma$,

$$m \cdot \nu(z) = \psi(z) \cdot \nu(z) + (\psi(0) - \psi(z)) \cdot \nu(z) \geq \gamma - \|\psi\|_{C^1} |z| \tag{3.13}$$

so that, if $\rho < \gamma/(4\|\psi\|_{C^1}(1+L))$, the condition $|m_\perp| \leq \frac{1}{2}\gamma$ implies

$$m_\Pi \cdot \nu(z) = m \cdot \nu(z) - m_\perp \cdot \nu(z) \geq \gamma - \frac{1}{4}\gamma - |m_\perp| \geq \frac{1}{4}\gamma. \tag{3.14}$$

By (3.12) and (3.9),

$$m_\Pi \cdot \nu(z) = \frac{1}{\sqrt{1+|DG|^2(z')}} \left[(e_n \cdot m) - (\hat{x} \cdot m)\hat{x}' \cdot DG(z')\right]. \tag{3.15}$$

With a slight abuse of notation we have used the dot to denote the inner product both in $\mathbb{R}^{n-1}$ and $\mathbb{R}^n$.

Let $\zeta(t) := (t\hat{x}', G(t\hat{x}'))$, for $t \in [0, |x'|]$. For $\mathcal{H}^{n-1}$-almost every choice of $x' \in B'_\rho$ we have that for $\mathcal{H}^1$-almost every t the function G is differentiable in $t\hat{x}'$. Clearly $\zeta(0) = 0$, $\zeta(|x'|) = x$, and ζ is Lipschitz with

$$D\zeta(t) = \hat{x} + (\hat{x}' \cdot DG(t\hat{x}'))e_n. \tag{3.16}$$

We define

$$m_\Pi^\perp := (e_n \otimes \hat{x} - \hat{x} \otimes e_n)m = (\hat{x} \cdot m)e_n - (e_n \cdot m)\hat{x} \in \Pi, \tag{3.17}$$

and compute

$$m_\Pi^\perp \cdot x = \int_0^{|x'|} m_\Pi^\perp \cdot D\zeta(t)dt = \int_0^{|x'|} \left[(\hat{x} \cdot m)(\hat{x}' \cdot DG(t\hat{x}')) - (e_n \cdot m)\right] dt. \tag{3.18}$$

Using first (3.15) and then (3.14),

$$m_\Pi^\perp \cdot x = -\int_0^{|x'|} m_\Pi \cdot \nu(\zeta(t))\sqrt{1+|DG|^2(t\hat{x}')}dt \leq -|x'|\frac{\gamma}{4}. \tag{3.19}$$

With $|m_\Pi^\perp| \leq 1$ and $m_\Pi^\perp \cdot m = 0$ (note that $m_\Pi^\perp \in \Pi$ and $m_\Pi^\perp \cdot m_\Pi = 0$) we obtain

$$|Px| \geq \left| x \cdot \frac{m_\Pi^\perp}{|m_\Pi^\perp|} \right| \geq |x \cdot m_\Pi^\perp| \geq \frac{\gamma}{4}|x'|. \tag{3.20}$$

Recalling that $|x| \leq |x'| + |G(x')| \leq (1+L)|x'|$, this concludes the proof of (3.8), and therefore of (3.6) (with $c = 4(1+L)/\gamma^2$). $\square$

We are now ready to prove Theorem 3.1. Before starting, we recall that by Brower's invariance of domain theorem any injective continuous map $f : E \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is open, in the sense that if E is open then $f(E)$ is open.

Proof of Theorem 3.1. *Step 1.* Let ψ be as in Lemma 3.2, ρ as in Lemma 3.3, and define $f : \partial\Omega \times \mathbb{R} \to \mathbb{R}^n$ by

$$f(x,t) := x + t\psi(x). \tag{3.21}$$

We claim that there are $\varepsilon > 0$ and $C > 0$, depending only on Ω, such that for all $x, y \in \partial\Omega$, all $t, s \in (-\varepsilon, \varepsilon)$,

$$|x-y| + |t-s| \leq C|f(x,t) - f(y,s)|. \tag{3.22}$$

In order to prove (3.22) we write

$$f(x,t) - f(y,s) = x - y + t\psi(x) - s\psi(y). \tag{3.23}$$

We shall choose $\varepsilon \leq \rho$. If $|x-y| \leq \rho$ we can use Lemma 3.3. Let P_y be the projection onto $\psi(y)^\perp$. Then

$$P_y(f(x,t) - f(y,s)) = P_y(x-y) + tP_y(\psi(x) - \psi(y)). \tag{3.24}$$

For the second term we use $|\psi(x) - \psi(y)| \leq \|\psi\|_{C^1}|x-y|$. For the first term, we use (3.6). We then obtain

$$\begin{aligned} |f(x,t) - f(y,s)| &\geq |P_y(x-y)| - |t|\,|\psi(x) - \psi(y)| \\ &\geq \frac{1}{c}|x-y| - \varepsilon\|\psi\|_{C^1}|x-y| \geq \frac{1}{2c}|x-y| \end{aligned} \tag{3.25}$$

provided that $\varepsilon \leq 1/(2c\|\psi\|_{C^1})$. To estimate $t-s$ we write (3.23) as

$$f(x,t) - f(y,s) = (t-s)\psi(x) + x - y + s(\psi(x) - \psi(y)) \tag{3.26}$$

so that

$$|t-s| \leq |f(x,t) - f(y,s)| + |x-y| + |s|\,\|\psi\|_{C^1}\,|x-y| \tag{3.27}$$

which, recalling that $|s|\|\psi\|_{C^1} \leq 1/(2c)$, together with (3.25) concludes the proof of (3.22) in this case.

Assume now that $|x-y|>\rho$, still with $|t|,|s|<\varepsilon$. Then (3.23) gives

$$|f(x,t)-f(y,s)|\geq|x-y|-|t|-|s|\geq\frac{1}{2}|x-y|+\frac{1}{2}\rho-2\varepsilon. \tag{3.28}$$

Choosing $\varepsilon\leq\rho/4$ and recalling (3.27), this concludes the proof of (3.22).

Step 2. We define $\omega:=f(\partial\Omega\times(-\varepsilon,\varepsilon))$ and check that $f|_{\partial\Omega\times(-\varepsilon,\varepsilon)}$ is bilipschitz. Let $y,y'\in\omega$. Then there are $x,x'\in\partial\Omega$, $t,t'\in(-\varepsilon,\varepsilon)$, such that

$$y=f(x,t),\qquad y'=f(x',t'), \tag{3.29}$$

which by (3.22) and (3.26) in Step 1 implies

$$\frac{1}{C}(|x-x'|+|t-t'|)\leq|y-y'|\leq C(|x-x'|+|t-t'|). \tag{3.30}$$

Hence $f|_{\partial\Omega\times(-\varepsilon,\varepsilon)}$ is bilipschitz. Let now $\Phi:\omega\to\omega$ be

$$\Phi(y):=f(f_x^{-1}(y),-f_t^{-1}(y)) \tag{3.31}$$

where f_x^{-1} and f_t^{-1} denote the components of f^{-1}. Obviously $\Phi(y)=y$ if $y\in\partial\Omega\subseteq\omega$. We check that Φ is bilipschitz. Indeed, arguing as above, setting

$$Y:=f(x,-t)=\Phi(y),\qquad Y':=f(x',-t')=\Phi(y'), \tag{3.32}$$

we get

$$\frac{1}{C}(|x-x'|+|t-t'|)\leq|Y-Y'|\leq C(|x-x'|+|t-t'|), \tag{3.33}$$

therefore Φ is bilipschitz.

It remains to show that ω is open if ε is sufficiently small. Assume $\varepsilon\leq\varepsilon_0/(1+L_0)$, with ε_0, L_0 the quantities introduced right after (3.2). Select $y\in\omega$, and let $x\in\partial\Omega$, $s\in(-\varepsilon,\varepsilon)$ be such that $y=f(x,s)$. Choose A_x, G_x as in (3.2) and let $h:B'_{\varepsilon_0/(1+L_0)}\times(-\varepsilon,\varepsilon)\to\omega$ be defined by

$$h(z',t):=f(A_x(z',G_x(z')),t). \tag{3.34}$$

The map h is injective and Lipschitz, and therefore open. Therefore ω contains an open neighborhood of $y=f(x,s)$. $\square$

Remark 3.4. If ψ were only L^∞, the map $f(x,t)$ may not be invertible in $\partial\Omega\times(-\varepsilon,\varepsilon)$, for all choices of $\varepsilon>0$. This happens for example if $\partial\Omega$ is (locally) the graph of the function $\chi_{(0,1)}(x)|x|^{1+\alpha}$ for $x\in(-1,1)$, where $0<\alpha<1$ (see (3.25)).

Theorem 3.5. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open Lipschitz set, $\theta > 0$, and $u \in SBV(\Omega;\mathbb{R}^m)$ such that $E_{|\cdot|^p,g_0}[u,\Omega] < \infty$, for some $p \geq 1$ and g_0 satisfying ($H_1^{g_0}$)-($H_2^{g_0}$). Then there are an open set Ω' with $\overline{\Omega} \subset \Omega'$ and $|\Omega'| \leq |\Omega| + \theta$, and a function $U \in SBV(\mathbb{R}^n;\mathbb{R}^m)$ such that $U = u$ on Ω, $|DU|(\partial\Omega) = 0$,*

$$\int_{\Omega'} |\nabla U|^p \, dx \leq \int_{\Omega} |\nabla u|^p \, dx + \theta \,, \tag{3.35}$$

and

$$\int_{\Omega' \cap J_U} g_0(|[U]|) d\mathcal{H}^{n-1} \leq \int_{J_u} g_0(|[u]|) d\mathcal{H}^{n-1} + \theta \,. \tag{3.36}$$

In particular,

$$E_{|\cdot|^p,g_0}[U,\Omega'] \leq E_{|\cdot|^p,g_0}[u,\Omega] + 2\theta. \tag{3.37}$$

If $\mathcal{H}^{n-1}(J_u) < \infty$, then additionally $\mathcal{H}^{n-1}(J_U) < \infty$ and $\mathcal{H}^{n-1}(J_U \cap \Omega' \setminus \Omega) < \theta$; if $\nabla u = 0$ $\mathcal{L}^n$-almost everywhere on Ω then also $\nabla U = 0$ $\mathcal{L}^n$-almost everywhere on Ω'. If $u \in W^{1,p}(\Omega;\mathbb{R}^m)$ then $U \in W^{1,p}(\Omega';\mathbb{R}^m)$.

Proof. We consider the open set ω and the bilipschitz map Φ provided by Theorem 3.1. Thus, [5, Theorem 3.16] yields that $u \circ \Phi^{-1} \in SBV(\omega \setminus \overline{\Omega};\mathbb{R}^m)$, with

$$|\operatorname{Lip}(\Phi)|^{1-n} \Phi_\# |D(u|_{\Omega\cap\omega})| \leq |D(u \circ \Phi^{-1})| \leq |\operatorname{Lip}(\Phi^{-1})|^{n-1} \Phi_\# |D(u|_{\Omega\cap\omega})| \,,$$

where $\Phi_\#$ denotes the push forward of measures. In particular, by the Coarea formula (cf. [5, Theorem 2.93]) we conclude that

$$\int_{\omega\setminus\overline{\Omega}} |\nabla(u \circ \Phi^{-1})|^p \, dx \leq C \int_{\Omega\cap\omega} |\nabla u|^p \, dx \,, \tag{3.38}$$

and by the Area formula (2.9)

$$\int_{J_{u\circ\Phi^{-1}}} g_0(|[u \circ \Phi^{-1}]|) d\mathcal{H}^{n-1} \leq C \int_{J_u \cap (\Omega\cap\omega)} g_0(|[u]|) d\mathcal{H}^{n-1} \,, \tag{3.39}$$

for a constant C depending only on n and $\operatorname{Lip}(\Phi^{-1})$.

Having fixed $\theta > 0$, up to restricting ω, we may assume that both right-hand sides of (3.38) and (3.39) are actually less than or equal to θ, and in addition that ω is Lipschitz with $|\omega \setminus \overline{\Omega}| \leq \theta$.

Then, to conclude, set $\Omega' := \Omega \cup \omega$ and

$$U(x) := \begin{cases} u(x) & x \in \Omega, \\ u(\Phi^{-1}(x)) & x \in \omega \setminus \overline{\Omega}, \\ 0 & x \in \mathbb{R}^n \setminus \overline{\Omega'}. \end{cases} \tag{3.40}$$

By construction $U = u$ on Ω. Moreover, recalling that $\Phi|_{\partial\Omega}$ is the identity and that ω is Lipschitz, [5, Corollary 3.89] implies that $U \in SBV(\mathbb{R}^n;\mathbb{R}^m)$, $|DU|(\partial\Omega) = 0$, and moreover that $DU \llcorner \Omega' = Du \llcorner \Omega + D(u \circ \Phi^{-1}) \llcorner (\omega \setminus \overline{\Omega})$. Finally, estimates (3.35) and (3.36) readily follows from (3.38) and (3.39).

In the case $\mathcal{H}^{n-1}(J_u) < \infty$ the additional estimate follows from the same proof, using $1 + g_0$ in place of g_0 in (3.39). Similarly, if either $\nabla u = 0$ or $J_u = \emptyset$, the same property is immediately inherited by U on $\Omega \cup \omega$. $\square$

3.2. Approximate regularity on an intermediate scale

Given a SBV function satisfying the hypotheses of Theorem 1.1, we show that at an intermediate scale, denoted by δ, the regular part of the gradient is uniformly approximately continuous, and the jump is approximately given by a fixed jump concentrated on a C^1 manifold. Having fixed g_0 satisfying $(\mathrm{H}_1^{g_0})$-$(\mathrm{H}_2^{g_0})$, for every $u \in SBV(\Omega;\mathbb{R}^m)$, we introduce the notation

$$\mu_u := g_0(|[u]|)\mathcal{H}^{n-1} \llcorner J_u \,. \tag{3.41}$$

The case of interest in what follows is when μ_u is a finite measure on Ω; in any case μ_u is concentrated on a σ-finite set with respect to $\mathcal{H}^{n-1} \llcorner J_u$. The main result is the following.

Proposition 3.6. *Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded, $p \in [1, \infty)$, and g_0 be satisfying $(H_1^{g_0})$-$(H_2^{g_0})$. There is a constant C depending on p, n and m, such that for every $u \in SBV(\Omega;\mathbb{R}^m)$ with $\mu_u(\Omega) < \infty$ and $\nabla u \in L^p(\Omega;\mathbb{R}^{m\times n})$, and for every $\theta > 0$, after fixing an orientation of J_u there is $\delta \in (0, \theta]$ such that, setting $A_\delta := \{z \in \Omega \cap \delta\mathbb{Z}^n : \mathrm{dist}(z, \partial\Omega) > \delta\sqrt{n}\}$ and $Q_z^* := z + (-\delta, \delta)^n$, the following holds: there are $R : A_\delta \to \mathrm{SO}(n)$, $s : A_\delta \to \mathbb{R}^m$, $\eta : A_\delta \to \mathbb{R}^{m\times n}$, $\varphi : A_\delta \to C_c^1(\mathbb{R}^{n-1})$, and $x : A_\delta \to \mathbb{R}^n$ such that, setting*

$$L_z := x_z + R_z\{(y', \varphi_z(y')) : y' \in \mathbb{R}^{n-1}\}, \tag{3.42}$$

one has $\|D\varphi_z\|_{L^\infty} \le \theta$ and

$$\sum_{z\in A_\delta} \int_{Q_z^*} |\nabla u - \eta_z|^p dx + \sum_{z\in A_\delta} \int_{Q_z^*\cap J_u\setminus L_z} g_0(|[u]|) d\mathcal{H}^{n-1}$$

$$+ \sum_{z\in A_\delta} \int_{Q_z^*\cap L_z} g_0(|[u]|)|\nu_u - R_z e_n| d\mathcal{H}^{n-1}$$

$$+ \sum_{z\in A_\delta} \int_{Q_z^*\cap L_z} g_0(|[u] - s_z|) d\mathcal{H}^{n-1} \le C\theta(1 + \mu_u(\Omega) + |\Omega|)\,. \tag{3.43}$$

If $\mathcal{H}^{n-1}(J_u) < \infty$ then additionally

$$\sum_{z\in A_\delta} \mathcal{H}^{n-1}(Q_z^* \cap (J_u \triangle L_z)) \le C\theta. \tag{3.44}$$

Before proving it we introduce a preliminary pointwise result for the jump part of the energy.

