Representing mathematical induction in proving processes

Bernardo Nannini

University of Florence, Department of Mathematics and Computer Science, Italy;
bernardo.nannini@unifi.it

Using a multimodal semiotic perspective, I investigate the production and use of signs in expert (doctoral) students involved in processes of constructing argumentations and proof by mathematical induction. Focusing on their speech, written inscriptions, and gestures, different categories of signs, related to mathematical induction, are identified and analysed. In this paper I show some paradigmatic examples of these signs.

Keywords: Mathematical induction, proving processes, semiotic bundle, gestures.

Introduction

Semiotic offers an interesting perspective for research in mathematics education, providing a window through which to observe and investigate several teaching-learning processes. Recently the analysis of signs has been enriched including the study of gestures. This has involved different areas of research, as well as the studies on argumentation and proof. Arzarello and Sabena, for instance, bring empirical evidence that "gestures may also play specific roles in providing a logical structure to argumentation" (2014, p. 99). Similarly, Krause observes that the gestures' production "may support the collective act of reasoning […]. It makes traceable how the argument was organized as logical inference" (2015, p.1432). Along the same line, Sabena, registers that the use of gestures "support[s] the students in structuring the entire argumentation at a global level" (2018, p. 556). The study presented in this paper is part of wider research, conducted with a semiotic perspective, on proving by mathematical induction processes of post-graduate, undergraduate and secondary school students.

The proving scheme of Mathematical Induction (MI) is, at the same time, important and useful for a mathematician, interesting from a logical point of view, but also extremely problematic from a didactic perspective. The difficulties involved can be observed across different levels of education, from secondary school students (Fischbein & Engels, 1989), to master's students in mathematics (Carotenuto et al., 2018). A problematic aspect for students is related to its justification, i.e. why, given a predicate P on the natural numbers, we can conclude that $\forall n\in\mathbb{N}$, P(n), by proving the base case P(0) and the inductive step $\forall n\in\mathbb{N}$, P(n)→P(n+1). Ernest (1984) affirms:

> Many students encountering the method of proof by induction wonder why this rather complex and seemingly arbitrary principle is adopted […] [It] is neither self evident nor a generalisation of previous more elementary experience. (p. 183).

Generally, a non-formal justification is that MI works as a "cascade" of infinite syllogisms (Poincaré, 1906, p. 9): from P(0) and P(0)→P(1) it follows P(1); from P(1) and P(1)→P(2) it follows P(2), and so on. Often, to provide an intuitive explanation for MI, teachers and textbooks describe it by using some images (for some examples, see Ernest, 1984). Perhaps one of the most known is the falling dominos analogy: given an infinite line of dominos (i.e. the whole set $\mathbb{N}$), by the fact that the first one is knocked over (i.e. the base case) and the fact that each couple of consecutive dominos is at the right distance so that if one domino falls, it will knock over the consecutive one (i.e. the inductive step), we can conclude that every domino after the first one in the line will fall (i.e. $\forall n \in \mathbb{N}$, P(n)).

Images (or, more generally, signs) like the cascade of syllogisms or the falling dominos, are often used by teachers and textbooks to introduce and describe MI to students. On the other side, similar signs might also be produced and used by subjects involved in proving by induction activities. In this paper, I focus on this second aspect, analysing the signs produced and used by students with experience of proving by induction during the resolution of problems which potentially involve MI.

Theoretical framework

A multimodal semiotic perspective

In order to take into consideration a wide spectrum of signs, in this study I adopt a multimodal semiotic perspective. Arzarello (2006) considers all the different kinds of sings involved in teaching and learning processes (verbal language, mathematical symbols, diagrams, sketches, gazes, gestures, etc.) as an inseparable unit. He defines a *semiotic bundle* as a dynamic structure composed by different *semiotic sets* together with the relationships between them. Within this perspective, a sign may have different components depending on the semiotic sets involved in it. For instance, a subject's spoken utterance with a simultaneous gesture, referring to a certain written inscription, can be seen as a unique sign made by three components (speech, gesture, and inscription).

Linking and iteration signs

Using this multimodal perspective, in a previous study (Antonini & Nannini, 2021), we analysed the semiotic bundle consisting of three semiotic sets (speech, written inscriptions and gestures) in some post-graduate students' processes involved in the generation of a conjecture and of a proof by MI. As a result, we identified two particular categories of signs which seem to play an important role in these processes:

- *Linking signs*: Signs produced or used to refer to two or more entities (objects, mathematical objects, problems, situations, etc.) and to their relationships, where these entities are seen in connection with two consecutive natural numbers.
- *Iteration signs:* Signs that refer to iteration, or that are composed by a repetition (in time or in space) of linking signs, or that refer to a repetition of them.

