Z. Angew. Math. Phys. (2026) 77:264

https://doi.org/10.1007/s00033-026-02918-2

Zeitschrift für angewandte Mathematik und Physik ZAMP

Damage-induced paths in the partially homogenized dynamics of complex networks

Paolo Maria Mariano and Marco Spadini

Abstract. We consider complex lattices whose elementary constituents are interacting sub-lattices. A Cauchy–Born-type matching rule at the sublattice level leads to a partially homogenized Hamiltonian description in which translational and coarse microstructural degrees of freedom coexist. Under T-periodic external forcing, we establish the existence of T-periodic motions under a suitable non-T-resonance condition. Structural damage modifies the Hamiltonian itself and induces a corresponding spectral evolution. For an FPUT-type chain with an additional internal degree of freedom, chosen as a prototype example, we show that monotone bond degradation can produce a transversal resonance crossing and a local bifurcation of periodic motions. The same transition is characterized through the Poincaré map and, under nonzero forcing, unfolds into a local fold of T-periodic solutions.

Mathematics Subject Classification. 37J06, 34C23, 37J12, 74A45.

Keywords. Lattice dynamics, Homogenization, Resonance, Recurrence, Damage.

1. Introduction

The dynamics of systems composed of interacting units endowed with internal structure arises in several areas of mechanics and materials science. Typical examples include molecular aggregates, polymeric chains, and multilevel architected materials (see, e.g., the treatises [7,16,17] for pertinent examples and general issues). In such systems, the collective behavior depends not only on interactions among neighboring units but also on the internal organization of each constituent. A description based solely on the positions of material units may therefore miss dynamically relevant degrees of freedom.

A standard route from discrete systems to continuum models is based on Cauchy–Born-type procedures [4,8,18]. When the internal structure of each constituent remains relevant at the scale of observation, however, complete homogenization may eliminate precisely the degrees of freedom responsible for part of the dynamics. Here we adopt instead a partial homogenization: The internal configuration of each sub-lattice is represented by a collective tensorial descriptor (see also [2] for an analogous coarse description of proteins), while the interactions among neighboring sub-lattices remain discrete. The resulting system combines translational and microstructural degrees of freedom within a finite-dimensional Hamiltonian structure.

The construction shares with objective structures [14] the use of collective variables to reduce a complex microscopic configuration, but its purpose is different. What we obtain from a partially homogenized lattice structure is the Hamiltonian of a discretized elastic complex body with affine structure (for the general model-buildign framework for the mechanics of complex bodies, see [15] and references therein). We are concerned with the dynamics generated by partial homogenization and, in particular, with the distinct effects of external periodic forcing and structural damage. Periodic forcing acts on a fixed Hamiltonian structure, whereas damage modifies the interaction potentials and consequently generates a path in the family of Hamiltonian systems.

Published online: 07 September 2026

For a T-periodic external perturbation, existence of T-periodic motions is obtained by degree-theoretical arguments [6] under a non-T-resonance condition involving the spectrum of the linearized autonomous dynamics. Specifically, the techniques we use have been developed in [9,10] (see also [3,11]). However, those works do not deal with Hamiltonian structures similar to those considered here and, above all, do not treat damage [3,9,10] or consider it in a way essentially numerical [11]. Damage changes this spectrum and may drive a characteristic frequency through a resonant value. We show that this mechanism can produce an actual dynamical transition rather than merely the loss of a sufficient existence condition.

To make the transition explicit, we specialize the partially homogenized architecture to a Fermi–Pasta–Ulam–Tsingou-type chain whose units possess an additional internal expansion degree of freedom. For a simple resonant mode, monotone degradation of the inter-unit stiffness gives a transversal spectral crossing whenever the critical mode effectively deforms the damaged bonds. A Lyapunov–Schmidt reduction then yields the local bifurcation equation (see the treatises [1,12] for related techniques) and shows explicitly how the internal stiffness enters both the resonant state and the nonlinear coefficient determining the emerging branch.

The same transition admits an intrinsic description through the Poincaré map. In the unforced problem, the spectral crossing corresponds exactly to a pair of Floquet multipliers reaching 1. For nonzero periodic forcing, the reduced equation provides the local unfolding of the autonomous bifurcation. When the forcing has a non-vanishing projection on the critical mode, the bifurcation is imperfect and gives a fold of T-periodic solutions. Thus, partial homogenization affects both the linear damage threshold and the nonlinear structure of the ensuing transition.

The paper is organized as follows: Section 2 introduces the partial homogenization of the complex lattice and the associated Hamiltonian structure. Section 3 distinguishes structural damage from periodic forcing. Section 4 recalls the degree-theoretical existence result for forced T-periodic motions. Section 5 analyzes damage-dependent spectral evolution and develops the structured FPUT reduction, the local bifurcation analysis and its Poincaré-map formulation. Conclusions are collected in Sect. 6.

Regarding notation, an interposed dot will denote duality pairing between an element of a linear space (whatever it is) and one of the dual space; it coincides with the scalar product when referred to orthonormal reference frames, a choice that we adopt here; thus, we avoid distinguishing between covariant and contravariant components. A superposed dot will indicate, as usual time derivative.

2. Partial homogenization of complex lattices

2.1. Complex lattices and matching rule

Consider a lattice formed by N interacting sub-lattices in three-dimensional space, each consisting of S connected point masses. The center of mass of the ith sub-lattice is located at $\bar{x}^{(i)}$ in a reference configuration, while $r^{(i;k)}$ denotes the reference position of its kth mass point in a local frame centered at $\bar{x}^{(i)}$. An indicative pictorial representation is shown in Fig. 1.

At time t, let $x^{(i)}(t)$ denote the position of the center of mass, with $x^{(i)}(0) = \bar{x}^{(i)}$, and let $y^{(i;k)}(t)$ be the position of the kth mass point. If $m^{(i;k)}$ and $p^{(i;k)}$ are its mass and momentum, respectively, the discrete Hamiltonian of the ith sub-lattice is:

$$H_{\mathrm{dis},i} := \sum_{k=1}^{S} \frac{p^{(i;k)} \cdot p^{(i;k)}}{2\, m^{(i;k)}} + U^{(i)}(y_S) + U^{(i)}_{\mathrm{ext}}(y_S),$$

where $y_S = (y^{(i;1)}, \ldots, y^{(i;S)})$ and

$$U^{(i)}(y_S) = \sum_{k<l} u(y^{(i;k)}, y^{(i;l)}) + \sum_{k<l<s} \bar{u}(y^{(i;k)}, y^{(i;l)}, y^{(i;s)}) + \cdots .$$

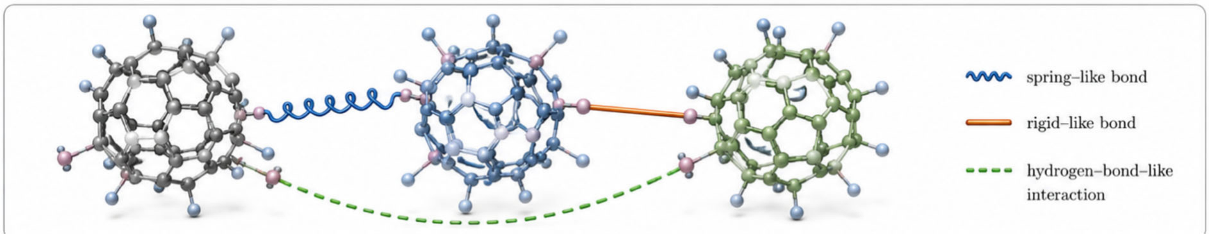

FIG. 1. Complex lattice of units with internal structure connected by different types of bonds. Links indicated as rigid are understood to possess hinged ends allowing relative rotations

We reduce the internal degrees of freedom of each sub-lattice through the following Cauchy–Born-type matching rule, prescribed only at sublattice level.