Lemma 3.7. *Let $\Omega \subseteq \mathbb{R}^n$ be open, and g_0 be satisfying ($H_1^{g_0}$)-($H_2^{g_0}$). Let $u \in SBV(\Omega;\mathbb{R}^m)$ with $\mu_u(\Omega) < \infty$. Then, for $\mathcal{H}^{n-1}$-almost every $x \in J_u$ there are $R_x \in \mathrm{SO}(n)$, $s_x \in \mathbb{R}^m \setminus \{0\}$, $\varphi_x \in C^1(\mathbb{R}^{n-1})$ such that $\varphi_x(0) = 0$, $D\varphi_x(0) = 0$, and, letting $L_x := x + R_x\{(y', \varphi_x(y')) : y' \in \mathbb{R}^{n-1}\}$,*

$$\lim_{r\to 0} \frac{1}{\mu_u(B_r(x))} \left[\int_{B_r(x)\cap J_u\setminus L_x} g_0(|[u]|) d\mathcal{H}^{n-1} + \int_{B_r(x)\cap L_x} \big(g_0(|[u] - s_x|) + g_0(|[u]|)|\nu_u - R_x e_n|\big)\, d\mathcal{H}^{n-1} \right] = 0. \tag{3.45}$$

If $\mathcal{H}^{n-1}(J_u) < \infty$, then additionally

$$\lim_{r\to 0} \frac{\mathcal{H}^{n-1}(B_r(x) \cap (J_u \triangle L_x))}{\mu_u(B_r(x))} = 0. \tag{3.46}$$

Proof. We first observe that for $\mathcal{H}^{n-1}$-almost every $x \in J_u$ by [5, Th. 2.83(i)] we have

$$\lim_{r\to 0} \frac{\mu_u(B_r(x))}{\omega_{n-1} r^{n-1}} = g_0(|[u](x)|) \neq 0, \tag{3.47}$$

therefore to prove (3.45) it suffices to show that for $\mathcal{H}^{n-1}$-almost every $x \in J_u$ there is L_x as stated such that, setting $s_x := [u](x)$, one has

$$\lim_{r\to 0} \frac{1}{r^{n-1}} \int_{B_r(x)\cap L_x} g_0(|[u] - s_x|) d\mathcal{H}^{n-1} = 0, \tag{3.48}$$

$$\lim_{r\to 0} \frac{1}{r^{n-1}} \int_{B_r(x)\cap J_u\setminus L_x} g_0(|[u]|) d\mathcal{H}^{n-1} = 0, \tag{3.49}$$

and

$$\lim_{r\to 0} \frac{1}{r^{n-1}} \int_{B_r(x)\cap L_x\cap J_u} |\nu_u - R_x e_n| d\mathcal{H}^{n-1} = 0. \tag{3.50}$$

Note that (3.48) implies (3.49). Indeed, subadditivity and monotonicity of g_0 imply $g_0(|s_x|) \le g_0(|[u]|) + g_0(|[u] - s_x|)$ and therefore

$$\begin{aligned}\mu_u(B_r(x) \setminus L_x) &= \mu_u(B_r(x)) - \mu_u(B_r(x) \cap L_x)\\ &\le \mu_u(B_r(x)) - g_0(|s_x|)\mathcal{H}^{n-1}(B_r(x) \cap L_x) + \int_{B_r(x)\cap L_x} g_0(|[u] - s_x|) d\mathcal{H}^{n-1}.\end{aligned} \tag{3.51}$$

As L_x is the graph of a C^1 function,

$$\lim_{r\to 0} \frac{\mathcal{H}^{n-1}(B_r(x) \cap L_x)}{\omega_n r^{n-1}} = 1. \tag{3.52}$$

We divide (3.51) by $\omega_{n-1} r^{n-1}$ and take the lim sup as $r \to 0$. Using (3.47) to estimate the first term, (3.52) for the second, and (3.48) for the third, we obtain

$$\limsup_{r\to 0} \frac{1}{\omega_{n-1} r^{n-1}} \mu_u(B_r(x) \setminus L_x) = 0, \tag{3.53}$$

which is (3.49). Therefore it remains to prove (3.48) and (3.50).

For any $j > 0$ let $A_j := \{x \in J_u : |[u](x)| \ge 2^{-j}\}$. As $u \in SBV(\Omega; \mathbb{R}^m)$, we have $\mathcal{H}^{n-1}(A_j) < \infty$, and A_j is countably $(n-1)$-rectifiable. Therefore for $\mathcal{H}^{n-1}$-almost every $x \in A_j$ there are R_x^j, φ_x^j as in the statement such that the corresponding set L_x^j obeys

$$\lim_{r\to 0} \frac{\mathcal{H}^{n-1}(B_r(x) \cap (A_j \triangle L_x^j))}{r^{n-1}} = 0 \tag{3.54}$$

and $\nu_u(x) = R_x^j e_n$. As $|[u]| \in L^1(\Omega; \mathcal{H}^{n-1} \llcorner J_u)$, and $\mathcal{H}^{n-1}(A_j)$ is finite, for $\mathcal{H}^{n-1}$-almost every $x \in A_j$

$$\lim_{r\to 0} \frac{1}{r^{n-1}} \int_{A_j\cap B_r(x)} |[u] - s_x| d\mathcal{H}^{n-1} = 0 \tag{3.55}$$

and similarly

$$\lim_{r\to 0} \frac{1}{r^{n-1}} \int_{A_j\cap B_r(x)} |\nu_u - \nu_u(x)| d\mathcal{H}^{n-1} = 0. \tag{3.56}$$

We recall that for $x \in J_u$ we defined $s_x = [u](x) \neq 0$. We first show that (3.54) and (3.55) imply

$$\lim_{r\to 0}\frac{1}{r^{n-1}}\int\limits_{L_x^j\cap B_r(x)}|[u]-s_x|d\mathcal{H}^{n-1}=0 \tag{3.57}$$

for $\mathcal{H}^{n-1}$-almost every $x\in A_j$. Indeed, we have

$$\int\limits_{L_x^j\cap B_r(x)}|[u]-s_x|d\mathcal{H}^{n-1}\le\int\limits_{A_j\cap B_r(x)}|[u]-s_x|d\mathcal{H}^{n-1}$$
$$+(|s_x|+2^{-j})\mathcal{H}^{n-1}((L_x^j\setminus A_j)\cap B_r(x))$$

and the conclusion then follows from (3.54) and (3.55).

We next show that (3.57) implies that for $\mathcal{H}^{n-1}$-almost every $x\in A_j$

$$\lim_{r\to 0}\frac{1}{r^{n-1}}\int\limits_{L_x^j\cap B_r(x)}g_0(|[u]-s_x|)d\mathcal{H}^{n-1}=0. \tag{3.58}$$

Indeed, using (2.2), namely that for every $\lambda>0$ there is $C_\lambda>0$ such that $g_0(t)\le\lambda+C_\lambda t$ for all $t\in[0,\infty)$, with (3.57) we obtain

$$\limsup_{r\to 0}\frac{1}{r^{n-1}}\int\limits_{B_r(x)\cap L_x^j}g_0(|[u]-s_x|)d\mathcal{H}^{n-1}$$
$$\le\lambda\lim_{r\to 0}\frac{\mathcal{H}^{n-1}(B_r(x)\cap L_x^j)}{r^{n-1}}=\lambda\omega_{n-1}. \tag{3.59}$$

Since λ was arbitrary this concludes the proof of (3.58).

Let $N\subseteq J_u$ be an $\mathcal{H}^{n-1}$-null set such that (3.58) holds for all $x\in J_u\setminus N$ and all j such that $x\in A_j$. For any $x\in J_u\setminus N$ we define L_x as L_x^j for the smallest $j\in N$ such that $x\in A_j$. This proves (3.48). Condition (3.50) follows similarly from (3.56) using $A_j\subseteq J_u$ and (3.54).

Assume now that $\mathcal{H}^{n-1}(J_u)<\infty$. Then we can replace (3.54) by

$$\lim_{r\to 0}\frac{\mathcal{H}^{n-1}(B_r(x)\cap(J_u\triangle L_x))}{r^{n-1}}=\lim_{r\to 0}\frac{\mathcal{H}^{n-1}(B_r(x)\cap(A_j\triangle L_x))}{r^{n-1}}=0. \tag{3.60}$$

The second equality leads as above to (3.45); from the first one, one immediately obtains (3.46) (using again (3.47)). $\square$

Proof of Proposition 3.6. As $\nabla u\in L^p(\Omega;\mathbb{R}^{m\times n})$, there is $f\in C^0(\overline{\Omega};\mathbb{R}^{m\times n})$ such that $\|\nabla u-f\|_{L^p(\Omega)}^p\le\theta$. Let $\delta>0$ be such that $|f(x)-f(y)|^p\le\theta$ for all $x,y\in\overline{\Omega}$ with $|x-y|\le\delta\sqrt{n}$. For any $z\in A_\delta$ we set $\eta_z:=f(z)$ and obtain

$$\sum_{z\in A_\delta}\int\limits_{Q_z^*}|\nabla u-\eta_z|^p dx\le 2^{p-1}\sum_{z\in A_\delta}\int\limits_{Q_z^*}(|\nabla u-f|^p+|f-\eta_z|^p)dx$$

$$\leq 2^{p+n}(1+|\Omega|)\theta \tag{3.61}$$

as any point $x \in \mathbb{R}^n$ belongs to at most 2^n of the cubes Q_z^*, $z \in A_\delta$. This treats the first term.

The jump terms are treated using Lemma 3.7. For μ_u-almost every $x \in \Omega$ there are $\hat{R}_x \in \mathrm{SO}(n)$, $\hat{s}_x \in \mathbb{R}^m \setminus \{0\}$, and $\hat{\varphi}_x \in C^1(\mathbb{R}^{n-1})$ as stated, we define $\hat{R}_x := \mathrm{Id}$, $\hat{s}_x := 0$, and $\hat{\varphi}_x := 0$ on the others. We recall that $\hat{L}_x = x + \hat{R}_x\{(y', \hat{\varphi}_x(y')) : y' \in \mathbb{R}^{n-1}\}$, and define, for any $x \in \Omega$, the measure

$$\begin{aligned} \hat{m}_x :=& g_0(|[u]|)\mathcal{H}^{n-1} \llcorner (J_u \setminus \hat{L}_x) + g_0(|[u]-\hat{s}_x|)\mathcal{H}^{n-1} \llcorner \hat{L}_x \\ &+ g_0(|[u]|)\, |\nu_u - \hat{R}_x e_n|\, \mathcal{H}^{n-1} \llcorner \hat{L}_x\,. \end{aligned} \tag{3.62}$$

By Lemma 3.7, for μ_u-almost every $x \in \Omega$

$$\lim_{r\to 0} \frac{\hat{m}_x(B_r(x))}{\mu_u(B_r(x))} = 0 \text{ and } \lim_{r\to 0} \|D\hat{\varphi}_x\|_{L^\infty(B'_{3r})} = 0 \tag{3.63}$$

(we write B' for balls in $\mathbb{R}^{n-1}$). We define, for $k \in \mathbb{N}_{>0}$,

$$\begin{aligned} E^{k,\theta} :=& \{x \in \Omega : \|D\hat{\varphi}_x\|_{L^\infty(B'_{\frac{3}{k}})} > \frac{1}{3}\theta\} \\ &\cup \{x \in \Omega : \exists r \in (0, \frac{1}{k}] \text{ with } \hat{m}_x(B_r(x) \cap \Omega) \geq \theta \mu_u(B_r(x) \cap \Omega)\}. \end{aligned} \tag{3.64}$$

Obviously $E^{k',\theta} \subseteq E^{k,\theta}$ if $k < k'$. By (3.63), for μ_u-almost every $x \in \Omega$ there is k such that $x \notin E^{k,\theta}$. Therefore

$$\mu_u(\bigcap_{k\in\mathbb{N}} E^{k,\theta}) = 0. \tag{3.65}$$

We select $k_\theta > 2/\theta$ such that $\mu_u(E^{k_\theta,\theta}) \leq \theta$, and assume that δ is such that $2\delta\sqrt{n} \leq 1/k_\theta$.

For some $(s, x, R, \varphi) : A_\delta \to \mathbb{R}^m \times \Omega \times \mathrm{SO}(n) \times C_c^1(\mathbb{R}^{n-1})$ (still to be defined) and any $z \in A_\delta$ we intend to estimate an error measure defined in analogy to (3.62) by $S_z := x_z + R_z\{(y', \varphi_z(y')) : y' \in \mathbb{R}^{n-1}\}$ and

$$m_z := g_0(|[u]|)\mathcal{H}^{n-1} \llcorner (J_u \setminus S_z) + \left[g_0(|[u]-s_z|) + g_0(|[u]|)|\nu_u - R_z e_n|\right]\mathcal{H}^{n-1} \llcorner S_z, \tag{3.66}$$

namely to prove

$$\sum_{z\in A_\delta} m_z(Q_z^*) \leq C\theta(1+\mu_u(\Omega))\,,$$

with a constant $C > 0$ depending on n, p.

Let $F := \{z \in A_\delta : Q_z^* \subseteq E^{k_\theta,\theta}\}$. If $z \in F$, we set $s_z := 0$, $x_z := z + 2\delta e_n$, $R_z := \mathrm{Id}$, $\varphi_z := 0$, so that $S_z \cap Q_z^* = \emptyset$ and $m_z \llcorner Q_z^* = \mu_u \llcorner Q_z^*$. Therefore

$$\sum_{z\in F} m_z(Q_z^*) = \sum_{z\in F} \mu_u(Q_z^*) \le 2^n \mu_u\Big(\bigcup_{z\in F} Q_z^*\Big) \le 2^n \mu_u(E^{k_\theta,\theta}) \le 2^n\theta. \tag{3.67}$$

Consider now $z \in A_\delta \setminus F$, and select $x_z \in Q_z^* \setminus E^{k_\theta,\theta}$. We set $s_z := \hat{s}_{x_z}$, $R_z := \hat{R}_{x_z}$, and

$$S_z := \hat{L}_{x_z} = x_z + R_z\{(y', \hat{\varphi}_{x_z}(y')) : y' \in \mathbb{R}^{n-1}\}. \tag{3.68}$$

Note that with this choice $m_z = \hat{m}_{x_z}$. Moreover, as $x_z \in Q_z^*$ we have $Q_z^* \subset B_{2\sqrt{n}\delta}(x_z)\cap\Omega$. As $2\delta\sqrt{n} \le 1/k_\theta$, $x_z \notin E^{k_\theta,\theta}$ implies $\hat{m}_{x_z}(B_{2\delta\sqrt{n}}(x_z)\cap\Omega) < \theta\mu_u(B_{2\delta\sqrt{n}}(x_z)\cap\Omega)$, so that

$$\sum_{z\in A_\delta\setminus F} m_z(Q_z^*) \le \sum_{z\in A_\delta\setminus F} \theta\mu_u(B_{2\delta\sqrt{n}}(x_z)\cap\Omega). \tag{3.69}$$

Each ball $B_{2\sqrt{n}\delta}(x_z)$ overlaps with a finite number $C(n)$ of cubes with center in $\delta\mathbb{Z}^n$ and side of length δ, which implies that they have finite overlap. Therefore

$$\sum_{z\in A_\delta\setminus F} m_z(Q_z^*) \le C\theta\mu_u(\Omega). \tag{3.70}$$

Recall that $\hat{\varphi}_{x_z} \in C^1(\mathbb{R}^{n-1})$ satisfies $\|D\hat{\varphi}_{x_z}\|_{L^\infty(B'_{3/k_\theta})} \le \frac{1}{3}\theta$ and $\hat{\varphi}_{x_z}(0) = 0$. We fix $\alpha_z \in C_c^1(B'_{3/k_\theta};[0,1])$ such that $\alpha_z = 1$ on B'_{1/k_θ} and $\|D\alpha_z\|_{L^\infty} \le \frac{2}{3}k_\theta$, and set $\varphi_z := \alpha_z\hat{\varphi}_{x_z}$. Then $\varphi_z \in C_c^1(\mathbb{R}^{n-1})$, with $\varphi_z = \hat{\varphi}_{x_z}$ on $B'_{1/k_\theta}(x_z)$. Using the bounds above, $\hat{\varphi}_{x_z}(0) = 0$ and the mean-value theorem, we obtain

$$\begin{aligned}\|D\varphi_z\|_{L^\infty(\mathbb{R}^{n-1})} &\le \|D\hat{\varphi}_{x_z}\|_{L^\infty(B'_{3/k_\theta})}\|\alpha_z\|_{L^\infty} + \|\hat{\varphi}_{x_z}\|_{L^\infty(B'_{3/k_\theta})}\|D\alpha_z\|_{L^\infty}\\ &\le \|D\hat{\varphi}_{x_z}\|_{L^\infty(B'_{3/k_\theta})} + \frac{3}{k_\theta}\|D\hat{\varphi}_{x_z}\|_{L^\infty(B'_{3/k_\theta})}\|D\alpha_z\|_{L^\infty} \le \theta.\end{aligned} \tag{3.71}$$

Combining this remark with the results in (3.61), (3.67), (3.70) gives the first assertion.

Assume now that additionally $\mathcal{H}^{n-1}(J_u) < \infty$. We proceed in the same way, replacing the measure $\hat{m}_x$ defined in (3.62) by $\hat{M}_x := \hat{m}_x + \mathcal{H}^{n-1}\llcorner(J_u\triangle\hat{L}_x)$ and μ_u by $\hat{\mu}_u := (g_0(|[u]|)+1)\mathcal{H}^{n-1}\llcorner J_u$. By (3.46) in Lemma 3.7 we obtain that (3.63) holds with $\hat{M}_x$ in place of $\hat{m}_x$, so that we can define $E^{k,\theta}$ with $\hat{M}_x$ and $\hat{\mu}_u$. Similarly, we consider in place of m_z defined in (3.66) the measure $M_z := m_z + \mathcal{H}^{n-1}\llcorner(J_u\triangle S_z)$. The rest of the proof is unchanged, replacing m_z by M_z and μ_u by $\hat{\mu}_u$. $\square$

4. Proof of the approximation theorem

4.1. Explicit construction on a single simplex

We show how to construct the piecewise affine approximation in a single simplex, assuming that the values at the vertices and the jumps on the sides are given. On each edge we shall use a function of the form illustrated on the right-hand side of Fig. 1. For

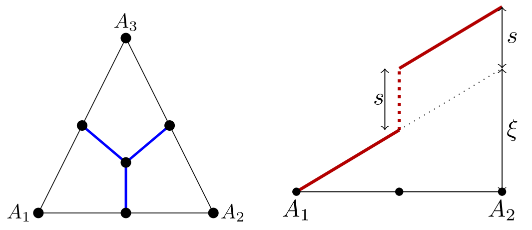

Fig. 1. Sketch of the construction in Proposition 4.1 in the 2d case. Left: decomposition of the triangle. The blue lines represent the jump set of v. Right: profile along a single edge. The parameter s denotes the jump in the middle, the parameter ξ the rest of the height change, which corresponds to the uniform slope in the rest. (For interpretation of the colors in the figure, the reader is referred to the web version of this article.)

simplicity we deal here only with scalar functions, the construction will then be applied componentwise.

