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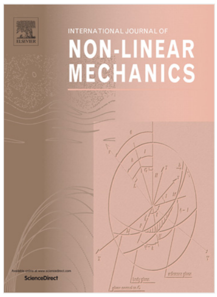


Flow of fluids with pressure-dependent viscosity down an incline: Long-wave linear stability analysis

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# Graphical Abstract

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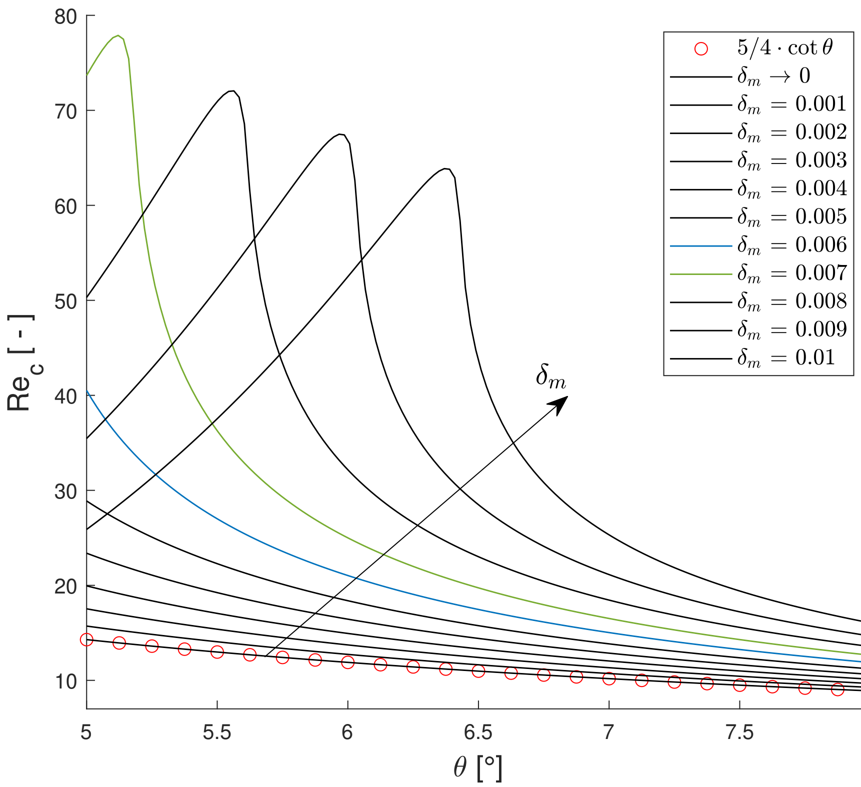

# Highlights

**Flow of fluids with pressure-dependent viscosity down an incline: Long-wave linear stability analysis**

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- We investigate the linear stability of a piezo-viscous fluid flowing down an incline.
- We perform the linear stability analysis using the long-wave approximation method.
- We discuss the effects of the material and geometrical parameters on the onset of instability.

# Flow of fluids with pressure-dependent viscosity down an incline: Long-wave linear stability analysis

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**Abstract**

In this paper, we investigate the linear stability of a gravity-driven fluid with pressure-dependent viscosity flowing down an inclined plane. The linear stability analysis is formulated using the long-wave approximation method. We show that the onset of instability occurs at a critical Reynolds number that depends on the material and geometrical parameters. Our results suggest that the dependence of the viscosity on pressure can influence the stability characteristics of the flow down an incline.

*Keywords:* Linear stability, Long-wave approximation method, Piezo-viscous fluids

## 1. Introduction

The flow of a fluid down an inclined plane occurs in various geophysical phenomena, industrial and everyday processes. Such flows can exhibit complex rheological behaviour and can involve complex phenomena and processes to model. Therefore, numerous theoretical, numerical, and experimental studies have been developed concerning the flows of complex fluids (e.g. [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]) and their stability (e.g. [15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37]) down an incline. There has been an increasing interest on fluids with pressure-dependent viscosities (see e.g. [38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 14, 48] and references therein). In particular, the problem of elastohydrodynamics is an example in which the dependence of viscosity on pressure is relevant [14, 43, 49]. The dependence of fluid viscosity on pressure was recognised several centuries ago, and the book of Bridgman [38] documents most of the works until 1931, highlighting that viscosity can change significantly with pressure [47]. In this paper we analyse the linear stability of a fluid whose viscosity depends on pressure, namely a piezo-viscous fluid [14]. In particular, we choose an exponential dependence of viscosity on pressure, i.e. the empirical relationship between viscosity and pressure as proposed by Barus in [50], namely