To give an example of linking and iterations signs we can interpret from a semiotic point of view the above-described image of the falling dominos. The whole image (either pictured or verbally described) is a rather complex sign composed by several other signs. Of those, one is the image of two dominos, one of each falling, potentially falling, or already fallen onto the other one. Out of the analogy this sign represents two propositions, P(n*) and P(n*+1), for which the truth of the first one implies the truth of the second one (i.e., an instance of the inductive step). This is an example of linking sign. Moreover, the whole line of dominos (fallen, falling, or standing), representing the infinite syllogisms obtained by base case and inductive step, is a repetition of linking signs and, globally, can be interpreted as an iteration sign.

In this paper I report on an explorative and qualitative study focusing on linking and iteration signs in post-graduate students' processes involved in problem solving activities. The study investigates the different characteristics that linking and iteration signs can have during the problem resolution process, aiming at a possible classification of them.

Methods

The study is based on interviews in which expert students were asked to solve some problems and then to speak about mathematical induction. Participants were 4 doctoral students in Mathematics. They were interviewed individually for approximately 80 minutes each. They were not aware of the focus of the study. Collected data consist of audio-video recordings and of the written inscriptions produced by the students. In this paper I will refer to the following two problems.

The chessboard problem: *"Consider a $2^n x 2^n$ chessboard. What is the maximum number of squares which can be tiled with L-shaped pieces composed of 3 squares each?"*. The solution, which can be proved by MI on n, is that it is possible to tile the entire $2^n x 2^n$ chessboard except for one square.

The false coin problem: "N identical coins are given. One of these, however, is false and it weighs less than the others. There is a traditional weighing scale at our disposition. What is, in function of N, the minimum number of weighings necessary to determine the false coin?". A partial solution for the problem is that, if $N=3^m$, then with m weightings it is possible to determine the false coin. Again, this can be proved by MI.

Preliminary findings

A classification for linking and iteration signs

With the analysis of the semiotic bundle produced and used by the students during the interviews, it was possible to identify three different categories of linking and iteration signs:

- Linking and iteration signs produced or used to refer to the (mathematical or not) objects described in the problem's text (functions, variables, chessboards, tiles, coins, etc.), to their properties, or to their mutual relationships. For these, I use the term *Ground-level* signs.
- Linking and iteration signs produced or used to refer to the proving scheme of mathematical induction itself, to its logical structure or to the justification of its validity. For these, I use the term *Meta-level* signs.
- Linking and iteration signs that could be seen simultaneously as ground and meta-level signs. This happens when a single component or different components of the same sign refer both to some objects of the problem and to mathematical induction itself. For this intermediate category, I use the term *Hybrid-level* signs.

In the following pages I will present some paradigmatic examples of these categories of signs. The students' names are pseudonyms. In the transcripts, with (*italics*) I describe gestures or inscriptions in the moment when they are made.

Ground-level signs

Lorenzo, in this part of the interview, is dealing with the false coin problem. After reading the text he claims to remember the solution of a similar problem in which nine weights are given, all identical except for one which is lighter. Lorenzo describes the solution of this second problem:

1 Lorenzo: In this case the game was: you split in three groups, then three coins, first group, three coins, second group, three coins, third group *(he writes '9' on the sheet, then he draws three arrows starting from it and pointing to the right, he then writes '3' at the end of each arrow and, finally, '1st', '2nd', and '3rd', Figure 1a)*. Then you say: we take two groups, we weigh them and since there are three possible results, that are one is lighter, the other is lighter, or even, I can determine in which group the lighter one is. Then I iterate the procedure on the others.

Later on, Lorenzo tries to generalise the just described solution for a group of n coins:

2 Lorenzo: So, the reasoning would be… n coins, I split in three groups (*he draws a point and three lines starting from it and pointing down on the sheet*), I select one of them with one weighing, I split in other three groups, (*he draws a second point at the end of a line and then three other lines starting from this new point*), I select one of them with another weighing (*again, he draws a third point at the end of a new line and then three other lines from this. At this point he has drawn a mathematical tree, Figure 1b*).