We consider points $A_1, \dots, A_{n+1} \in \mathbb{R}^n$ such that their convex envelope, the simplex $T := \mathrm{conv}(\{A_1, \dots, A_{n+1}\})$, has positive measure. The basic construction is outlined in general for values $u_1, \dots, u_{n+1} \in \mathbb{R}$ of the function on the vertices, and jumps $s_{ij} \in \mathbb{R}$ on the (oriented) edges, with $s_{ij} = -s_{ji}$ (which obviously implies $s_{ii} = 0$). We then define the average gradients on the edges $\xi_{ij} := u_j - u_i - s_{ij}$ (and therefore $\xi_{ii} = 0$). The definition of ξ implies that whenever $\{i, j, k\} \subseteq \{1, \dots, n\}$ then

$$\xi_{ij} + \xi_{jk} + \xi_{ki} + s_{ij} + s_{jk} + s_{ki} = 0. \tag{4.1}$$

The compatibility conditions arising from longer paths are not independent, as each path can be written as a concatenation of triangles. On the edge joining A_i with A_j, we require our target function to take the form (see Fig. 1)

$$\begin{aligned} v(A_i + t(A_j - A_i)) =& u_i + t\xi_{ij} + s_{ij}\chi_{t>1/2} \\ =& u_i + t(u_j - u_i) + s_{ij}(\chi_{t>1/2} - t). \end{aligned} \tag{4.2}$$

Proposition 4.1. *Let $A_1, \dots, A_{n+1} \in \mathbb{R}^n$ be such that $T := \mathrm{conv}(\{A_1, \dots, A_{n+1}\})$ has positive measure. There is a decomposition of T into $n+1$ closed polyhedra $T_1, \dots, T_{n+1}$ with disjoint interior such that the following holds. Let $u_1, \dots, u_{n+1} \in \mathbb{R}$, fix $s \in \mathbb{R}_* := \mathbb{R}^{(n+1)^2}_{\mathrm{skw}}$, and define $\xi \in \mathbb{R}_*$ by $\xi_{ij} + s_{ij} = u_j - u_i$. Then there is $v : T \to \mathbb{R}$ affine in each $T_j \setminus \bigcup_{i \neq j} \partial T_i$ such that*

$$|\nabla v| \le C \frac{\mathrm{diam}(T)^{n-1}}{|T|} |\xi| , \qquad |[v]| \le 3|s|, \tag{4.3}$$

$$\mathcal{H}^{n-1}(J_v \cap T) \le \mathcal{H}^{n-1}(\partial T), \tag{4.4}$$

and with

$$v(A_i + t(A_j - A_i)) = u_i + t\xi_{ij} + s_{ij}\chi_{t>1/2} \tag{4.5}$$

for all $i < j \in \{1, \dots, n+1\}$, $t \in [0,1]$. The constant C depends only on n. The function v depends linearly on $\{u_i\} \cup \{s_{ij}\}$.

The function v on a face of T does not depend on the opposing vertex. Precisely, for any k, if $x \in \mathrm{conv}(\{A_1, \dots, A_{n+1}\} \setminus \{A_k\})$ then $v(x)$ depends only on A_i, u_i for $i \neq k$ and on s_{ij} for $i, j \neq k$.

Proof. We first observe that each point $x \in T$ can be uniquely represented as $x = \sum_{i=1}^{n+1} \lambda_i A_i$ for some $\lambda \in \Lambda := \{\lambda \in [0,1]^{n+1} : \sum_{i=1}^{n+1} \lambda_i = 1\}$. We define the polyhedra T_j by

$$T_j := \{\sum_i \lambda_i A_i : \lambda \in \Lambda, \lambda_j \geq \lambda_i \text{ for } i \neq j\}, \tag{4.6}$$

see Fig. 1(left) (in the case that T is regular, this amounts to the Voronoi decomposition of T). We remark that the condition $\lambda_i \leq \lambda_j = 1 - \sum_{k \neq j} \lambda_k$ for all $i \neq j$ is equivalent to

$$2\lambda_i \leq 1 - \sum_{k \neq i,j} \lambda_k \quad \text{ for all } i \neq j\,, \tag{4.7}$$

so that

$$x \in T_j \iff x = A_j + \sum_{i \neq j} \lambda_i (A_i - A_j) \quad \lambda \in \Lambda \text{ as in (4.7)}\,.$$

We define $v_j : T_j \to \mathbb{R}$ by

$$v_j(x) := \hat{v}_j(\mathbb{A}_j^{-1}(x - A_j))\,,$$

where $\mathbb{A}_j$ is the matrix with columns given by $A_i - A_j$ for $i \neq j$, and $\hat{v}_j : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$\hat{v}_j(\lambda_1, \dots, \lambda_{j-1}, \lambda_{j+1}, \dots, \lambda_{n+1}) := u_j + \sum_{i \neq j} \lambda_i \xi_{ji} \tag{4.8}$$

so that $v_j(\sum_i \lambda_i A_i) = u_j + \sum_{i \neq j} \lambda_i \xi_{ji}$. We define v by setting

$$v := v_j \text{ in } T_j \setminus \bigcup_{i<j} T_i. \tag{4.9}$$

Obviously $A_j \in T_j$ and $v(A_j) = u_j$. Further, for any j the function v is affine in $T_j \setminus \bigcup_{i<j} \partial T_i$, with

$$\nabla v(x) = (\mathbb{A}_j^{-1})^T \nabla \hat{v}_j (\mathbb{A}_j^{-1}(x - A_j)) = (\mathbb{A}_j^{-1})^T (\xi_{ji})_{i \neq j}$$

for all x inside T_j, and therefore

$$|\nabla v| \le \|\mathbb{A}_j^{-1}\|_{op} |(\xi_{ji})_{i\neq j}| \quad \text{on } T_j$$

from which we infer that

$$|\det \mathbb{A}_j||\nabla v| \le \|\mathrm{cof}\,\mathbb{A}_j\|_{op}|\xi| \quad \text{on } T_j\,.$$

By definition of $\mathbb{A}_j$, it holds $|\det \mathbb{A}_j| = n!|T|$. As a cofactor is a homogeneous polynomial of degree $n-1$, one obtains $\|\mathrm{cof}\,\mathbb{A}_j\|_{op} \le C\mathrm{diam}(T)^{n-1}$, for some dimensional constant $C < \infty$. This proves the first bound in (4.3).

To conclude that $v \in SBV(T)$, with the claimed estimates, we note that since by construction v is affine on each T_j, it jumps only on the points $x = \sum_i \lambda_i A_i \in \partial T_j \cap \partial T_k$ for some $j \neq k$. Necessarily $\lambda_j = \lambda_k$, and the conditions $\xi_{ij} + s_{ij} = u_j - u_i$, $\sum_i \lambda_i = 1$ with $\lambda_i \in [0,1]$, the antisymmetry of ξ, and the compatibility condition in (4.1) imply that

$$\begin{aligned}
(v_j - v_k)(\sum_i \lambda_i A_i) &= u_j - u_k + \lambda_k \xi_{jk} - \lambda_j \xi_{kj} + \sum_{i\notin\{j,k\}} \lambda_i(\xi_{ji} - \xi_{ki}) \\
&= \xi_{kj} + s_{kj} - (\lambda_k + \lambda_j)\xi_{kj} + \sum_{i\notin\{j,k\}} \lambda_i(\xi_{ji} + \xi_{ik}) \\
&= s_{kj} + \sum_{i\notin\{j,k\}} \lambda_i(\xi_{kj} + \xi_{ji} + \xi_{ik}) \\
&= s_{kj} - \sum_{i\notin\{j,k\}} \lambda_i(s_{kj} + s_{ji} + s_{ik}) \\
&= (\lambda_j + \lambda_k)s_{kj} - \sum_{i\notin\{j,k\}} \lambda_i(s_{ji} + s_{ik}).
\end{aligned} \tag{4.10}$$

Therefore $v \in SBV(T)$ with $|[v]| \le 3|s|$ and

$$\mathcal{H}^{n-1}(J_v) \le \sum_{i\neq j} \mathcal{H}^{n-1}(\partial T_i \cap \partial T_j) \le \mathcal{H}^{n-1}(\partial T)\,. \tag{4.11}$$

The last inequality is proven in (4.19) below. This concludes the proof of (4.3) and (4.4). Condition (4.5) follows directly from the definition above.

By construction, it is clear that v does not depend on the vertex A_k on the opposing face F_k, since on F_k we have $\lambda_k = 0$ and $F_k \cap T_k = \emptyset$.

It remains to prove the geometric inequality that was used in the last step of (4.11). By Fubini's theorem one easily checks the following: Consider a set α of $k+1$ points in $\mathbb{R}^n$, $0 \le k < n$. Then for any $x \in \mathbb{R}^n$ one has

$$\mathcal{H}^{k+1}(\mathrm{conv}(\alpha \cup \{x\})) = \frac{1}{k+1}\mathcal{H}^k(\mathrm{conv}(\alpha)) \cdot \mathrm{dist}(x, \text{aff-span}(\alpha)), \tag{4.12}$$

where aff-span(α) is the smallest affine space that contains α (if $k = 0$ then $\mathrm{conv}(\alpha) = \text{aff-span}(\alpha) = \alpha$ and $\mathcal{H}^0(\mathrm{conv}(\alpha)) = 1$).

Fix now $i \neq j \in \{1, \dots, n+1\}$, and consider $\partial T_i \cap \partial T_j$. Then

$$
\begin{aligned}
\partial T_i \cap \partial T_j =& \{\sum_{p=1}^{n+1} \lambda_p A_p : \lambda \in \Lambda, \lambda_i = \lambda_j = \max_p \lambda_p\} \\
=& \{2\lambda_i \frac{A_i + A_j}{2} + \sum_{p \neq \{i,j\}} \lambda_p A_p : \lambda \in \Lambda, \lambda_i = \lambda_j = \max_p \lambda_p\} \\
\subseteq& \{\sum_{p=1}^{n} \lambda_p^* A_p^* : \lambda^* \in \Lambda^*, \lambda_1^* = \max_p \lambda_p^*\},
\end{aligned}
\tag{4.13}
$$

where $\Lambda^* := \{\lambda^* \in [0,1]^n : \sum_{p=1}^n \lambda_p^* = 1\}$, $A_1^* := \frac{A_i+A_j}{2}$ and $\{A_p^*\}_{p=2,\dots,n}$ is any relabeling of the $n-1$ points in $\alpha_{ij} := \{A_1, \dots, A_{n+1}\} \setminus \{A_i, A_j\}$ (the inclusion in the last step follows from the fact that $\lambda_1^* = 2\lambda_i \geq 2\lambda_p^*$ for all $p > 1$). By symmetry, all n sets $\{\lambda^* \in \Lambda^* : \lambda_i^* = \max_p \lambda_p^*\}$ have the same area, and as they are disjoint up to $\mathcal{H}^{n-1}$-dimensional null sets we obtain

$$
\mathcal{H}^{n-1}(\partial T_i \cap \partial T_j) \leq \frac{1}{n} \mathcal{H}^{n-1}\big(\mathrm{conv}\{A_p^*\}\big) = \frac{1}{n} \mathcal{H}^{n-1}\Big(\mathrm{conv}\big(\alpha_{ij} \cup \{\frac{A_i + A_j}{2}\}\big)\Big) \tag{4.14}
$$

so that (4.12) gives

$$
\mathcal{H}^{n-1}(\partial T_i \cap \partial T_j) \leq \frac{1}{n(n-1)} \mathcal{H}^{n-2}(\mathrm{conv}(\alpha_{ij})) \cdot \mathrm{dist}\Big(\frac{A_i + A_j}{2}, \text{aff-span}(\alpha_{ij})\Big). \tag{4.15}
$$

By convexity

$$
\begin{aligned}
&\mathrm{dist}\Big(\frac{A_i + A_j}{2}, \text{aff-span}(\alpha_{ij})\Big) \\
&\quad \leq \frac{1}{2} \mathrm{dist}(A_i, \text{aff-span}(\alpha_{ij})) + \frac{1}{2} \mathrm{dist}(A_j, \text{aff-span}(\alpha_{ij}))\,.
\end{aligned}
\tag{4.16}
$$

Let $F_i := \mathrm{conv}(\{A_1, \dots, A_{n+1}\} \setminus \{A_i\}) = \mathrm{conv}(\alpha_{ij} \cup \{A_j\})$ be the face opposite to the vertex A_i. By (4.12),

$$
\mathcal{H}^{n-1}(F_i) = \frac{1}{n-1} \mathcal{H}^{n-2}(\mathrm{conv}(\alpha_{ij})) \cdot \mathrm{dist}(A_j, \text{aff-span}(\alpha_{ij})). \tag{4.17}
$$

Combining (4.15), (4.16) and (4.17) gives

$$
\mathcal{H}^{n-1}(\partial T_i \cap \partial T_j) \leq \frac{1}{2n} \mathcal{H}^{n-1}(F_i) + \frac{1}{2n} \mathcal{H}^{n-1}(F_j). \tag{4.18}
$$

We sum over all pairs (i, j) with $i \neq j$ and obtain

$$
\sum_{i \neq j} \mathcal{H}^{n-1}(\partial T_i \cap \partial T_j) \leq \sum_{i=1}^{n+1} \sum_{j \neq i} \frac{1}{n} \mathcal{H}^{n-1}(F_i) = \sum_{i=1}^{n+1} \mathcal{H}^{n-1}(F_i) = \mathcal{H}^{n-1}(\partial T) \tag{4.19}
$$

which concludes the proof. □

4.2. Projection on piecewise affine functions

In this section, we use Proposition 4.1 to construct a good piecewise affine interpolation of any vectorial function $u \in SBV_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^m)$ over a suitable partition of $\mathbb{R}^n$ in simplexes. First, Lemma 4.2 states the general properties of the chosen partition. Proposition 4.1 then can be applied componentwise in each simplex of a suitable shift of the partition. The resulting interpolation can be interpreted as a projection of u over piecewise affine functions and enjoys good energy estimates, see Proposition 4.3.

Lemma 4.2. *For any $n \geq 1$ there is a countable set of simplexes $\mathcal{T}_0 \subseteq \mathcal{P}(\mathbb{R}^n)$ such that, denoting by $\mathrm{Vert}(\tau_0)$ the set of vertices of $\tau_0 \in \mathcal{T}_0$, one has:*

(i) $\#\mathrm{Vert}(\tau_0) = n+1$; $|\tau_0| > 0$ *for all* $\tau_0 \in \mathcal{T}_0$;
(ii) $\tau_0 \cap \tau_0' = \mathrm{conv}(\mathrm{Vert}(\tau_0) \cap \mathrm{Vert}(\tau_0'))$, *in particular* $|\tau_0 \cap \tau_0'| = 0$ *if* $\tau_0 \neq \tau_0'$;
(iii) $\bigcup_{\tau_0 \in \mathcal{T}_0} \tau_0 = \mathbb{R}^n$;
(iv) *For any* $\tau_0 \in \mathcal{T}_0$ *there is* $z \in \mathbb{Z}^n$ *such that* $\mathrm{Vert}(\tau_0) \subseteq \{z + \sum_i \lambda_i e_i : \lambda_i \in \{0,1\}, i \in \{1,\dots,n\}\}$, *with the* e_i*'s the canonical basis vectors;*
(v) *If* $\tau_0 \in \mathcal{T}_0$, *then* $\tau_0 + 2e_i \in \mathcal{T}_0$, *with* e_i *any of the canonical basis vectors.*

We recall that $\mathrm{conv}(\emptyset) = \emptyset$. Condition (i) and condition (ii) with $\tau_0 = \tau_0'$ imply that τ_0 is a closed simplex. Condition (iv) implies that for all $\varepsilon > 0$, the rescaled simplex $\varepsilon\tau_0$ has diameter at most $\varepsilon\sqrt{n}$, and together with condition (i) that its volume is at least $\varepsilon^n/n!$ (indeed, it is $1/n!$ times the determinant of a matrix with entries in $\{-\varepsilon, 0, \varepsilon\}$). The last two imply that this is a refinement of the natural subdivision of $\mathbb{R}^n$ into unitary cubes, with period $[0,2]^n$.

Proof. This can be obtained taking any partition, as for example the Freudenthal partition, of $[0,1]^n$, reflecting this along the coordinate axes to obtain a partition of $[-1,1]^n$, and then extending periodically. □

In the rest of this section we define for any $\varepsilon > 0$ and $\zeta \in B_\varepsilon$ a projection $\Pi_{\varepsilon,\zeta}$ on the space of functions that are affine on each polyhedron in a refinement of $\zeta + \varepsilon\mathcal{T}_0$. The projection will be used on functions $u \in SBV(\mathbb{R}^n;\mathbb{R}^m)$. As it depends on point values, we shall only obtain a well-defined result for values of the translation ζ outside a null set. The null set, however, depends on u. To avoid this, the precise definition is given not on equivalence classes but on functions from $\mathbb{R}^n$ to $\mathbb{R}^m$. In order to understand the key properties, it is useful to consider first the action of $\Pi_{\varepsilon,\zeta}$ on elements of $SBV(\mathbb{R}^n;\mathbb{R}^m)$ (or, equivalently, $SBV_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^m)$, since $\Pi_{\varepsilon,\zeta}$ is local).