$$\mu^*(p^*) = \mu_0^* e^{\delta^*(p^* - p_0^*)}, \tag{1}$$

where, by denoting the "*" as dimensional quantities, $\mu_0^*$ is the viscosity at the reference pressure $p_0^*$ and $\delta^*$ is the pressure coefficient. A constitutive relation of the implicit type (there exists an implicit relation e.g. between quantities such as stress, strain, velocity gradient) is used to describe such complex fluids [14, 47, 51, 52, 53, 54, 55, 56]. In the work [14], the flow of a piezo-viscous fluid down an inclined surface in different flow regimes was investigated using the lubrication theory. Such work was extended in [11], within the context of the lubrication approximation, by carrying out an analysis of the flow of a fluid with pressure- and shear-rate-dependent viscosity down an inclined plane. The authors of [55] analysed the stability of the Rayleigh–Bénard convection for a fluid with temperature- and pressure-dependent viscosity. The viscosity was assumed to be an analytical function of temperature and pressure in the context of a generalisation of the Oberbeck–Boussinesq approximation. In particular, the thermal-convection in a fluid with viscosity that depends on both the temperature and pressure was investigated, showing that the linear and non-linear stability coincide. Here, we follow the approach described in [18, 19, 57] to perform a linear stability analysis of a fluid with pressure-dependent viscosity down an incline using the long-wave approximation method. The pioneering works for the analysis of the stability analysis of the flow down an inclined plane are given in [58] and [59] for the case of Newtonian fluids. In particular, by using the long-wave approximation method, a proportionality relation between the so-called critical Reynolds number $\mathsf{Re}_c$ and the tilt angle $\theta$ was shown in [58] and [59]. This relation was later experimentally validated in [60]. The long-wave approximation method consists in the assumption of disturbances of long wavelength and, thus, small wave number $\alpha$, providing a reliable estimation of the critical parameters for the onset of instability. To the best of the authors' knowledge, the analysis of the onset of instability of piezo-viscous fluids down an incline using the long-wave approximation method has never been performed before, and this motivates our investigations. The paper is organised as follows: in Section 2 we present the mathematical problem with the main features of a flow with pressure-dependent viscosity down an incline. In Section 3, we formulate the differential system governing the linear stability analysis using the long-wave approximation method. In Section 4 and 5, we then report the results and some concluding remarks.

## 2. The mathematical problem

Let us consider a fluid flowing down an incline as depicted in Fig. 1 and we denote throughout the paper the "*" as a dimensional quantity. We suppose that the domain of the flow is given by

$$D = \left\{(x^*, y^*) \in \mathbb{R}^2 | 0 \leq x^* \leq L^*, 0 \leq y^* \leq h^*(x^*, t^*)\right\},$$

where $\theta \in (0, \pi/2)$ is the tilt angle, $L^*$ is the length of the domain and $y^* = h^*(x^*, t^*)$ is the upper free boundary (not a priori known) and $H^* = \max h^*$.

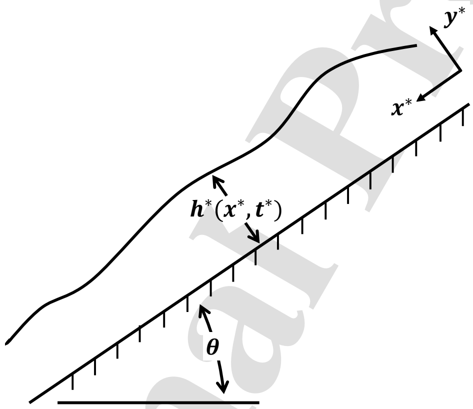



Figure 1: Diagram of the reference framework.

We assume that the Cauchy stress tensor, $\mathbb{T}^*$, is

$$\mathbb{T}^* = -p^*\mathbb{I} + \mathbb{S}^*, \tag{2}$$

where $\mathbb{S}^*$ is the deviatoric part is given by

$$\mathbb{S}^* = 2\mu^*(p^*)\mathbb{D}^*, \tag{3}$$

where $\mu^*(p^*)$ is the fluid viscosity as a function of the pressure $p^*$ given by (1) and $\mathbb{D}^* = 1/2\left(\nabla^*\mathbf{v}^* + \nabla^*\mathbf{v}^{*T}\right)$.

The governing equations for the two-dimensional incompressible flow, $\mathbf{v}^* = u^*\mathbf{e}_x + v^*\mathbf{e}_y$, are in dimensional form

$$\begin{cases} \rho^*\dot{\mathbf{v}}^* = -\nabla p^* + \mathrm{div}\,(\mathbb{S}^*) + \mathbf{g}^*, \\ \\ \mathrm{div}\,(\mathbf{v}^*) = 0, \end{cases} \tag{4}$$

where $\dot{(\cdot)}$ is the material derivative, $\mathbf{g}^* = \rho^* g^* \sin\theta \mathbf{e}_x - \rho^* g^* \cos\theta \mathbf{e}_y$, $g^*$ is gravity and $\rho^*$ is the constant material density. We consider the non-slip and impermeability conditions

$$u^* = v^* = 0, \quad \text{on} \quad y^* = 0, \tag{5}$$

and the kinematical-dynamical conditions

$$\begin{cases} h^*_{t^*} + u^* h^*_{x^*} = v^*, & y^* = h^*, \\ \\ \mathbb{T}^*\mathbf{n} = 0, & y^* = h^*, \end{cases} \tag{6}$$

where $\mathbf{n}$ is the outer normal and

$$(\cdot)_{t^*} = \frac{\partial(\cdot)}{\partial t^*}, \quad (\cdot)_{x^*} = \frac{\partial(\cdot)}{\partial x^*}, \quad (\cdot)_{y^*} = \frac{\partial(\cdot)}{\partial y^*}.$$

Following [18, 19] and exploiting $(4)_2$, we rewrite $(6)_1$ as

$$h^*_{t^*} + \left(\int_0^{h^*} u^* dy^*\right)_{x^*} = 0. \tag{7}$$

We introduce the following dimensionless quantities

$$\begin{aligned} &\mathbf{x} = \frac{\mathbf{x}^*}{H^*}, \, \mathbf{v} = \frac{\mathbf{v}^*}{U^*_{ref}}, \, t = \frac{U^*_{ref}}{H^*} t^*, \, p = \frac{H^*}{\mu^*_0 U^*_{ref}} (p^* - p^*_0), \\ &\mu(p) = \frac{\mu^*(p^*)}{\mu^*_0}, \, \mathbb{D} = \frac{H^*}{U^*_{ref}} \mathbb{D}^*, \, \mathbb{S} = \frac{H^*}{\mu^*_0 U^*_{ref}} \mathbb{S}^*, \end{aligned} \tag{8}$$

where $U^*_{ref}$ is the reference velocity to be defined. Introducing the Reynolds number