Assumption 1. *(Matching rule)* There exists a field

$$(x,t) \longmapsto \nu(x,t) \in \mathrm{Hom}(\mathbb{R}^3,\mathbb{R}^3)$$

differentiable in time such that

$$\begin{aligned} y^{(i;k)} &= \nu(x^{(i)}(t),t) r^{(i;k)} =: \nu^{(i)} r^{(i;k)}, \\ p^{(i;k)} &= m^{(i;k)} \dot{\nu}^{\flat}(x^{(i)}(t),t) r^{(i;k)} =: m^{(i;k)} \dot{\nu}^{\flat}_{(i)} r^{(i;k)}. \end{aligned}$$

The second-rank tensor $\nu^{(i)}$ is a collective descriptor of the morphology of the ith sub-lattice. Thus, each internally discrete unit is replaced by a homogeneously deformable element, while the interactions among different units remain discrete, as shown in Fig. 2.

Setting $\mu^{(i)} := \dot{\nu}^{(i)\flat}$, the partially homogenized Hamiltonian associated with the ith unit is obtained by imposing $H_{\mathrm{cont},i} = H_{\mathrm{dis},i}$ under the matching rule. It is

$$H_{\mathrm{cont},i}(\nu^{(i)},\mu^{(i)};r_S) = \frac{\mu^{(i)}\cdot\mu^{(i)}}{2}\cdot\sum_{k=1}^{S} r^{(i;k)}\otimes r^{(i;k)} + U^{(i)}(\nu^{(i)};r_S) + U^{(i)}_{\mathrm{ext}}(\nu^{(i)};r_S),$$

where $r_S = (r^{(i;1)},\ldots,r^{(i;S)})$. The matching rule therefore homogenizes the internal motion of each unit without homogenizing the lattice formed by their centers.

Assume that each sub-lattice contains a mass point at its center and that neighboring units interact through links joining these points. Denoting by $\bar{m}^{(i)}$ the total mass associated with the ith center and by

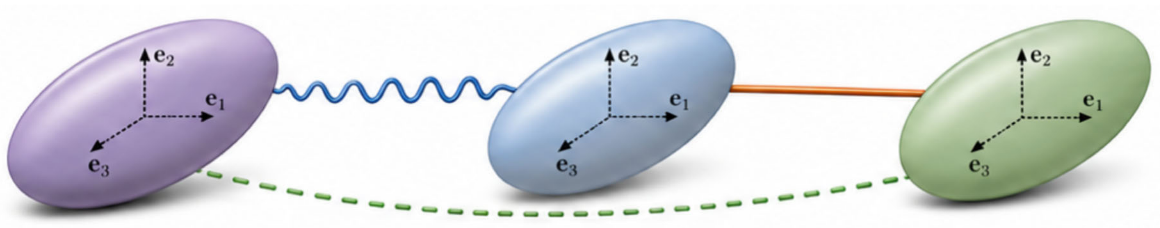

FIG. 2. Partially homogenized description: each sub-lattice is represented by a homogeneously deformable unit whose internal morphology is described by $\nu^{(i)}$

$p^{(i)}$ its momentum, the full Hamiltonian $\mathcal{H}$ of the partially homogenized lattice is

$$\mathcal{H} = \sum_{i=1}^{N} \frac{p^{(i)} \cdot p^{(i)}}{2\bar{m}^{(i)}} + \sum_{i=1}^{N} H_{\mathrm{cont},i}(\nu^{(i)}, \mu^{(i)}) + \widehat{U}(x_N) + \widehat{U}_{\mathrm{ext}}(x_N),$$

where $x_N = (x^{(1)}, \ldots, x^{(N)})$ and, in general,

$$\widehat{U}(x_N) = \sum_{k<l} u_G(x^{(k)}, x^{(l)}) + \sum_{k<l<s} \bar{u}_G(x^{(k)}, x^{(l)}, x^{(s)}) + \cdots .$$

The associated Hamilton equations are:

$$\dot{x}^{(i)} = p^{(i)}, \quad \dot{p}^{(i)} = -\partial_{x^{(i)}}\left(U^{(i)} + U^{(i)}_{\mathrm{ext}} + \widehat{U} + \widehat{U}_{\mathrm{ext}}\right),$$
$$\dot{\nu}^{(i)} = \mu^{(i)}, \quad \dot{\mu}^{(i)} = -\partial_{\nu^{(i)}}\left(U^{(i)} + U^{(i)}_{\mathrm{ext}}\right).$$

The resulting system is discrete with respect to the centers $x^{(i)}$ and homogenized only with respect to the internal configuration of each unit.

2.2. Pair interactions

For the remainder of the paper, we work considering orthonormal reference frames, so that $\mu^{\sharp} = \mu$ and $p^{\sharp} = p$, and restrict attention to pair interactions between neighboring units:

$$\widehat{U}(x_N) = \sum_{i=1}^{N-1} u_G(x^{(i+1)} - x^{(i)}).$$

Writing

$$\xi = (x, \nu),$$

the autonomous interaction potential will be denoted by $V(\xi)$. The corresponding autonomous force field is:

$$F(\xi) = -\nabla_{\xi} V(\xi).$$

Two distinct perturbations of this partially homogenized Hamiltonian system will be considered. The first is a T-periodic external forcing. The second is structural damage, intended in a parametric sense.

3. T-periodic forcing and related T-periodic solutions

Set $z = (x, p, \nu, \mu)$. We consider an external perturbation $\mathcal{P}(t, z)$ such that

$$\mathcal{P}(t, z) = \lambda \Phi(z, \omega t), \qquad \Phi(z, \theta + 2\pi) = \Phi(z, \theta), \qquad T = \frac{2\pi}{\omega},$$

where λ is the forcing amplitude. For example,

$$\mathcal{P}(t, z) = \lambda \left[\sum_{i=1}^{N} W_i(\nu_{(i)}) + \Psi(x_N)\right] \cos(\omega t).$$

The corresponding Hamiltonian is

$$\mathcal{H}_f(t, z) = \mathcal{H}(z) + \lambda \Phi(z, \omega t),$$

and Hamilton's equations read

$$\dot{z} = J\nabla_z \mathcal{H}_f(t, z) = J\nabla_z \mathcal{H}(z) + \lambda J \nabla_z \Phi(z, \omega t).$$

We seek motions satisfying $z(t + T) = z(t)$.

3.1. A degree-theoretical viewpoint

Let φ_T^λ denote the time-T flow map associated with

$$\dot{z} = J\nabla_z \mathcal{H}_f(t, z).$$

A T-periodic solution is a fixed point of the Poincaré map, namely

$$\varphi_T^\lambda(z_0) = z_0.$$

For T-periodic forcing, which always occurs in the autonomous case or when $\lambda = 0$, defining

$$P_\lambda(z) := \varphi_T^\lambda(z) - z,$$

one seeks zeros of P_λ. If $\Omega \subset \mathbb{R}^m$ is bounded and open,

$$P_\lambda(z) \neq 0 \quad \text{on } \partial\Omega, \qquad \deg(P_\lambda, \Omega, 0) \neq 0,$$

then the forced system admits at least one T-periodic solution with the initial datum in Ω. We use below standard properties of the Brouwer degree; see [6]. The continuation result needed here follows from the general theory developed in [9,10].