In the first part of the excerpt (line 1), when describing the solution of the problem for a group of nine coins, Lorenzo produces a sign (the inscription in Figure 1a, together with his speech) which represents how the group of 9 coins is linked to three groups of 3 coins each, namely a linking sign. With this sign, Lorenzo represents the first step of the solution of the problem (to divide the group in three subgroups and to determine which one contains the false coins). Afterwards (line 2), Lorenzo describes a possible strategy to solve the general problem, and to do this he draws an inscription which is composed by the repetition of the previous linking sign, this time without any indication to the number of coins (Figure 1b). This is an iteration sign and allows him to describe the iterative solution of the problem.

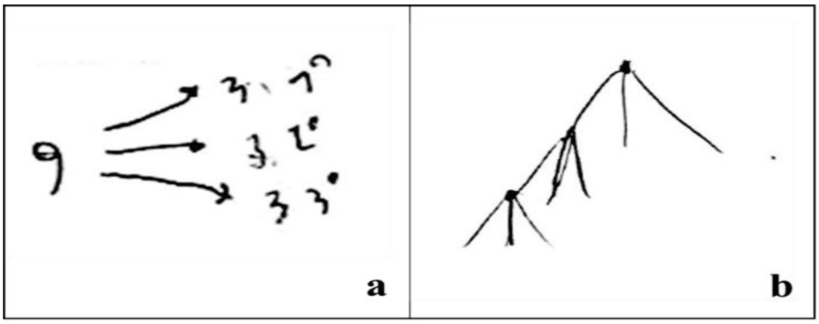

Figure 1: Lorenzo's inscriptions in line 1, Figure a, and in line 2, Figure b

In the remaining part of the interview related to this problem, Lorenzo explores the mathematical tree that he drew in order to find, in function of n, its height (which corresponds to the number of weightings necessary to determine the false coin). Both the just presented linking and iteration signs are ground-level signs because they refer to different groups of coins and the relationships between them.

Meta-level signs

Guido, at the end of the interview, says to be convinced of the validity of MI as a proving scheme. Then he justifies his answer.

1 Guido: You do the base case, which is true, and you verify it (*he puts the right hand in front of him at the level of the table, touching his leg with the four fingers, Figure 2a*).

2 Guido: Then (*he makes two consecutive arc-shaped gestures rotating the right hand in the air keeping thumb and pointing finger at a constant distance and moving the hand from left to right, Figures 2b/c*) the inductive step (*he repeats the previous gestures a second time*) guarantees that it is always true (*he moves rapidly the right hand starting from his leg to an up-right direction in the air, Figures 2d/e*).

Guido's speech alone does not seem to provide a justification for the validity of MI. He only says that after the case base is proved true (line 1), the inductive step assures the truth of the proposition for every natural number (line 2). However, if we look at his gestures, we notice that his discourse is enriched by other semantic elements. The base case is represented by a point on Guido's leg, which he touches with his right hand (Figure 2a). Before saying "the inductive step", Guido makes a rather complex gesture: he moves his right hand in the air from left to right and simultaneously he rotates it twice, keeping thumb and pointing finger at a constant distance, forming two arcs (Figures 2b/c). He then repeats the same gesture while saying "the inductive step".