$$\mathsf{Re} = \frac{\rho^* U^*_{ref} H^*}{\mu^*_0}, \tag{9}$$

and using the adimensionalization (8), system (4) and the constitutive law (3) become

$$\begin{cases} \mathsf{Re}\,\dot{\mathbf{v}} = -\nabla p + \mathrm{div}\mathbb{S} + \mathbf{g}, \\ \\ \mathrm{div}(\mathbf{v}) = 0, \end{cases} \tag{10}$$

$$\mathbb{S} = 2\mu(p)\mathbb{D}, \tag{11}$$

respectively, where $\mathbf{g} = \xi\mathbf{e}_x + \xi\cot\theta\mathbf{e}_y$,

$$\xi = \frac{\mathsf{Re}}{\mathsf{Fr}^2}\sin\theta = \frac{\rho^* g^* {H^*}^2}{\mu_0^* U_{ref}^*}\sin\theta, \tag{12}$$

and

$$\mathsf{Fr}^2 = \frac{{U_{ref}^*}^2}{g^* H^*}, \tag{13}$$

is Froude number. From (9) and (12) we have

$$\begin{aligned} H^* &= \left(\frac{\xi{\mu_0^*}^2}{{\rho^*}^2 g^* \sin\theta}\mathsf{Re}\right)^{1/3}, \\ U_{ref}^* &= \frac{\mu_0^*}{\rho^* H^*}\mathsf{Re} = \left(\frac{g^*\mu_0^*\sin\theta}{\xi\rho^*}\right)^{1/3}\mathsf{Re}^{2/3}. \end{aligned} \tag{14}$$

We look for a one dimensional laminar stationary flow, namely a solution in the form

$$\boldsymbol{v} = u(y)\mathbf{e}_x, \tag{15}$$

and $h = 1$, so that system (10) reduces to

$$\begin{cases} 0 = -p_x + S_{12,y} + \xi, \\ \\ 0 = -p_y - \xi\cot\theta. \end{cases} \tag{16}$$

System (16), entails

$$p(y) = \xi\cot\theta\,(1-y)\,, \tag{17}$$

and, by noting that $4||\mathbb{D}||^2 = u'(y)^2$, we have

$$u'(y) = \xi(1-y)e^{-\delta\xi(1-y)}, \tag{18}$$

where

$$\delta(\Lambda,\xi,\mathsf{Re}) = \delta^* \frac{\mu_0^* U_{ref}^*}{H^*} \cot\theta \underset{(14)}{=} \delta^* \mu_0^{*2/3} \rho^{*1/3} g^{*1/3} (\sin\theta)^{2/3} \cot\theta \frac{\mathsf{Re}^{1/3}}{\xi^{2/3}} = \Lambda \frac{\mathsf{Re}^{1/3}}{\xi^{2/3}}, \quad (19)$$

$$\begin{cases} \Lambda = \delta_m \Theta(\theta), \\ \delta_m = \delta^* \mu_0^{*2/3} \rho^{*1/3} g^{*1/3}, \\ \Theta(\theta) = (\sin\theta)^{2/3} \cot\theta, \end{cases} \quad (20)$$

and (1), (8), (11) have been exploited. From (18) and recalling that $u(0) = 0$, we have

$$u(y) = \frac{[1 + \xi\delta(1-y)]\, e^{-\delta\xi(1-y)} - (1+\xi\delta)\, e^{-\delta\xi}}{\xi\delta^2}, \quad (21)$$

that is, using (19),

$$u(y) = \frac{\left[1 + \Lambda\,(\xi\mathsf{Re})^{1/3}\,(1-y)\right] e^{-\Lambda(\xi\mathsf{Re})^{1/3}(1-y)}}{\Lambda^2 \xi^{-1/3} \mathsf{Re}^{2/3}} - \frac{\left[1 + \Lambda\,(\xi\mathsf{Re})^{1/3}\right] e^{-\Lambda(\xi\mathsf{Re})^{1/3}}}{\Lambda^2 \xi^{-1/3} \mathsf{Re}^{2/3}}. \quad (22)$$

By requiring $u(1) = 1$, we obtain

$$\mathcal{F}(\xi,\mathsf{Re}) = \frac{1 - (\xi\delta + 1) e^{-\delta\xi}}{\xi\delta^2} = \frac{1 - \left[\Lambda\,(\xi\mathsf{Re})^{1/3} + 1\right] e^{-\Lambda(\xi\mathsf{Re})^{1/3}}}{\Lambda^2 \xi^{-1/3} \mathsf{Re}^{2/3}} = 1. \quad (23)$$

Figure 2 displays different velocity profiles by varying the values of $\xi \in (0.1, 5)$ in (21) and that the velocity field of the classical Newtonian flow is recovered when $\xi = 2$ (red line). In fact, by applying the product rule of limits and the Hospital's rule, we obtain that $\mathcal{F}(\mathsf{Re},\xi) \to \xi/2$ when $\Lambda \to 0$ and, thus, from (23) we have $\xi = 2$ when $\Lambda \to 0$ for any $\mathsf{Re}$. Figure 3a shows the plot $\mathcal{F}(\mathsf{Re},\xi)$ for $\Lambda = 0.63$, i.e. $\delta^* = 0.01$ kg/(ms$^2$), and $\theta = 5°$. In particular,