Write

$$\xi := (x, \nu), \qquad z = (\xi, \dot{\xi}),$$

using local coordinates for the microstructural variables when necessary, and set

$$\lambda\mathcal{K}(t, \xi, \dot{\xi}) := -\partial_\xi \mathcal{P}(t, z).$$

The periodically forced equations can then be written as:

$$\ddot{\xi} = \mathcal{F}(\xi) + \lambda\mathcal{K}(t, \xi, \dot{\xi}), \qquad \mathcal{F}(\xi) = -\nabla_\xi \mathcal{V}(\xi),$$

where

$$\mathcal{V}(\xi) = \sum_{i=1}^{N} U_{(i)}(\nu_{(i)}; r_S) + \widehat{U}(x_N).$$

3.2. Non-T-resonance

Let ξ_0 be an equilibrium of the autonomous system,

$$\mathcal{F}(\xi_0) = 0,$$

and let

$$A := D\mathcal{F}(\xi_0) = -D^2_{\xi\xi}\mathcal{V}(\xi_0).$$

The spectrum of A determines the characteristic frequencies associated with both translational and microstructural degrees of freedom. If

$$\det D^2_{\xi\xi}\mathcal{V}(\xi_0) \neq 0,$$

then ξ_0 is isolated and

$$\deg(\mathcal{F}, U, 0) \neq 0$$

for every sufficiently small neighborhood U of ξ_0.

The equilibrium is non-T-resonant [5] if

$$-\left(\frac{2\pi n}{T}\right)^2 \notin \sigma(A) \qquad \forall n \in \mathbb{Z}. \tag{1}$$

Under this condition, the linearized periodic problem has no non-trivial T-periodic solution.

Theorem 2. *Let ξ_0 be an isolated equilibrium of the autonomous partially homogenized lattice and assume that, for some bounded open neighborhood U of ξ_0,*

$$\deg(\mathcal{F}, U, 0) \neq 0.$$

Let

$$\mathcal{K} : \mathbb{R} \times U \times \mathbb{R}^m \longrightarrow \mathbb{R}^m$$

be continuous and T-periodic in the first variable. Then, the system

$$\ddot{\xi} = \mathcal{F}(\xi) + \lambda \mathcal{K}(t, \xi, \dot{\xi})$$

admits a connected set Γ of nontrivial T-forced pairs such that

$$\overline{\Gamma} \cap \left(\{0\} \times \{\bar{\xi} : \mathcal{F}(\bar{\xi}) = 0\}\right) \neq \emptyset.$$

If, in addition, ξ_0 satisfies the non-T-resonance condition (1), then Γ is not contained in the slice $\lambda = 0$. Consequently, there exists $\lambda^* > 0$ such that, for every $0 < \lambda < \lambda^*$, the system admits at least one T-periodic solution.

The condition (1) is therefore sufficient for continuation of forced T-periodic motions from an isolated equilibrium. In the next section, we examine how structural damage modifies this condition and what occurs when a damage-dependent characteristic frequency crosses a T-resonant value.

4. Damage-induced spectral evolution and periodic bifurcation

4.1. Damage

We describe damage by an observable scalar parameter

$$\mathfrak{d} = \mathfrak{d}(t) \in [0, 1], \qquad \dot{\mathfrak{d}}(t) \geq 0$$

(for related general issues, see the treatise [13]). For a fixed damage level $\mathfrak{d} \in [0, 1]$, we write the Hamiltonian as

$$\mathcal{H}(z, \mathfrak{d}) = \sum_{i=1}^{N} \frac{|p_{(i)}|^2}{2\bar{m}_{(i)}} + \sum_{i=1}^{N} \mathcal{H}_{\mathrm{cont},i}(\nu_{(i)}, \mu_{(i)}; \mathfrak{d}) + \widehat{U}(x_N; \mathfrak{d}).$$

Structural degradation may affect both intra-unit and inter-unit interactions. For instance,

$$\begin{aligned} U_{(i)}(\nu_{(i)}; r_S, \mathfrak{d}) &= U^0_{(i)}(\nu_{(i)}; r_S) + \mathcal{D}_{(i)}(\nu_{(i)}; r_S, \mathfrak{d}), \\ \widehat{U}(x_N; \mathfrak{d}) &= \widehat{U}^0(x_N) + \widehat{\mathcal{D}}(x_N, \mathfrak{d}), \end{aligned}$$

where $\mathcal{D}$ and $\widehat{\mathcal{D}}$ indicate the perturbations of the potentials, due to damage. For localized damage, one may take

$$\widehat{\mathcal{D}}(x_N, \mathfrak{d}) = \sum_{j \in \mathcal{I}_{\mathfrak{d}}} \mathfrak{d}\, \Delta u_j(x_{(j+1)} - x_{(j)}), \qquad D^2 \Delta u_j(0) \leq 0,$$

where $\mathcal{I}_{\mathfrak{d}}$ is the set of damaged bonds. Together with irreversibility, a sufficient condition for energetic degradation is

$$\dot{\mathfrak{d}} \geq 0, \qquad \partial_{\mathfrak{d}} \mathcal{H}(z, \mathfrak{d}) \leq 0.$$

Hence, along a damage path,

$$\mathcal{H}(z(T), \mathfrak{d}(T)) - \mathcal{H}(z(0), \mathfrak{d}(0)) = \int_0^T \partial_{\mathfrak{d}} \mathcal{H}(z(t), \mathfrak{d}(t)) \dot{\mathfrak{d}}(t)\, \mathrm{d}t \leq 0.$$

Thus, $\mathfrak{d}$ parametrizes a family of Hamiltonian structures. Along a solution of the corresponding non-autonomous Hamiltonian system,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_{\mathfrak{d}}(z(t),t) = \partial_t \mathcal{H}_{\mathfrak{d}}(z(t),t) = \partial_{\mathfrak{d}} \mathcal{H}(z(t),\mathfrak{d}(t))\,\dot{\mathfrak{d}}(t).$$

The damage variable is treated parametrically throughout the present work; no autonomous evolution law for $\mathfrak{d}$ is introduced.

4.2. Damage-dependent resonant set

Let

$$\mathcal{F}(\xi,\mathfrak{d}) := -D_\xi \mathcal{V}(\xi,\mathfrak{d}).$$

Assume that, for some $\mathfrak{d}_0$, there exists an equilibrium

$$\mathcal{F}(\xi_{\mathfrak{d}_0},\mathfrak{d}_0) = 0, \qquad \det D_\xi \mathcal{F}(\xi_{\mathfrak{d}_0},\mathfrak{d}_0) \neq 0.$$

The implicit function theorem gives, locally around $\mathfrak{d}_0$, a smooth branch

$$\mathfrak{d} \longmapsto \xi_{\mathfrak{d}}, \qquad \mathcal{F}(\xi_{\mathfrak{d}},\mathfrak{d}) = 0.$$

Along this branch define

$$A(\mathfrak{d}) := D_\xi \mathcal{F}(\xi_{\mathfrak{d}},\mathfrak{d}) = -D^2_{\xi\xi}\mathcal{V}(\xi_{\mathfrak{d}},\mathfrak{d}).$$

The spectrum of $A(\mathfrak{d})$ varies continuously with $\mathfrak{d}$. For a prescribed period T, introduce the resonant damage set

$$\mathcal{R}_T := \left\{ \mathfrak{d} \in [0,1] : \det\left[A(\mathfrak{d}) + \left(\frac{2\pi n}{T}\right)^2 I \right] = 0 \text{ for some } n \in \mathbb{Z} \right\}.$$

For every fixed $\mathfrak{d} \notin \mathcal{R}_T$ such that

$$\deg(\mathcal{F}(\cdot,\mathfrak{d}),U,0) \neq 0,$$

the continuation theorem of Sect. 3.1 applies. Thus, sufficiently small T-periodic forcing admits at least one T-periodic response. We now examine what occurs at a point of $\mathcal{R}_T$ for an explicit partially homogenized lattice.