Let $u \in SBV(\mathbb{R}^n;\mathbb{R}^m)$. For any couple of vertices $a \neq b \in \mathrm{Vert}(\tau)$ of a simplex $\tau \in \zeta + \varepsilon\mathcal{T}_0$, we consider the slice $u_a^{b-a}(t) := u(a+t(b-a))$. For $\mathcal{L}^{2n}$-almost every pair (a,b) we have $u_a^{b-a} \in SBV(\mathbb{R};\mathbb{R}^m)$ with

$$u(b) - u(a) = \int_0^1 (u_a^{b-a})'(t)dt + \sum_{t\in(0,1)\cap J_{u_a^{b-a}}} [u_a^{b-a}](t) \tag{4.20}$$

with

$$(u_a^{b-a})'(t) = \nabla u(a+t(b-a))(b-a)$$

and

$$[u_a^{b-a}](t) = [u](a+t(b-a))\mathrm{sgn}(\nu_{a+t(b-a)} \cdot (b-a))$$

(see [5, Sect. 3.11] or [11, Th. 4.1(a)]). The parameters s and ξ entering the piecewise affine construction in Proposition 4.1 are then defined from the two components of (4.20). Precisely, we define the jump over an edge $[a,b]$ by

$$s_{[a,b]} := \sum_{t\in(0,1)\cap J_{u_a^{b-a}}} [u](a+t(b-a))\mathrm{sgn}(\nu_{a+t(b-a)} \cdot (b-a)) \tag{4.21}$$

(setting it to zero if the sum does not converge or is not defined) and correspondingly the integral of the absolutely continuous part of the gradient by

$$\xi_{[a,b]} := u(b) - u(a) - s_{[a,b]}. \tag{4.22}$$

For $\mathcal{L}^{2n}$-almost all pairs (a,b)

$$\xi_{[a,b]} = \int_0^1 \nabla u(a+t(b-a))(b-a)dt. \tag{4.23}$$

By monotonicity and subadditivity of g_0,

$$g_0(|s_{[a,b]}|) \leq \sum_{t\in(0,1)\cap J_{u_a^{b-a}}} g_0(|[u](a+t(b-a))|)\,. \tag{4.24}$$

Similarly, for $\mathcal{L}^{2n}$-almost all pairs (a,b), using (4.23) and Jensen's inequality,

$$\left|\frac{\xi_{[a,b]}}{|b-a|}\right|^p \leq \int_0^1 |\nabla u(a+t(b-a))|^p dt. \tag{4.25}$$

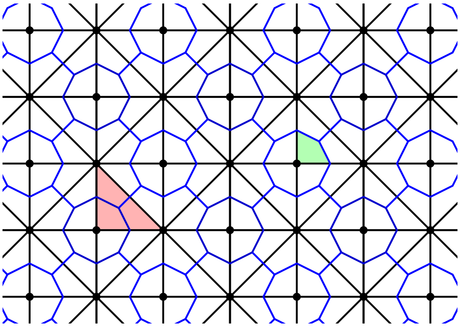

Fig. 2. Sketch of the construction for Proposition 4.3. The dots mark the points on which $\Pi_{\varepsilon,\zeta} f$ coincides with f, the black triangles (one of them is colored red) are the elements of $\mathcal{T}_{\varepsilon,\zeta}$, on which Proposition 4.1 is applied. The blue segments are the (eventual) discontinuities introduced by the construction of Proposition 4.1 and delimit the polygons which compose $\mathcal{T}^*_{\varepsilon,\zeta}$. The function $\Pi_{\varepsilon,\zeta} u$ is affine on the smaller polyhedra (one of them is colored green). (For interpretation of the colors in the figure, the reader is referred to the web version of this article.)

In Proposition 4.3 we will turn both estimates (4.24) and (4.25) into estimates relating the energy over shifts of the segment, averaged over all possible shifts of size less than ε, and integrals over J_u and Ω, respectively.

Proposition 4.3. *There is a locally finite subdivision of $\mathbb{R}^n$ into countably many essentially disjoint polyhedra, $\mathcal{T}_0^*$, finer than $\mathcal{T}_0$ and with the same periodicity, and $C > 0$ such that, for any $\varepsilon > 0$ and $\zeta \in B_\varepsilon$, to any function $f : \mathbb{R}^n \to \mathbb{R}^m$ one can associate a function $\Pi_{\varepsilon,\zeta} f : \mathbb{R}^n \to \mathbb{R}^m$, affine in the interior of each element of $\mathcal{T}^*_{\varepsilon,\zeta} := \zeta + \varepsilon\mathcal{T}_0^*$, so that the following holds:*

(i) *If either $\tau_0 \in \mathcal{T}_0$ or $\tau_0 \in \mathcal{T}_0^*$ then* $\operatorname{diam}(\tau_0) \le \sqrt{n}$ *and* $|\tau_0| \ge 1/C$*;*
(ii) *$\Pi_{\varepsilon,\zeta}$ is a projection, in the sense that $\Pi_{\varepsilon,\zeta}\Pi_{\varepsilon,\zeta} f = \Pi_{\varepsilon,\zeta} f$ for all f, and it commutes with translations, in the sense that $\Pi_{\varepsilon,\zeta}[f(\cdot - \zeta)] = [\Pi_{\varepsilon,0} f](\cdot - \zeta)$;*
(iii) *One has $\Pi_{\varepsilon,\zeta} f \in SBV_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^m)$ and, with $\mathcal{T}_{\varepsilon,\zeta} := \zeta + \varepsilon\mathcal{T}_0$,*

$$|D\Pi_{\varepsilon,\zeta} f|\Big(\bigcup_{\tau\in\mathcal{T}_{\varepsilon,\zeta}} \partial\tau\Big) = 0. \tag{4.26}$$

If $u \in SBV_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^m)$, then for $\mathcal{L}^n$-almost every $\zeta \in B_\varepsilon$ one has

$$\mathcal{H}^{n-1}\Big(J_u \cap \bigcup_{\tau\in\mathcal{T}_{\varepsilon,\zeta}} \partial\tau\Big) = 0.$$

(iv) *The function $\Pi_{\varepsilon,\zeta} f$ on a set ω depends only on the value of f on the set $(\omega)_{\varepsilon\sqrt{n}}$.*
(v) *If $A : \mathbb{R}^n \to \mathbb{R}^m$ is affine and $\lambda \in \mathbb{R}$, then for any function f one has $\Pi_{\varepsilon,\zeta}(\lambda f + A) = \lambda(\Pi_{\varepsilon,\zeta} f) + A$; if $u, v \in SBV_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^m)$ then for almost every $\zeta \in B_\varepsilon$ one has $\Pi_{\varepsilon,\zeta}(u + v) = \Pi_{\varepsilon,\zeta} u + \Pi_{\varepsilon,\zeta} v$;*

(vi) *For any $\eta \in \mathbb{R}^{m\times n}$ and $\tau_0 \in \mathcal{T}_0$, one has for any $u \in SBV_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^m)$ that*

$$\fint_{B_\varepsilon}\Big(\int_{\zeta+\varepsilon\tau_0} |\nabla\Pi_{\varepsilon,\zeta}u-\eta|^p dx\Big)d\zeta \le C\int_{(\varepsilon\tau_0)_{c_*\varepsilon}} |\nabla u-\eta|^p dx, \tag{4.27}$$

$$\fint_{B_\varepsilon}\Big(\int_{J_{\Pi_{\varepsilon,\zeta}u}\cap(\zeta+\varepsilon\tau_0)} g_0(|[\Pi_{\varepsilon,\zeta}u]|)d\mathcal{H}^{n-1}\Big)d\zeta \le C\int_{J_u\cap(\varepsilon\tau_0)_{c_*\varepsilon}} g_0(|[u]|)d\mathcal{H}^{n-1}, \tag{4.28}$$

$$\fint_{B_\varepsilon}\mathcal{H}^{n-1}(J_{\Pi_{\varepsilon,\zeta}u}\cap(\zeta+\varepsilon\tau_0))d\zeta \le C\mathcal{H}^{n-1}(J_u\cap(\varepsilon\tau_0)_{c_*\varepsilon}), \tag{4.29}$$

$$\fint_{B_\varepsilon}\Big(\int_{\zeta+\varepsilon\tau_0} |\Pi_{\varepsilon,\zeta}u-u|dx\Big)d\zeta \le C\varepsilon|Du|((\varepsilon\tau_0)_{c_*\varepsilon}), \tag{4.30}$$

and, for every $n-1$-rectifiable set Σ,

$$\fint_{B_\varepsilon}\Big(\int_{\Sigma} |\Pi_{\varepsilon,\zeta}u|d\mathcal{H}^{n-1}\Big)d\zeta \le \frac{Ck_\Sigma}{\varepsilon}\|u\|_{L^1((\Sigma)_{c_*\varepsilon})} + Ck_\Sigma|Du|((\Sigma)_{2c_*\varepsilon}), \tag{4.31}$$

where

$$k_\Sigma := \sup_{r>0,x\in\mathbb{R}^n}\frac{\mathcal{H}^{n-1}(\Sigma\cap B_r(x))}{r^{n-1}}. \tag{4.32}$$

Here $c_* \in [1+\sqrt{n},\infty)$ is a constant that depends only on n; C may depend on n, m, p.

(vii) *If $u \in SBV_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^m)$ and $\nabla u = 0$ $\mathcal{L}^n$-almost everywhere then for almost every $\zeta \in B_\varepsilon$ one has $\nabla\Pi_{\varepsilon,\zeta}u = 0$ $\mathcal{L}^n$-almost everywhere. In particular, if $u = \chi_E$ for some set E then there is a countable union of polygons $F_{\varepsilon,\zeta}$ such that $\Pi_{\varepsilon,\zeta}u = \chi_{F_{\varepsilon,\zeta}}$. If $\mathcal{H}^{n-1}(J_u) = 0$ then for almost every $\zeta \in B_\varepsilon$ one has $\mathcal{H}^{n-1}(J_{\Pi_{\varepsilon,\zeta}u}) = 0$.*

Condition (4.27) easily implies that for any Borel set $\omega \subset \mathbb{R}^n$ and any η

$$\fint_{B_\varepsilon}\Big(\int_{\omega} |\nabla\Pi_{\varepsilon,\zeta}u-\eta|^p dx\Big)d\zeta \le C\int_{(\omega)_{2c_*\varepsilon}} |\nabla u-\eta|^p dx. \tag{4.33}$$

Indeed, it suffices to sum (4.27) over all $\tau_0 \in \mathcal{T}_0$ such that there is $\zeta \in B_\varepsilon$ with $(\zeta + \varepsilon\tau_0)\cap\omega \ne \emptyset$, which implies $\varepsilon\tau_0 \subseteq (\omega)_{(1+\sqrt{n})\varepsilon}$. Analogous observations hold for (4.28), (4.29) and (4.30).

We remark that (4.31) fails if we remove the derivative term in the right-hand side. Consider for example the sequence $u_j(x) := \frac{1}{j}\langle jx_1\rangle$, where $\langle x\rangle := x - \lfloor x\rfloor$ denotes the fractional part of $x \in \mathbb{R}$, which converges uniformly to 0 as $j \to \infty$. As $\nabla u_j e_1 = 1$ almost everywhere, for any ε and ζ we have $\partial_1\Pi_{\varepsilon,\zeta}u_j = 1$ almost everywhere, and since $\Pi_{\varepsilon,\zeta}u_j$

is piecewise affine on a scale ε we obtain $\|\Pi_{\varepsilon,\zeta}u_j\|_{L^\infty} \geq \frac{1}{2}\varepsilon$, which does not depend on j. Similarly, one cannot estimate $\Pi_{\varepsilon,\zeta}u$ in L^1 only in terms of the L^1 norm of u.

Proof. The grid $\mathcal{T}_0^*$ is defined decomposing each simplex $\tau_0 \in \mathcal{T}_0$ as in Proposition 4.1. The projection $\Pi_{\varepsilon,\zeta}f$ is defined by application of the construction in Proposition 4.1 componentwise in each simplex $\tau \in \mathcal{T}_{\varepsilon,\zeta} = \zeta + \varepsilon\mathcal{T}_0$.

Precisely, let $\tau = \zeta + \varepsilon\tau_0$, for some $\tau_0 \in \mathcal{T}_0$, and let $\{w_1, \dots, w_{n+1}\} := \mathrm{Vert}(\tau)$ be its vertices. In order to define the cumulated jump over the edge $[w_i, w_j]$ we consider the slice $v_{ij}^f(t) := f(w_i + t(w_j - w_i))$, for $t \in [0,1]$. If $v_{ij}^f \in SBV((0,1);\mathbb{R}^m)$ then we set

$$s_{ij}^f := \sum_{t\in(0,1)\cap J_{v_{ij}^f}} [v_{ij}^f](t), \tag{4.34}$$

otherwise we set $s_{ij}^f := 0$. The function $\Pi_{\varepsilon,\zeta}f$ is then defined in τ using Proposition 4.1 componentwise. As discussed above, if $f = u \in SBV_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^m)$ then for almost every choice of ζ one has that $v_{ij}^u \in SBV((0,1);\mathbb{R}^m)$ for all choices of τ_0 and of i, j.

(i): The upper bound on the diameter follows from Lemma 4.2 and the fact that $\mathcal{T}_0^*$ is a refinement of $\mathcal{T}_0$. The lower bound on the volume follows from the fact that both grids are locally finite and periodic.

(ii): Assume for simplicity that f is scalar. For any τ and $w_1, \dots, w_{n+1}$ as above, one easily obtains that $(\Pi_{\varepsilon,\zeta}f)(w_i) = f(w_i)$. Let s_{ij}^f be defined as in (4.34). By (4.5) the function $v_{ij}^{\Pi_{\varepsilon,\zeta}f}$ has a unique jump point in $(0,1)$, which is located at $1/2$, and the amplitude of the jump is exactly s_{ij}^f. Therefore $s_{ij}^{\Pi_{\varepsilon,\zeta}f} = s_{ij}^f$ and $\Pi_{\varepsilon,\zeta}$ is a projection. The relation to translations follows observing that for any $\tau_0 \in \mathcal{T}_0$ the vertices of $\zeta + \varepsilon\tau_0$ are translated with respect to the vertices of $\varepsilon\tau_0$ by ζ.

(iii): Let $\tau \neq \tau' \in \mathcal{T}_{\varepsilon,\zeta}$ be such that $\mathcal{H}^{n-1}(\partial\tau \cap \partial\tau') > 0$, so that by Lemma 4.2(ii) $\partial\tau \cap \partial\tau' = \mathrm{conv}(\mathrm{Vert}(\tau) \cap \mathrm{Vert}(\tau'))$. Proposition 4.1 implies that $\Pi_{\varepsilon,\zeta}f|_{\partial\tau\cap\partial\tau'}$ only depends on the in-plane vertices $\mathrm{Vert}(\tau) \cap \mathrm{Vert}(\tau')$, on the values of f on such vertices, and on the jumps s_σ on the in-plane edges $\sigma \subset \partial\tau\cap\partial\tau'$. Hence $|D\Pi_{\varepsilon,\zeta}f|(\bigcup_{\tau\in\mathcal{T}_{\varepsilon,\zeta}} \partial\tau) = 0$. The second condition follows from the fact that $\mathcal{H}^{n-1} \llcorner J_u$ is σ-finite.

(iv): Given a set $\omega \subset \mathbb{R}^n$, the function $\Pi_{\varepsilon,\zeta}f|_\omega$ only depends on the values of f on the vertices of the simplexes intersecting ω. Since their diameter is by construction not greater than $\varepsilon\sqrt{n}$, $\Pi_{\varepsilon,\zeta}f|_\omega$ only depends on the value of f on the neighborhood $(\omega)_{\varepsilon\sqrt{n}}$.

(v): It follows from the fact that the function constructed in Proposition 4.1 depends linearly on the prescribed values on the vertices u_i and jumps s_{ij}.

(vi): By (v), it suffices to prove the first bound in the case $\eta = 0$. Let $\tau_0 \in \mathcal{T}_0$, and $\{w_1, \dots, w_{n+1}\} = \mathrm{Vert}(\varepsilon\tau_0)$. For any $\zeta \in B_\varepsilon$, by the uniform estimate in (4.3) we have

$$\int_{\zeta+\varepsilon\tau_0} |\nabla\Pi_{\varepsilon,\zeta}u|^p dx \leq C\varepsilon^n \sum_{i,j} \left|\frac{\xi_{\zeta+[w_i,w_j]}}{|w_i - w_j|}\right|^p. \tag{4.35}$$

Next, we claim that for all i, j

$$\int_{B_\varepsilon} \left| \frac{\xi_{\zeta+[w_i,w_j]}}{|w_i - w_j|} \right|^p d\zeta \le \int_{B_{(1+\sqrt{n})\varepsilon}(w_i)} |\nabla u|^p dx. \tag{4.36}$$

Indeed, starting from (4.25) and integrating over all translations $\zeta \in B_\varepsilon$ we get, setting $\ell := w_j - w_i$,

$$\begin{aligned}\int_{B_\varepsilon} \left| \frac{\xi_{\zeta+[w_i,w_j]}}{|\ell|} \right|^p d\zeta &\le \int_{B_\varepsilon} \int_0^1 |\nabla u(\zeta + w_i + t\ell)|^p dt d\zeta \\ &= \int_0^1 \int_{B_\varepsilon(w_i+t\ell)} |\nabla u(x)|^p dx dt \le \int_{B_{(1+\sqrt{n})\varepsilon}(w_i)} |\nabla u|^p dx\end{aligned}$$

since $|\ell| = |w_i - w_j| \le \varepsilon\sqrt{n}$. Therefore, from (4.35) and (4.36) we conclude

$$\begin{aligned}&\fint_{B_\varepsilon} \Big(\int_{\zeta+\varepsilon\tau_0} |\nabla \Pi_{\varepsilon,\zeta} u|^p dx \Big) d\zeta \\ &\le C \sum_i \int_{B_{(1+\sqrt{n})\varepsilon}(w_i)} |\nabla u|^p dx \le C \int_{(\varepsilon\tau_0)_{(1+\sqrt{n})\varepsilon}} |\nabla u|^p dx,\end{aligned}$$

which concludes the proof of (4.27).