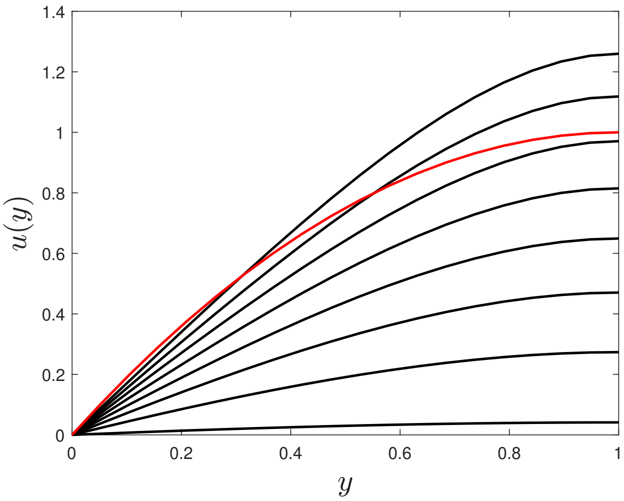
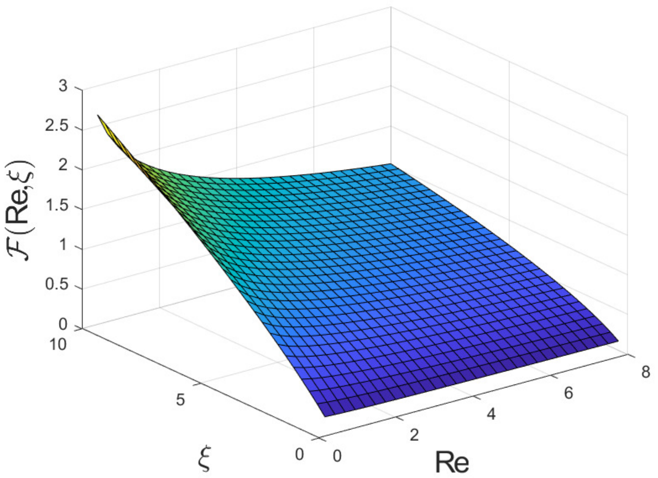



Figure 2: Plot of $u(y)$ for different values of $\xi \in (0.1, 5)$ with $\mathsf{Re} = 1$, $\Lambda = 0.63$ (i.e. $\delta^* = 0.01$ kg/(ms$^2$)), and $\theta = 5°$. In particular, the red line represents the case of no pressure-dependent viscosity, i.e. $\xi = 2$ for any $\mathsf{Re}$, showing that the velocity field of the classical Newtonian flow is recovered.

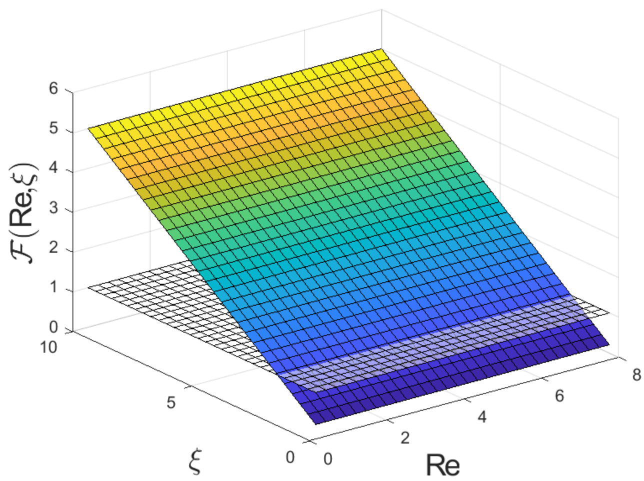
as expected, we obtain a one-to-one relation between $\mathsf{Re}$ and $\xi$ and we denote by $\hat{\xi}(\mathsf{Re})$ the unique solution to (23) so that $u(1) = 1$. In fact, for given values of the geometrical and material parameters, from the normalization of $u$ we obtain a unique value of $\xi$, denoted as $\hat{\xi}$, for any $\mathsf{Re}$. Moreover, in Figure 3b the evolution of $\mathcal{F}(\mathsf{Re}, \xi)$ for $\Lambda \to 0$, i.e. $\delta^* \to 0$ kg/(ms$^2$), is reported showing that the classical Newtonian case (no pressure-dependent viscosity) is retrieved.

## 3. Differential system governing linear stability analysis: Long-wave approximation method

In this section, we consider the basic laminar flow (15), i.e. $h(x,t) = h_b$, with $h_b = 1$, $\mathbf{v}_b = u_b(y)\mathbf{e}_x$, where $u_b(y)$ is given by (22), $p = p_b(y) = \hat{\xi}\cot\theta(1-y)$. We perturb the basic flow by superimposing a “small” 2D disturbance in the




(a) Pressure-dependent viscosity.




(b) No pressure-dependent viscosity.

Figure 3: Plot of $\mathcal{F}(\mathsf{Re}, \xi)$ for (a) $\Lambda = 0.63$, i.e. $\delta^* = 0.01$ kg/(ms$^2$), and (b) $\Lambda \to 0$, i.e. $\delta^* \to 0$ kg/(ms$^2$) (no pressure-dependent viscosity). In particular, for both cases we have assumed $\theta = 5°$.

form of travelling wave as in [18, 19, 24]

$$\begin{cases} h = 1 + \hat{h}e^{i\alpha(x-ct)}, \\ \\ \mathbf{v} = \mathbf{v}_b(y) + \hat{\mathbf{v}} = \left(u_b\left(y\right) + \hat{u}(y)e^{i\alpha(x-ct)}\right)\mathbf{e}_x + \hat{v}(y)e^{i\alpha(x-ct)}\mathbf{e}_y, \\ \\ p = p_b(y) + \hat{p}(y)e^{i\alpha(x-ct)}, \end{cases} \tag{24}$$

and

$$\mathbb{D} = \mathbb{D}_b + \hat{\mathbb{D}}e^{i\alpha(x-ct)}, \quad \mathbb{S} = \mathbb{S}_b + \hat{\mathbb{S}}e^{i\alpha(x-ct)}, \tag{25}$$