5. An explicit bifurcation result in a special lattice structure

Throughout this section, we take $\lambda = 0$ until Sect. 5.3, where nonzero forcing is considered.

Consider a one-dimensional specialization of the architecture introduced above. We consider a periodic chain of N units, with indices understood modulo N,

$$u_{i+N} = u_i, \qquad a_{i+N} = a_i,$$

and admissible wave numbers $q = 2\pi\ell/N$. The ith unit has a translational displacement u_i and a scalar internal variable a_i describing its expansion; in other words, here $\nu = aI$, with $a \in \mathbb{R}$ and I the second-rank unit tensor. In the one-dimensional setting, a_i represents a relative internal displacement along the direction of the gross motion and is therefore unaffected by superposed rigid translations of the chain.

We define the effective elongation of the bond joining units i and $i+1$ by

$$\varepsilon_i := u_{i+1} - u_i - c(a_i + a_{i+1}),$$

where $c > 0$ measures the coupling between translation and internal expansion. The inter-unit interaction is taken of FPUT-$\alpha\beta$ type,

$$W_{\mathfrak{d}}(\varepsilon) = \frac{1}{2}k(\mathfrak{d})\varepsilon^2 + \frac{\alpha}{3}\varepsilon^3 + \frac{\beta}{4}\varepsilon^4,$$

while the internal elastic energy is

$$U(a) = \frac{1}{2}\kappa a^2.$$

Here $k(\mathfrak{d}) > 0$ is the differentiable damage-dependent bond stiffness, $\kappa > 0$ the internal stiffness, $m > 0$ the translational mass, and $J > 0$ the inertia associated with the internal variable. The Hamiltonian is

$$\mathcal{H}_{\mathfrak{d}} = \sum_i \left[\frac{m}{2}\dot{u}_i^2 + \frac{J}{2}\dot{a}_i^2 + \frac{\kappa}{2}a_i^2 + W_{\mathfrak{d}}(\varepsilon_i) \right].$$

The equations of motion are

$$\begin{aligned} m\ddot{u}_i &= W_{\mathfrak{d}}'(\varepsilon_i) - W_{\mathfrak{d}}'(\varepsilon_{i-1}), \\ J\ddot{a}_i &= -\kappa a_i + c\big[W_{\mathfrak{d}}'(\varepsilon_i) + W_{\mathfrak{d}}'(\varepsilon_{i-1})\big]. \end{aligned}$$

Since $W_{\mathfrak{d}}'(0) = 0$, the configuration $u_i = a_i = 0$ is an equilibrium for every $\mathfrak{d}$.

Linearization around this equilibrium gives

$$\begin{aligned} m\ddot{u}_i &= k(\mathfrak{d})\big[u_{i+1} - 2u_i + u_{i-1} - c(a_{i+1} - a_{i-1})\big], \\ J\ddot{a}_i &= -\kappa a_i + c\,k(\mathfrak{d})\big[u_{i+1} - u_{i-1} - c(a_{i+1} + 2a_i + a_{i-1})\big]. \end{aligned}$$

For a Bloch mode

$$u_i = \widehat{u}\,e^{\mathrm{i}(qi - \Omega t)}, \qquad a_i = \widehat{a}\,e^{\mathrm{i}(qi - \Omega t)},$$

a phase change in the internal amplitude reduces the spectral problem to

$$\big(K_q(\mathfrak{d}) - \Omega^2 M\big)v = 0, \qquad v = \begin{pmatrix} \widehat{u} \\ \widehat{a} \end{pmatrix}.$$

with

$$M = \begin{pmatrix} m & 0 \\ 0 & J \end{pmatrix}, \qquad K_q(\mathfrak{d}) = \begin{pmatrix} 4k(\mathfrak{d})s_q^2 & 4ck(\mathfrak{d})s_q c_q \\ 4ck(\mathfrak{d})s_q c_q & \kappa + 4c^2 k(\mathfrak{d})c_q^2 \end{pmatrix},$$

where

$$s_q := \sin\frac{q}{2}, \qquad c_q := \cos\frac{q}{2}.$$

Introducing

$$A_q := \frac{4k(\mathfrak{d})s_q^2}{m}, \qquad B_q := \frac{\kappa + 4c^2 k(\mathfrak{d})c_q^2}{J}, \qquad G_q := \frac{4ck(\mathfrak{d})s_q c_q}{\sqrt{mJ}},$$

the two dispersion branches are

$$\Omega_{\pm}^2(q,\mathfrak{d}) = \frac{1}{2}\left[A_q + B_q \pm \sqrt{(A_q - B_q)^2 + 4G_q^2} \right]. \tag{2}$$

Thus, partial homogenization produces two coupled spectral branches: one associated predominantly with translation and the other with the internal degree of freedom.

Let $\Omega_j^2(q,\mathfrak{d})$, $j \in \{+,-\}$, be a simple eigenvalue and let v_j be mass-normalized:

$$K_q(\mathfrak{d})v_j = \Omega_j^2(q,\mathfrak{d})Mv_j, \qquad v_j^T M v_j = 1.$$

The stiffness matrix may be written as:

$$K_q(\mathfrak{d}) = k(\mathfrak{d})\,r_q^T r_q + \kappa e_2 e_2^T, \qquad r_q = (2s_q, 2cc_q), \qquad e_2 = (0,1)^T.$$

Hence, if damage acts only through $k(\mathfrak{d})$,

$$K_q'(\mathfrak{d}) = k'(\mathfrak{d})\,r_q^T r_q.$$

Differentiating the generalized eigenvalue equation and using $v_j^T M v_j = 1$ gives

$$\frac{\mathrm{d}}{\mathrm{d}\mathfrak{d}}\Omega_j^2(q,\mathfrak{d}) = v_j^T K_q'(\mathfrak{d})v_j = k'(\mathfrak{d})|r_q v_j|^2. \tag{3}$$

Indeed,

$$K_q' v_j + K_q v_j' = (\Omega_j^2)' M v_j + \Omega_j^2 M v_j',$$

and multiplication by v_j^T, together with $v_j^T K_q = \Omega_j^2 v_j^T M$, cancels the terms containing v_j'.

Proposition 1. *Assume that $k'(\mathfrak{d}_*) \neq 0$ and that, for some q, j and $n_* \geq 1$,*

$$\Omega_j(q, \mathfrak{d}_*) = \frac{2\pi n_*}{T}.$$

If

$$r_q v_j(q, \mathfrak{d}_*) \neq 0,$$

the resonance crossing is transversal:

$$\frac{d}{d\mathfrak{d}} \Omega_j^2(q, \mathfrak{d}) \bigg|_{\mathfrak{d}=\mathfrak{d}_*} \neq 0.$$

If, in addition, $k'(\mathfrak{d}) < 0$, every simple mode that effectively deforms the damaged bonds has a strictly decreasing squared frequency as damage increases.

Proof. By (3),

$$\frac{\mathrm{d}}{\mathrm{d}\mathfrak{d}} \Omega_j^2(q, \mathfrak{d}) \bigg|_{\mathfrak{d}_*} = k'(\mathfrak{d}_*) |r_q v_j(q, \mathfrak{d}_*)|^2.$$

The right-hand side is nonzero under the stated assumptions. If $k' < 0$, it is strictly negative whenever $r_q v_j \neq 0$. □

The transversality condition is therefore not an additional spectral assumption for this lattice. It follows directly from monotone degradation of the inter-unit stiffness, provided the critical mode effectively strains the damaged bonds.