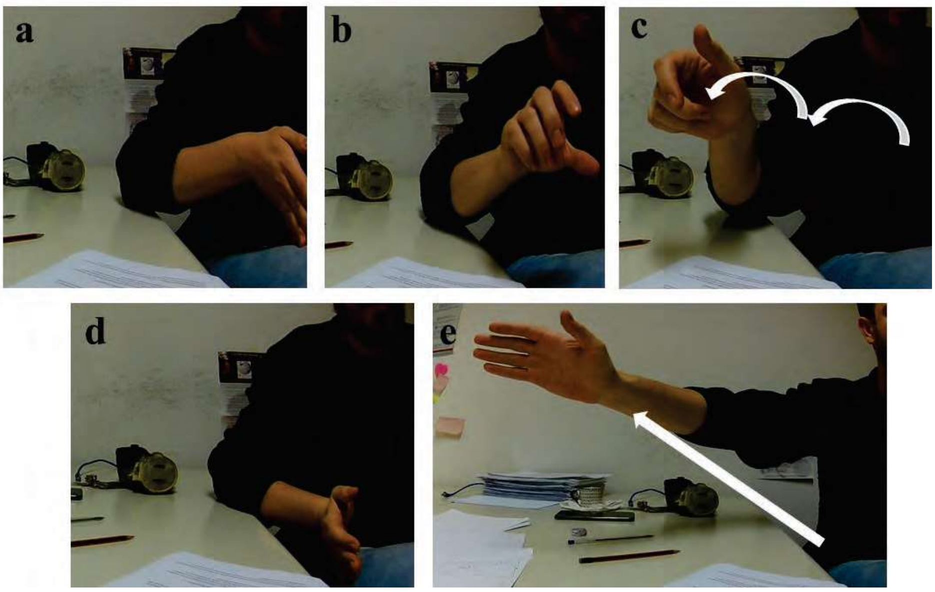

Figure 2: Guido's gestures. In Figure a, it is showed the "case base gesture", in Figures b/c the "inductive step gesture", and in Figures d/e the final gesture of line 2. The white arrows summarise the gestures' trajectories

These gestures are interesting because with them Guido seems to represent the inductive step itself as a series of arcs in the air. Each arc-gesture could indicate, metaphorically, the link between two cases of the proposition to be proved by induction (the implication $P(n) \rightarrow P(n+1)$). These are examples of meta-level linking signs. Finally, while saying "it is always true", Guido performs a new gesture which is a composition of the previous ones, but now more rapid and contracted: He starts touching his leg (as when he was referring to the base case), and then he moves fast to an up-right direction in the air. This time the hand moves in a straight line without shaping any arcs (Figures 2d/e). With this gesture he seems to describe the whole iteration which, starting from the base case and by successively applying the inductive step, allows to conclude the truth of P(n) for all the natural numbers greater than the base. This is an example of a meta-level iteration sign.

Hybrid-level signs

Silvio, in this part of the interview, is dealing with the chessboard problem. Until this moment he has explored the problem, finding that for the chessboards corresponding to n=0, n=1, and n=2, it is possible to create a tessellation which leaves out only one little square and he has conjectured that the same thing is possible for every $2^n x 2^n$ chessboard. Then he has observed that a $2^n x 2^n$ chessboard is composed by four $2^{n-1} x 2^{n-1}$ chessboards. This seems to suggest to him a possible solving strategy:

1	Silvio:	I think that one could do something like by induction (*keeping the pen in his right hand, he rapidly draws some circles in the air, Figures 3a/b)*.
2	Silvio:	Because, since in the case zero I have only one little square and it remains out (*with the pen he touches the sheet where he previously wrote* ‘n=0→0 tiles’, *Figure 3c*),
3	Silvio:	then, let’s say, in a sequential way this little square will always remain out (*with the pointing finger of his right hand, he draws some circles in the air whilst moving up his hand, Figures 3d/e*).

In the remaining part of the interview related to this problem, Silvio tries to construct a proof by MI for his conjecture.

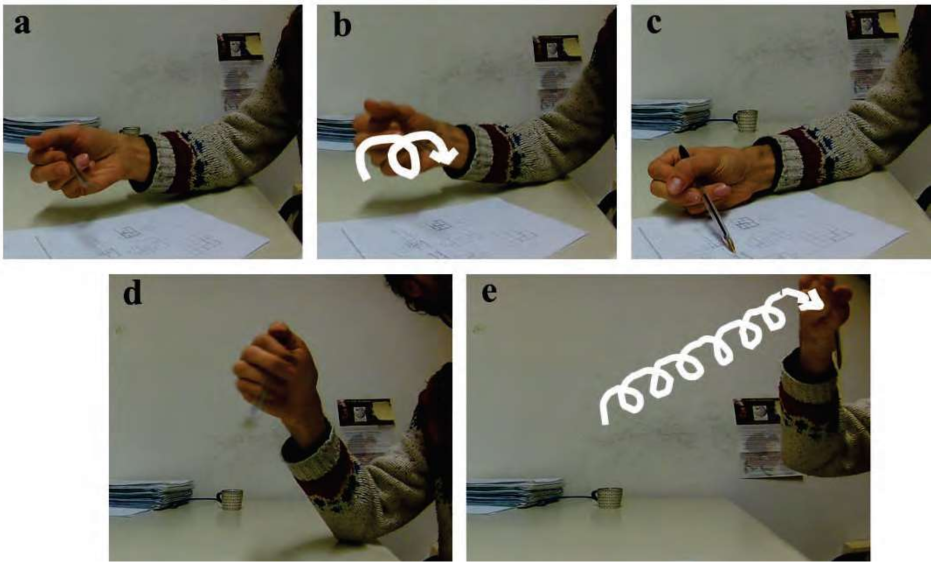

Figure 3: Silvio’s gestures. In Figures a/b it is showed the gesture of line 1, in Figure c the gesture of line 2, and in Figures d/e the gesture of line 3. The white arrows summarise the gestures’ trajectories