where $\alpha \in \mathbb{R}$ is the wave number, $c \in \mathbb{C}$ is the complex wave speed and the notation $\hat{(\cdot)}$ represents the amplitude of the infinitesimal disturbance. Moreover, the viscosity can be rewritten in terms of the perturbed pressure as

$$\begin{aligned} \mu(p) = e^{\delta(p_b(y)+\hat{p}(y)e^{i\alpha(x-ct)})} = e^{\delta p_b(y)}e^{\delta\hat{p}(y)e^{i\alpha(x-ct)}} &\approx e^{\delta p_b(y)}\left(1 + \delta\hat{p}(y)e^{i\alpha(x-ct)}\right) \\ &= \mu_b(y) + \hat{\mu}(y)e^{i\alpha(x-ct)}, \end{aligned} \tag{26}$$

where

$$\begin{cases} \mu_b(y) = e^{\delta p_b(y)} = e^{\delta\hat{\xi}(1-y)} \underset{(19)}{=} e^{\Lambda(\hat{\xi}\mathsf{Re})^{1/3}(1-y)}, \\ \\ \hat{\mu}(y) = \delta\mu_b(y)\hat{p}(y). \end{cases} \tag{27}$$

Then, we express the velocity field in terms of the stream function

$$\hat{\psi}(x, y, t) = \phi(y)e^{i\alpha(x-ct)},$$

as

$$\begin{aligned} \hat{u}(y)e^{i\alpha(x-ct)} &= \hat{\psi}_y = \phi'\left(y\right)e^{i\alpha(x-ct)}, \\ \hat{v}(y)e^{i\alpha(x-ct)} &= -\hat{\psi}_x = -i\alpha\phi(y)e^{i\alpha(x-ct)}, \end{aligned} \tag{28}$$

where, here and in the sequel, $(\cdot)'$ denotes the differentiation w.r.t. $y$. Inserting the perturbations (24)-(28) into system (10), we obtain

$$\begin{cases} \mathsf{Re}\left(-i\alpha c\phi' + i\alpha u_b\phi' - i\alpha u_b'\phi\right) = -i\alpha\hat{p} + i\alpha\hat{S}_{11} + \hat{S}_{12}', \\ \\ \mathsf{Re}\left(\alpha^2 c\phi + \alpha^2\phi\right) = -\hat{p}' + i\alpha\hat{S}_{12} + \hat{S}_{22}'. \end{cases} \tag{29}$$

Since we have

$$\begin{aligned}\mathbb{S} = 2\mu(y)\mathbb{D} = 2\left(\mu_b(y) + \hat{\mu}(y)e^{i\alpha(x-ct)}\right)\left(\mathbb{D}_b + \hat{\mathbb{D}}e^{i\alpha(x-ct)}\right)\\ \approx 2\mu_b(y)\mathbb{D}_b + 2\mu_b(y)\left(\hat{\mathbb{D}} + \delta\hat{p}(y)\mathbb{D}_b\right)e^{i\alpha(x-ct)}\\ = \mathbb{S}_b + \hat{\mathbb{S}}e^{i\alpha(x-ct)},\end{aligned} \quad (30)$$

we can write

$$\hat{\mathbb{S}} = 2\left(\mu_b(y)\hat{\mathbb{D}} + \delta\mu_b(y)\hat{p}(y)\mathbb{D}_b\right)$$

$$= \begin{pmatrix} 2i\alpha\mu_b(y)\hat{u}(y) & \mu_b(y)(\hat{u}'(y) + i\alpha\hat{v}(y)) + \delta\mu_b(y)\hat{p}(y)u_b'(y) \\ \mu_b(y)(\hat{u}'(y) + i\alpha\hat{v}(y)) + \delta\mu_b(y)\hat{p}(y)u_b'(y) & 2\mu_b(y)\hat{v}'(y) \end{pmatrix}$$

$$= \begin{pmatrix} 2i\alpha\mu_b(y)\phi'(y) & \mu_b(y)(\phi''(y) + \alpha^2\phi(y)) + \delta\mu_b(y)\hat{p}(y)u_b'(y) \\ \mu_b(y)(\phi''(y) + \alpha^2\phi(y)) + \delta\mu_b(y)\hat{p}(y)u_b'(y) & -2i\alpha\mu_b(y)\phi'(y) \end{pmatrix}. \quad (31)$$

Therefore, system (29) becomes

$$\begin{cases} i\alpha\mathsf{Re}\left[(-c+u_b)\,\phi'(y) - u_b'(y)\phi(y)\right] = -i\alpha\hat{p}(y) \\ \qquad +i\alpha\left(2i\alpha\mu_b(y)\phi'(y)\right) \\ \qquad +\left[\mu_b(y)\left(\phi''(y) + \alpha^2\phi(y)\right)\right]' \\ \qquad +\delta\left(\mu_b(y)u_b'(y)\hat{p}(y)\right)', \\ \\ \alpha^2\mathsf{Re}\,(c+1)\,\phi(y) = -\hat{p}'(y) + i\alpha\mu_b(y)(\phi''(y) + \alpha^2\phi(y)) \\ \qquad +i\alpha\delta\mu_b(y)u_b'(y)\hat{p}(y) \\ \qquad -2i\alpha\left(\mu_b(y)\phi'(y)\right)', \end{cases} \quad (32)$$

which, following [18, 19, 24], has to be solved by coupling it with the perturbed boundary conditions, which, recalling conditions (5) and (6), are

$$\begin{cases} \phi(0) = 0, \\ \\ \phi'(0) = 0, \\ \\ \hat{h}(1-c) = -\phi(1), \\ \\ \hat{h}p_b'(1) + \hat{p}(1) - \hat{S}_{22}(1) = 0, \\ \\ \hat{h}S_{b_{12}}'(1) + \hat{S}_{12}(1) = 0. \end{cases} \tag{33}$$