5.1. Local bifurcation at a resonant damage level

Fix a simple mode satisfying Proposition 1 and set

$$\omega_* := \Omega_j(q, \mathfrak{d}_*) = \frac{2\pi n_*}{T}, \qquad \gamma := \mathfrak{d} - \mathfrak{d}_*,$$

where $\mathfrak{d}_*$ is the resonant damage level. We write the autonomous equations as

$$M\ddot{y} + K(\mathfrak{d})y + \alpha B^T (By)^2 + \beta B^T (By)^3 = 0,$$

where $y = (u, a)$ and B is the linear bond operator,

$$(By)_i = u_{i+1} - u_i - c(a_i + a_{i+1}),$$

with powers of By understood componentwise.

We work in the space of even T-periodic functions, which removes the sine–cosine degeneracy associated with time translations. The critical mode is represented by

$$\phi(t) = v_* \cos(\omega_* t).$$

Let

$$\mathscr{L}_* := M \frac{d^2}{dt^2} + K(\mathfrak{d}_*).$$

We assume that

$$\ker \mathscr{L}_* = \operatorname{span}\{\phi\}$$

and that $\mathscr{L}_*$ is invertible on its orthogonal complement. In particular, this excludes resonances of the zero-frequency and second-harmonic components generated below by the quadratic nonlinearity. The rigid translational mode is removed by fixing the mean displacement.

Let Π denote the orthogonal projection onto span$\{\phi\}$ and $Q = I - \Pi$. We seek solutions in the form

$$y = \mathsf{a}\phi + w, \qquad \mathsf{a} \in \mathbb{R}, \qquad Qw = w,$$

where a is the scalar amplitude coordinate along $\ker \mathscr{L}_*$. The complementary equation

$$Q\Big[M\ddot{y} + K(\mathfrak{d})y + \alpha B^T(By)^2 + \beta B^T(By)^3\Big] = 0$$

has, by the implicit function theorem, a unique smooth solution

$$w = w(\mathsf{a}, \gamma), \qquad w(0, \gamma) = 0, \qquad D_{\mathsf{a}} w(0, 0) = 0,$$

for (a, γ) in a sufficiently small neighborhood of $(0, 0) \in \mathbb{R}^2$. In general,

$$w(\mathsf{a}, \gamma) = O(|\gamma||\mathsf{a}| + \mathsf{a}^2),$$

whereas at the resonant damage level

$$w(\mathsf{a}, 0) = \mathsf{a}^2 w_2 + O(|\mathsf{a}|^3).$$

Set $b := Bv_*$. At order a^2 and $\gamma = 0$, the complementary equation gives

$$\mathscr{L}_* w_2 = -\alpha B^T b^2 \cos^2(\omega_* t) = -\frac{\alpha}{2} B^T b^2 \big[1 + \cos(2\omega_* t)\big].$$

Therefore,

$$w_2(t) = -\frac{\alpha}{2}\left[R_0 B^T b^2 + R_2 B^T b^2 \cos(2\omega_* t)\right],$$

where

$$R_0 := K(\mathfrak{d}_*)^{-1}, \qquad R_2 := \big[K(\mathfrak{d}_*) - 4\omega_*^2 M\big]^{-1},$$

the first inverse being taken after removal of the rigid translational mode.

Since

$$\frac{1}{T}\int_0^T \cos^2(\omega_* t)\, dt = \frac{1}{2}, \qquad \frac{1}{T}\int_0^T \cos^4(\omega_* t)\, dt = \frac{3}{8},$$

and

$$\frac{1}{T}\int_0^T \cos^2(\omega_* t)\cos(2\omega_* t)\, dt = \frac{1}{4},$$

the parameter-dependent linear term is

$$\Gamma\gamma\mathsf{a}, \qquad \Gamma = \frac{1}{2} v_*^T K'(\mathfrak{d}_*) v_* = \frac{1}{2} k'(\mathfrak{d}_*)|r_q v_*|^2 \neq 0.$$

The direct cubic term gives

$$\frac{3\beta}{8}\sum_i b_i^4 \mathsf{a}^3,$$

while insertion of w_2 into the quadratic interaction gives

$$-\alpha^2\left[\frac{1}{2}\langle b^2, BR_0B^Tb^2\rangle + \frac{1}{4}\langle b^2, BR_2B^Tb^2\rangle\right]\mathsf{a}^3.$$

Thus, the reduced equation has the form

$$\Gamma\gamma\mathsf{a} + \mathcal{N}_3\mathsf{a}^3 + o(|\gamma\mathsf{a}| + |\mathsf{a}|^3) = 0, \tag{4}$$

with

$$\mathcal{N}_3 = \frac{3\beta}{8}\sum_i b_i^4 - \alpha^2\left[\frac{1}{2}\langle b^2, BR_0B^Tb^2\rangle + \frac{1}{4}\langle b^2, BR_2B^Tb^2\rangle\right]. \tag{5}$$

Theorem 3. *Under the preceding assumptions, let $\mathcal{N}_3 \neq 0$. Then, the equilibrium at $\mathfrak{d} = \mathfrak{d}_*$ is a local bifurcation point of non-trivial T-periodic motions. More precisely,*

$$\mathsf{a}^2 = -\frac{\Gamma}{\mathcal{N}_3}(\mathfrak{d} - \mathfrak{d}_*) + o(|\mathfrak{d} - \mathfrak{d}_*|).$$

Hence, the sign of $\Gamma/\mathcal{N}_3$ determines on which side of $\mathfrak{d}_*$ the non-trivial periodic branch exists.

Proof. For $\mathsf{a} \neq 0$, division of (4) by a gives

$$\Gamma\gamma + \mathcal{N}_3\mathsf{a}^2 + o(|\gamma| + \mathsf{a}^2) = 0.$$

Setting $s = \mathsf{a}^2$, the derivative with respect to γ at $(s, \gamma) = (0, 0)$ is $\Gamma \neq 0$. The implicit function theorem therefore yields

$$\gamma = -\frac{\mathcal{N}_3}{\Gamma}s + o(s),$$

which is equivalent to the stated expansion. □

5.2. Explicit nonlinear coefficient and role of the internal stiffness

We now specialize the coefficient $\mathcal{N}_3$ in (5) to a spatial Fourier mode and reduce the resolvent contributions to explicit scalar expressions. We restrict attention to a generic admissible wave number q for which the Fourier modes 0, q and $2q$ are distinct and no aliasing occurs in the quadratic and quartic sums below. Let $\widehat{v}_* \in \mathbb{R}^2$ be the mass-normalized eigenvector associated with the critical branch,

$$K_q(\mathfrak{d}_*)\widehat{v}_* = \omega_*^2 M\widehat{v}_*, \qquad \widehat{v}_*^T M\widehat{v}_* = 1,$$

and choose the corresponding real mode so that its bond deformation is

$$b_i = (Bv_*)_i = \sqrt{\frac{2}{N}}\,\rho_q \cos\left(q\left(i + \frac{1}{2}\right)\right), \qquad \rho_q := r_q\widehat{v}_*.$$

Then,

$$\sum_{i=1}^{N} b_i^2 = \rho_q^2, \qquad \sum_{i=1}^{N} b_i^4 = \frac{3\rho_q^4}{2N},$$

and

$$b_i^2 = \frac{\rho_q^2}{N}\left[1 + \cos\left(2q\left(i + \frac{1}{2}\right)\right)\right].$$