Analogously, using (iii), (4.3), and (4.4), we get

$$\int_{J_{\Pi_{\varepsilon,\zeta} u} \cap (\zeta+\varepsilon\tau_0)} g_0(|[\Pi_{\varepsilon,\zeta} u]|) d\mathcal{H}^{n-1} \le C\varepsilon^{n-1} \sum_{i,j} g_0(|s_{\zeta+[w_i,w_j]}|), \tag{4.37}$$

by monotonicity and subadditivity of g_0, where as before the w_i are the vertices of $\varepsilon\tau_0$. We claim that

$$\int_{B_\varepsilon} g_0(|s_{\zeta+[w_i,w_j]}|) d\zeta \le C\varepsilon \int_{B_{(2+\sqrt{n})\varepsilon}(w_i) \cap J_u} g_0(|[u]|) d\mathcal{H}^{n-1}. \tag{4.38}$$

Indeed, we start from (4.24), integrate over translations, and separate the component ζ_ℓ along ℓ from the orthogonal ones, which we denote by ζ', so that $\zeta = \zeta' + \zeta_\ell \ell/|\ell|$,

$$\begin{aligned}\int_{B_\varepsilon} g_0(|s_{\zeta+[w_i,w_j]}|)d\zeta &\leq \int_{B_\varepsilon} \sum_{t\in(0,1)} g_0(|[u](\zeta+w_i+t\ell)|)d\zeta \\ &\leq \int_{-\varepsilon}^{\varepsilon}\int_{B'_\varepsilon} \sum_{t\in(0,1)} g_0(|[u](\zeta'+w_i+t\ell+\zeta_\ell\frac{\ell}{|\ell|})|)d\zeta' d\zeta_\ell \qquad (4.39)\\ &\leq 2\varepsilon \int_{B'_\varepsilon} \sum_{t\in(-\varepsilon,\varepsilon(1+\sqrt{n}))} g_0(|[u](\zeta'+w_i+t\frac{\ell}{|\ell|})|)d\zeta'.\end{aligned}$$

Using the Coarea formula, [5, Eq. (2.72)] with $M = n$, $N = k = n-1$, $g = g_0([u])\chi_{w_i+B'_\varepsilon\times(-\varepsilon,\varepsilon(1+\sqrt{n}))}$, and f the orthogonal projection onto $\ell^\perp$, we obtain

$$\int_{B_\varepsilon} g_0(|s_{\zeta+[w_i,w_j]}|)d\zeta \leq 2\varepsilon \int_{J_u\cap B_{(2+\sqrt{n})\varepsilon}(w_i)} g_0(|[u]|)\left|\nu_u\cdot\frac{\ell}{|\ell|}\right|d\mathcal{H}^{n-1}. \qquad (4.40)$$

Clearly, (4.38) easily follows from (4.40).

Hence, by (4.37) and (4.38)

$$⨍_{B_\varepsilon} \int_{J_{\Pi_{\varepsilon,\zeta}u}\cap(\zeta+\varepsilon\tau_0)} g_0(|[\Pi_{\varepsilon,\zeta}u]|)d\mathcal{H}^{n-1}d\zeta \leq C\int_{J_u\cap(\varepsilon\tau_0)_{(2+\sqrt{n})\varepsilon}} g_0(|[u]|)d\mathcal{H}^{n-1} \qquad (4.41)$$

which concludes the proof of (4.28).

The proof of (4.29) is similar. Let $g_1 : [0,\infty) \to [0,\infty)$ be defined by $g_1(0) = 0$, $g_1(s) = 1$ for $s \neq 0$. The derivation of (4.37) above uses only (iii), (4.3), (4.4), and the fact that g_0 is nondecreasing and subadditive, and g_1 has the same properties. By (4.34), $s^u_{ij} = 0$ if v^u_{ij} does not jump on $[0,1]$, thus we obtain instead of (4.37) the estimate

$$\int_{J_{\Pi_{\varepsilon,\zeta}u}\cap(\zeta+\varepsilon\tau_0)} g_1(|[\Pi_{\varepsilon,\zeta}u]|)d\mathcal{H}^{n-1} \leq C\varepsilon^{n-1}\sum_{i,j} g_1(|s_{\zeta+[w_i,w_j]}|). \qquad (4.42)$$

The rest of the computation leading to (4.41) is unchanged. This proves (4.29).

Next, we estimate the L^1 distance of u from $\Pi_{\varepsilon,\zeta}u$. By (4.3) for $\tau_0 \in \mathcal{T}_0$ one has the pointwise estimate

$$\begin{aligned}&⨍_{B_\varepsilon}\int_{\zeta+\varepsilon\tau_0} |\Pi_{\varepsilon,\zeta}u - u|dx\,d\zeta \\ &\quad\leq ⨍_{B_\varepsilon}\int_{\zeta+\varepsilon\tau_0} \sum_i |u(\zeta+w_i)-u(x)|dx\,d\zeta + C\varepsilon^n ⨍_{B_\varepsilon}\sum_{i,j}|\xi_{\zeta+[w_i,w_j]}|d\zeta.\end{aligned}$$

We observe that for all choices of i, ζ and x we have $x \in \zeta+\varepsilon\tau_0 \subseteq (\varepsilon\tau_0)_\varepsilon$ and $\zeta+w_i \in \zeta+\varepsilon\tau_0 \subseteq (\varepsilon\tau_0)_\varepsilon$. Therefore, each addend in the first term can be estimated using Poincaré's inequality for BV functions by

$$\fint_{B_\varepsilon} \int_{\zeta+\varepsilon\tau_0} |u(\zeta + w_i) - u(x)| dx\, d\zeta \le \frac{1}{|B_\varepsilon|} \int_{(\varepsilon\tau_0)_\varepsilon} \int_{(\varepsilon\tau_0)_\varepsilon} |u(y) - u(y')| dy dy'$$

$$\le 2\frac{|(\varepsilon\tau_0)_\varepsilon|}{|B_\varepsilon|} \int_{(\varepsilon\tau_0)_\varepsilon} |u(y) - \bar{u}| dy \le C\varepsilon |Du|((\varepsilon\tau_0)_\varepsilon),$$

where $\bar{u}$ denotes the average of u in $(\varepsilon\tau_0)_\varepsilon$. For the second one, we write using (4.36) with $p = 1$

$$\varepsilon^n \fint_{B_\varepsilon} \sum_{i,j} |\xi_{\zeta+[w_i,w_j]}| d\zeta \le C\varepsilon \int_{(\varepsilon\tau_0)_{C\varepsilon}} |\nabla u| dx.$$

Combining the two gives (4.30).

Finally, we prove (4.31): for any $\tau_0 \in \mathcal{T}_0$ and $\zeta \in B_\varepsilon$, we have $\mathcal{H}^{n-1}(\Sigma \cap (\zeta + \varepsilon\tau_0)) \le Ck_\Sigma \varepsilon^{n-1}$. As the map $\Pi_{\varepsilon,\zeta} u$ is affine on each element of $\zeta + \varepsilon\mathcal{T}_0^*$, we have

$$\|\Pi_{\varepsilon,\zeta} u\|_{L^\infty(\zeta+\varepsilon\tau_0)} \le \frac{C}{\varepsilon^n} \|\Pi_{\varepsilon,\zeta} u\|_{L^1(\zeta+\varepsilon\tau_0)}. \tag{4.43}$$

We sum over all τ_0 such that $\zeta + \varepsilon\tau_0$ intersects Σ and obtain

$$\int_\Sigma |\Pi_{\varepsilon,\zeta} u| d\mathcal{H}^{n-1} \le \frac{Ck_\Sigma}{\varepsilon} \|\Pi_{\varepsilon,\zeta} u\|_{L^1((\Sigma)_{\sqrt{n}\varepsilon})} \tag{4.44}$$

for a.e. $\zeta \in B_\varepsilon$. Then we use (4.30) and a triangular inequality to conclude.

(vii): If $\nabla u = 0$ $\mathcal{L}^n$-almost everywhere, then (4.27) with $\eta = 0$ implies $\nabla \Pi_{\varepsilon,\zeta} u = 0$ $\mathcal{L}^n$-almost everywhere for $\mathcal{L}^n$-almost every ζ. If u takes values in $\{0, 1\}$, then for each application of Proposition 4.1 we have that $\xi_{ij} = 0$, therefore the constructed function is piecewise constant and takes values in the set $\{u_1, \ldots, u_{n+1}\} \subseteq \{0, 1\}$.

Finally, if $\mathcal{H}^{n-1}(J_u) = 0$ then necessarily $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{R}^m)$. In turn, the slices of u are Sobolev functions for $\mathcal{L}^n$-almost every $\zeta \in B_\varepsilon$, so that $J_{\Pi_{\varepsilon,\zeta} u} = \emptyset$, in turn implying that $\Pi_{\varepsilon,\zeta} u$ is actually continuous and in $W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{R}^m)$ (alternatively, this follows also from (4.29)). $\square$

4.3. Global construction

We are now ready to establish Theorem 1.1. The proof contains two different scales, denoted by δ and ε in the following. The scale δ is the one at which the function u has approximately regular jump and gradient, and is identified in Proposition 3.6. The second scale $\varepsilon \ll \delta$, used for the construction in Proposition 4.3, is the one on which we construct a piecewise affine approximation of u. This is achieved in each cube at scale δ by applying Proposition 4.3 to the extensions, obtained via Theorem 3.5, of u itself restricted to domains separated by the regular part of its jump set. In turn,

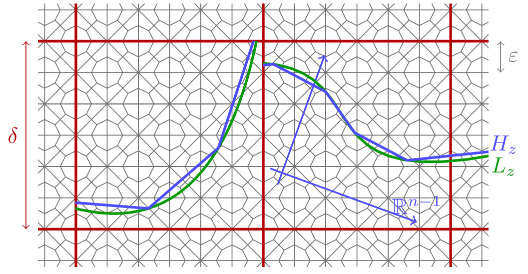

Fig. 3. Sketch of the grids used in the proof of Theorem 1.1. The grid $(\mathcal{T}', V')$ is taken in $\mathbb{R}^{n-1}$, it is then rotated. Similarly, H_z and L_z are graphs of ψ_z and φ_z respectively, in these rotated coordinates (in the picture for H_z we have set $\beta = 0$ for the sake of simplicity, see (4.58).) (For interpretation of the colors in the figure, the reader is referred to the web version of this article.)

the regularization L_z of the jump set J_u will be separately approximated using piecewise affine elements in $\mathbb{R}^{n-1}$ using again the scale ε. Fig. 3 shows a sketch of the construction, the different parts will become clear during the proof.

Proof of Theorem 1.1. Let $u \in SBV(\Omega; \mathbb{R}^m)$ and $\theta \in (0, \frac{1}{2}]$. To simplify the notation we work at fixed θ, and in the end choose a sequence $\theta_j \to 0$. It is not restrictive to assume additionally that

$$t \le g_0(t) \text{ for all } t \in [0, \infty); \tag{4.45}$$

indeed, this follows by proving first the theorem with $g_0(t) + t$ in place of g_0 and then deducing the statement for g_0 as a by-product. By (2.2), for any $\lambda > 0$ (fixed below, it will depend on θ and δ but not on ε and δ'; $\lambda = \theta\delta$ will do) there is $C_\lambda \ge 1$ such that

$$g_0(t) \le \lambda + C_\lambda t \text{ for all } t \in [0, \infty). \tag{4.46}$$

This will be used to estimate terms of the form $g_0(|[w]|)$ on sets of finite $n-1$-dimensional measure in terms of the jump.

Step 1: Choice of the scale δ on which u is regular and of the sets A_δ, A^*_δ.

By Theorem 3.5 we can assume that $u \in SBV(\mathbb{R}^n; \mathbb{R}^m)$ with $|Du|(J_u \cap \partial\Omega) = 0$,

$$\int_{\Omega'} |\nabla u|^p \, dx \le \int_{\Omega} |\nabla u|^p \, dx + \theta \,, \tag{4.47}$$

and

$$\int_{\Omega' \cap J_u} g_0(|[u]|) d\mathcal{H}^{n-1} \le \int_{\Omega \cap J_u} g_0(|[u]|) d\mathcal{H}^{n-1} + \theta \,, \tag{4.48}$$

for some bounded open set Ω' with $\overline{\Omega} \subset \Omega'$ and $|\Omega'| \le 2|\Omega|$. We choose $\delta_0 \in (0, \theta]$ such that $3\delta_0\sqrt{n} \le \operatorname{dist}(\Omega, \partial\Omega')$ and

$$\int_{(\partial\Omega)_{3\delta_0\sqrt{n}}} |\nabla u|^p dx + \mu_u((\partial\Omega)_{3\sqrt{n}\delta_0}) \le \theta, \tag{4.49}$$

where $\mu_u := g_0(|[u]|)\mathcal{H}^{n-1} \llcorner J_u$ as in (3.41). If $\mathcal{H}^{n-1}(J_u \cap \Omega) < \infty$, then Theorem 3.5 gives $\mathcal{H}^{n-1}(J_u) < \infty$ and we may also require

$$\mathcal{H}^{n-1}(J_u \cap (\partial\Omega)_{3\sqrt{n}\delta_0}) \le \theta. \tag{4.50}$$

By Proposition 3.6, used with δ_0 in place of θ, there is $\delta \in (0, \delta_0] \subseteq (0, \theta]$ such that, with $A_\delta := \{z \in (\delta\mathbb{Z}^n) \cap \Omega : \operatorname{dist}(z, \partial\Omega) > \delta\sqrt{n}\}$, there are $R : A_\delta \to \mathrm{SO}(n)$, $s : A_\delta \to \mathbb{R}^m$, $\eta : A_\delta \to \mathbb{R}^{m\times n}$, $\varphi : A_\delta \to C_c^1(\mathbb{R}^{n-1})$, $x : A_\delta \to \mathbb{R}^n$ which, setting

$$L_z := x_z + R_z\{(y', \varphi_z(y')) : y' \in \mathbb{R}^{n-1}\} \tag{4.51}$$

and $Q_z^* := z + (-\delta, \delta)^n$, satisfy $\|D\varphi_z\|_{L^\infty} \le \theta$ and

$$\begin{aligned}\sum_{z\in A_\delta} \int_{Q_z^*} |\nabla u - \eta_z|^p dx + \sum_{z\in A_\delta} \int_{Q_z^*\cap J_u \setminus L_z} g_0(|[u]|) d\mathcal{H}^{n-1} \\ + \sum_{z\in A_\delta} \int_{Q_z^*\cap L_z} [g_0(|[u] - s_z|) + g_0(|[u]|)|\nu_u - R_z e_n|] d\mathcal{H}^{n-1} \le C\theta\,.\end{aligned} \tag{4.52}$$

Here and in what follows we do not explicitly indicate the dependence of C on $\mu_u(\Omega)$ and $|\Omega|$. If $\mathcal{H}^{n-1}(J_u) < \infty$ we have in addition

$$\sum_{z\in A_\delta} \mathcal{H}^{n-1}(Q_z^* \cap (J_u \triangle L_z)) \le C\theta. \tag{4.53}$$

We further define $A_\delta^* := \{z \in (\delta\mathbb{Z})^n : \operatorname{dist}(z, \partial\Omega) \le \delta\sqrt{n}\}$. We observe that $Q_z^* \subseteq B_{\delta\sqrt{n}}(z)$, so that by (4.49)

$$\begin{aligned}\sum_{z\in A_\delta^*} \int_{Q_z^*} |\nabla u|^p dx + \sum_{z\in A_\delta^*} \int_{Q_z^*\cap J_u} g_0(|[u]|) d\mathcal{H}^{n-1} \\ \le C\Big(\int_{(\partial\Omega)_{2\delta\sqrt{n}}} |\nabla u|^p dx + \mu_u((\partial\Omega)_{2\delta\sqrt{n}})\Big) \le C\theta\,.\end{aligned} \tag{4.54}$$

For $\gamma \in B_{\delta/4}$ and $z \in A_\delta \cup A_\delta^*$ we define $Q_z^\gamma := \gamma + z + (-\delta/2, \delta/2)^n \subseteq Q_z^*$, and observe that since $2\delta\sqrt{n} \le 2\delta_0\sqrt{n} \le \operatorname{dist}(\Omega, \partial\Omega')$ we have $\Omega \subset \bigcup_{z\in A_\delta\cup A_\delta^*} \overline{Q_z^\gamma} \subset \Omega'$. Further, for $\mathcal{L}^n$-almost every choice of $\gamma \in B_{\delta/4}$ we have

$$\mathcal{H}^{n-1}\left(J_u \cap \bigcup_{z\in A_\delta\cup A_\delta^*} \partial Q_z^\gamma\right) = 0 \quad \text{and} \quad \mathcal{H}^{n-1}\left(\bigcup_{z\in A_\delta} (L_z \cap \partial Q_z^\gamma)\right) = 0. \tag{4.55}$$

This follows from the fact that $\mathcal{H}^{n-1} \llcorner J_u$ and $\mathcal{H}^{n-1} \llcorner \bigcup_{z \in A_\delta} L_z$ are σ-finite. In the rest of the proof γ is a fixed value with property (4.55) and we write Q_z in place of Q_z^γ.