After some algebra, system (32) and conditions (33) can be rewritten as

$$\begin{cases} \mu_b(y)\phi'''(y) = (\alpha A_{01}(y) + \alpha^2 A_{02}(y) \\ \qquad\qquad +\alpha^3 A_{03}(y))\, \phi(y) + (\alpha A_{11}(y) \\ \qquad\qquad +\alpha^2 A_{12}(y))\, \phi'(y) + (\alpha A_2(y) \\ \qquad\qquad +\mu_b'(y))\, \phi''(y) + \hat{p}(y) \left[\alpha A_p(y) \right. \\ \qquad\qquad \left. +\delta \left(-u_b'(y)\mu_b'(y) - \mu_b(y) u_b''(y)\right)\right], \\ \\ \hat{p}'(y) = (\alpha^2 B_{01}(y) + \alpha^3 B_{02})\, \phi(y) + \alpha B_1(y)\phi'(y) \\ \qquad\qquad +\alpha B_2(y)\phi''(y) + \alpha B_p(y)\hat{p}(y), \\ \\ \phi(0) = \phi'(0) = 0, \\ \\ \phi''(1) + \phi(1)\left(\alpha^2 - \dfrac{\hat{\xi}}{c-1}\right) = 0, \\ \\ -\dfrac{\hat{\xi}}{c-1}\cot\theta\phi(1) + 2i\alpha\phi'(1) + \hat{p}(1) = 0, \end{cases} \tag{34}$$

where

$$\begin{cases} A_{01} = -iu_b'(y)\mathsf{Re}, \\ \\ A_{02} = \mathsf{Re}\delta(c+1) - \mu_b'(y), \\ \\ A_{03} = -i\delta\mu_b^2(y)u_b'(y), \\ \\ A_{11} = -i\left[-2\delta\mu_b'(y)\mu_b(y)u_b'(y) + \mathsf{Re}\left(c - u_b(y)\right)\right], \\ \\ A_{12} = \mu_b(y), \\ \\ A_2 = i\delta u_b'(y)\mu_b^2(y), \\ \\ A_p = i\left(1 - u_b'^2(y)\mu_b^2(y)\right), \end{cases} \tag{35}$$

and

$$\begin{cases} B_{01} = -\mathsf{Re}\left(c+1\right), \\ \\ B_{02} = i\mu_b(y), \\ \\ B_1 = -2i\mu_b'(y), \\ \\ B_2 = -i\mu_b(y), \\ \\ B_p = i\delta\mu_b(y)u_b'(y). \end{cases} \tag{36}$$

It is worth highlighting that the classical Orr-Sommerfeld equation and the corresponding boundary conditions for Newtonian case without the pressure-dependent viscosity are retrieved for $\delta \to 0$ (i.e., when $\delta_m \to 0$), and thus $\hat{\mu}(y) = 0$ (e.g., see [18, 19, 24] when $q(y) = 2$ and $s(y) = 1$). Now, following [18, 19, 57], we consider disturbances of long wavelength $\lambda = 2\pi/\alpha$ (i.e. $\lambda \gg 1$ and $\alpha \ll 1$) and we look for solutions of the eigenvalue problem expanding $\phi$ and $c$ in powers of $\alpha$, namely

$$\begin{aligned}
&\phi(y) = \phi_0(y) + \alpha\phi_1(y) + \alpha^2\phi_2(y) + \cdots , \\
&c = c_0 + \alpha c_1 + \alpha^2 c_2 + \cdots .
\end{aligned} \tag{37}$$

By inserting (37) into (34)-(36) and expanding in terms of $\alpha$, at the zero-th order we get

$$\begin{cases}
(\mu_b(y)\phi_0'''(y))' = \hat{p}_0(y)\delta\left(-u_b'(y)\mu_b'(y) - u_b''(y)\mu_b(y)\right), \\
\hat{p}_0(y) = 0, \\
\phi_0(0) = \phi_0'(0) = 0, \\
\phi_0''(1) + \phi_0(1)\dfrac{\hat{\xi}}{c_0 - 1}, \\
\phi_0(1)\dfrac{\hat{\xi}\cot\theta}{c_0 - 1} + \phi_0(1) = 0,
\end{cases} \tag{38}$$

whose solution is

$$\begin{cases}
\phi_0(y) = \hat{p}_0 m_0(y) + \dfrac{y^2}{2}, \\
\hat{p}_0 = \dfrac{\cot\theta}{(c_0 - 1)(1 + m_0(1))}, \\
c_0 = \left(\cot\theta\; m_0(1) + \dfrac{1 - \cot\theta\; m_0''}{2}\right)\hat{\xi} + 1,
\end{cases} \tag{39}$$

where

$$m_0(y) = \delta \int_0^y \left( \int_0^{\tilde{y}} \left( \frac{1}{\mu_b(\tilde{\tilde{y}})} \int_0^{\tilde{\tilde{y}}} m_{01}(z)dz \right) d\tilde{\tilde{y}} \right) d\tilde{y}, \tag{40}$$

with

$$m_{01}(z) = -u_b'(z)\mu_b'(z) - u_b''(z)\mu_b(z). \tag{41}$$

Since the zeroth order analysis leads to a real eigenvalue, such analysis does not provide sufficient information regarding stability. Thus, we proceed with investigations of the first order as in [18, 19]. At the first order, by considering (38), we have