Introducing the normalized bond modes

$$(e_0)_i = \frac{1}{\sqrt{N}}, \qquad (e_{2q})_i = \sqrt{\frac{2}{N}}\cos\left(2q\left(i + \frac{1}{2}\right)\right),$$

we obtain

$$b^2 = \frac{\rho_q^2}{\sqrt{N}}e_0 + \frac{\rho_q^2}{\sqrt{2N}}e_{2q}. \tag{6}$$

Set $k_* := k(\mathfrak{d}_*)$. For a spatial wave number p and temporal frequency ν, define

$$r_p := \left(2\sin\frac{p}{2},\, 2c\cos\frac{p}{2}\right), \qquad K_p := k_* r_p^T r_p + \kappa e_2 e_2^T, \qquad e_2 = (0, 1)^T,$$

and

$$S(p, \nu; k_*, \kappa) := r_p(K_p - \nu^2 M)^{-1} r_p^T. \tag{7}$$

Since the linearized lattice is diagonal in spatial Fourier modes, (6) gives

$$\langle b^2, BR_0B^Tb^2\rangle = \frac{\rho_q^4}{N}\left[S(0,0;k_*,\kappa) + \frac{1}{2}S(2q,0;k_*,\kappa)\right],$$
$$\langle b^2, BR_2B^Tb^2\rangle = \frac{\rho_q^4}{N}\left[S(0,2\omega_*;k_*,\kappa) + \frac{1}{2}S(2q,2\omega_*;k_*,\kappa)\right].$$

Substitution into (5) yields

$$\mathcal{N}_3 = \frac{\rho_q^4}{N}\left[\frac{9}{16}\beta - \alpha^2 Q(k_*,\kappa)\right], \tag{8}$$

where

$$Q(k_*,\kappa) := \frac{1}{2}S(0,0;k_*,\kappa) + \frac{1}{4}S(0,2\omega_*;k_*,\kappa) + \frac{1}{4}S(2q,0;k_*,\kappa) + \frac{1}{8}S(2q,2\omega_*;k_*,\kappa). \tag{9}$$

Thus, the direct FPUT-β contribution and the quadratic feedback generated by the FPUT-α interaction are separated explicitly.

Writing

$$s_p := \sin\frac{p}{2}, \qquad c_p := \cos\frac{p}{2},$$

direct inversion of the 2×2 matrix gives

$$S(p,\nu;k_*,\kappa) = \frac{4\big[s_p^2(\kappa - J\nu^2) - c^2 m\nu^2 c_p^2\big]}{(4k_* s_p^2 - m\nu^2)(\kappa - J\nu^2) - 4k_* c^2 m\nu^2 c_p^2}. \tag{10}$$

The four contributions entering (9) are therefore

$$S_{00} = \frac{4c^2}{\kappa + 4c^2 k_*}, \tag{11}$$
$$S_{02} = \frac{4c^2}{\kappa + 4c^2 k_* - 4J\omega_*^2}, \tag{12}$$
$$S_{20} = \frac{1}{k_*}, \tag{13}$$
$$S_{22} = \frac{\sin^2 q\,(\kappa - 4J\omega_*^2) - 4c^2 m\omega_*^2\cos^2 q}{(k_*\sin^2 q - m\omega_*^2)(\kappa - 4J\omega_*^2) - 4k_* c^2 m\omega_*^2\cos^2 q}. \tag{14}$$

Consequently,

$$Q(k_*,\kappa) = \frac{2c^2}{\kappa + 4c^2k_*} + \frac{c^2}{\kappa + 4c^2k_* - 4J\omega_*^2} + \frac{1}{4k_*} + \frac{1}{8}S_{22}. \tag{15}$$

The preceding expressions are evaluated at a resonant state. Hence, k_* and κ are not independent if the critical frequency ω_* and wave number q are prescribed. Indeed, the T-resonance condition

$$\det\left(K_q - \omega_*^2 M\right) = 0$$

is equivalent to

$$(4k_* s_q^2 - m\omega_*^2)(\kappa - J\omega_*^2) - 4k_* c^2 m\omega_*^2 c_q^2 = 0. \tag{16}$$

Whenever

$$s_q^2(\kappa - J\omega_*^2) - c^2\, m\omega_*^2 c_q^2 \neq 0,$$

this relation can be solved explicitly as

$$k_*(\kappa) = \frac{m\omega_*^2(\kappa - J\omega_*^2)}{4\big[s_q^2(\kappa - J\omega_*^2) - c^2 m\omega_*^2 c_q^2\big]}. \tag{17}$$

Thus, variations of the internal stiffness along the resonant family are accompanied by the corresponding variations of the damaged bond stiffness and, through $k_* = k(\mathfrak{d}_*)$, of the resonant damage value.

For a fixed resonant state (k_*, κ) with $\rho_q \neq 0$, equation (8) gives

$$\operatorname{sgn} \mathcal{N}_3 = \operatorname{sgn} \left[\frac{9}{16}\beta - \alpha^2 Q(k_*, \kappa) \right].$$

Defining

$$\beta_c(k_*, \kappa) := \frac{16}{9}\alpha^2 Q(k_*, \kappa),$$

we have

$$\mathcal{N}_3 < 0 \iff \beta < \beta_c, \qquad \mathcal{N}_3 > 0 \iff \beta > \beta_c.$$

Hence, $\beta \neq \beta_c$ guarantees the non-degeneracy condition $\mathcal{N}_3 \neq 0$ required in Theorem 3. For monotone degradation, $k'(\mathfrak{d}_*) < 0$, Proposition 1 gives $\Gamma < 0$, so the two signs of $\mathcal{N}_3$ correspond to periodic branches emerging on opposite sides of the resonant damage level.

The internal degree of freedom therefore enters the transition at two distinct levels. Through the dispersion relation it contributes to the location of the resonant state, while through the resolvents in (9) it contributes to the nonlinear modal feedback and hence to the side on which the periodic branch emerges.

5.3. Poincaré map and nonzero forcing

The bifurcation described above admits an intrinsic formulation through the Poincaré map introduced in Sect. 3.1. For fixed $(\lambda, \mathfrak{d})$, let

$$P_{\lambda,\mathfrak{d}}(z) := \varphi^T_{\lambda,\mathfrak{d}}(z) - z.$$

A T-periodic motion corresponds to a zero of $P_{\lambda,\mathfrak{d}}$. If $z_{\lambda,\mathfrak{d}}$ is such a zero, the linearized fixed-point problem is governed by

$$D_z P_{\lambda,\mathfrak{d}}(z_{\lambda,\mathfrak{d}}) = M_{\lambda,\mathfrak{d}} - I, \qquad M_{\lambda,\mathfrak{d}} := D_z \varphi^T_{\lambda,\mathfrak{d}}(z_{\lambda,\mathfrak{d}}),$$

where $M_{\lambda,\mathfrak{d}}$ is the monodromy operator. Hence, the loss of local invertibility occurs precisely when

$$1 \in \sigma(M_{\lambda,\mathfrak{d}}).$$

It is therefore natural to introduce the critical set

$$\mathscr{C}_T := \left\{ (\lambda, \mathfrak{d}, z) : P_{\lambda,\mathfrak{d}}(z) = 0, \quad 1 \in \sigma\big(D_z \varphi^T_{\lambda,\mathfrak{d}}(z)\big) \right\}.$$

The resonant damage set $\mathcal{R}_T$ introduced above is recovered from the $\lambda = 0$ section of $\mathscr{C}_T$ by restricting to the equilibrium branch $z = (\xi_{\mathfrak{d}}, 0)$.