Step 2: Approximation of the interface.

Let $\varepsilon \in (0, \frac{\delta}{2})$. For every $z \in A_\delta$, we let

$$L_z^+ := x_z + R_z\{(y', y_n) : y' \in \mathbb{R}^{n-1},\ y_n > \varphi_z(y')\}, \tag{4.56}$$

so that $L_z = \partial L_z^+$, and then let $L_z^- := \mathbb{R}^n \setminus L_z \setminus L_z^+$. Fix a triangulation $(\mathcal{T}', V')$ of $\mathbb{R}^{n-1}$, with $V' = \varepsilon\mathbb{Z}^{n-1}$, as in Lemma 4.2. We define $\psi_z : \mathbb{R}^{n-1} \to \mathbb{R}$ setting $\psi_z = \varphi_z$ on V', and ψ_z affine in each element of $\mathcal{T}'$.

We stress that the triangulation $(\mathcal{T}', V')$ used above to approximate L_z is not related to the triangulation $(\mathcal{T}_{\varepsilon,\zeta}, V_{\varepsilon,\zeta})$ used in Proposition 4.3 for the definition of $\Pi_{\varepsilon,\zeta}$. The usage of the same scale ε for both triangulations is only to avoid having one more small parameter. In any case, it would be crucial for both scales to be much smaller than δ.

We claim that there is a modulus of continuity ω_ε, infinitesimal as $\varepsilon \downarrow 0$, such that for all $z \in A_\delta$ we have

$$\|\varphi_z - \psi_z\|_{L^\infty(\mathbb{R}^{n-1})} \le \varepsilon\omega_\varepsilon \text{ and } \|D\varphi_z - D\psi_z\|_{L^\infty(\mathbb{R}^{n-1})} \le \omega_\varepsilon. \tag{4.57}$$

In the following we shall assume that ε is sufficiently small to ensure $\omega_\varepsilon \le \theta$. To prove (4.57), we observe that since $\varphi_z \in C_c^1(\mathbb{R}^{n-1})$ there is a modulus of continuity $\hat\omega_\varepsilon$ such that $|D\varphi_z(y) - D\varphi_z(\tilde y)| \le \hat\omega_\varepsilon$ whenever $|y - \tilde y| \le \varepsilon\sqrt{n}$. As there are finitely many choices of z, we can assume that $\hat\omega_\varepsilon$ does not depend on z. Consider now an element $\tau' \in \mathcal{T}'$. For every edge (a, b) of τ' we have $D\psi_z|_{\tau'}(b - a) = \varphi_z(b) - \varphi_z(a)$, so that

$$|(D\psi_z|_{\tau'} - D\varphi_z(b))(b - a)| \le \hat\omega_\varepsilon |b - a|$$

for all edges (a, b) of τ'. As the simplexes τ' are uniformly nondegenerate this implies $|D\psi_z - D\varphi_z| \le C\hat\omega_\varepsilon$, for some constant C depending only on n. This proves (4.57).

Using this interpolation and a shift $\beta \in (-\varepsilon, \varepsilon)$ we define the set

$$H_z^+ := x_z + R_z\{(y', y_n) : y' \in \mathbb{R}^{n-1},\ y_n > \psi_z(y') + \beta\}, \tag{4.58}$$

which is a polyhedral approximation of L_z^+, and $H_z := \partial H_z^+$, $H_z^- := \mathbb{R}^n \setminus H_z \setminus H_z^+$ (see Fig. 3). We choose β such that

$$\mathcal{H}^{n-1}\left(\bigcup_{z \in A_\delta} (H_z \cap \partial Q_z)\right) = 0. \tag{4.59}$$

Condition (4.59), which holds for almost all β, will be needed to estimate the (unilateral) $\mathcal{H}^{n-1}$-difference between the jump of the approximation and J_u, see text after (4.77) and the proof of (4.131).

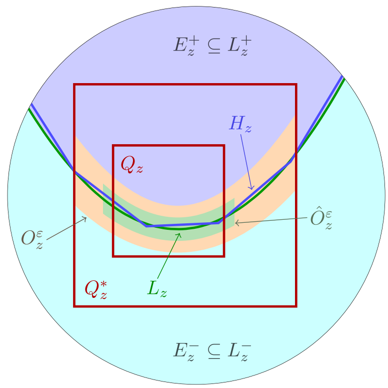

Fig. 4. Sketch of geometry in the construction of U_z^+ and U_z^- in Step 3 in the proof of Theorem 1.1, assuming $n = 2$, $R_z = \mathrm{Id}$, and that L_z is the graph of a parabola. The set E_z^+ is the area above L_z (inside the ball B_z), O_z^ε is a neighborhood of L_z intersected with the larger cube Q_z^*, and $\hat{O}_z^\varepsilon$ is a smaller neighborhood of L_z intersected with a neighborhood of Q_z. The set H_z is an approximation of L_z with the graph of a piecewise affine function, and the part inside Q_z belongs to $\hat{O}_z^\varepsilon$ (in the picture for H_z we have set $\beta = 0$ for the sake of simplicity, see (4.58)). Fig. 3 shows how this construction interacts with the rest. (For interpretation of the colors in the figure, the reader is referred to the web version of this article.)

Step 3: Construction of $w_{z,\zeta}$ and w_ζ. In this step we define an approximation $w_{z,\zeta}$ on each cube Q_z, for $z \in A_\delta$ and $\zeta \in B_\varepsilon$. This requires two different extensions of u on different sets, a sketch is given in Fig. 4.

If $L_z \cap Q_z^* = \emptyset$ we set $w_{z,\zeta} = \Pi_{\varepsilon,\zeta} u$. The other case is more complex. We pick $y_z \in L_z \cap Q_z^*$, let $B_z := B_{3\sqrt{n}\delta}(y_z)$, so that $Q_z^* \subset\subset B_z \subset \Omega'$, and consider the sets $E_z^\pm := B_z \cap L_z^\pm$. One checks that, since φ_z is θ-Lipschitz with $\theta \le \frac{1}{2}$, the sets $E_z^\pm$ are Lipschitz with a constant C. For this it is convenient to use that a bounded open set A is Lipschitz if and only if there are a nontrivial open one-sided cone E and $r > 0$ such that $B_r(x) \cap (x + E) \subset A$ and $B_r(x) \cap (x - E) \cap A = \emptyset$ for all $x \in \partial A$, see [5, Remark after Def. 2.60].

Let $M > 0$ be fixed, it will be chosen below depending only on the dimension n. By Theorem 3.5 there are $U_z^\pm \in SBV(\mathbb{R}^n; \mathbb{R}^m)$ which extend the restriction of u to $E_z^\pm$, respectively. In particular, we have $U_z^\pm = u$ on $E_z^\pm$, and $|DU_z^+|(\partial E_z^+) = |DU_z^-|(\partial E_z^-) = 0$. For ε sufficiently small, letting $O_z^\varepsilon := (L_z)_{2M\varepsilon} \cap Q_z^*$, from $\bigcap_{\varepsilon>0} O_z^\varepsilon = L_z \cap Q_z^* \subset \partial E_z^+$ we obtain

$$E_{|\cdot|^p, g_0}[U_z^+, O_z^\varepsilon] + |DU_z^+|(O_z^\varepsilon) + \|\nabla u\|^p_{L^p(O_z^\varepsilon)} + |Du|(O_z^\varepsilon \setminus L_z) \le \frac{\delta^n \theta}{C_\lambda} \tag{4.60}$$

for all z, and the same for U_z^-. If $\mathcal{H}^{n-1}(J_u) < \infty$, then we also have for $\varepsilon > 0$ small enough

$$\sum_{z \in A_\delta} \mathcal{H}^{n-1}(J_{U_z^+} \cap O_z^\varepsilon) + \mathcal{H}^{n-1}(J_{U_z^-} \cap O_z^\varepsilon) \le C\theta. \tag{4.61}$$

The function $w_{z,\zeta}$ will be defined as a discretization of U_z^+ on $H_z^+ \cap Q_z$, and similarly with the other sign. By Proposition 4.3(iv) it depends on $U_z^\pm$ on a small neighborhood of the sets $H_z^\pm \cap Q_z$, and by Proposition 4.3(vi) the relevant properties of $w_{z,\zeta}$ can be estimated by corresponding properties of $U_z^\pm$ on $2c_*\varepsilon$-neighborhoods of $H_z^\pm \cap Q_z$ (see also (4.33)). Therefore we need to estimate $U_z^\pm$ on these sets.

Let $\hat{O}_z^\varepsilon := (L_z)_{M\varepsilon} \cap (Q_z)_{M\varepsilon}$. From (4.56), (4.57) and (4.58) we obtain $H_z \subseteq (L_z)_{2\varepsilon}$ and $(H_z^+)_{2c_*\varepsilon} \subseteq L_z^+ \cup (L_z)_{(2+2c_*)\varepsilon}$, which imply

$$(H_z^+ \cap Q_z)_{2c_*\varepsilon} \subseteq (L_z^+ \cap Q_z^*) \cup \hat{O}_z^\varepsilon \tag{4.62}$$

and the same for the other sign, provided that $M \geq 2 + 2c_*$ and ε is sufficiently small. Recalling (4.52), (4.60) in particular implies

$$\begin{aligned}
&\sum_{z\in A_\delta} \int_{J_{U_z^+} \cap (H_z^+\cap Q_z)_{2c_*\varepsilon}} g_0(|[U_z^+]|)d\mathcal{H}^{n-1} \\
&+ \sum_{z\in A_\delta} \int_{J_{U_z^-} \cap (H_z^-\cap Q_z)_{2c_*\varepsilon}} g_0(|[U_z^-]|)d\mathcal{H}^{n-1} \leq C\theta
\end{aligned} \tag{4.63}$$

and

$$\sum_{z\in A_\delta} \int_{(H_z^+\cap Q_z)_{2c_*\varepsilon}} |\nabla U_z^+ - \eta_z|^p dx + \int_{(H_z^-\cap Q_z)_{2c_*\varepsilon}} |\nabla U_z^- - \eta_z|^p dx \leq C\theta. \tag{4.64}$$

We next estimate the difference between u, U_z^+ and U_z^- around H_z, this will be important after (4.107) (cf. (4.111)-(4.112)). We pick $x_1, \ldots, x_K \in L_z$ such that

$$L_z \cap Q_z \subseteq \bigcup_i B_i, \qquad B_i := B_{M\varepsilon}(x_i), \tag{4.65}$$

and

$$\hat{O}_z^\varepsilon \subseteq \bigcup_i B_i^*, \qquad B_i^* := B_{2M\varepsilon}(x_i). \tag{4.66}$$

We can pick the points so that the larger balls B_i^* have finite overlap uniformly in ε, namely $\sum_{i=1}^K \chi_{B_i^*} \leq C$, and that they are all contained in $(Q_z)_{3M\varepsilon}$. This implies in particular $K \leq \frac{C\delta^{n-1}}{\varepsilon^{n-1}}$, for a constant C. For ε sufficiently small, $B_i^* \subseteq O_z^\varepsilon$. As φ_z is $\frac{1}{2}$-Lipschitz, the sets $B_i^* \cap L_z^+$ and $B_i^* \cap L_z^-$ are uniformly Lipschitz, therefore there is a constant C such that for any i there are h_i^+, $h_i^- \in \mathbb{R}^m$ with

$$\frac{1}{\varepsilon}\|u - h_i^+\|_{L^1(B_i^*\cap L_z^+)} + \|Tu - h_i^+\|_{L^1(B_i^*\cap L_z;\mathcal{H}^{n-1})} \leq C|Du|(B_i^* \cap L_z^+). \tag{4.67}$$

In (4.67) we write Tu for the inner trace on the boundary of $B_i^* \cap L_z^+$, which on $B_i^* \cap L_z$ coincides with u^+. The corresponding estimate holds with the other sign (then with $Tu = u^-$). This in particular implies

$$\int_{B_i^* \cap L_z} |[u] - (h_i^+ - h_i^-)| d\mathcal{H}^{n-1} \leq C|Du|(B_i^* \setminus L_z). \tag{4.68}$$

We observe that $\mathrm{Lip}\,(\varphi_z) \leq \frac{1}{2}$ also implies for some constant C

$$\frac{\varepsilon^n}{C} \leq |B_i^* \cap L_z^+|, \qquad \frac{\varepsilon^n}{C} \leq |B_i^* \cap L_z^-|, \tag{4.69}$$

as well as

$$\frac{\varepsilon^{n-1}}{C} \leq \mathcal{H}^{n-1}(B_i \cap L_z) \leq C\varepsilon^{n-1}, \quad \frac{\varepsilon^{n-1}}{C} \leq \mathcal{H}^{n-1}(B_i \cap H_z) \leq C\varepsilon^{n-1}. \tag{4.70}$$

By Poincaré's inequality applied to U_z^+ on B_i^*, using $u = U_z^+$ on $B_i^* \cap L_z^+$, (4.67) and (4.69),

$$\|U_z^+ - h_i^+\|_{L^1(B_i^*)} \leq C\varepsilon|DU_z^+|(B_i^*) \tag{4.71}$$

and analogously for U_z^- and h_i^-, so that

$$\int_{B_i^*} |U_z^+ - U_z^- - h_i^+ + h_i^-| dx \leq C\varepsilon(|DU_z^+| + |DU_z^-|)(B_i^*). \tag{4.72}$$

Finally, a direct application of Poincaré's inequality to $U_z^+ - u$ on B_i^*, using $u = U_z^+$ on $B_i^* \cap L_z^+$ and (4.69), leads to

$$\int_{B_i^*} |U_z^+ - u| dx \leq C\varepsilon(|DU_z^+| + |Du|)(B_i^*), \tag{4.73}$$

obviously the same holds for U_z^-. Summing (4.73) over all balls shows that

$$\|U_z^+ - u\|_{L^1(\hat{O}_z^\varepsilon)} \leq C\varepsilon(|DU_z^+| + |Du|)(O_z^\varepsilon) \tag{4.74}$$

and the same for U_z^-. Summing instead (4.73) only over the balls with centers contained in $(\partial Q_z)_{3M\varepsilon} \cap L_z$ gives

$$\|U_z^+ - u\|_{L^1((\partial Q_z)_{M\varepsilon} \cap (L_z)_{M\varepsilon})} \leq C\varepsilon\big(|DU_z^+| + |Du|\big)\,((\partial Q_z)_{5M\varepsilon} \cap O_z^\varepsilon), \tag{4.75}$$

and the same bound for U_z^-.

For $z \in A_\delta$ with $L_z \cap Q_z^* \neq \emptyset$ we define $w_{z,\zeta} : \mathbb{R}^n \to \mathbb{R}^m$ by

$$w_{z,\zeta} := \begin{cases} \Pi_{\varepsilon,\zeta} U_z^+ & \text{in } H_z^+, \\ \Pi_{\varepsilon,\zeta} U_z^- & \text{in } H_z^-, \end{cases} \tag{4.76}$$

we recall that if instead $L_z \cap Q_z^* = \emptyset$ we had set $w_{z,\zeta} = \Pi_{\varepsilon,\zeta} u$. In both cases, the function $w_{z,\zeta}$ is piecewise affine. For any function f, we have for almost all $\zeta \in B_\varepsilon$

$$\mathcal{H}^{n-1}(J_{\Pi_{\varepsilon,\zeta} f} \cap H_z) = \mathcal{H}^{n-1}(J_{\Pi_{\varepsilon,\zeta} f} \cap \partial Q_z) = 0\,, \tag{4.77}$$

and in particular this holds for U_z^+ and U_z^-. With this choice, and recalling (4.59), $J_{w_{z,\zeta}} \cap \overline{Q}_z$ splits (up to $\mathcal{H}^{n-1}$-null sets) into the disjoint union of $J_{\Pi_{\varepsilon,\zeta} U_z^+} \cap H_z^+ \cap Q_z$, $J_{\Pi_{\varepsilon,\zeta} U_z^-} \cap H_z^- \cap Q_z$, and a subset of $H_z \cap Q_z$, with

$$[w_{z,\zeta}] = \Pi_{\varepsilon,\zeta} U_z^+ - \Pi_{\varepsilon,\zeta} U_z^-, \quad \mathcal{H}^{n-1}\text{-a.e. on } H_z \cap Q_z. \tag{4.78}$$

By (4.57) and the fact that φ_z is θ-Lipschitz we also obtain

$$|\nu_{w_{z,\zeta}} - R_z e_n| \le C\theta \quad \mathcal{H}^{n-1}\text{-a.e. on } Q_z^* \cap H_z \cap J_{w_{z,\zeta}}. \tag{4.79}$$

If $\mathcal{H}^{n-1}(J_u) < \infty$ then using (4.29) on U_z^+ (cf. the discussion to get (4.33)) leads to

$$\fint_{B_\varepsilon} \mathcal{H}^{n-1}(\overline{Q}_z \cap H_z^+ \cap J_{\Pi_{\varepsilon,\zeta} U_z^+}) d\zeta \le C \sum_{z \in A_\delta} \mathcal{H}^{n-1}(J_{U_z^+} \cap (Q_z \cap H_z^+)_{2c_*\varepsilon}), \tag{4.80}$$

and the same for U_z^-. Using (4.62) in the first step, and in the second one that $U^+ = u$ in the open set $L_z^+ \cap Q_z$ and that $\hat{O}_z^\varepsilon \subseteq O_z^\varepsilon$,

$$\begin{aligned} &\fint_{B_\varepsilon} \mathcal{H}^{n-1}(\overline{Q}_z \cap H_z^+ \cap J_{\Pi_{\varepsilon,\zeta} U_z^+}) d\zeta \\ &\le C\mathcal{H}^{n-1}(J_{U_z^+} \cap L_z^+ \cap Q_z^*) + C\mathcal{H}^{n-1}(J_{U_z^+} \cap \hat{O}_z^\varepsilon) \\ &\le C\mathcal{H}^{n-1}(J_u \cap L_z^+ \cap Q_z^*) + C\mathcal{H}^{n-1}(J_{U_z^+} \cap O_z^\varepsilon), \end{aligned} \tag{4.81}$$

and the same for U_z^-. Finally, since L_z, L_z^+, L_z^- are mutually disjoint summing over all $z \in A_\delta$ and using (4.53) and (4.61) we obtain

$$\begin{aligned} &\fint_{B_\varepsilon} \sum_{z \in A_\delta} \mathcal{H}^{n-1}(\overline{Q}_z \cap H_z^+ \cap J_{\Pi_{\varepsilon,\zeta} U_z^+}) + \mathcal{H}^{n-1}(\overline{Q}_z \cap H_z^- \cap J_{\Pi_{\varepsilon,\zeta} U_z^-}) d\zeta \\ &\le C \sum_{z \in A_\delta} \mathcal{H}^{n-1}(Q_z^* \cap J_u \setminus L_z) + \mathcal{H}^{n-1}(O_z^\varepsilon \cap J_{U_z^+}) + \mathcal{H}^{n-1}(O_z^\varepsilon \cap J_{U_z^-}) \le C\theta. \end{aligned} \tag{4.82}$$

We define $w_\zeta^0 := \Pi_{\varepsilon,\zeta} u$ and then $w_\zeta \in SBV(\Omega_Q; \mathbb{R}^m)$, where $\Omega_Q := \mathrm{int}(\bigcup_{z \in A_\delta \cup A_\delta^*} \overline{Q}_z)$, by setting $w_\zeta := w_{z,\zeta}$ on Q_z if $z \in A_\delta$, and $w_\zeta := w_\zeta^0$ if $z \in A_\delta^*$. For any $\zeta \in B_\varepsilon$, the function w_ζ is piecewise affine and obeys property (i). This concludes the construction of w_ζ.