$$\begin{cases} \mu_b(y)\phi_1'''(y) = A_{01}(y)\phi_0(y) + A_{11}(y)\phi_0'(y) \\ \qquad\qquad +A_2(y)\phi_0''(y) - \mu_b'(y)\phi_1''(y) + \hat{p}_0\alpha A_p(y) \\ \qquad\qquad +\delta\hat{p}_1(y)\left(-u_b'(y)\mu_b'(y) - u_b''(y)\mu_b(y)\right), \\ \\ \hat{p}_1'(y) = B_1(y)\phi_0'(y) + B_2(y)\phi_0''(y) + B(y)\hat{p}_0, \\ \\ \phi_1(0) = \phi_1'(0) = 0, \\ \\ \phi_1''(1) + \phi_0(1)\dfrac{\hat{\xi}}{c_0 - 1}c_1, \\ \\ \hat{\xi}\cot\theta\phi_0(1)\dfrac{c_1}{(c_0-1)^2} + 2i\phi_0'(1) + \hat{p}_1(1) = 0, \end{cases} \tag{42}$$

whose solution is

$$\begin{cases} \phi_1(y) = M_0(y) + D_1 M_1(y), \qquad D_1 \in \mathbb{R}, \\ \\ \hat{p}_1(y) = P_1(y) + K_1, \qquad K_1 \in \mathbb{R}, \end{cases} \tag{43}$$

where

$$\begin{aligned} M_0(y) = \int_0^y \int_0^{\tilde{y}} \frac{1}{\mu_b(\tilde{\tilde{y}})} \int_0^{\tilde{\tilde{y}}} & [A_{01}(z)\phi_0(z) + A_{11}(z)\phi_0'(z) \\ & +A_2(z)\phi_0''(z) - \mu_b'(z)\phi_1''(z) + \hat{p}_0 A_p(z) \\ & +\delta\hat{p}_1(z)m_{01}(z)]\, dz d\tilde{\tilde{y}} d\tilde{y}, \end{aligned} \tag{44}$$

$$M_1(y) = \int_0^y \int_0^{\tilde{y}} \frac{1}{\mu_b(\tilde{\tilde{y}})} d\tilde{\tilde{y}} d\tilde{y}, \tag{45}$$

$$P_1(y) = \int_0^y \left(B_1(z)\phi_0'(z) + B_2(z)\phi_0''(z) + B_p(z)\hat{p}_0\right) dz, \tag{46}$$

$$K_1 = \frac{-2i\phi_0'(1) - P_1(1) + \cot(\theta)M_0''(1) - 2\cot(\theta)M_0(1)}{1 - \cot(\theta)m_0''(1) + 2\cot(\theta)m_0(1)}. \quad (47)$$

Moreover, we impose $\phi(1) = \phi_0(1)$ and, consequently, $\phi_i(1) = 0$ $i = 1, 2, ...$, so that the solution (43) is non-trivial provided

$$c_1 = -\frac{\phi_1''(1)}{\phi_0(1)}\frac{(c_0-1)^2}{\hat{\xi}} = i\mathcal{G}(\mathsf{Re}, \Lambda), \quad (48)$$

where

$$\begin{aligned}\phi_1''(1) &= M_0''(1) + D_1 M_1''(1)\\ &= \frac{1}{\mu_b(1)}\int_0^1 [A_{01}(z)\phi_0(z) + A_{11}(z)\phi_0'(y) + A_2(z)\phi_0''(z)\\ &\qquad -\mu_b'(z)\phi_1''(z) + \hat{p}_0 A_p(z) + \delta\hat{p}_1(z)m_{01}(z)]\,dz + \frac{D_1}{\mu_b(1)}, \quad (49)\end{aligned}$$

$$D_1 = -\frac{M_0(1)}{M_1(1)}, \quad (50)$$

and $\phi_0$, $\hat{p}_0$, and $c_0$ are given by (39).
We recall that the critical value of $\mathsf{Re}$, denoted as $\mathsf{Re}_c$, can be found as zeros of the imaginary part of $c$, namely $\Im(c) = \Im(c_1) = \mathcal{G}(\mathsf{Re}, \Lambda) = 0$, once the material and geometrical characteristics are prescribed. In particular, the $\alpha$th mode is stable when $\mathsf{Re} < \mathsf{Re}_c$ and is unstable when $\mathsf{Re} > \mathsf{Re}_c$.

## 4. Results

The critical value of the Reynolds number, $\mathsf{Re_c}$, is computed by solving (23) and finding zeros of (48) with MATLAB® 2022a, using the function FZERO.
The variation of the critical Reynolds number, $\mathsf{Re}_c$, with respect of the tilt angle $\theta$ for different values of $\delta_m$ is depicted in 4. The classical Newtonian case is recovered when $\delta_m \to 0$. In particular, the values of $\mathsf{Re}_c$ coincide with $5/4\cot\theta$ when $\delta_m \to 0$ as in [18, 19], see Figure 4. Moreover, an increase of $\theta$ induces a flow destabilization [18, 19] at a given $\delta_m$ when $\delta_m < \delta_{m,c}$ with $\delta_{m,c} \in (0.0066, 0.0067]$, see Fig.s 4 and 5. In fact, $\mathsf{Re}_c$ is a decreasing function of $\theta$ at a given $\delta_m$ when $\delta_m < \delta_{m,c}$. In a recent work, the two-dimensional linear stability of a regularized Casson [19] fluid flowing down an incline by using the long-wave approximation method was studied. In