For the autonomous problem, restrict attention to the two-dimensional phase-space block associated with the critical mode. Its monodromy matrix is

$$M_{0,\mathfrak{d}} = \begin{pmatrix} \cos(\Omega T) & \sin(\Omega T)/\Omega \\ -\Omega \sin(\Omega T) & \cos(\Omega T) \end{pmatrix}, \qquad \Omega = \Omega_j(q, \mathfrak{d}).$$

Consequently,

$$\Omega_j(q, \mathfrak{d}_*) = \frac{2\pi n_*}{T} \iff M_{0,\mathfrak{d}_*} = I \iff 1 \in \sigma(M_{0,\mathfrak{d}_*}).$$

Thus, the spectral definition of the resonant damage set and the fixed-point formulation give the same critical condition along the equilibrium branch.

The corresponding Floquet multipliers are

$$\gamma_\pm(\mathfrak{d}) = e^{\pm i\Omega_j(q,\mathfrak{d})T}.$$

By Proposition 1,

$$\Omega_j'(\mathfrak{d}_*) = \frac{k'(\mathfrak{d}_*)|r_q v_*|^2}{2\omega_*},$$

and therefore

$$\gamma'_{\pm}(\mathfrak{d}_*) = \pm i \, \frac{Tk'(\mathfrak{d}_*)|r_q v_*|^2}{2\omega_*} \neq 0. \tag{18}$$

Equivalently,

$$\left.\frac{dM_{0,\mathfrak{d}}}{d\mathfrak{d}}\right|_{\mathfrak{d}_*} = \frac{Tk'(\mathfrak{d}_*)|r_q v_*|^2}{2\omega_*} \begin{pmatrix} 0 & 1/\omega_* \\ -\omega_* & 0 \end{pmatrix}.$$

Thus, the mechanical transversality condition of Proposition 1,

$$k'(\mathfrak{d}_*)|r_q v_*|^2 \neq 0,$$

is precisely the condition that the critical Floquet multipliers reach 1 transversally:

$$\gamma_{\pm}(\mathfrak{d}_*) = 1, \qquad \gamma'_{\pm}(\mathfrak{d}_*) \neq 0.$$

Before phase reduction, the autonomous Hamiltonian problem has two critical real directions, corresponding to the sine and cosine phases of the resonant mode. This is consistent with the simultaneous arrival of the conjugate Floquet multipliers at 1. Restriction to even T-periodic motions removes the phase degeneracy and reduces the critical kernel to one dimension. The scalar bifurcation equation (4) is therefore the phase-fixed local form of the singular Poincaré fixed-point problem.

Consider now $\lambda \neq 0$. Let ψ_* span the cokernel of the phase-fixed linearized fixed-point problem, with a fixed normalization, and define

$$F_* := -\left\langle \psi_*, \, \partial_\lambda P_{\lambda,\mathfrak{d}_*}(0)|_{\lambda=0} \right\rangle.$$

Here the origin denotes the equilibrium $z = 0$ at $(\lambda, \mathfrak{d}) = (0, \mathfrak{d}_*)$. If $\mathcal{U}_*(t, s)$ denotes the evolution operator of the autonomous linearization along this equilibrium, differentiation of the flow with respect to λ gives

$$\partial_\lambda P_{\lambda,\mathfrak{d}_*}(0)|_{\lambda=0} = \int_0^T \mathcal{U}_*(T, s) J \nabla_z \Phi(0, \omega s)\, ds,$$

and consequently

$$F_* = -\int_0^T \langle \psi_*, \mathcal{U}_*(T, s) J \nabla_z \Phi(0, \omega s) \rangle \, ds.$$

Thus, $F_* \neq 0$ means that the periodic perturbation has a non-vanishing component in the critical reduced equation.

Lyapunov–Schmidt reduction of

$$P_{\lambda,\mathfrak{d}}(z) = 0$$

around $(\lambda, \mathfrak{d}, z) = (0, \mathfrak{d}_*, 0)$ gives the smooth scalar equation

$$g(\mathsf{a}, \gamma, \lambda) = \Gamma \gamma \mathsf{a} + \mathcal{N}_3 \mathsf{a}^3 - \lambda F_* + R(\mathsf{a}, \gamma, \lambda) = 0, \qquad \gamma := \mathfrak{d} - \mathfrak{d}_*, \tag{19}$$

where

$$R = O\big(\gamma^2 \mathsf{a} + |\gamma||\mathsf{a}|^3 + |\mathsf{a}|^5 + |\lambda|(|\gamma| + \mathsf{a}^2) + \lambda^2\big).$$

For $\lambda = 0$, the reduced equation retains the factor a and therefore the trivial branch $\mathsf{a} = 0$. If $F_* \neq 0$, the term $-\lambda F_*$ removes this factor for $\lambda \neq 0$. In this precise sense, (19) is an imperfect unfolding of the autonomous bifurcation. By smoothness of the reduced map, the corresponding derivative estimates for R hold as well.

For fixed $\lambda \neq 0$, we call $(\mathsf{a}_\mathrm{f}, \gamma_\mathrm{f})$ a non-degenerate fold of the reduced problem if

$$g(\mathsf{a}_\mathrm{f}, \gamma_\mathrm{f}, \lambda) = 0, \qquad \partial_\mathsf{a} g(\mathsf{a}_\mathrm{f}, \gamma_\mathrm{f}, \lambda) = 0, \tag{20}$$

and

$$\partial_\gamma g(\mathsf{a}_\mathrm{f}, \gamma_\mathrm{f}, \lambda) \neq 0, \qquad \partial_{\mathsf{aa}} g(\mathsf{a}_\mathrm{f}, \gamma_\mathrm{f}, \lambda) \neq 0.$$

The first two conditions identify a singular point of the projection of the local solution curve onto the damage parameter, while the last two are the usual non-degeneracy conditions for a fold.

To determine this singular set for small nonzero λ, introduce

$$\mathsf{a} = |\lambda|^{1/3}\eta, \qquad \gamma = |\lambda|^{2/3}\delta.$$

After division by $|\lambda|$, (19) becomes

$$\Gamma\delta\eta + \mathcal{N}_3\eta^3 - \operatorname{sgn}(\lambda)F_* + o(1) = 0$$

in C^1 on bounded (η, δ)-sets. The fold condition $\partial_{\mathsf{a}} g = 0$, after the corresponding rescaling, gives in the limit

$$\Gamma\delta + 3\mathcal{N}_3\eta^2 = 0.$$

Hence, the limiting fold system is

$$\Gamma\delta\eta + \mathcal{N}_3\eta^3 - \operatorname{sgn}(\lambda)F_* = 0, \qquad \Gamma\delta + 3\mathcal{N}_3\eta^2 = 0.$$

Its unique solution with $\eta \neq 0$ satisfies

$$\eta_{\mathrm{f}}^3 = -\frac{\operatorname{sgn}(\lambda)F_*}{2\mathcal{N}_3}, \qquad \delta_{\mathrm{f}} = -\frac{3\mathcal{N}_3}{\Gamma}\eta_{\mathrm{f}}^2.$$

The Jacobian of the limiting system with respect to (η, δ) is

$$\begin{pmatrix} 0 & \Gamma\eta_{\mathrm{f}} \\ 6\mathcal{N}_3\eta_{\mathrm{f}} & \Gamma \end{pmatrix},$$

whose determinant is

$$-6\Gamma\mathcal{N}_3\eta_{\mathrm{f}}^2 \neq 0.$$

The implicit function theorem therefore gives, for each fixed sign of λ, a unique nearby solution of the fold equations for sufficiently small $|\lambda|$. Moreover,

$$\partial_\gamma g = \Gamma\mathsf{a} + o(|\mathsf{a}|), \qquad \partial_{\mathsf{aa}} g = 6\mathcal{N}_3\mathsf{a} + o(|\mathsf{a}|)$$

along this solution, so both quantities are nonzero for sufficiently small $\lambda \neq 0$. The resulting singularity is therefore a non-degenerate fold in the sense of (20).