Step 4: Estimates on w_ζ and ∇w_ζ.

We first check that we have not added too much jump on the boundary between adjacent cubes by replacing w^0_ζ by $w_{z,\zeta}$. Fix $z \in A_\delta$. By (4.59) we have $\mathcal{H}^{n-1}(H_z \cap \partial Q_z) = 0$, and by (4.77) for any f we have $\mathcal{H}^{n-1}(J_{\Pi_{\varepsilon,\zeta} f} \cap \partial Q_z) = 0$ for almost every ζ. As H_z, H_z^+, H_z^- are a partition of $\mathbb{R}^n$, for almost every ζ the functions w^0_ζ and $w_{z,\zeta}$ have a well-defined trace on ∂Q_z, and (4.76) implies

$$\int_{\partial Q_z} |w^0_\zeta - w_{z,\zeta}| d\mathcal{H}^{n-1} = \int_{\partial Q_z \cap H_z^+} |\Pi_{\varepsilon,\zeta} u - \Pi_{\varepsilon,\zeta} U_z^+| d\mathcal{H}^{n-1} + \int_{\partial Q_z \cap H_z^-} |\Pi_{\varepsilon,\zeta} u - \Pi_{\varepsilon,\zeta} U_z^-| d\mathcal{H}^{n-1}. \tag{4.83}$$

By estimate (4.31) with $\Sigma = \partial Q_z \cap H_z^+$,

$$\fint_{B_\varepsilon} \int_{\partial Q_z \cap H_z^+} |w^0_\zeta - w_{z,\zeta}| d\mathcal{H}^{n-1} d\zeta \leq \frac{C}{\varepsilon} \|u - U_z^+\|_{L^1((\partial Q_z \cap H_z^+)_{c_*\varepsilon})} + C|D(u - U_z^+)|((\partial Q_z \cap H_z^+)_{2c_*\varepsilon}). \tag{4.84}$$

Recalling that $U_z^+ = u$ on E_z^+ and (4.62), both domains can be restricted to $(\partial Q_z)_{M\varepsilon} \cap (L_z)_{M\varepsilon}$. The first term is estimated by (4.75), and we conclude

$$\fint_{B_\varepsilon} \int_{\partial Q_z} |w^0_\zeta - w_{z,\zeta}| d\mathcal{H}^{n-1} d\zeta \leq C(|Du| + |DU_z^+| + |DU_z^-|)((\partial Q_z)_{5M\varepsilon} \cap O_z^\varepsilon) \tag{4.85}$$

so that, using (4.55),

$$\limsup_{\varepsilon \to 0} \sum_{z \in A_\delta} \fint_{B_\varepsilon} \int_{\partial Q_z} |w^0_\zeta - w_{z,\zeta}| d\mathcal{H}^{n-1} d\zeta = 0. \tag{4.86}$$

We next address the L^1 convergence. For any $z \in A_\delta$ for which $L_z \cap Q_z^* \neq \emptyset$, by the definition of w_ζ and Proposition 4.3(v), we have for $\mathcal{L}^n$-almost every $\zeta \in B_\varepsilon$

$$\begin{aligned} \int_{Q_z} |w_\zeta - u| dx &= \int_{Q_z \cap H_z^+} |\Pi_{\varepsilon,\zeta} U_z^+ - u| dx + \int_{Q_z \cap H_z^-} |\Pi_{\varepsilon,\zeta} U_z^- - u| dx \\ &\leq \int_{Q_z \cap H_z^+} \big(|\Pi_{\varepsilon,\zeta} U_z^+ - U_z^+| + |U_z^+ - u|\big) dx \\ &\quad + \int_{Q_z \cap H_z^-} \big(|\Pi_{\varepsilon,\zeta} U_z^- - U_z^-| + |U_z^- - u|\big)\, dx. \end{aligned} \tag{4.87}$$

We recall that $U_z^+ = u$ on $Q_z \cap H_z^+ \setminus \hat{O}_z^\varepsilon \subseteq E_z^+$ and (4.74) to obtain

$$\begin{aligned} \int\limits_{Q_z \cap H_z^+} |U_z^+ - u| dx \le & \|U_z^+ - u\|_{L^1(\hat{O}_z^\varepsilon)} \\ & \le C\varepsilon(|DU_z^+| + |DU_z^-| + |Du|)(O_z^\varepsilon). \end{aligned} \tag{4.88}$$

With (4.30) in Proposition 4.3,

$$\begin{aligned} \fint\limits_{B_\varepsilon} \int\limits_{Q_z \cap H_z^+} |\Pi_{\varepsilon,\zeta} U_z^+ - U_z^+| dx\, d\zeta \le & C\varepsilon |DU_z^+|((Q_z \cap H_z^+)_{2c_*\varepsilon}) \\ & \le C\varepsilon(|Du|(Q_z^*) + |DU_z^+|(O_z^\varepsilon)) \end{aligned} \tag{4.89}$$

and an analogous estimate holds for the term with the other sign. Recalling (4.60) we obtain

$$\sum_{z \in A_\delta :\, L_z \cap Q_z \neq \emptyset} \fint\limits_{B_\varepsilon} \int\limits_{Q_z} |w_\zeta - u| dx\, d\zeta \le C\theta \tag{4.90}$$

for ε sufficiently small. If $L_z \cap Q_z^* = \emptyset$ or $z \in A_\delta^*$ the computation is simpler as $w_\zeta = w_\zeta^0 = \Pi_{\varepsilon,\zeta} u$, and only (4.89) with u in place of U_z^+ appears. From this we conclude that

$$\fint\limits_{B_\varepsilon} \int\limits_{\Omega} |w_\zeta - u| dx\, d\zeta \le C\theta. \tag{4.91}$$

Moreover, from $w_\zeta = w_\zeta^0$ on $Q_z \setminus \hat{O}_z^\varepsilon$, from (4.60) and estimate (4.27) in Proposition 4.3(vi) we deduce

$$\begin{aligned} & \fint\limits_{B_\varepsilon} \int\limits_{\Omega} |\nabla w_\zeta - \nabla u|^p dx\, d\zeta \\ & \le C \fint\limits_{B_\varepsilon} \sum_{z \in A_\delta \cup A_\delta^*} \Big(\int\limits_{Q_z} |\nabla w_\zeta^0 - \nabla u|^p dx + \int\limits_{\hat{O}_z^\varepsilon} (|\nabla w_\zeta|^p + |\nabla u|^p) dx \Big) d\zeta \\ & \le C \sum_{z \in A_\delta \cup A_\delta^*} \int\limits_{(Q_z)_{2c_*\varepsilon}} |\nabla u - \eta_z|^p dx + C\theta \le C\theta, \end{aligned} \tag{4.92}$$

where the last estimate follows from (4.52), and the constant $C > 0$ depends on u, on the dimension n, on p and on Ω.

Step 5: Definition of the deformation Φ.

We select $\delta' \in (0,\delta)$ and define $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\Phi(x) := x + \sum_{z \in A_\delta : L_z \cap Q_z^* \neq \emptyset} \alpha_z(x) R_z \left(\psi_z((R_z^T(x-x_z))') + \beta - \varphi_z((R_z^T(x-x_z))') \right) e_n, \tag{4.93}$$

where $(R_z^T x)'$ denotes the first $n-1$ components of the vector $R_z^T x$ and we fixed a function $\alpha_z \in C_c^\infty(Q_z;[0,1])$ with $\alpha_z = 1$ on $Q_z'' := z + \gamma + (-\frac{\delta+\delta'}{4}, \frac{\delta+\delta'}{4})^n$, $|D\alpha_z| \leq 6/(\delta-\delta')$. Since α_z, ψ_z, φ_z are Lipschitz, so is Φ. By the definition of α_z we have $\Phi(x) = x$ for $x \notin \Omega$, and by (4.57) $|\Phi(x) - x| \leq 2\varepsilon$ for all x. For ε sufficiently small, $\Phi(Q_z) = Q_z$ for all $z \in \delta\mathbb{Z}^n$ by construction, and if we define $Q_z' := z + \gamma + (-\frac{\delta'}{2}, \frac{\delta'}{2})^n \subset\subset Q_z'' \subset\subset Q_z$, from (4.51), (4.57) and (4.58) we obtain

$$H_z \cap Q_z' = \Phi(L_z) \cap Q_z', \text{ and } \Phi^{-1}(H_z) \cap Q_z' = L_z \cap Q_z'. \tag{4.94}$$

In order to show that Φ is invertible with Lipschitz inverse it suffices to prove that $D\Phi$ is uniformly close to the identity. Indeed, using (4.57) we obtain

$$\begin{aligned} \|D\Phi - \mathrm{Id}\|_\infty \leq & \max_{z \in A_\delta} \|D\psi_z - D\varphi_z\|_\infty + \max_{z \in A_\delta} \|D\alpha_z\|_\infty (\|\psi_z - \varphi_z\|_\infty + |\beta|) \\ \leq & \omega_\varepsilon + \frac{12\varepsilon}{\delta - \delta'} . \end{aligned} \tag{4.95}$$

In particular, if ε is sufficiently small on a scale depending on δ and δ', we can ensure

$$\|D\Phi - \mathrm{Id}\|_\infty \leq \frac{1}{2}\theta. \tag{4.96}$$

Therefore,

$$(1 - \frac{1}{2}\theta)|x-y| \leq |\Phi(x) - \Phi(y)| \leq (1+\theta)|x-y|,$$

which implies that Φ is globally bilipschitz. Property (iii) follows.

Step 6: Estimate of the jump energy.

We are now able to estimate the energy of the jump contribution. We start to decompose it as

$$\begin{aligned} & \fint_{B_\varepsilon} \int_{\Omega \cap (J_u \cup \Phi^{-1}(J_{w_\zeta}))} g_0(|[u] - [w_\zeta] \circ \Phi|) \, d\mathcal{H}^{n-1} d\zeta \\ \leq & \fint_{B_\varepsilon} \sum_{z \in A_\delta \cup A_\delta^*} \int_{(J_u \cup \Phi^{-1}(J_{w_\zeta})) \cap Q_z} g_0(|[u] - [w_\zeta] \circ \Phi|) \, d\mathcal{H}^{n-1} d\zeta \\ & + \fint_{B_\varepsilon} \sum_{z \in A_\delta} \int_{\partial Q_z} g_0(|w_{z,\zeta} - w_\zeta^0|) d\mathcal{H}^{n-1} d\zeta \end{aligned} \tag{4.97}$$

where we separated the boundary contributions from the ones in the interior, then used $\Phi(x) = x$ on ∂Q_z, (4.55) to infer that $[u] = 0$ almost everywhere on ∂Q_z, (4.77) to infer $[w_{z,\zeta}] = [w_z^0] = 0$ almost everywhere on ∂Q_z, and finally used $|[w_\zeta]| \le |w_{z,\zeta} - w_\zeta^0| + |w_{z',\zeta} - w_\zeta^0|$ and subadditivity of g_0 on $\partial Q_z \cap \partial Q_{z'}$ for $z, z' \in A_\delta$, $|[w_\zeta]| = |w_{z,\zeta} - w_\zeta^0|$ on $\partial Q_z \cap \partial Q_{z'}$ for $z \in A_\delta$, $z' \in A_\delta^*$, and $|[w_\zeta]| = 0$ on $\partial Q_z \cap \partial Q_{z'}$ for $z, z' \in A_\delta^*$.

We start from the boundary term. From (4.46) we get

$$\begin{aligned}
&\fint_{B_\varepsilon} \sum_{z \in A_\delta} \int_{\partial Q_z} g_0(|w_{z,\zeta} - w_\zeta^0|) d\mathcal{H}^{n-1} d\zeta \\
&\qquad \le \# A_\delta \mathcal{H}^{n-1}(\partial Q_z)\lambda + C_\lambda \fint_{B_\varepsilon} \sum_{z \in A_\delta} \int_{\partial Q_z} |w_{z,\zeta} - w_\zeta^0| d\mathcal{H}^{n-1} d\zeta \\
&\qquad \le C\frac{\lambda}{\delta} + C\theta, \qquad (4.98)
\end{aligned}$$

by (4.86) and choosing ε sufficiently small. For $\lambda \le \delta\theta$ the entire term is bounded by $C\theta$.

We now turn to the first term of (4.97). We start from $z \in A_\delta$. Splitting

$$\begin{aligned}
J_u \cup \Phi^{-1}(J_{w_\zeta}) =& (J_u \cup \Phi^{-1}(J_{w_\zeta})) \cap (L_z \cup \Phi^{-1}(H_z)) \\
&\cup (J_u \cup \Phi^{-1}(J_{w_\zeta})) \setminus (L_z \cup \Phi^{-1}(H_z))
\end{aligned}$$

and using the subadditivity of g_0 to estimate the integral on the second set, we get

$$\fint_{B_\varepsilon} \int_{(J_u \cup \Phi^{-1}(J_{w_\zeta})) \cap Q_z} g_0(|[u] - [w_\zeta] \circ \Phi|)\, d\mathcal{H}^{n-1}\, d\zeta \le \mathrm{I}_z + \mathrm{II}_z + \mathrm{III}_z \qquad (4.99)$$

with

$$\begin{aligned}
\mathrm{I}_z :=& \fint_{B_\varepsilon} \int_{(L_z \cup \Phi^{-1}(H_z)) \cap Q_z} g_0(|[u] - [w_\zeta] \circ \Phi|)\, d\mathcal{H}^{n-1} d\zeta, \\
\mathrm{II}_z :=& \int_{J_u \cap Q_z \setminus L_z} g_0(|[u]|)\, d\mathcal{H}^{n-1}, \\
\mathrm{III}_z :=& \fint_{B_\varepsilon} \int_{\Phi^{-1}(J_{w_\zeta} \setminus H_z) \cap Q_z} g_0(|[w_\zeta] \circ \Phi|)\, d\mathcal{H}^{n-1}\, d\zeta.
\end{aligned} \qquad (4.100)$$

We start from I_z. We add and subtract s_z, write

$$\mathrm{I}_z \le \fint_{B_\varepsilon} \int_{(L_z \cup \Phi^{-1}(H_z)) \cap Q_z} g_0(|[u] - s_z|) + g_0(|[w_\zeta] \circ \Phi - s_z|)\, d\mathcal{H}^{n-1} d\zeta \qquad (4.101)$$

and observe that

$$g_0(|[u]-s_z|)\chi_{(\Phi^{-1}(H_z)\setminus L_z)\cap Q_z} \le g_0(|[u]|)\chi_{(J_u\setminus L_z)\cap Q_z} + g_0(|s_z|)\chi_{(\Phi^{-1}(H_z)\setminus L_z)\cap Q_z},$$

and similarly for the other term. Therefore

$$\mathrm{I}_z \le \mathrm{I}_z^1 + \mathrm{I}_z^2 + \mathrm{I}_z^3 + \mathrm{II}_z + \mathrm{III}_z, \tag{4.102}$$

where

$$\mathrm{I}_z^1 := \int_{L_z\cap Q_z^*} g_0(|[u]-s_z|)\, d\mathcal{H}^{n-1}, \tag{4.103}$$

$$\mathrm{I}_z^2 := \fint_{B_\varepsilon} \int_{\Phi^{-1}(H_z)\cap Q_z} g_0(|[w_\zeta]\circ\Phi - s_z|)\, d\mathcal{H}^{n-1} d\zeta, \tag{4.104}$$

and

$$\mathrm{I}_z^3 := g_0(|s_z|)\mathcal{H}^{n-1}((L_z \triangle \Phi^{-1}(H_z))\cap Q_z). \tag{4.105}$$