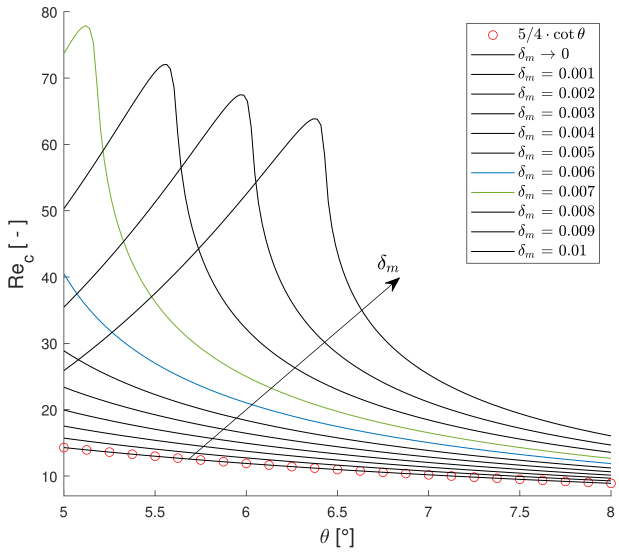



Figure 4: Evolution of the critical Reynolds number, $\mathsf{Re}_c$, with respect to the tilt angle, $\theta$, with different values of the "material" parameter $\delta_m$. The empty circles represent the case of the classical Newtonian fluid. The trend of $\mathsf{Re}_c$ is a non-monotone function of $\delta_m$ and $\theta$ between the blue line ($\delta_m = 0.006$) and the green line ($\delta_m = 0.007$).

particular, it was shown that for the regularized Casson flow an increase in the material parameters of the fluid induces a stabilizing effect. Although we are comparing different models, for $\delta_m < \delta_{m,c}$, we found that, similarly to [19], $\mathsf{Re}_c$ increases with increasing values of the "material" parameter $\delta_m$. Unexpectedly, for values of $\delta_m$ above $\delta_{m,c}$ we have that the evolution of $\mathsf{Re}_c$ is not monotonically

- increasing as $\delta_m$ increases, and
- decreasing as $\theta$ increases,

see Fig. 4 and 5. In particular, Figure 5 shows that $\delta_{m,c} \in (0.0066, 0.0067]$. Thus, the dependence of viscosity on pressure (the pressure coefficient is proportional to $\delta_m$, see formulas (1), (19), and (20)) leads to a stabilizing effect on the flow with respect to the classical Newtonian case when $\delta_m < \delta_{m,c}$ at a given $\theta$.

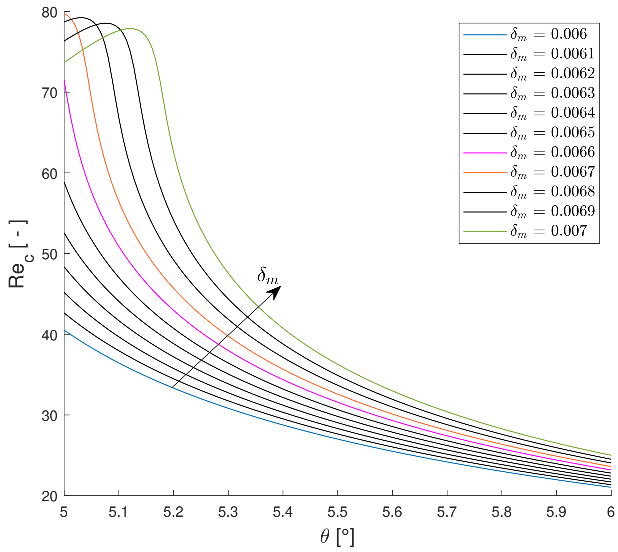



Figure 5: Zoom of the evolution of the critical Reynolds number, $\mathsf{Re}_c$, with respect to the tilt angle, $\theta$, with different values of $\delta_m \in [0.006, 0.007]$, showing that the critical value of $\delta_m$ is $\delta_{m,c} \in (0.0066, 0.0067]$, i.e. between the magenta and the orange line.

## 5. Conclusions

In this paper we have theoretically investigated the stability analysis of a fluid with pressure-dependent viscosity down an inclined plane. In particular, we have selected an exponential dependence of viscosity on pressure as proposed by Barus in [50]. Following [18, 19, 57], we have analysed the linear stability using the long-wave approximation method. Our results show the existence of a critical Reynolds number, $\mathsf{Re}_c$, which depends on the tilt angle, $\theta$, and the material parameters similarly to [18, 19, 24]. The classical Newtonian case (used as a benchmark in this paper) has been retrieved. In fact, the classical proportionality relation between the critical Reynolds number, $\mathsf{Re}_c$, and the tilt angle, $\theta$, has been recovered when the "material" parameter $\delta_m \to 0$. Although we are using a different model, similar results to [19] have been obtained when $\delta_m$ is below of a critical value $\delta_{m,c}$. In particular, when $\delta_m < \delta_{m,c}$, $\mathsf{Re}_c$ is a decreasing function of the tilt angle for a given $\delta_m$ (destabilizing effect on the flow), while $\mathsf{Re}_c$ is an increasing function of $\delta_m$ for a given $\theta$ (stabilizing effect on the flow). Thus, the increase of $\delta_m$ can lead to an increasingly stable effect when $\delta_m < \delta_{m,c}$ for a given $\theta$. Unexpectedly, our results show that, when $\delta_m > \delta_{m,c}$, $\mathsf{Re}_c$ is a non-monotone decreasing function of the tilt angle at a given $\delta_m$ and it is a non-monotone increasing function of $\delta_m$ at a given $\theta$. Since the pressure coefficient, $\delta^*$, in the viscosity expression (see formulas (1), (19), and (20)) is proportional to $\delta_m$, the dependence of viscosity on pressure can influence the stability properties of the fluid flowing down the incline. As next step it would be extremely interesting to deepen such stability characteristics when the fluid has pressure-dependent viscosity with further more exhaustive stability analyses through the comparison of theoretical, numerical, and experimental studies.

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## Authors declarations

*Declaration of competing interest*

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## Data Availability Statement

No data was used for the research described in the article.

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**Declaration of interests**

☒ The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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