Theorem 4. *Assume the hypotheses of Theorem 3 and let $F_* \neq 0$. For each fixed sign of λ and sufficiently small $\lambda \neq 0$, the Poincaré fixed-point problem possesses a unique non-degenerate fold in the scaling neighborhood corresponding to $(\eta_{\mathrm{f}}, \delta_{\mathrm{f}})$. Its amplitude and damage level satisfy*

$$\mathsf{a}_{\mathrm{f}}(\lambda) = \left(-\frac{\lambda F_*}{2\mathcal{N}_3}\right)^{1/3} + o(|\lambda|^{1/3}),$$

and

$$\mathfrak{d}_{\mathrm{f}}(\lambda) - \mathfrak{d}_* = -\frac{3\mathcal{N}_3}{\Gamma}\left(-\frac{\lambda F_*}{2\mathcal{N}_3}\right)^{2/3} + o(|\lambda|^{2/3}).$$

In particular,

$$|\mathfrak{d}_{\mathrm{f}}(\lambda) - \mathfrak{d}_*| = O(|\lambda|^{2/3}).$$

Proof. After the above rescaling, the fold equation and its amplitude derivative converge in C^1 to the limiting system, whose solution $(\eta_{\mathrm{f}}, \delta_{\mathrm{f}})$ is non-degenerate. For each fixed sign of λ, the non-vanishing Jacobian allows the implicit function theorem to be applied for sufficiently small $|\lambda|$, yielding a unique solution

$$\eta(\lambda) = \eta_{\mathrm{f}} + o(1), \qquad \delta(\lambda) = \delta_{\mathrm{f}} + o(1)$$

in a neighborhood of $(\eta_{\mathrm{f}}, \delta_{\mathrm{f}})$. The derivative estimates above verify the fold non-degeneracy conditions. Returning to (a, γ) gives the stated asymptotic expansions. □

The Poincaré formulation therefore completes the relation between the autonomous and forced problems. At $\lambda = 0$, damage moves a pair of Floquet multipliers transversally to 1 and, after phase reduction, produces the bifurcation described by Theorem 3. When the forcing has a non-vanishing critical projection, the autonomous reduced equation undergoes the imperfect unfolding described above and develops a non-degenerate fold. Its damage level is displaced from the autonomous resonant level by an amount of order $|\lambda|^{2/3}$.

6. Conclusions

We have considered complex lattices whose elementary units are sublattices, which we homogenize, based on the Cauchy–Born rule, obtaining a structure that describes both translational and coarse microstructural dynamics. Within the resulting Hamiltonian setting, periodic forcing and structural damage play distinct roles: the former acts on a fixed Hamiltonian structure, whereas the latter modifies the interaction potentials and hence the spectrum of the linearized system.

For fixed damage levels satisfying a non-T-resonance condition, degree-theoretical arguments provide T-periodic responses under sufficiently small periodic forcing. Damage may, however, drive a characteristic frequency through a resonant value.

For an FPUT-type specialization with an additional internal expansion degree of freedom, we have shown that monotone bond degradation produces a transversal spectral crossing whenever the critical mode effectively deforms the damaged bonds. A local reduction then yields a branch of T-periodic motions and identifies the nonlinear coefficient governing its character. The internal degree of freedom affects both the resonant damage threshold and the nonlinear modal feedback.

The same transition is recovered through the Poincaré map: At zero forcing, the critical Floquet multipliers reach 1 transversally, while nonzero forcing unfolds the autonomous bifurcation into a local fold of T-periodic solutions. Thus, structural damage can induce a genuine dynamical transition by modifying the spectral and nonlinear structure of the partially homogenized lattice.

Acknowledgements

This work belongs to activities of the research group “Theoretical Mechanics” in the “Centro di Ricerca Matematica Ennio De Giorgi” of the Scuola Normale Superiore in Pisa. We acknowledge the support of GNFM-INDAM and GNAMPA-INDAM.

Author contributions The authors share the responsibility on all aspects concerning this paper.

Funding Open access funding provided by Università degli Studi di Firenze within the CRUI-CARE Agreement. Not applicable.

Data Availability Statement No datasets were generated or analyzed during the current study.

Declarations

Conflict of interest The authors have no interests that are directly or indirectly related to the submitted work.

References

[1] Ambrosetti, A., Prodi, G.: A Primer in Nonlinear Analysis. Cambridge University Press, Cambridge (1993)
[2] Bacci, M., Mariano, P.M.: Discrete-to-continuum description of protein dynamics: models and comparative simulations. Mech. Mat. **154**, 103711 (2021)
[3] Bisconti, L., Spadini, M.: About the notion of non-T-resonance and applications to topological multiplicity results for ODEs on differentiable manifolds. Math. Meth. Appl. Sci. **38**, 4760–4773 (2015)
[4] Born, M., Huang, K.: Dynamical theory of crystal lattices. Oxford University Press, Oxford (1954)
[5] Cronin, J.: Differential Equations, Introduction and Qualitative Theory. Marcel Dekker Inc., New York (1994)
[6] Dinca, G., Mawhin, J.: Brouwer Degree - The core of nonlinear analysis. Birkhäuser, Cham (2021)
[7] Ermentrout, G.B., Terman, D.H.: Mathematical Foundations of Neuroscience. Springer, Berlin (2010)
[8] Ericksen, J.L.: On the Cauchy-Born rule. Math. Mech. Solids **13**, 199–220 (2008)
[9] Furi, M., Spadini, M.: Branches of forced oscillations for periodically perturbed second order autonomous ODEs on manifolds. J. Diff. Eq. **154**, 96–106 (1999)
[10] Furi, M., Pera, M.P., Spadini, M.: Forced oscillations on manifolds and multiplicity results for periodically perturbed autonomous systems. J. Comp. Appl. Math. **113**, 241–254 (2010)
[11] Galano, L., Mariano, P.M., Spadini, M.: Finite length Fermi-Pasta-Ulam-Tsingou-type chains: periodic forced dynamics and interaction inhomogeneity. Comm. Nonlinear Sci. Num. Sim. **151**, 109051 (2025)
[12] Guckenheimer, J., Holmes, P.: Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields. Springer, Berlin (1983)
[13] Krajcinovic, D.: Damage Mechanics. North-Holland, Amsterdam (1996)
[14] James, R.D.: Objective structures. J. Mech. Phys. Solids **54**, 2354–2390 (2006)
[15] Mariano, P.M.: Topological features in continuum mechanics: from simple to complex bodies. Proc. Royal Soc. A **480**, 20230919 (2024)
[16] Phillips, R.: Crystals, Defects and Microstructures Modeling Across Scales. Cambridge University Press, Cambridge (2001)
[17] Weinan, E.: Principles of Multiscale Modeling. Cambridge University Press, Cambridge (2011)
[18] Weinan, E., Ming, P.: Cauchy-Born rule and the stability of crystalline solids: static problems. Arch. Rational Mech. Anal. **183**, 241–297 (2007)

Paolo Maria Mariano
DICEA
University of Florence
via Santa Marta 3
501295 Firenze
Italy
e-mail: paolomaria.mariano@unifi.it

Marco Spadini
DIMAI
Università di Firenze
via Santa Marta 3
50139 Firenze
Italy
e-mail: marco.spadini@unifi.it

(Received: June 15, 2026; revised: August 26, 2026; accepted: August 28, 2026)