First note that

$$\sum_{z\in A_\delta} (\mathrm{I}_z^1 + \mathrm{II}_z) \le C\theta \tag{4.106}$$

thanks to (4.52). For I_z^2 we use first the Area formula (2.9), then (4.95) and (4.78) to obtain

$$\mathrm{I}_z^2 \le 2 \fint_{B_\varepsilon} \int_{H_z\cap Q_z} g_0(|\Pi_{\varepsilon,\zeta}(U_z^+ - U_z^-) - s_z|)\, d\mathcal{H}^{n-1} d\zeta. \tag{4.107}$$

We cover $H_z \cap Q_z$ with the balls B_i introduced in (4.65), and start from estimating the term

$$\mathrm{I}_z^2(B_i) := \fint_{B_\varepsilon} \int_{H_z\cap B_i} g_0(|\Pi_{\varepsilon,\zeta}(U_z^+ - U_z^-) - s_z|)\, d\mathcal{H}^{n-1} d\zeta. \tag{4.108}$$

By subadditivity,

$$\begin{aligned} &g_0(|\Pi_{\varepsilon,\zeta}(U_z^+ - U_z^-) - s_z|) \\ &\quad \le g_0(|h_i^+ - h_i^- - s_z|) + g_0(|\Pi_{\varepsilon,\zeta}(U_z^+ - U_z^-) - h_i^+ + h_i^-|). \end{aligned} \tag{4.109}$$

The first term, using (4.70) twice and subadditivity, leads to

$$\begin{aligned}\int_{H_z\cap B_i} g_0(|h_i^+ - h_i^- - s_z|)d\mathcal{H}^{n-1} &\leq C\varepsilon^{n-1} g_0(|h_i^+ - h_i^- - s_z|)\\ &\leq C\int_{L_z\cap B_i}[g_0(|[u]-s_z|)+g_0(|h_i^+ - h_i^- - [u]|)]d\mathcal{H}^{n-1},\end{aligned}\tag{4.110}$$

where the first integral is controlled by I_z^1. Using (4.46) in the second term of (4.109) and the second term of (4.110), for any $\lambda > 0$ we have

$$\begin{aligned}\mathrm{I}_z^2(B_i) \leq & C\int_{L_z\cap B_i} g_0(|[u]-s_z|)d\mathcal{H}^{n-1} + C\lambda\varepsilon^{n-1}\\ & + C_\lambda \int_{L_z\cap B_i} |h_i^+ - h_i^- - [u]|\, d\mathcal{H}^{n-1}\\ & + C_\lambda \fint_{B_\varepsilon}\int_{H_z\cap B_i} |\Pi_{\varepsilon,\zeta}(U_z^+ - U_z^- - h_i^+ + h_i^-)|\, d\mathcal{H}^{n-1}\, d\zeta.\end{aligned}\tag{4.111}$$

The term in the second line can be estimated with (4.68). For the last line we use first (4.31) and then (4.72), and obtain

$$\begin{aligned}&\fint_{B_\varepsilon}\int_{H_z\cap B_i} |\Pi_{\varepsilon,\zeta}(U_z^+ - U_z^- - h_i^+ + h_i^-)| d\mathcal{H}^{n-1}\, d\zeta\\ &\leq \frac{C}{\varepsilon}\int_{B_i^*} |U_z^+ - U_z^- - h_i^+ + h_i^-|dx + C(|DU_z^+| + |DU_z^-|)(B_i^*)\\ &\leq C(|DU_z^+| + |DU_z^-|)(B_i^*).\end{aligned}\tag{4.112}$$

Using that $\mathrm{Lip}\,(\varphi_z) \leq \frac{1}{2}$, (4.70) and $\sum_{i=1}^K \chi_{B_i^*} \leq C$, summing over i yields

$$\mathrm{I}_z^2 \leq C\mathrm{I}_z^1 + C\lambda\delta^{n-1} + C_\lambda |Du|(O_z^\varepsilon \setminus L_z) + C_\lambda(|DU_z^+| + |DU_z^-|)(O_z^\varepsilon),\tag{4.113}$$

so that by (4.60) summing on $z \in A_\delta$ we find for $\lambda \leq \delta\theta$ and ε sufficiently small

$$\sum_{z\in A_\delta} I_z^2 \leq C\theta\,.$$

We next turn to I_z^3, and observe that by (4.94) and (4.95)

$$\begin{aligned}\mathcal{H}^{n-1}((L_z\triangle\Phi^{-1}(H_z))\cap Q_z) \leq & \mathcal{H}^{n-1}(L_z\cap Q_z\setminus Q_z') + \mathcal{H}^{n-1}(\Phi^{-1}(H_z)\cap Q_z\setminus Q_z')\\ \leq & \mathcal{H}^{n-1}(L_z\cap Q_z\setminus Q_z') + 2\mathcal{H}^{n-1}(H_z\cap Q_z\setminus Q_z').\end{aligned}\tag{4.114}$$

Therefore

$$\mathrm{I}_z^3 \le 4 \max_{z\in A_\delta} g_0(|s_z|)\mathcal{H}^{n-1}\left((H_z \cup L_z) \cap (Q_z \setminus Q_z')\right).$$

As $\mathcal{H}^{n-1}(\bigcup_{z\in A_\delta}(H_z \cup L_z) \cap Q_z) < \infty$ and $\bigcup_{\delta'<\delta} Q_z' = Q_z$, choosing δ' sufficiently close to δ we have

$$\sum_{z\in A_\delta} \mathcal{H}^{n-1}\left((H_z \cup L_z) \cap (Q_z \setminus Q_z')\right) \le \theta \text{ and } \sum_{z\in A_\delta} \mathrm{I}_z^3 \le \theta. \tag{4.115}$$

Similarly, if $\mathcal{H}^{n-1}(J_u) < \infty$, for δ' sufficiently close to δ we have

$$\mathcal{H}^{n-1}(J_u \cap \bigcup_{z\in A_\delta} (Q_z \setminus Q_z')) \le \theta. \tag{4.116}$$

For III_z we use the Area formula (2.9) and (4.95). We obtain

$$\mathrm{III}_z \le 2 \fint_{B_\varepsilon} \int_{J_{w_{z,\zeta}} \cap Q_z \setminus H_z} g_0(|[w_{z,\zeta}]|)\, d\mathcal{H}^{n-1}\, d\zeta. \tag{4.117}$$

Therefore $\mathrm{III}_z \le 2\mathrm{III}_z^+ + 2\mathrm{III}_z^-$, with

$$\mathrm{III}_z^+ := \fint_{B_\varepsilon} \int_{J_{\Pi_{\varepsilon,\zeta} U_z^+} \cap Q_z \cap H_z^+} g_0(|[\Pi_{\varepsilon,\zeta} U_z^+]|)\, d\mathcal{H}^{n-1}\, d\zeta, \tag{4.118}$$

and similarly for III_z^-. We use inequalities (4.28) to infer

$$\sum_{z\in A_\delta} \mathrm{III}_z^+ \le C \sum_{z\in A_\delta} \int_{J_{U_z^+} \cap (H_z^+ \cap Q_z)_{2c_*\varepsilon}} g_0(|[U_z^+]|)\, d\mathcal{H}^{n-1}, \tag{4.119}$$

and the same for III_z^-. Both can be estimated via (4.63), we conclude that

$$\sum_{z\in A_\delta} \mathrm{III}_z \le C\theta. \tag{4.120}$$

We next treat the bulk term in (4.97) in the case $z \in A_\delta^*$. Since $\Phi(x) = x$ and $w_\zeta = w_\zeta^0$ on Q_z, recalling Proposition 4.3 (iii),

$$\fint_{B_\varepsilon} \int_{(J_u \cup J_{w_\zeta^0}) \cap Q_z} g_0(|[u] - [w_\zeta^0]|)\, d\mathcal{H}^{n-1}\, d\zeta \le \mathrm{IV}_z + \mathrm{V}_z, \tag{4.121}$$

where

$$\begin{aligned} \mathrm{IV}_z &:= \int_{J_u \cap Q_z} g_0(|[u]|)\, d\mathcal{H}^{n-1}, \\ \mathrm{V}_z &:= \fint_{B_\varepsilon} \int_{J_{w^0_\zeta} \cap Q_z} g_0(|[w^0_\zeta]|)\, d\mathcal{H}^{n-1}\, d\zeta. \end{aligned} \tag{4.122}$$

As for III^+_z, we use inequality (4.28) in Proposition 4.3 and obtain

$$\mathrm{V}_z \le C \int_{J_u \cap (Q_z)_{2c_*\varepsilon}} g_0(|[u]|)\, d\mathcal{H}^{n-1} \tag{4.123}$$

so that

$$\sum_{z \in A^*_\delta} (\mathrm{IV}_z + \mathrm{V}_z) \le C\mu_u\big(\bigcup_{z \in A^*_\delta} (Q_z)_{2c_*\varepsilon}\big) \le C\mu_u((\partial\Omega)_{3\sqrt{n}\delta}) \le C\theta, \tag{4.124}$$

for ε sufficiently small, where in the last step we used (4.49). Combining the previous estimates, (4.97) yields

$$\fint_{B_\varepsilon} \int_{\Omega \cap (J_u \cup \Phi^{-1}(J_{w_\zeta}))} g_0(|[u] - [w_\zeta] \circ \Phi|)\, d\mathcal{H}^{n-1} d\zeta \le C\theta\,. \tag{4.125}$$

We claim next that for ε sufficiently small

$$\fint_{B_\varepsilon} \int_{\Omega \cap (J_u \cup \Phi^{-1}(J_{w_\zeta}))} g_0(|[u]| + |[w_\zeta] \circ \Phi|)\, \big|\nu_u - \nu_{w_\zeta} \circ \Phi\big|\, d\mathcal{H}^{n-1} d\zeta \le C\theta\,. \tag{4.126}$$

Thanks to subadditivity and monotonicity of g_0, (4.125) implies that it suffices to prove

$$\fint_{B_\varepsilon} \int_{\Omega \cap (J_u \cap \Phi^{-1}(J_{w_\zeta}))} g_0(|[u]|)\, \big|\nu_u - \nu_{w_\zeta} \circ \Phi\big|\, d\mathcal{H}^{n-1} d\zeta \le C\theta\,. \tag{4.127}$$

Similarly, by (4.106), (4.115) and (4.124) it suffices to prove

$$\sum_{z \in A_\delta} \fint_{B_\varepsilon} \int_{Q_z \cap L_z \cap \Phi^{-1}(H_z)} g_0(|[u]|)|\nu_u - \nu_{w_\zeta} \circ \Phi| d\mathcal{H}^{n-1}\, d\zeta \le C\theta\,. \tag{4.128}$$

From (4.79) we obtain that $|\nu_{w_\zeta} \circ \Phi - R_z e_n| \le \theta$ almost everywhere on $Q_z \cap \Phi^{-1}(H_z)$. The claim follows then from (4.52) and integrability of $g_0(|[u]|)$.

Step 7: Choice of ζ, conclusion of the proof.

From (4.91), (4.92), (4.125) and (4.126), it is easy to check that there is a subset $\tilde{B} \subset B_\varepsilon$, with $|\tilde{B}|/|B_\varepsilon| > 1/2$, such that for all $\zeta \in \tilde{B}$ (4.77) holds and we have

$$\int_\Omega |w_\zeta - u| dx \le C\theta,$$

$$\int_\Omega |\nabla w_\zeta - \nabla u|^p dx \le C\theta,$$

$$\int_{\Omega \cap (J_u \cup \Phi^{-1}(J_{w_\zeta}))} g_0(|[u] - [w_\zeta] \circ \Phi|) \, d\mathcal{H}^{n-1} \le C\theta,$$

$$\int_{\Omega \cap (J_u \cup \Phi^{-1}(J_{w_\zeta}))} g_0(|[u]| + |[w_\zeta] \circ \Phi|) \, |\nu_u - \nu_{w_\zeta} \circ \Phi| \, d\mathcal{H}^{n-1} \le C\theta.$$

If $\mathcal{H}^{n-1}(J_u) < \infty$ then, using (4.82), we can choose $\tilde{B}$ so that additionally

$$\sum_{z \in A_\delta} \mathcal{H}^{n-1}(\overline{Q}_z \cap J_{w_\zeta} \setminus H_z) \le C\theta,$$

which by the Area formula (2.9) as usual implies

$$\sum_{z \in A_\delta} \mathcal{H}^{n-1}(\overline{Q}_z \cap \Phi^{-1}(J_{w_\zeta} \setminus H_z)) \le C\theta. \tag{4.129}$$

Properties (ii), (iii) and (v) follow; (i) and (iv) had already been proven. Property (vii) is immediate.

It remains to prove (vi). We assume that $\mathcal{H}^{n-1}(J_u) < \infty$ and start from a bound on $\Phi^{-1}(J_{w_\zeta}) \setminus J_u$. We split the jump set of w_ζ into the contribution inside each cube Q_z, for $z \in A_\delta^* \cup A_\delta$, and then for each $z \in A_\delta$, we split the jump set J_{w_ζ} into the part in H_z and the rest. We obtain

$$\begin{aligned} \mathcal{H}^{n-1}(\Omega \cap \Phi^{-1}(J_{w_\zeta}) \setminus J_u) \le & \sum_{z \in A_\delta^*} \mathcal{H}^{n-1}(\overline{Q}_z \cap \Phi^{-1}(J_{w_\zeta})) \\ & + \sum_{z \in A_\delta} \Big(\mathcal{H}^{n-1}(\overline{Q}_z \cap \Phi^{-1}(H_z) \setminus J_u) + \mathcal{H}^{n-1}(\overline{Q}_z \cap \Phi^{-1}(J_{w_\zeta} \setminus H_z)) \Big). \end{aligned} \tag{4.130}$$

In the first term in the second line, we use (4.59) to drop the part on ∂Q_z and then separate the contributions inside and outside L_z. Equation (4.94) ensures that $Q_z' \cap \Phi^{-1}(H_z) \setminus L_z = \emptyset$. We obtain

$$\begin{aligned} & \sum_{z \in A_\delta} \mathcal{H}^{n-1}(\overline{Q}_z \cap \Phi^{-1}(H_z) \setminus J_u) \\ & \le \sum_{z \in A_\delta} \Big(\mathcal{H}^{n-1}(Q_z \cap L_z \setminus J_u) + \mathcal{H}^{n-1}(Q_z \cap \Phi^{-1}(H_z) \setminus Q_z') \Big) \le C\theta, \end{aligned} \tag{4.131}$$

where the last inequality follows from (4.53), (4.115) and the Area formula (2.9). The other two terms in (4.130) can be bounded by (4.129) and (4.50), and we conclude

$$\mathcal{H}^{n-1}(\Omega \cap \Phi^{-1}(J_{w_\varsigma}) \setminus J_u) \le C\theta. \tag{4.132}$$

The converse inequality is proven by a different argument based on lower semicontinuity. As discussed in the first lines of the proof, taking a sequence $\theta_j \to 0$ we obtain a sequence w_j which has all stated properties, except that (vi) is replaced by the weaker assertion

$$\limsup_j \mathcal{H}^{n-1}(\Omega \cap \Phi_j^{-1}(J_{w_j}) \setminus J_u) = 0. \tag{4.133}$$

In particular w_j converges to u in $L^1(\Omega;\mathbb{R}^m)$, and since ∇w_j converges to ∇u strongly in $L^p(\Omega;\mathbb{R}^{m\times n})$, with $p \in [1,\infty)$ given, there is a function $f : \mathbb{R}^{m\times n} \to [0,\infty)$ with superlinear growth at infinity such that

$$\limsup_j \int_\Omega f(\nabla w_j)dx + \mathcal{H}^{n-1}(\Omega \cap J_{w_j}) < \infty \tag{4.134}$$

(if $p > 1$, then $f(\xi) := |\xi|^p$ itself will do; if $p = 1$, de la Vallée-Poussin Theorem gives the conclusion). By the SBV closure and lower semicontinuity theorem [5, Theorem 4.7], we deduce

$$\mathcal{H}^{n-1}(\Omega \cap J_u) \le \liminf_j \mathcal{H}^{n-1}(\Omega \cap J_{w_j}) = \liminf_j \mathcal{H}^{n-1}(\Omega \cap \Phi_j^{-1}(J_{w_j})), \tag{4.135}$$

where the last equality can be obtained from the Area formula and (iv). By additivity of $\mathcal{H}^{n-1}$,

$$\begin{aligned}\mathcal{H}^{n-1}(\Omega \cap J_u \setminus \Phi_j^{-1}(J_{w_j})) =& \mathcal{H}^{n-1}(\Omega \cap J_u) - \mathcal{H}^{n-1}(\Omega \cap \Phi_j^{-1}(J_{w_j})) \\ &+ \mathcal{H}^{n-1}(\Omega \cap \Phi_j^{-1}(J_{w_j}) \setminus J_u).\end{aligned} \tag{4.136}$$

Using first (4.135) and then (4.133),

$$\limsup_j \mathcal{H}^{n-1}(\Omega \cap J_u \setminus \Phi_j^{-1}(J_{w_j})) \le \limsup_j \mathcal{H}^{n-1}(\Omega \cap \Phi_j^{-1}(J_{w_j}) \setminus J_u) = 0, \tag{4.137}$$

which concludes the proof. $\square$

Data availability

No data was used for the research described in the article.

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