The analysis of this excerpt reveals the presence of a hybrid-level sign. Firstly, Silvio claims that a possible solution could be obtained doing “something like by induction” (line 1). Whilst saying this, he makes a gesture which seems to refer to the iterative structure of induction itself (Figures 3a/b). Considering the bundle (his utterance together with the gesture) we can interpret this as a meta-level iteration sign. In lines 2-3, Silvio clarifies what he meant with “something like induction”. He says that the chessboard corresponding to n=0 is composed only by one little square which thus cannot be tiled. Then he says that from this, "in a sequential way", it could be possible to show that the little square will remain out of the tessellation for all the bigger chessboards as well. In describing this strategy, Silvio is apparently using his previous observation of the fact that it is possible to construct a 2^nx2^n chessboard with four 2^{n-1}x2^{n-1} chessboards. Silvio's discourse contains elements which refer to some objects of the problem ("this little square will always remain out"), showing that with his speech he is referring to chessboards and tessellations. However, observing the whole semiotic bundle, we can see that he is also referring to the structure of the solving strategy itself. He firstly touches the sheet where the inscription for the 1x1 chessboard is, which he calls the "case zero" (Figure 3c). Then, while saying "in a sequential way", he draws several circles in the air whilst moving up the hand (Figures 3d/e). Note that now there is not any reference to chessboards or tessellations. This gesture, in fact, seems to repeat the "something like by induction" gesture previously made in line 1, but now it is longer (both in time and space). If we consider the whole movement of his right hand (Figures 3c/d/e), we can see it starting from the sheet, concretely touching with the pen the inscriptions which refer to the chessboards, and then moving up, drawing a sort of helix in the air, as representing the iterative structure of the reasoning by induction itself. This is an iteration signs which starts as a ground-level sign and then becomes a meta-level sign. This is therefore an example of a hybrid-level sign.

Concluding remarks

In the first two excerpts, I showed some examples of linking and iteration signs, both ground and meta-level. These two categories of signs apparently involve different aspects of the problem solving and proving processes. The ground-level signs seem to have an important role in the resolution of the problem, allowing the subject to recognise that it could be solved with an iterative procedure. The meta-level signs, instead, are related to the description of the logical structure of a generic proof by MI. We can in fact observe that the first two excerpts were taken from two very different moments of the interviews. Lorenzo's example, through which some ground-level signs were shown, is part of his initial exploration of the false coin problem. On the other side, Guido's excerpt, containing some examples of meta-level signs, refers to the final part of his interview in which, after having solved some problems (some of those by induction), he is explaining why he is convinced of the validity of MI. However, as the third excerpt has shown, it was also possible to observe some meta-level signs in other phases of the interviews, in particular during the exploration of a problem as well. When Silvio produces the meta-level iteration sign, in fact, he is still exploring the chessboard problem, and he has not found a way to tessellate a chessboard using the tessellation of the previous chessboard (a sort of inductive step). Nevertheless, he recognises a parallelism between his possible iterative solution to the problem and the proving scheme of MI. When this happens, he produces a hybrid-level sign: an iteration sign referring both to some problem's proper elements (ground-level), and to the structure of MI itself (meta-level). The production of these signs highlights a crucial moment in Silvio's problem solving process. In this moment, in fact, Silvio seems to see in his own argumentation the structure of a proof by MI (he says "something like by induction"). Subsequently, in fact, he decides to prove by MI his conjecture. In other terms, the presence of these signs seems to reveal a continuity between the process of generation of the conjecture and the subsequent construction of a proof for it. It could be interesting to further investigate this last aspect within the framework of the Cognitive Unity (Boero et al.,1996), focusing on the role of hybrid-level and meta-level signs in the transition between the argumentation supporting a conjecture and a subsequent proof by MI. Further research is also necessary to investigate the production and use of linking and iteration signs (either ground, meta, or hybrid-level) in less expert students.

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